Feynman Integrals For Heavy Quark Physics

Long-Bin Chen

2018-03-29

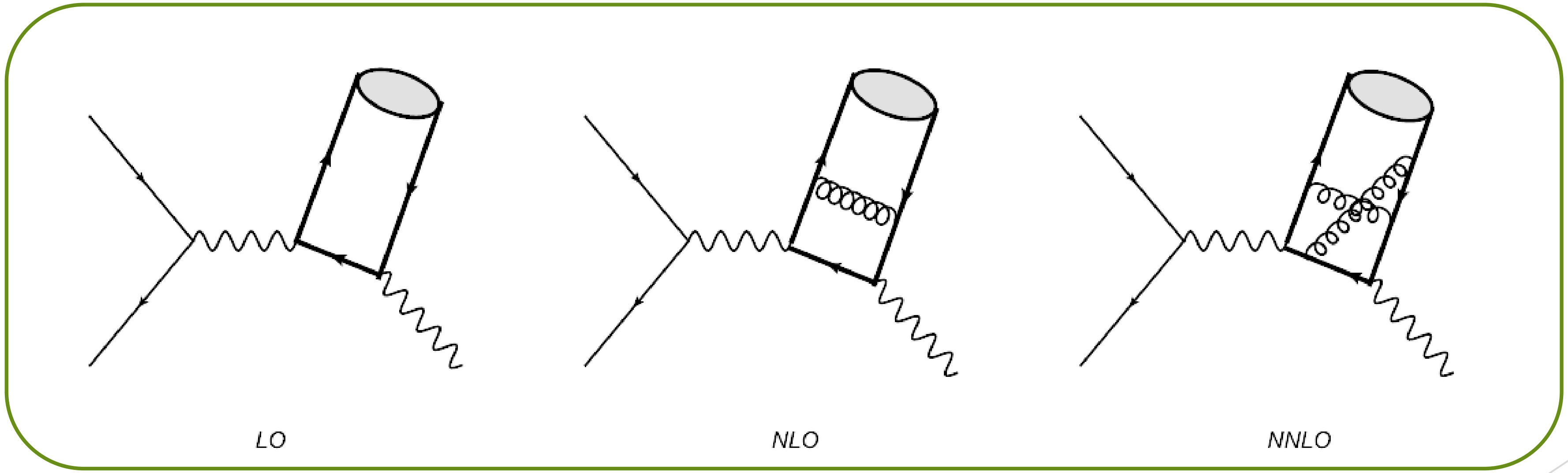
Based on:
JHEP 06,025(2017)
JHEP 02,066(2018)
arXiv: 1712.03516

Outline

- I. Two-loop master integrals for heavy quarkonium exclusive production
- II. Two-loop master integrals for heavy-to-light form factors of two different massive fermions

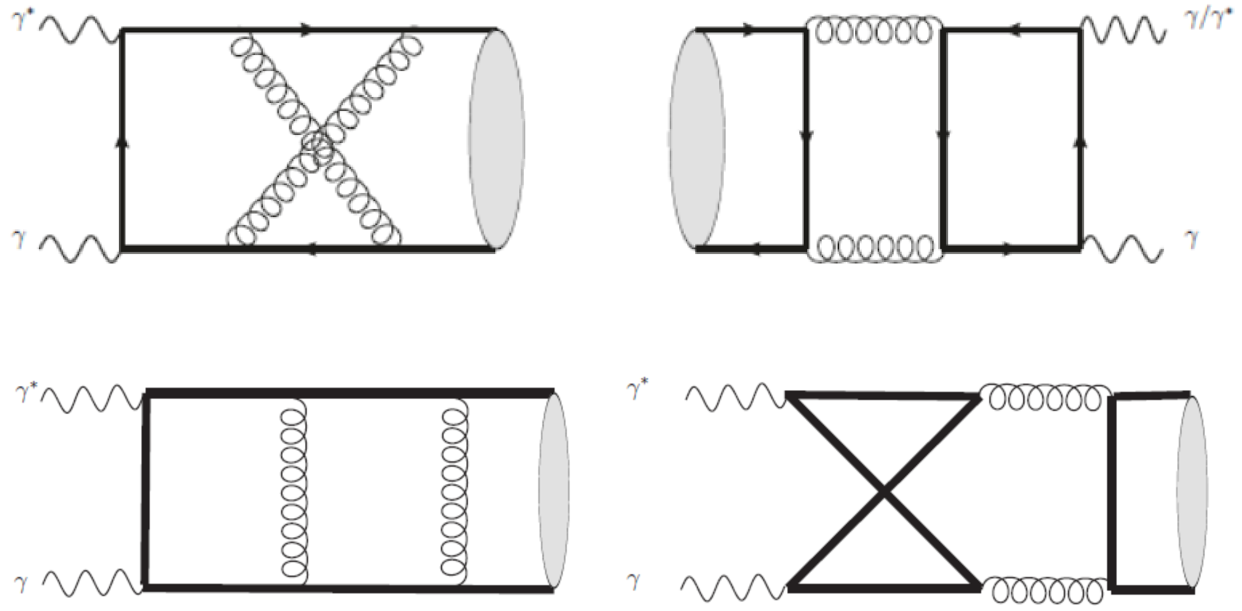
I. Two-loop master integrals for heavy quarkonium exclusive production

NNLO QCD corrections for photon+eta_c/eta_b exclusive production at electron-positron collider (NRQCD)

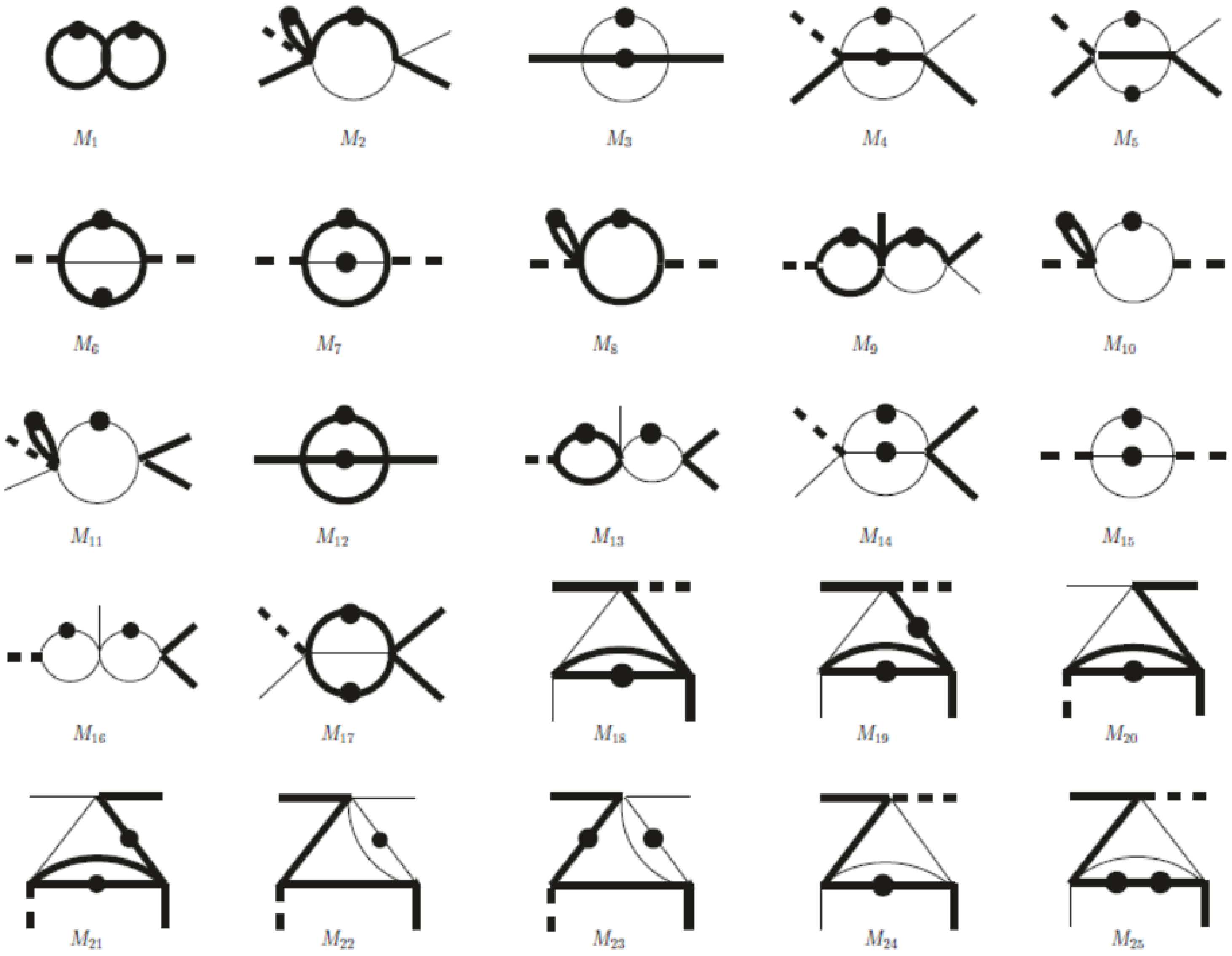


Sample of Feynman Diagrams

Typical NNLO Feynman diagrams

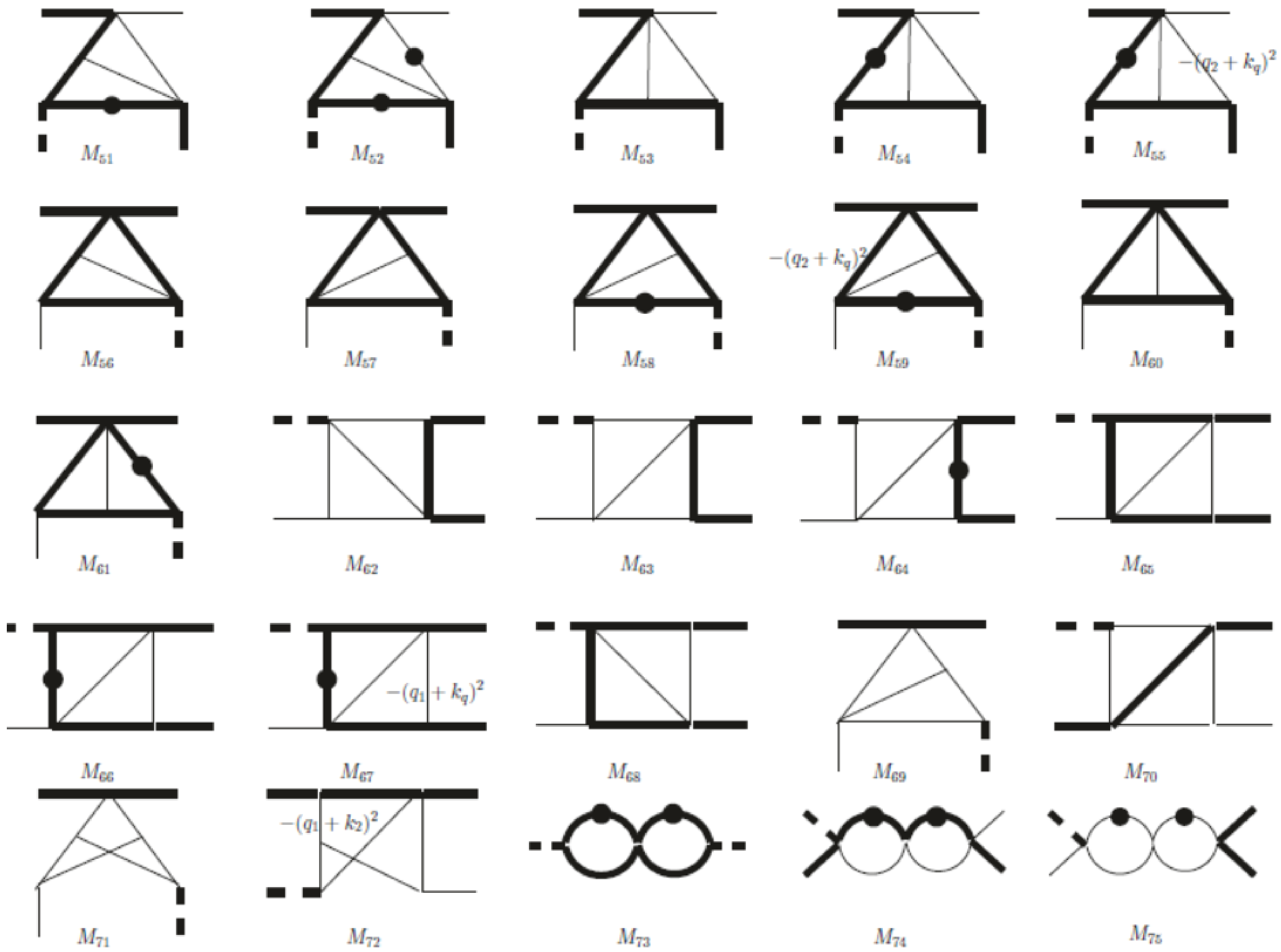


Two Scales: (s,m_q)
Amp can be reduced to 133 MI



IBP: FIRE

Master Integrals (Polylogarithms-86)



Master Integrals (Polylogarithms)

Method of differential equations(DE):

$$\partial_m f(\epsilon, x_n) = A_m(\epsilon, x_n) f(\epsilon, x_n)\,,$$

$$\partial_n A_m - \partial_m A_n + [A_n, A_m] = 0$$

$$\epsilon = (4 - d)/2$$

A. Kotikov, *Differential equations method: New technique for massive Feynman diagrams calculation*, *Phys.Lett.* **B254** (1991) 158–164.

A. Kotikov, *Differential equation method: The Calculation of N point Feynman diagrams*, *Phys.Lett.* **B267** (1991) 123–127.

Imrovement(1999)

Differential equations for two loop four point functions.
T. Gehrmann, E. Remiddi.
Nucl. Phys. B580 (2000) 485-518 .
arXiv: **hep-ph/9912329**

Improvement(2013)

Multiloop integrals in dimensional regularization made simple.
Johannes M. Henn.
Phys. Rev. Lett. 110 (2013) 251601.
arXiv:1304.1806

Canonical Basis

$$F_7 = \epsilon^2 \sqrt{ss}\sqrt{ss - 2m_q^2}(M_6 + 2M_7)/2\,,$$

$$F_8 = \epsilon^2 \sqrt{ss}\sqrt{ss - 2m_q^2}\, M_8\,,$$

$$F_9 = \epsilon^2 \,(ss - m_q^2)\sqrt{ss}\sqrt{ss - 2m_q^2}\, M_9\,,$$

$$F_{10} = \epsilon^2\, ss\, M_{10}\,,$$

$$F_{11} = \epsilon^2\, 2m_q^2\, M_{11}\,,$$

$$F_{12} = \epsilon^2\, m_q^2\, M_{12}\,,$$

$$F_{13} = \epsilon^2\, m_q^2 \sqrt{ss}\sqrt{ss - 2m_q^2}\, M_{13}\,,$$

$$F_{40} = \epsilon^2 \left[\frac{\sqrt{ss}(3ss - 8m_q^2)}{\sqrt{ss - 2m_q^2}} M_{40} + 2m_q^2(ss - 2m_q^2)\left(1 - \frac{2\sqrt{ss - 2m_q^2}}{\sqrt{ss}}\right) M_{39} \right.$$
$$+ 2\epsilon \left(\frac{(3ss - 4m_q^2)\sqrt{ss - 2m_q^2}}{\sqrt{ss}} - 3(ss - 2m_q^2) \right) M_{38}$$
$$\left. + \frac{2m_q^2}{\sqrt{ss}\sqrt{ss - 2m_q^2}} M_{17} \right],$$

Tansform the DEs into canonical form

$$\mathrm{d}\;\mathbf{F} = \epsilon\,(\mathrm{d}\;\mathbf{A})\,\mathbf{F}\,.$$

For Integrals in non-elliptic sectors:

$$\mathrm{d}\;\mathbf{A} = \sum_{k=1}^{9} \mathrm{R}_k\;\mathrm{d}\;\log(l_k)\,,$$

$$l_k \in \left\{ x_n, x_n-1, x_n+1, x_n-i, x_n+i, x_n+\frac{1-\sqrt{3}i}{2}, x_n+\frac{1+\sqrt{3}i}{2}, x_n+\frac{1}{3}, x_n+3 \right\}$$

R_k are rational matrices

CANONICA
Christoph Meyer
arXiv:1705.06252

Special Functions：Goncharov Polylogarithms

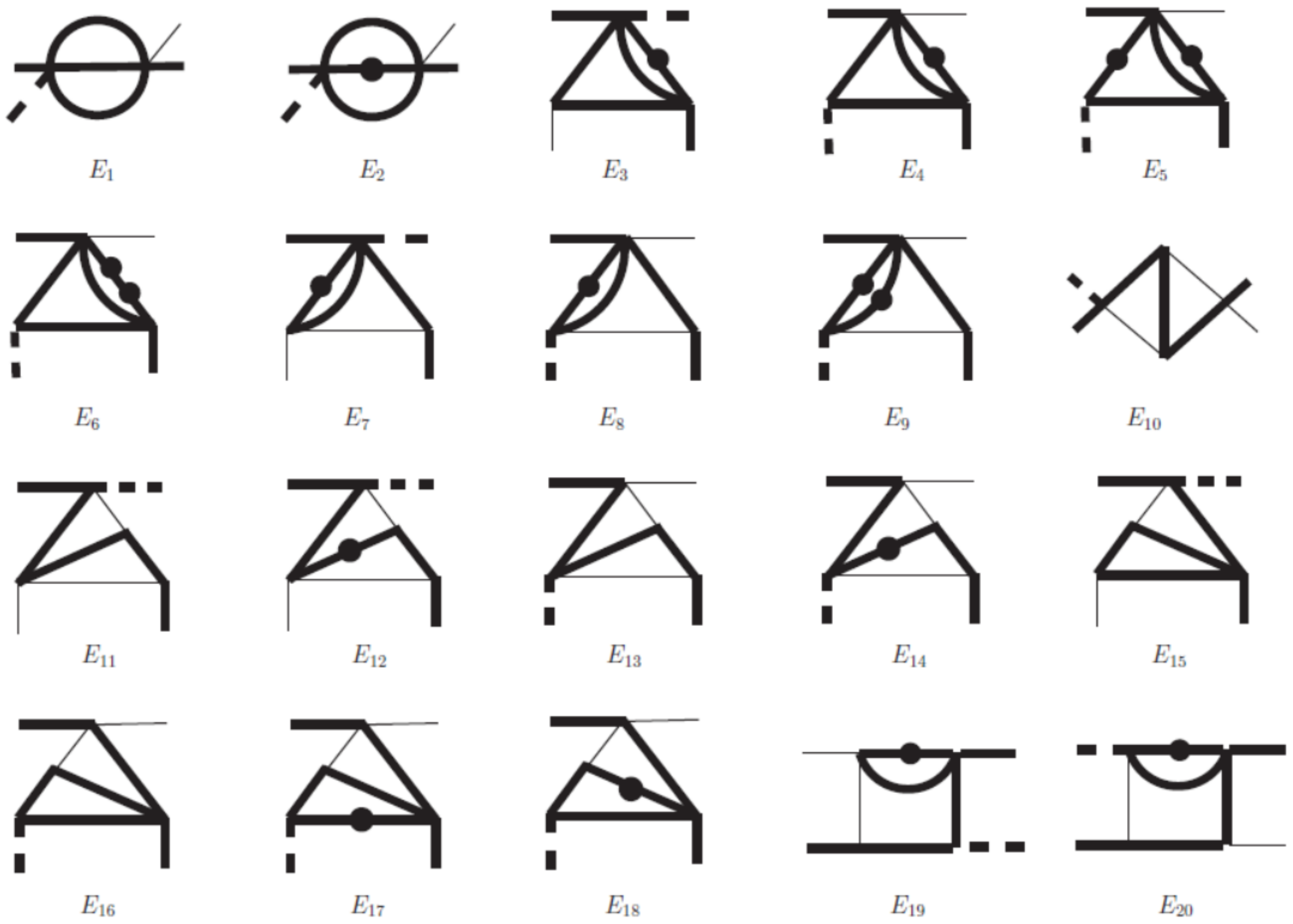
$$G_{a_1,a_2,\ldots,a_n}(x) \equiv \int_0^x \frac{\mathrm{d}t}{t-a_1} G_{a_2,\ldots,a_n}(x)\,,$$

$$G_{\vec{0}_n}(x) \equiv \frac{1}{n!}\log^n x\,.$$

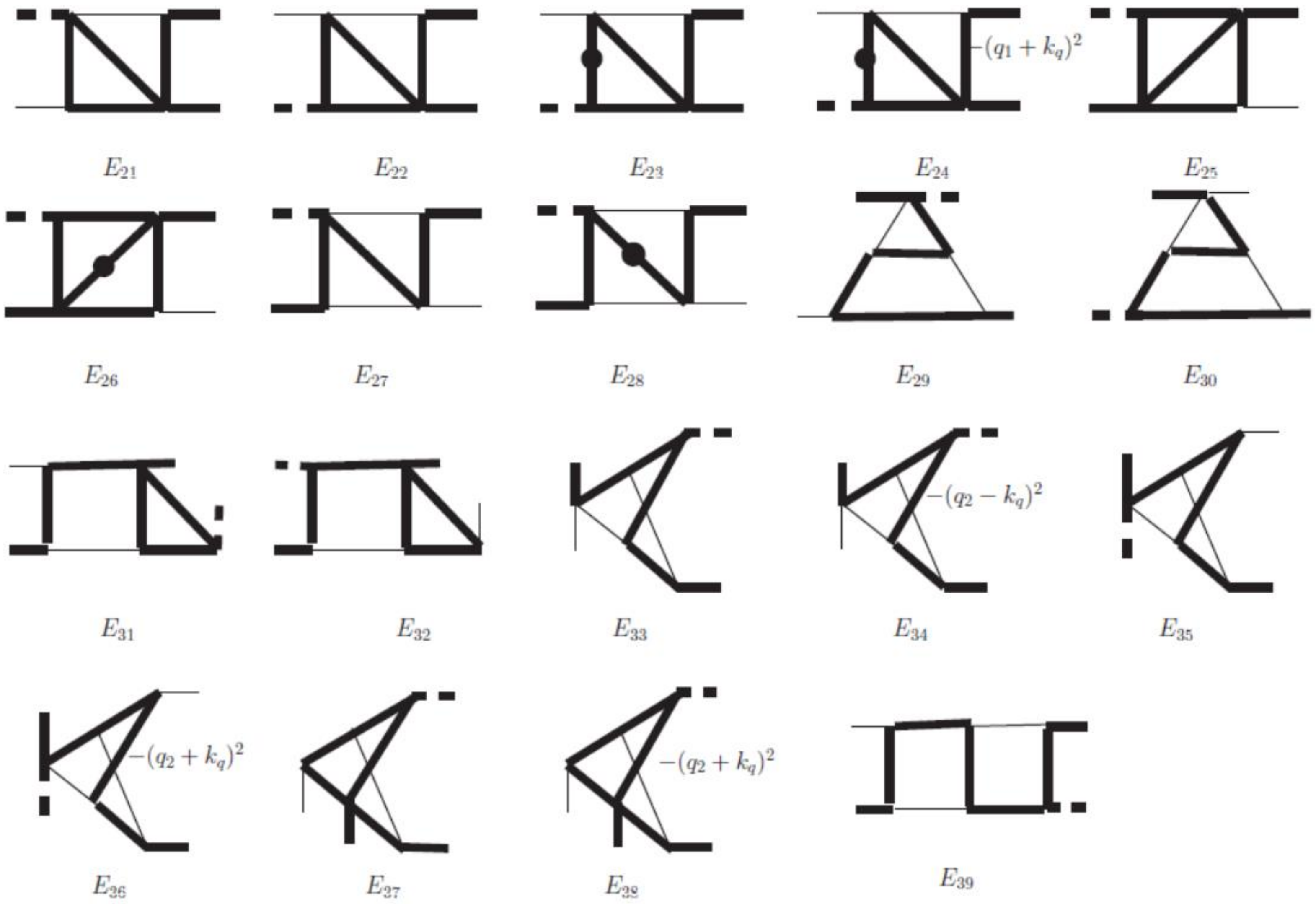
Shuffle Rules:

$$G_{a_1,\ldots,a_m}(x) G_{b_1,\ldots,b_n}(x) = \sum_{c\in a \text{III} b} G_{c_1,c_2,\ldots,c_{m+n}}(x)\,.$$

Numerical evaluation:
GINAC



Elliptic Sector I



Elliptic Sector I

Complete Elliptic Integrals of First and Second Type

$$K(x) = \int_0^1 \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-x\,t^2)}}\,, \qquad E(x) = \int_0^1 \frac{\sqrt{1-x\,t^2}}{\sqrt{1-t^2}}\mathrm{d}t$$

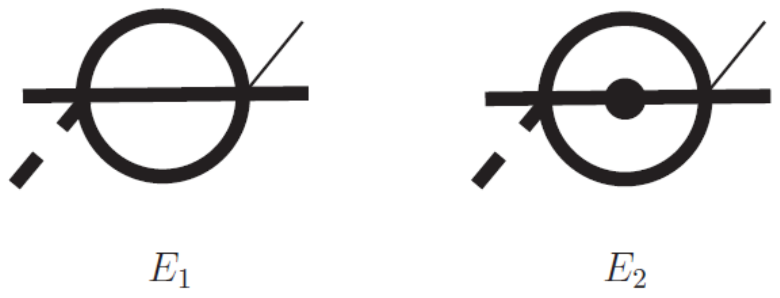
$$\frac{\mathrm{d}K(x)}{\mathrm{d}x} = \frac{E(x)-(1-x)K(x)}{2(1-x)x}\,,$$

$$\frac{\mathrm{d}E(x)}{\mathrm{d}x} = \frac{E(x)-K(x)}{2x}\,.$$

Legendre Relation

$$K(x)K(1-x) - K(x)E(1-x) - E(x)K(1-x) = -\frac{\pi}{2}$$

Two-loop Massive Sunrise Integrals:



Ettore Remiddi, Lorenzo Tancredi.
Nucl.Phys. B907 (2016) 400-444.
arXiv:1602.01481

Luise Adams and Stefan Weinzierl.
arXiv:1802.05020

Basis For Massive Sunrise Integrals

$$A_1 = \epsilon^2 \frac{12m_q^2((1-2\epsilon)(2(2-3\epsilon)E_1 + 2(ss+2m_q^2)E_2) - (ss-4m_q^2)F_1/\epsilon^2)}{(ss-2m_q^2)(ss-10m_q^2)}$$

$$A_2 = \epsilon^2 \frac{1}{m_q^2(ss-2m_q^2)(ss-10m_q^2)} \left(-8(1-2\epsilon)(2-3\epsilon)((1-4\epsilon)ss^2 + 4(11\epsilon-4)ss\,m_q^2\right.$$
$$+4(3-10\epsilon)m_q^4)E_1 - 8(1-2\epsilon)((2\epsilon-1)ss^3 - 6(7\epsilon-2)ss^2\,m_q^2 + 12(20\epsilon-7)s$$
$$-8(25\epsilon-8)m_q^6)E_2 - 4((1-4\epsilon)ss^3 + 2(22\epsilon-3)ss^2m_q^2$$
$$\left.-4(3+10\epsilon)ss\,m_q^4 + 8m_q^6)F_1/\epsilon^2\right) \,.$$

Only A_1 are neccessary

DE for $\mathrm{A}'_i(i = 3\ldots 39)$

$$\frac{\mathrm{d}\,\mathbf{A}'}{\mathrm{d}\,ss} = \epsilon(\mathbf{W}\cdot\mathbf{A}' + \mathbf{Y}\cdot\mathbf{F}) + (\epsilon\mathbf{Q_1} + \mathbf{Q_2})\mathrm{A}_1 + \mathbf{Q_3}\mathrm{A}_2$$

$$\frac{\mathrm{d}\,\mathrm{A}_1}{\mathrm{d}\,ss} = \frac{-(ss-m_q)^2 + 14(ss-m_q^2)m_q^2 + 3m_q^4}{2(ss-m_q^2)(ss-2m_q^2)(ss-10m_q^2)}\mathrm{A}_1 - \frac{2\epsilon}{ss-10m_q^2}\mathrm{A}_1 - \frac{3m_q^4}{2(ss-m_q^2)(ss-2m_q^2)(ss-10m_q^2)}\mathrm{A}_2\ .$$

Shift of basis (algebraic)

$$\mathrm{A}'_i \to \mathrm{A}'_i + b_i(ss)\mathrm{A}_1 \equiv \mathrm{A}_i\ (i = 3\ldots 39)$$

$$\frac{\mathrm{dA}_5}{\mathrm{d}x} = \epsilon\frac{33\mathrm{A}_4 + 6\mathrm{A}_5 - 6\mathrm{A}_6 - 4F_8 + 21F_{12}}{4x} + \epsilon\frac{\mathrm{A}_5 + 2F_8}{x-1}$$
$$+\epsilon\frac{9\mathrm{A}_4 - 2\mathrm{A}_5 - 2\mathrm{A}_6 + 9F_{12}}{x-3} + \epsilon\frac{9\mathrm{A}_4 + 2\mathrm{A}_5 - 2\mathrm{A}_6 + 9F_{12}}{x-\frac{1}{3}}$$
$$+\epsilon\frac{1}{6}\left(\frac{1}{x^2} + \frac{28}{x} - \frac{40}{(x-1)^2} + \frac{80}{x-3} - \frac{80}{x-\frac{1}{3}} + 1\right)A_1$$
$$-\frac{4}{3}\left(\frac{5}{(x-1)^2} + \frac{1}{x}\right)\mathrm{A}_1\ ,$$
$$\frac{\mathrm{dA}_9}{\mathrm{d}x} = \epsilon\frac{6\mathrm{A}_8 + 3\mathrm{A}_9 + 2F_7}{x} - \epsilon\frac{4\mathrm{A}_9 + F_7}{x-1} - \epsilon\frac{2\mathrm{A}_9}{x+1}$$
$$+\epsilon\frac{1}{3}\left(\frac{4}{x^2} + \frac{160}{(x-1)^2} + \frac{52}{x} + 4\right)\mathrm{A}_1$$
$$-\frac{8}{3}\left(\frac{5}{(x-1)^2} + \frac{1}{x}\right)\mathrm{A}_1\ .$$

$$\frac{ss}{m_q^2} = -\frac{(1-x)^2}{2x}$$

Homogeneous terms For A_5 and A_9 are in canonical form

All DE of basis (except A_5,A_9) are in canonical form

Elliptic Sector II

$$B_7 = \epsilon^4 (ss - 2m_q^2)^2 C_7 ,$$

$$B_8 = \epsilon^4 \frac{ss^2 - 4ssm_q^2 + 20m_q^4}{ss - 2m_q^2} m_q^4 C_8$$

Andreas von Manteuffel, Lorenzo Tancredi
JHEP 1706 (2017) 127.
e-Print: arXiv:1701.0590

$$\frac{\mathrm{d}^2 B_7}{dss^2} - \frac{ss^2 - 4ssm_q^2 - 12m_q^4}{(ss-2m_q^2)(ss^2-4ssm_q^2+20m_q^4)}\frac{\mathrm{d}B_7}{\mathrm{d}ss}$$

$$-\frac{16m_q^4}{(ss-2m_q^2)^2(ss^2-4ssm_q^2+20m_q^4)}B_7 = N(\epsilon, ss, m_q^2)$$

$$v = \frac{-i(ss-2m_q^2)}{4m_q^2}$$

$$\frac{\mathrm{d}^2 B_7}{dv^2} - \frac{1+v^2}{v(1-v^2)}\frac{\mathrm{d}B_7}{\mathrm{d}v} + \frac{1}{v^2(1-v^2)}B_7 = 0$$

Homogeneous Solutions

$$y_1(v) = vK(v^2), \qquad y_2(v) = vK(1-v^2)$$

II. Two-loop master integrals for heavy-to-light form factors of two different massive fermions

Massive quark decays to massive quark

$$(t \to b + W^+(l + \bar{\nu}), b \to c + l + \bar{\nu})$$

Massive lepton decays to massive lepton

$$(\mu \to e + \nu_\mu + \bar{\nu}_e, \tau \to \mu + \nu_\tau + \bar{\nu}_\mu)$$

[1] K.G. Chetyrkin, R. Harlander, T. Seidensticker and M. Steinhauser, *Second order QCD corrections to* $\Gamma(t \to Wb)$, *Phys. Rev.* **D 60** (1999) 114015 [hep-ph/9906273] [INSPIRE].

[2] I.R. Blokland, A. Czarnecki, M. Slusarczyk and F. Tkachov, *Heavy to light decays with a two loop accuracy*, *Phys. Rev. Lett.* **93** (2004) 062001 [hep-ph/0403221] [INSPIRE].

[3] A. Czarnecki, M. Ślusarczyk and F.V. Tkachov, *Enhancement of the hadronic b quark decays*, *Phys. Rev. Lett.* **96** (2006) 171803 [hep-ph/0511004] [INSPIRE].

[4] M. Brucherseifer, F. Caola and K. Melnikov, $\mathcal{O}(\alpha_s^2)$ *corrections to fully-differential top quark decays*, *JHEP* **04** (2013) 059 [arXiv:1301.7133] [INSPIRE].

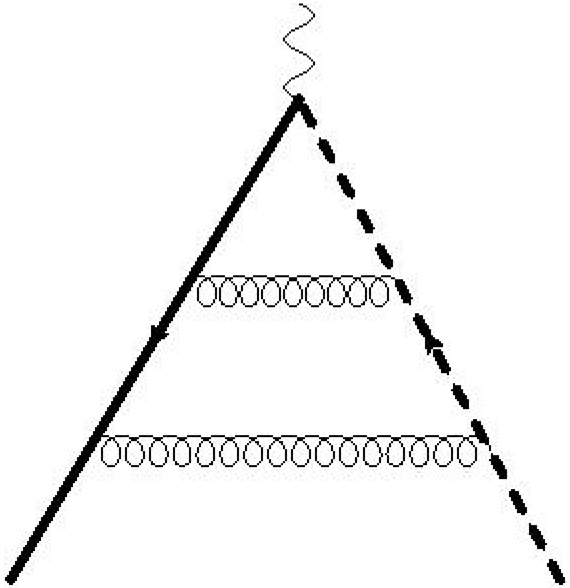
[5] J. Gao, C.S. Li and H.X. Zhu, *Top quark decay at next-to-next-to leading order in QCD*, *Phys. Rev. Lett.* **110** (2013) 042001 [arXiv:1210.2808] [INSPIRE].

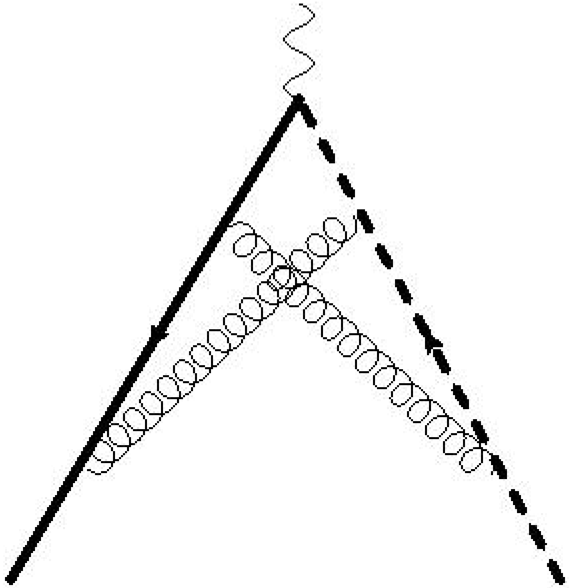
[6] R. Bonciani and A. Ferroglia, *Two-loop QCD corrections to the heavy-to-light quark decay*, *JHEP* **11** (2008) 065 [arXiv:0809.4687] [INSPIRE].

………………

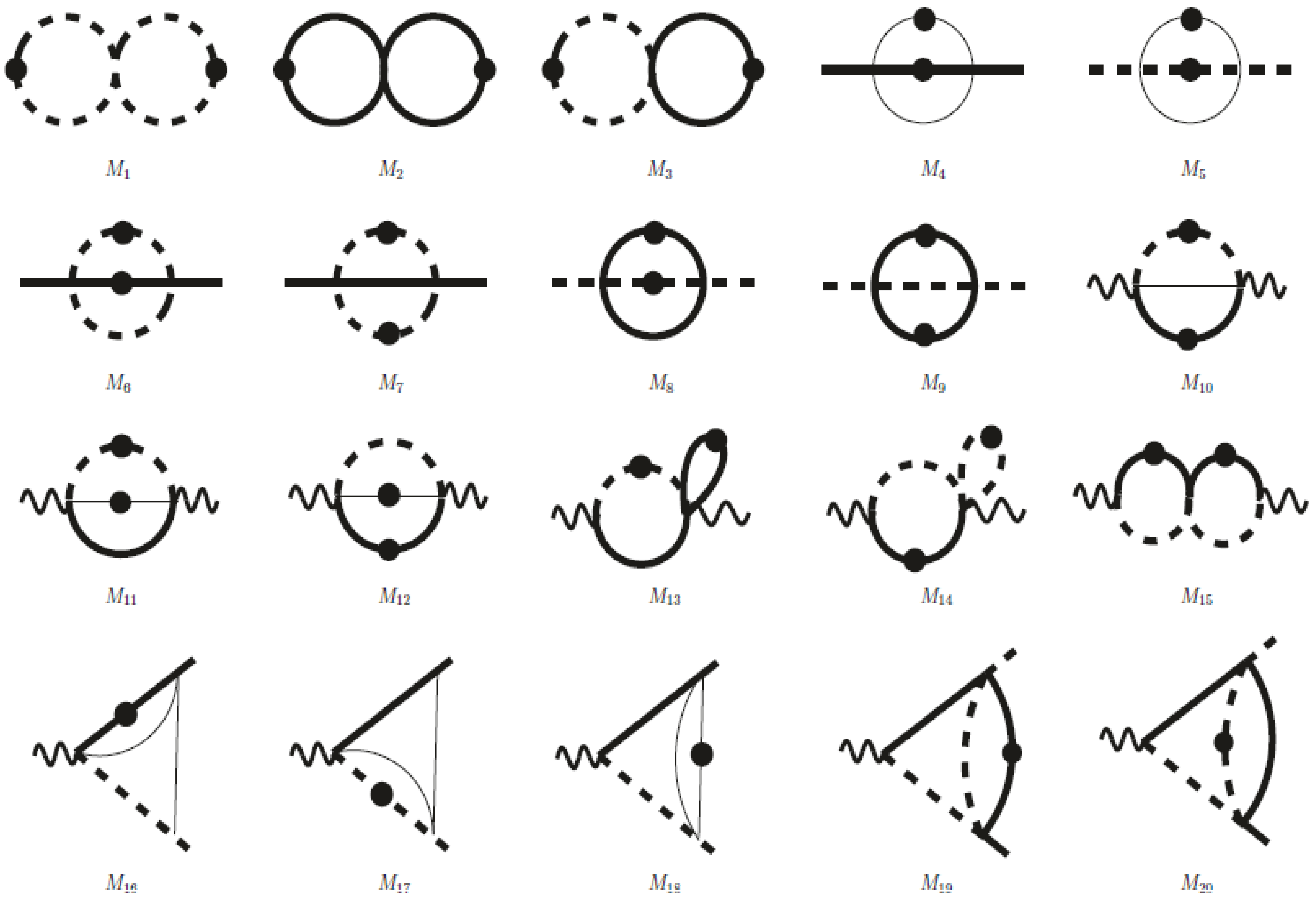
$$\ln^2\left(\frac{m_{\text{heavy}}}{m_{\text{light}}}\right) \qquad \ln\left(\frac{m_{\text{heavy}}}{m_{\text{light}}}\right)$$

Large logarithms will appear when calculating the differential decay rates

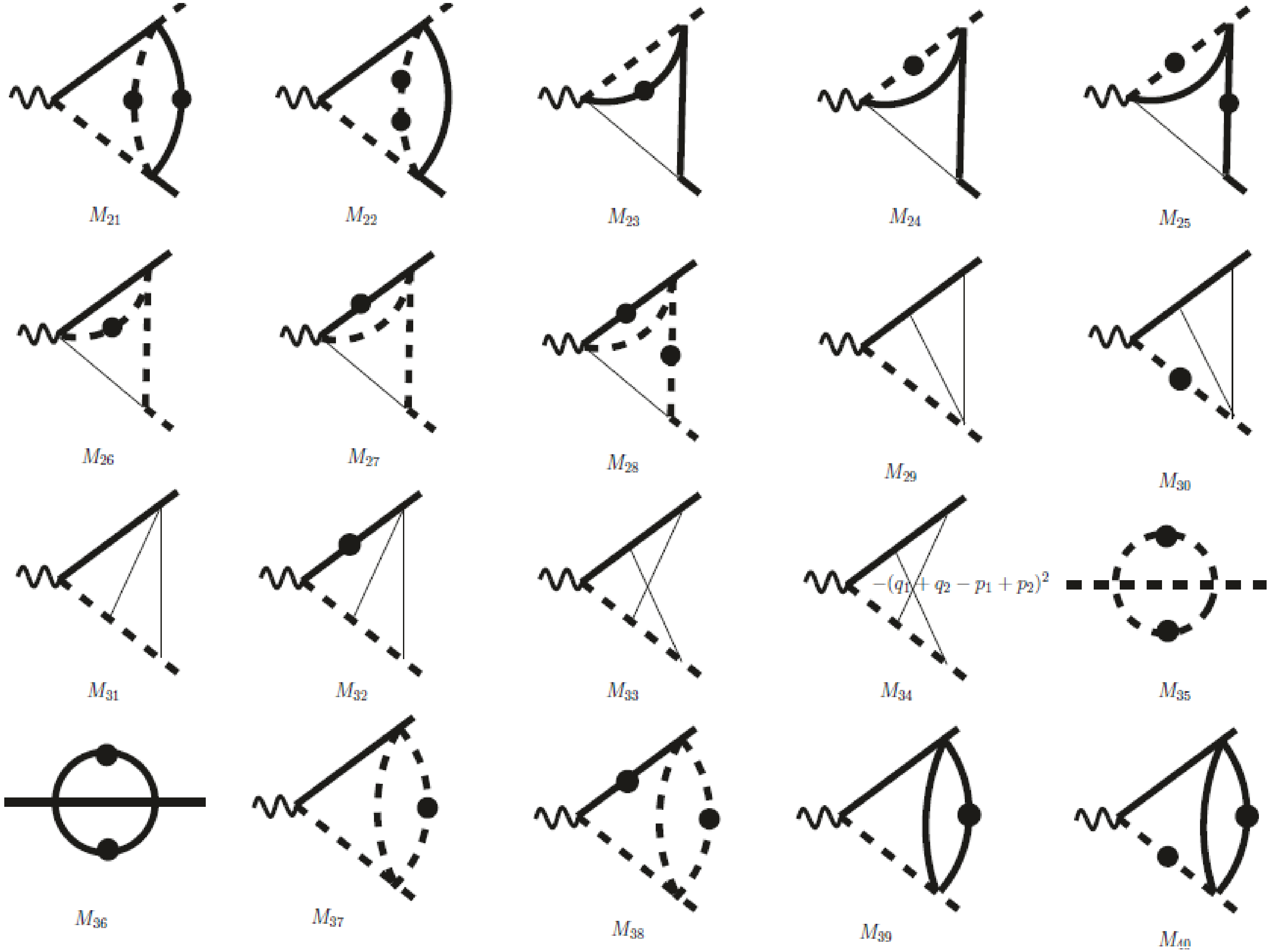




Sample of Two-loop Feynman diagrams



40 Master Integrals
Solid: Heavy; Dash solid: Light;
Thick: massless.



Three scales: S,m1,m2

$$F_{19} = \epsilon^3 \sqrt{s-(m_1+m_2)^2}\sqrt{s-(m_1-m_2)^2}\, M_{19}\,,$$

$$F_{20} = \epsilon^3 \sqrt{s-(m_1+m_2)^2}\sqrt{s-(m_1-m_2)^2}\, M_{20}\,,$$

$$F_{21} = \epsilon^2 \Big(2s\left(\epsilon(M_{19}+2M_{20}) - m_2^2 M_{21} - 2m_1^2 M_{22}\right) \\ +2(m_2^2-m_1^2)(\epsilon(M_{19}+2M_{20}) + m_2^2 M_{21} - 2m_1^2 M_{22})\Big) + 2\frac{m_2}{m_1}F_9\,,$$

$$F_{22} = \epsilon^2 \Big(2(m_1^2-m_2^2)(2\epsilon(M_{19}+2M_{20}) + (m_1^2+m_2^2-s)M_{21} - 4m_1^2 M_{22})\Big) \\ +2\frac{m_1}{m_2}F_7 - 2\frac{m_2}{m_1}F_9\,,$$

$$F_{23} = \epsilon^3 \sqrt{s-(m_1+m_2)^2}\sqrt{s-(m_1-m_2)^2}\, M_{23}\,,$$

$$F_{24} = \epsilon^3 \sqrt{s-(m_1+m_2)^2}\sqrt{s-(m_1-m_2)^2}\, M_{24}\,,$$

Canonical Basis

Differential Equations In Canonical Form:

$$\mathrm{d}\mathbf{F}(x, y; \epsilon) = \epsilon\, \mathrm{d}\, \tilde{A}(x, y)\, \mathbf{F}(x, y; \epsilon)$$

$$\begin{aligned}\tilde{A}(x, y) = {} & A_1 \ln(x) + A_2 \ln(x+1) + A_3 \ln(x-1) + A_4 \ln(x+y) + A_5 \ln(x-y) \\ & + A_6 \ln(x\,y+1) + A_7 \ln(x\,y-1) + A_8 \ln(y) + A_9 \ln(y+1) + A_{10} \ln(y-1) \\ & + A_{11} \ln(x^2\,y - 2x + y) + A_{12} \ln(x^2 - 2y\,x + 1).\end{aligned}$$

A_i are rational matrices

$$s = m_1^2 \frac{(x-y)(1-x\,y)}{x}, \quad \text{and} \quad m_2 = m_1\, y.$$

Results and Check:

$$F_{33} = \epsilon^3 \left[2G_{0,0,0}(x) - 2G_{0,1,0}(x) - 2G_{0,-1,0}(x) + \frac{1}{6}\pi^2 G_0(x) + \zeta(3) \right] + \mathcal{O}(\epsilon^4)$$

$$M_{33}^{\text{SecDec}}(-5.4, 1.0, 0.2) = \frac{-0.4466129 \pm 0.0000004}{\epsilon} - 0.507366 \pm 0.000006,$$

$$M_{33}^{\text{FIESTA}}(-5.4, 1.0, 0.2) = \frac{-0.446613 \pm 0.000005}{\epsilon} - 0.507387 \pm 0.000049,$$

$$M_{33}^{\text{Ours}}(-5.4, 1.0, 0.2) = \frac{-0.4466129967\dots}{\epsilon} - 0.5073683817\dots.$$

Thanks！