Feynman Integrals 2

$$I(d; s_1 \dots s_n; m_1^2, \dots, m_r^2) =$$

$$= \int \prod_{j=1}^{\ell} D^d k_j \quad \frac{S_1^{d_1} \cdots S_\sigma^{d_\sigma}}{D_1^{b_1} \cdots D_\tau^{b_\tau}} \qquad S_j = k_i \cdot p_k$$

$$= k_i \cdot k_k$$

IBPs $\longrightarrow$ MASTER INTEGRALS

which are BASIS

① F.I. are homogeneous functions

$$I(d; \lambda s_1, \dots, \lambda s_n; \lambda m_1^2, \dots, \lambda m_r^2) =$$

$$= (\lambda)^\alpha I(d; s_1, \dots, s_n; m_1^2, \dots, m_r^2)$$

$$I = \int \prod_j D^d k_j \quad \frac{S_1 \cdots S_n}{D_1 \cdots D_r}$$

$$\lambda = \frac{1}{m_r^2}$$

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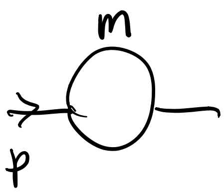
1 Loop Bubble



$$= \int D^d k \; \frac{1}{(k^2+m^2)\left((k-p)^2+m^2\right)}$$

$$= F(d; \underline{\underline{p^2}}; \underline{\underline{m^2}}) = F\left(d; \frac{p^2}{m^2}\right) \qquad m^2 = 1$$

Feynman Integrals depend on dimensionless ratios

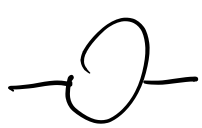
Work in EUCLIDEAN KINEMATICS

$p^\mu \in \mathbb{R}^d$ $\quad F(d; p^2, m^2)$ $\quad p^2 > 0$ Euclidean

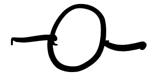
$p^2 \to -s < 0$
Minkowski kin.

Feynman Prescription $\underline{s \to s + i\varepsilon}$

Analytic continuation at the end!

 $= \int D^d k \, \frac{1}{D_1^{n_1} D_2^{n_2}}$ IBPs

2 MIs  Tadpole $= T(m^2) = \int \frac{D^d k}{k^2 + m^2}$

 $= \int D^d k \, \frac{1}{D_1 D_2}$

$$T(m^2) = \int \frac{D^d k}{k^2+m^2} = \mu(d) \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2+m^2}$$

$$= \mu(d) \frac{4\pi^{\frac{d}{2}}}{(2\pi)^d} \Gamma\left(\frac{6-d}{2}\right) \frac{(m^2)^{\frac{d-2}{2}}}{(d-2)(d-4)}$$

poles in $d=2$
$d=4$

$$\mu(d) = \frac{(2\pi)^d}{4\pi^{\frac{d}{2}}} \frac{1}{\Gamma\left(\frac{6-d}{2}\right)}$$

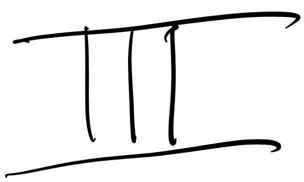
$$T(m^2) = \frac{(m^2)^{\frac{d-2}{2}}}{(d-2)(d-4)}$$

V

Compute the 1 loop Bubble

$$F(p^2, m^2) = \int D^d k \frac{1}{(k^2+m^2)((k-p)^2+m^2)}$$

$$\frac{1}{AB} = \int_0^1 dx \, \frac{1}{[Ax + B(1-x)]^2}$$ Feynman Parameters



7 propagators

$\Rightarrow$ 6 Feynman Parameters

Differential Equations Method

Kotikov '90

Remiddi '99

$F(p^2, m^2) \Rightarrow$ I would like to derive a differential equation in $\frac{\partial}{\partial p^2}$

$$\frac{\partial}{\partial p^2} F = \, ?$$

$$p^2 = p^\mu p_\mu \qquad \frac{\partial p^2}{\partial p_\mu} = 2p^\mu$$

$$p_\mu \frac{\partial p^2}{\partial p_\mu} = 2p^2$$

$$\frac{\partial}{\partial p^2} = \frac{1}{2p^2}\left[p_\mu \frac{\partial}{\partial p_\mu}\right]$$

$$\frac{\partial}{\partial p^2} F(p^2, m^2) = \frac{1}{2p^2} p_\mu \frac{\partial}{\partial p_\mu} \int D^d k \frac{1}{(k^2+m^2)((k-p)^2+m^2)}$$

$$= \frac{1}{2p^2} p_\mu \int D^d k \frac{\partial}{\partial p_\mu}\left(\frac{1}{(k^2+m^2)((k-p)^2+m^2)}\right)$$

$$\frac{\partial}{\partial p_\mu}\left(\frac{1}{(k-p)^2+m^2}\right) = \frac{2(k-p)^\mu}{\left[(k-p)^2+m^2\right]^2}$$

$$p_\mu \frac{\partial}{\partial p_\mu}\left(\frac{1}{(\ldots)}\right) = \frac{2k\cdot p - 2p^2}{\left[(k-p)^2+m^2\right]^2} = \frac{2k\cdot p - 2p^2}{D_2^2}$$

$$D_1 = k^2 + m^2 \qquad D_2 = (k-p)^2 + m^2$$

$$p_\mu \frac{\partial}{\partial p_\mu} \frac{1}{D_2} = \frac{1}{D_2^2}\left[D_1 - D_2 - p^2\right]$$

$$\frac{\partial}{\partial p^2} \int \frac{D^d k}{D_1 D_2} = \frac{1}{2p^2} \int D^d k \, \frac{\left[D_1 - D_2 - p^2\right]}{D_1 D_2^2}$$

$$= \frac{1}{2p^2}\left[\int \frac{D^d k}{D_2^2} - \int \frac{D^d k}{D_1 D_2} - p^2 \int \frac{D^d k}{D_1 D_2^2}\right]$$

$$I(n_1, n_2) = \int \frac{D^d k}{D_1^{n_1} D_2^{n_2}}$$

$$\frac{\partial}{\partial p^2} \underset{\substack{\Updownarrow \\ F(p^2, m^2)}}{I(1,1)} = \frac{1}{2p^2}\left[\underset{\substack{\Updownarrow \\ T(m^2)}}{I(0,2)} - \underset{\substack{\uparrow \\ F}}{I(1,1)}\right] - \frac{1}{2} \underset{\substack{\Updownarrow \\ F,\, T}}{I(1,2)}$$

$$I(0,2) = -\frac{(d-2)}{2m^2}\, T(m^2)$$

$$I(1,2) = -\frac{d-2}{2m^2(p^2+4m^2)}\, T(m^2) - \frac{(d-3)}{p^2+4m^2}\, F(p^2,m^2)$$

Plug this into expression for $\frac{\partial}{\partial p^2} F(p^2,m^2)$

$$\frac{\partial}{\partial p^2} F(p^2,m^2) = \frac{1}{2}\left(\frac{d-3}{p^2+4m^2} - \frac{1}{p^2}\right) F(p^2,m^2)$$

$$- \frac{(d-2)}{p^2(p^2+4m^2)}\, T(m^2)$$

@ 1 loop there is at MOST 1 MI per Topology

$$\underbrace{\frac{d}{dp^2} F = H \cdot F + G}$$

1st solve Homogeneous Eq. $\frac{d}{dp^2} F_H = H \cdot F_H$

$$F = F_H \cdot f \longrightarrow \frac{df}{dp^2} = \frac{G}{F_H}$$

$$\longrightarrow f(p^2) = \int^{p^2} dt \, \frac{G(t)}{F_H(t)} + C$$

For the Bubble:

① Homogeneous Eq

$$\frac{d}{dp^2} F_H = \frac{1}{2}\left(\frac{d-3}{p^2+4m^2} - \frac{1}{p^2}\right) F_H$$

$$F_H = \sqrt{\frac{(p^2+4m^2)^{d-3}}{p^2}}$$

② Non-Homogeneous Solution

$$F(p^2,m^2) = -(d-2)\,T(m^2)\sqrt{\frac{(p^2+4m^2)^{d-3}}{p^2}}\left[\int_0^{p^2} dt\, \frac{t^{-1/2}}{(t+4m^2)^{\frac{d-1}{2}}} + c\right]$$

How do I fix the BOUNDARY CONSTANT c ?

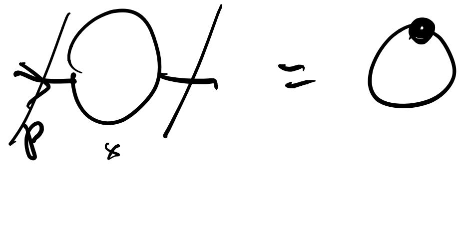
Study limit $p^2 \to 0$

In Euclidean kinematics this means $p^\mu \to 0$

$$\lim_{p^2\to 0} \int \frac{D^d k}{(k^2+m^2)((k-p)^2+m^2)} = \int \frac{D^d k}{(k^2+m^2)^2} = \underline{\underline{I(2,0)}}$$

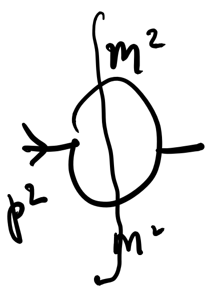
$$\overset{\text{IBPs}}{=} \underline{-\frac{(d-2)}{2m^2}\,T(m^2)}$$

Boundary Value!



$$\frac{d}{dp^2}F(p^2,m^2) = \frac{1}{2}\left[\frac{d-3}{p^2+4m^2} - \frac{1}{p^2}\right]F(p^2,m^2)$$

$$- \frac{d-2}{p^2(p^2+4m^2)}\,T(m^2)$$



$p^2 \to -4m^2$ $\qquad$ $p^2 \to 0$

$\underline{s = +4m^2}$ $\qquad$ <u>has to be regular</u>

$p^2=0$ PSEUDO-THRESHOLD regular! $\leftarrow$

$p^2=-4m^2$ THRESHOLD ! $\sim \log(s-4m^2)$

$$p^2\,\cancel{\frac{d}{dp^2}F} = \left[\frac{d-3}{2}\cancel{\frac{p^2}{p^2+4m^2}} - \frac{1}{2}\right]F - \frac{d-2}{p^2+4m^2}\,T$$

$$-\frac{1}{2}F(0,m^2) - \frac{d-2}{4m^2}T(m^2) = 0 \Rightarrow \underbrace{F(0,m^2) = -\frac{d-2}{2m^2}T(m^2)}$$

I can get the boundary value FOR FREE without doing any computation, by IMPOSING REGULARITY on the PSEUDO THRESHOLDS

Final solution is $(c=0)$

$$F(p^2,m^2) = -\frac{(m^2)^{\frac{d-2}{2}}}{(d-4)}\sqrt{\frac{(p^2+4m^2)^{d-3}}{p^2}}\int_0^{p^2} dt\, \frac{t^{-1/2}}{(t+4m^2)^{\frac{d-1}{2}}}$$

rescale $p^2 = m^2\xi$ $\quad t = m^2 y$ $\quad \xi, y$ dimensionless

$$F(x,m^2) = -\frac{(m^2)^{\frac{d-4}{2}}}{d-4}\sqrt{\frac{(\xi+4)^{d-3}}{\xi}}\int_0^{\xi} dy\, \frac{y^{-1/2}}{(y+4)^{\frac{d-1}{2}}}$$

$$F(x, m^2) = -(m^2)^{\frac{d-4}{2}} \left[\frac{2^{2-d}}{d-4} \sqrt{(\xi+4)}^{\,d-3} \; {}_2F_1\left(\frac{1}{2}, \frac{d-1}{2}, \frac{3}{2}, -\frac{\xi}{4}\right) \right]$$

$${}_2F_1(a, b, c, z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(b-c)} \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}\, dx$$

Solution for general "d" is very difficult to obtain, usually <u>IMPOSSIBLE</u> !

$\varepsilon = \frac{4-d}{2}$ Laurent Coefficients

what I want is to obtain

$$F(p^2, m^2) = \sum_{j=-1}^{\infty} (d-4)^j F^{(j)}(p^2, m^2)$$

Compute Laurent Expansion : ($m^2 = 1$)

$$T(1) = \frac{1}{(d-2)(d-4)}$$

$$F(p^2) = \frac{1}{(d-4)} F^{(-1)}(p^2) + F^{(0)}(p^2) + (d-4) F^{(1)}(p^2) + \ldots$$

plug expansion into the differential Equation

$$\frac{d}{dp^2} F^{(-1)}(p^2) = \frac{1}{2}\underbrace{\left(\frac{1}{p^2+4} - \frac{1}{p^2}\right)}\, F^{(-1)}(p^2) - \underbrace{\frac{1}{p^2(p^2+4)}}_{\text{Tadpole}}$$

$$\frac{d}{dp^2} F^{(0)}(p^2) = \overbrace{\frac{1}{2}\left(\frac{1}{p^2+4} - \frac{1}{p^2}\right)}\, F^{(0)}(p^2) + \underbrace{\frac{1}{2}\,\frac{1}{(p^2+4)}\, F^{(-1)}(p^2)}$$

$$\frac{d}{dp^2} F^{(n)}(p^2) = \frac{1}{2}\left(\frac{1}{p^2+4} - \frac{1}{p^2}\right) F^{(n)}(p^2) + \underbrace{\frac{1}{2}\frac{1}{(p^2+4)} F^{(n-1)}(p^2)}$$

Chained system of differential equations

Let's solve it

. Homogeneous part ($d = 4$)

$$\frac{d}{dp^2} F_H(p^2) = \frac{1}{2}\left(\frac{1}{p^2+4} - \frac{1}{p^2}\right) F_H(p^2)$$

$$F_H(p^2) = \sqrt{\frac{p^2+4}{p^2}} \quad \text{Homogeneous solution}$$

solve the Equations :

$$F^{(-1)}(p^2) = -\sqrt{\frac{p^2+4}{p^2}}\left[\underbrace{\int_0^{p^2} dt \sqrt{\frac{t}{t+4}} \frac{1}{t(t+4)}} + C^{(-1)}\right]$$

$\vdots$

$$F^{(n)}(p^2) = \frac{1}{2}\sqrt{\frac{p^2+4}{p^2}}\left[\int_0^{p^2} dt \underbrace{\sqrt{\frac{t}{t+4}}\,\frac{1}{t+4}}_{\text{kernel}} F^{(n-1)}(t) + C^{(n)}\right]$$

we get solution "naturally" in terms of

ITERATED INTEGRALS

General theory Chen '77

LANDAU CHANGE OF VARIABLES

$$p^2 = m^2\frac{(1-x)^2}{x} = \frac{(1-x)^2}{x}$$

$$x = \frac{\sqrt{p^2+4m^2}-\sqrt{p^2}}{\sqrt{p^2+4m^2}+\sqrt{p^2}} \quad !$$

Rederive DIFFERENTIAL EQS directly in the variable x (<u>Landau Variable</u>)

$$\frac{d}{dx}F(x) = \left[\frac{1}{1+x} + \frac{1}{1-x} + (d-4)\left(\frac{1}{1+x} - \frac{1}{2x}\right)\right] F(x)$$

$$+ \frac{1}{2(d-4)} \underbrace{\left(\frac{1}{1+x} + \frac{1}{1-x}\right)}_{\text{Tadpole}}$$

$$\frac{d}{dx}F_H(x) = \left[\frac{1}{1+x} + \frac{1}{1-x}\right] F_H(x)$$

$$F_H(x) = \left[\frac{1+x}{1-x}\right]$$

Expand Eq in $(d-4)$

$$\frac{d}{dx} F^{(n)}(x) = \left(\frac{1}{1+x} + \frac{1}{1-x} \right) F^{(n)}(x) + \left(\frac{1}{1+x} - \frac{1}{2x} \right) F^{(n-1)}(x)$$

$$F^{(n)}(x) = \left(\frac{1+x}{1-x} \right) \cdot M^{(n)}(x)$$

$$\begin{cases} \dfrac{dM^{(n)}}{dx} = \left(\dfrac{1}{1+x} - \dfrac{1}{2x} \right) M^{(n-1)}(x) \\[2ex] \dfrac{dM^{(-1)}}{dx} = \dfrac{1}{(1+x)^2} \end{cases}$$

coming from the TADPOLE

$$\int dx \left(\frac{1}{1+x} ; \frac{1}{x} \right) f(x)$$

my new iterated integrals !