An Introduction To Resummation : LECTURE I

- QCD has a logarithmic sensitivity to regimes in which the radiation ~~pro~~ probes the corners of the available phase space.

  e.g.

  - UV limit (virtual corrections): logarithmic divergences handled by renormalization

    ⟶ running coupling $\alpha_s(\mu^2)$

  - collinear limit (real and virtual corrections): singular behaviour when radiation propagates collinearly to the emitter.

    ⟶ renormalization of PDFs and DGLAP evolution equations: running parton densities $q(x, \mu^2)$

  - soft limit (real and virtual corrections): singular behaviour when radiation propagates with very low momentum (energy)

- The concept of running coupling or running PDF is closely related to that of resummation

  → Renormalization acts as a cut off for the theory at scales above (below) μ^2 in the UV (collinear) limit, respectively.

  → The divergences still manifest themselves in form of logarithms in $\alpha_s(\mu^2)$ and $q(x, \mu^2)$

  → when the renormalization (factorization) scale probes the UV (non perturbative) limit such logarithms must be resummed at all orders.

⊕ Unlike ~~in~~ the above examples (α_s and DGLAP), usually physical observables are devised in such a way that IR and C divergences cancel between real and virtual corrections in perturbation theory:

KLN theorem & IRC safety

⊕ However, divergences of IRC nature can still appear in form of logarithms in regimes where the real and virtual radiation is highly unbalanced.

e.g.

→ the real-virtual unbalance can be related to the very nature of the process under consideration:

e.g. ~~...~~

⊙ the final state is produced at threshold, i.e. ~~soft~~ radiation (real) is very suppressed and constrained to be soft. An example is given by the ~~large~~ small-x limit of some AP splitting functions (note: soft limit at $x \to 0$ here!)

⊕ We try to probe a very soft parton inside a hadron. Real soft radiation is favoured w.r.t. virtual corrections → high-energy logarithms and BFKL regime.

An example is given by the ~~small~~ large-x limit of some AP splitting functions. (note: soft limit at $x \to 0$!)

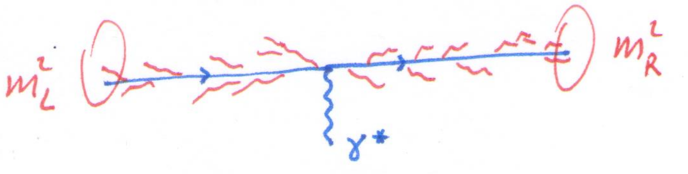
subject of these lectures.

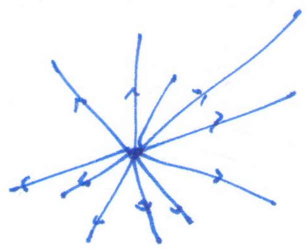
→ $\boxed{\text{The real-virtual unbalance can be induced by a specific measurement, where the observable probes the end points of the phase space.}}$

A concrete example: the Thrust in e^+e^-

$$T = \max_{\vec{n}} \frac{\sum_i |\vec{p}_i \cdot \vec{n}|}{\sum_i |\vec{p}_i|}$$

$T = 1$ $\qquad\qquad$ $T = 1/2$





$Q^2 \gg m_{R,L} \gg \Lambda_{QCD}^2$

pencil-like event $\qquad\qquad$ spherical event

This event shape has been widely studied at LEP, due to its sensitivity to α_s: departure from the two-jet configuration is proportional to α_s

$$\frac{1}{\sigma_0}\frac{d\sigma}{dT} = \delta(1-T) + \boxed{\frac{\alpha_s(\mu^2)}{2\pi}} A(T) + \frac{\alpha_s^2(\mu^2)}{(2\pi)^2} B(T) + \dots$$

To measure the departure from the 2-jet limit, it is therefore convenient to define $\tau = 1-T$, so that the two-jet limit is recovered as $\tau \to 0$.

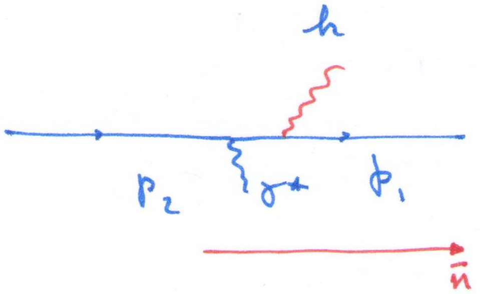
This limit is interesting since as $\tau \to 0$ the real radiation is forced to be either soft or colinear to the Born partons, hence leaving a potential unbalance between the real and virtual corrections.

Let's study this limit more in detail.

We first simplify the observable to work in the $\tau \to 0$ limit, i.e. soft and/or collinear emissions.

It is useful to choose a parametrization for the momenta. Let's denote by p_1 and p_2 the two Born momenta, and by k_i any extra emission.

In the $\tau \to 0$ limit, the thrust axis $\bar{n}$ is aligned with the direction of the Born $q\bar{q}$ system.



We denote by p and $\bar{p}$ the momenta of the Born quarks prior to any radiation. At the Born level $p = p_1$ and $\bar{p} = p_2$, while this is not true at higher orders. We parametrize the real emissions as $(2p\cdot\bar{p} = Q^2)$

$$\boxed{k_i = z_i^{(1)}\, p + z_i^{(2)}\, \bar{p} + k_{\perp i} \,, \quad \text{with } k_{\perp i} \text{ spacelike and } k_{\perp i}^2 = -k_{ti}^2}$$

if k_i is collinear to p, $z_i^{(1)} > z_i^{(2)}$, and viceversa if k_i is collinear to $\bar{p}$.

We have

$p^\mu = Q\, n^\mu$

$\bar{p}^\mu = Q\, \bar{n}^\mu$

$$\tau = \min_{\bar{n}} \frac{\sum_i |\vec{k}_i \cdot \vec{n}| + |\vec{p}_1 \cdot \vec{n}| + |\vec{p}_2 \cdot \vec{n}|}{Q} \longleftarrow \text{massless partons}$$

thrust is additive

$$\simeq \sum_{i\in \mathcal{H}_p} \frac{k_{ti}^2}{Q^2 z_i^{(1)}} + \sum_{i\in \mathcal{H}_{\bar{p}}} \frac{k_{ti}^2}{Q^2 z_i^{(2)}} + \frac{|\sum_{i\in \mathcal{H}_p} \vec{k}_{ti}|^2}{Q^2 (1 - \sum_{i\in\mathcal{H}_p} z_i^{(1)})} + \frac{|\sum_{i\in \mathcal{H}_{\bar{p}}} \vec{k}_{ti}|}{Q^2 (1 - \sum_{i\in\mathcal{H}_{\bar{p}}} z_i^{(2)})}$$

radiation collinear to p — radiation collinear to $\bar{p}$ — contribution of p_1 — contribution of p_2

For a single emission collinear to p, we thus have

$$\tau = \frac{k_t^2}{Q^2 z^{(1)}(1-z^{(1)})}$$

Let's now compute the LO distribution. It is more convenient to work with the integrated (cumulative) distribution, that is

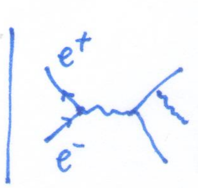
$$\Sigma(\tau) = \int_0^\tau d\tau' \frac{1}{\sigma_0}\frac{d\sigma}{d\tau'}$$

In the collinear limit, the matrix element squared for a real emission reads ($\overline{MS}$ coupling)

$$\underbrace{[d\Phi_B]}_{\text{PS of the Born system}}\underbrace{[dk]}_{\text{PS of the extra radiation}} |M(p_1,p_2,k)|^2 = [d\Phi_B]\,|M(p_1,p_2)|^2 \times 2\frac{dk_t}{k_t}\frac{dz^{(1)}}{z^{(1)}}\frac{d\phi}{2\pi}\frac{\alpha_s(Q^2)}{2\pi}\frac{e^{\varepsilon\gamma_E}}{\Gamma(1-\varepsilon)}\frac{z^{(1)}P_{gq}(z^{(1)})}{(k_t^2)^\varepsilon}$$

where $P_{gq}(z) = C_F\left(\frac{1+(1-z)^2}{z} - \varepsilon z\right)$ | soft limit at $z\to 0$ in this parametrization

The real correction thus becomes



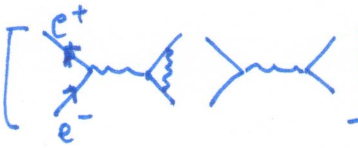
$$\left|\;\cdot\;\right|^2 \longrightarrow 2\sigma_0\int\frac{dk_t}{k_t}\frac{dz^{(1)}}{z^{(1)}}\frac{d\phi}{2\pi}\frac{\alpha_s(Q^2)}{\pi}\frac{e^{\varepsilon\gamma_E}}{\Gamma(1-\varepsilon)}\frac{z^{(1)}P_{gq}(z^{(1)})}{(k_t^2)^\varepsilon}\Theta\left(\tau-\frac{k_t^2}{Q^2 z^{(1)}(1-z^{(1)})}\right)$$

2: contribution from emission collinear to $\bar{p}$.

$$= \sigma_0\frac{\alpha_s(Q^2)}{2\pi}C_F\times\left[\boxed{\frac{2}{\varepsilon^2}+\frac{3}{\varepsilon}} + 3\ln\frac{1}{\tau} - 2\ln^2\frac{1}{\tau} + 7 - \frac{5}{6}\pi^2\right]$$

this is valid up to corrections that are regular (τ-suppressed) in the limit $\tau\to 0$.

the corresponding virtual correction reads

$$2\,\mathrm{Re}\Big[$$  $$\Big] \longrightarrow \sigma_0 C_F\left[\boxed{-\frac{2}{\varepsilon^2}-\frac{3}{\varepsilon}}-8+\frac{7}{6}\pi^2\right]\frac{\alpha_s(Q^2)}{2\pi}$$

In the sum of real and virtual corrections poles cancel due to KLN theorem (thrust is IRC safe!).

However, they leave behind logarithmic terms that diverge as $\tau \to 0$.

$$\Sigma(\tau) = \sigma_0 C_F\Big(\underbrace{-2L^2}_{\text{double logarithm}}+\underbrace{3L}_{\text{single logarithm}}+\dots\Big)\frac{\alpha_s}{2\pi} \qquad L = \ln\frac{1}{\tau}$$

double logarithm: when the emission is <u>both</u> soft ($z^{(1)} \to 0$) and colinear ($k_t \to 0$)

single logarithm: when the emission is collinear ($k_t \to 0$), but not soft

Physically, these logarithms manifest the fact that when $\tau \to 0$ real emissions are heavily constrained, therefore very unbalanced with the virtual corrections.

This implies that for $\tau \to 0$ limits the divergences are again manifest (due to virtual corrections), and the only way to cancel them is through including real radiation at all orders in the strong coupling:

<u>RESUMMATION</u>

An exact all-order calculation is clearly not feasible, and one needs to find approximate ways of getting to the result.

For instance, it is important to define a concept of logarithmic accuracy, to understand what approximations can be effectively ~~done~~ made.

FIXED-ORDER EXPANSION

$\mathcal{O}(\alpha_s)$	$\alpha_s L^2$	$\alpha_s L$	α_s		
$\mathcal{O}(\alpha_s^2)$	$\alpha_s^2 L^4$	$\alpha_s^2 L^3$	$\alpha_s^2 L^2$	$\alpha_s^2 L$	α_s^2
$\mathcal{O}(\alpha_s^3)$	$\alpha_s^3 L^6$	$\alpha_s^3 L^5$	$\alpha_s^3 L^4$	$\alpha_s^3 L^3$	....
⋮					

relative order α_s

relative order $1/L$

RESUMMED EXPANSION

In the singular limit, we can reorganize the perturbative series in terms of a new expansion parameter:

$1/L$ instead of α_s, in the above example

More commonly, the logarithmic counting is defined at the level of the logarithm of $\Sigma(\tau)$

$$\ln \Sigma(\tau) \sim \text{"}\underbrace{\boxed{\alpha_s^n L^{n+1}}}_{\text{LL}}\text{"} + \text{"}\underbrace{\boxed{\alpha_s^n L^{n}}}_{\text{NLL}}\text{"} + \text{"}\underbrace{\boxed{\alpha_s^n L^{n-1}}}_{\text{NNLL}}\text{"} + \dots$$

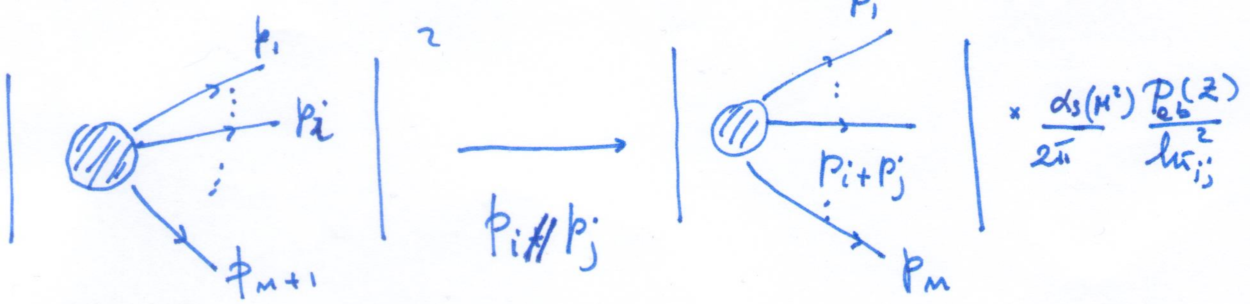
Step one towards resummation: IRC limit of the squared amplitudes

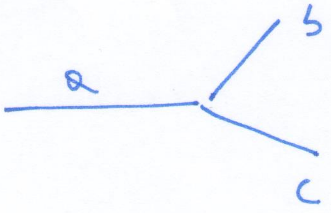
To perform an all-order calculation, we cannot afford having to compute a new amplitude at each new order in α_s. This would be computationally unsustainable.

Instead, one can use the fact that in the IRC limits the QCD amplitudes factorise in terms of somewhat universal kernels.

The all-order squared amplitude, with a given logarithmic accuracy, can therefore be obtained by iterating such kernels to all perturbative orders.

As we saw in other lectures (e.g. Parton showers), the amplitudes squared factorise in the collinear limit (see also previous example on Thrust)



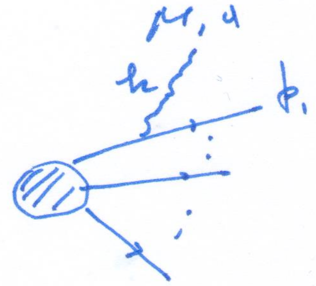


$$P_{ba}(z) \rightarrow \begin{cases} P_{gq}(z) = \frac{1+(1-z)^2}{z} C_F \\ P_{qg}(z) = \left(z^2+(1-z)^2\right) T_R \\ P_{gg}(z) = \left(\frac{z}{1-z} + \frac{1-z}{z} + z(1-z)\right) C_A \end{cases}$$

In this limit the radiation probability is entirely described in terms of universal objects: the AP splitting functions. Higher order corrections do not modify the factorisation properties (some exceptions exist).

We then consider the emission of soft radiation, where things get a little more complicated.

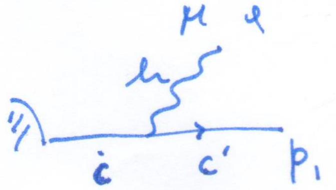
Let's start with the emission of a gluon off a quark



The amplitude is a matrix in colour space

$$\mathcal{M}(\{p_i\}, k) \underset{k\to 0}{\simeq} ig\,\varepsilon_\mu(k)\, \underbrace{J^{\mu a}(k)}_{\text{colour does not show full factorization.}}\, \mathcal{M}(\{p_i\})$$

To obtain $J^{\mu a}(k)$, we consider the diagram (ig factor understood)



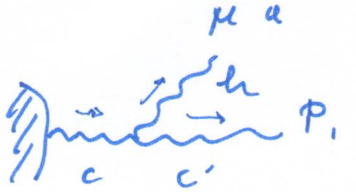
$$\simeq \frac{(\not{p}_1 + \not{k} + m)}{(p_1+k)^2 - m^2}\, \gamma^\mu t^a_{cc'}\, u(p_1)$$

$$\simeq \frac{(\not{p}_1 + m)}{2p_1\cdot k}\, \gamma^\mu t^a_{cc'}\, u(p_1) = \frac{2p_1^\mu + \overbrace{\gamma^\mu(-\not{p}_1 + m)}^{\text{on-shell condition}}}{2p_1\cdot k}\, t^a_{cc'}\, u(p_1)$$

$$= \frac{2p_1^\mu t^a_{cc'} u(p_1)}{2p_1\cdot k} \longrightarrow J^{\mu a}(k) = t^a_{cc'} \frac{p_1^\mu}{p_1\cdot k} \rightarrow \text{eikonal factor}$$

An emission off a gluon leads to the same factor, provided we replace $t^a_{cc'}$ by $if_{cac'}$. In general, the colour charge is as follows

$$T^a_{cc'} = \begin{cases} t^a_{cc'} & \text{outgoing } q \text{ or incoming } \bar{q} \\ -t^{+a}_{cc'} & \text{outgoing } \bar{q} \text{ or incoming } q \\ if_{cac'} & \text{gluon} \end{cases}$$



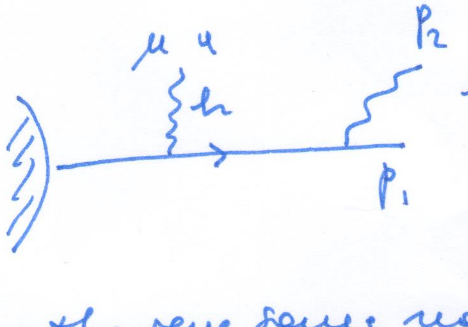
$$(p_1-k)\cdot\varepsilon(p_1+k)\,\varepsilon^{\mu}(p_1)$$
$$+\,(p_1+2k)\cdot\varepsilon(p_1)\,\varepsilon^{\mu}(p_1+k)$$
$$-\,\varepsilon(p_1+k)\cdot\varepsilon(p_1)\,(2p_1+k)^{\mu}$$

$$\overset{k\to 0}{\simeq}\quad \overset{=0}{\cancel{p_1\cdot\varepsilon(p_1)}}\,\varepsilon^{\mu}(p_1)+\overset{=0}{\cancel{p_1\cdot\varepsilon(p_1)}}\,\varepsilon^{\mu}(p_1)-\varepsilon^2(p_1)\,2p_1^{\mu}$$

(in physical gauge $\varepsilon\cdot p=0$ and $\varepsilon^2=-1$)

$= 2p_1^{\mu}$ → same result as before.

Internal lines that never go onshell do not lead to a soft singularity



$$\overset{k\to 0}{\sim}\quad \frac{1}{(p_1+p_2)^2-m^2}$$ <u>regular</u>

For the very same reason, the 4-gluon vertex is not singular when only a single gluon goes <u>soft</u>.

This easily generalises to the amplitude for a soft emission off any hard leg as

(proof in discussion session)

$$\mathcal{M}^{a,\mu}(\{p_i\},k) = ig\,\underbrace{\sum_i T_i^a\,\frac{p_i^{\mu}}{p_i\cdot k}}_{J^{\mu a}(k)}\,\mathcal{M}(\{p_i\})$$

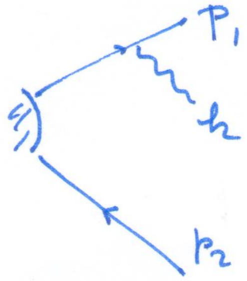
which defines the eikonal current.
Gauge invariance follows trivially from colour conservation

$$k\cdot J^a(k) = \sum_i T_i^a = 0$$

Squaring the amplitude and colinear approximation.

$-g^2 J^{\mu a}(k)\, J_\mu^{a}(k)$ has a very involved structure already in the single-gluon case, due to the pattern of colour interference.

To see this, we consider a process with 2 emitting legs:



$$-g^2 J^{\mu a}(k)\, J_\mu^{a}(k) = -g^2 \Big[T_1^a\cdot T_1^a \frac{\overbrace{p_1^2}^{=0}}{(p_1\cdot k)^2} + T_2^a\cdot T_2^a \frac{\overbrace{p_2^2}^{=0}}{(p_2\cdot k)^2}$$

$$+ T_1^a\cdot T_2^a \frac{p_1\cdot p_2}{(p_1\cdot k)(p_2\cdot k)} + T_2^a\cdot T_1^a \frac{p_1\cdot p_2}{(p_1\cdot k)(p_2\cdot k)} \Big]$$ Only interference survives

$$= g^2\, 2 C_F \frac{p_1\cdot p_2}{(p_1\cdot k)(p_2\cdot k)}$$ $\longrightarrow$ the simple colour structure allows one to simplify the squared amplitude

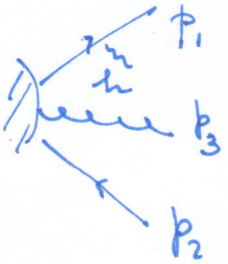
$\uparrow$

$T_1 + T_2 = 0$ (colour conservation)

$\sum_a T_1^{a\,2} = C_F$

Similarly, with 3 emitting legs one finds

$(T_i + T_j)^2 = T_k^2 \rightarrow 2T_i\cdot T_j = T_k^2 - T_i^2 - T_j^2$



$$-g^2 J^{\mu a}(k)\, J_\mu^{a}(k) = 2\Big(T_1^a\cdot T_2^a \frac{p_1\cdot p_2}{(p_1\cdot k)(p_2\cdot k)} + T_1^a\cdot T_3^a \frac{p_1\cdot p_3}{(p_1\cdot k)(p_3\cdot k)} + T_2^a\cdot T_3^a \frac{p_2\cdot p_3}{(p_2\cdot k)(p_3\cdot k)} \Big)(-g^2) \longrightarrow$$

$$= -g^2 \Big[(C_A - 2C_F) \frac{p_1 \cdot p_2}{(p_1 \cdot k)(p_2 \cdot k)} + (C_F - C_F - C_A) \frac{p_1 \cdot p_3}{(p_1 \cdot k)(p_3 \cdot k)} +$$

$$+ (C_F - C_F - C_A) \frac{p_2 \cdot p_3}{(p_2 \cdot k)(p_3 \cdot k)} \Big] =$$

$$= 2g^2 \Big[(C_F - C_A/2) \frac{p_1 \cdot p_2}{(p_1 \cdot k)(p_2 \cdot k)} + \frac{C_A}{2} \frac{p_1 \cdot p_3}{(p_1 \cdot k)(p_3 \cdot k)} + \frac{C_A}{2} \frac{p_2 \cdot p_3}{(p_1 \cdot k)(p_3 \cdot k)} \Big]$$

$q\bar{q}$ dipole qg dipole $\bar{q}g$ dipole

Sum of three dipoles with respective colour factors.

→ 4 and more hard legs.
In this case is not possible anymore to express the colour interference pattern $T_i^a \cdot T_j^a$ in terms of simple Casimir operators.
The squared matrix element thus depends on the precise colour flow (configuration) of the hard scattering reaction. Each flow has to be treated separately, one has to deal with a (colour-) matrix of squared amplitudes (e.g. $gg \to t\bar{t}$)

e.g. 4 legs

$$-g^2 J^{\mu\mu}(k)\, J_{\mu\mu}(k) = -g^2 \sum_{i\neq j}^{4} T_i^a \cdot T_j^a \; \frac{p_i \cdot p_j}{(p_i \cdot k)(p_j \cdot k)}$$

parametrise massless emissions' momenta and recast it as

$$= -g^2 \; 2 \underbrace{\sum_{i<j}}_{\text{two sums}} \frac{T_i^a \cdot T_j^a}{E_k^2} \underbrace{\frac{(1-\cos|\bar\theta_{ij}|)}{(1-\cos|\bar\theta_{ik}|)(1-\cos|\bar\theta_{jk}|)}}_{W_{ij}} \qquad (A)$$

the angles are vectors in the Transverse plane, i.e.

$$|\bar\theta_{ij}|^2 = \theta_i^2 + \theta_j^2 - 2\theta_i\theta_j \cos\phi_{ij}$$

Things are much simplified if we integrate inclusively over the azimuthal angle of the emission k.

I start by decomposing W_{ij} into two pieces

colinear singularity along p_i

$$W_{ij} = W_{ij}^{[i]} + W_{ij}^{[j]}$$

colinear singularity along p_j

$$W_{ij}^{[i]} = \frac{1}{2}\left(W_{ij} + \frac{1}{(1-\cos|\bar\theta_{ik}|)} - \frac{1}{(1-\cos|\bar\theta_{jk}|)}\right)$$

(same for $W_{ij}^{[j]}$)

$$\int_0^{2\pi} \frac{d\phi_{ik}}{2\pi} W_{ij}^{[i]} = \frac{1}{1-\cos|\bar\theta_{ik}|}\, \Theta(|\bar\theta_{ij}| - |\bar\theta_{ik}|)$$

integrate at fixed $|\bar\theta_{ik}|$

Angular-ordered picture

↳ coherent branching

to see this, it is convenient to express $W_{ij}^{[i]}$ in a different parametrisation:

$$k = E_k (1, \sin\theta_k \sin\phi_k, \sin\theta_k \cos\phi_k, \cos\theta_k)$$
$$p_i = E_i (1, \sin\theta_i \sin\phi_i, \sin\theta_i \cos\phi_i, \cos\phi_i)$$

from which we get

$$1-\cos|\bar\theta_{jk}| = 1 - \cos\theta_j \cos\theta_k - \sin\theta_j \sin\theta_k \cos(\overbrace{\phi_j - \phi_k}^{\phi_{jk}})$$

If we choose to parametrize the angles with respect to direction $\bar{p}_i$, then ($|\bar{\theta}| = \theta$ for brevity)

$$1 - \cos\theta_{jk} = 1 - \cos\theta_{ij}\cos\theta_{ik} - \sin\theta_{ij}\sin\theta_{ik}\cos(\phi_{ij} - \phi_{ik})$$

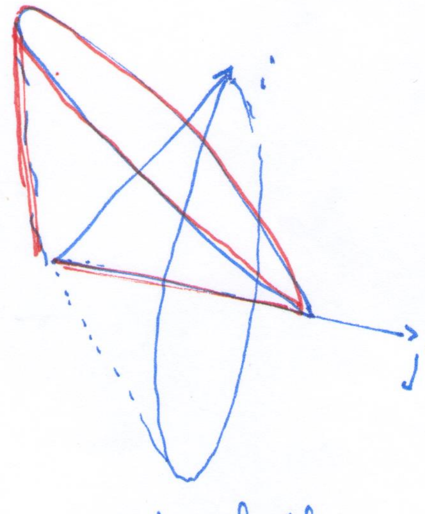
we can set $\phi_{ij} = 0$ through a simple rotation. So that

$$W_{ij}^{[i]} = \frac{1}{2}\frac{1}{1-\cos\theta_{ik}}\left[1 + \frac{-\cos\theta_{ij} + \cos\theta_{ik}}{1 - \cos\theta_{ij}\cos\theta_{ik} - \sin\theta_{ij}\sin\theta_{ik}\cos\phi_{ik}}\right]$$

$$\int_0^{2\pi}\frac{d\phi_{ik}}{2\pi} W_{ij}^{[i]} = \frac{1}{1-\cos\theta_{ik}}\left[1 + \mathrm{sign}[\cos\theta_{ij} - \cos\theta_{ik}]\right]$$

$$= \frac{1}{1-\cos\theta_{ik}}\,\Theta(\theta_{ij} - \theta_{ik})$$

Interference is destructive outside of the cone.



This approximation allows us to get rid of the colour interference, and replace it with angular ordering (azimuthal average)

$$-g^2 J^{\mu a}(k) J^a_{\mu}(k) \simeq -\frac{2g^2}{E_k^2}\sum_{i\neq j}\frac{1}{1-\cos\theta_{ik}}\left(-T_i^a \cdot T_i^a\right)\Theta(\theta_{ij} - \theta_{ik})$$

two sums

with respect to the closest hard parton

$$= \frac{2g^2}{E_k^2}\sum_{i\neq j} C_i \frac{1}{1-\cos\theta_{ik}}\,\Theta(\theta_{ij} - \theta_{ik})$$

this is a great simplification since it gets rid of the colour problem.
However, we are unfortunately not allowed to integrate over ϕ_{ik} for a generic observable!

Nevertheless, we notice that by taking the collinear approximation of $W_{ij}^{[i]}$ we obtain

$$-g^2 J^{\mu a}(k) J_\mu^{a}(k) = -2g^2 \sum_{i<j} \frac{T_i^a \cdot T_j^a}{E_k^2} \left(W_{ij}^{[i]} + W_{ij}^{[j]} \right)$$

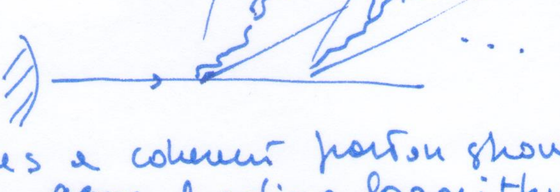
$$W_{ij}^{[i]} \underset{\substack{\uparrow \\ k // p_i \; (\cos\theta_{ij} \simeq \cos\theta_{jk})}}{\simeq} \frac{1}{1-\cos\theta_{ik}} \Theta(\theta_{ij} - \theta_{ik})$$

← the same simplification occurs.

Therefore, in the collinear and soft approximation, where the leading singularities are, colour interference can be replaced by angular ordering.

Can we use this approximation to resum the leading logarithmic contribution in *any* process?

To answer this question we have to deal with multigluon emissions:
one can easily iterate the angular ordering argument for subsequent gluon branchings

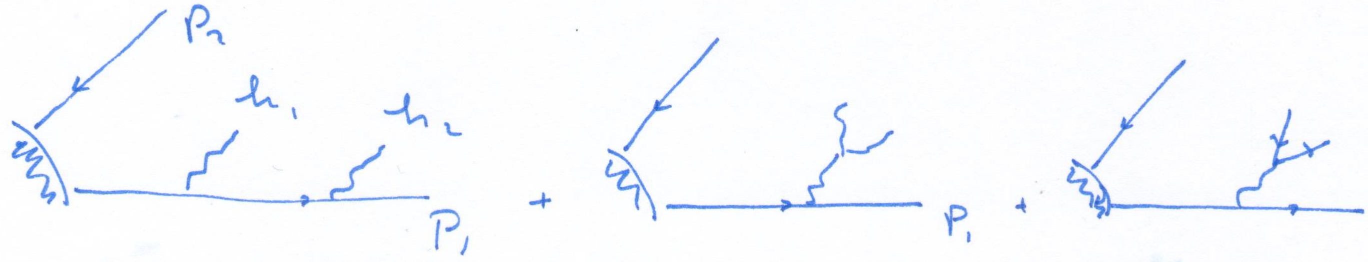


However, as we will see in a second, this *is not* a correct treatment for the gluon branchings.

This picture defines a coherent parton shower, that can be used to resum the ~~...~~ leading logarithms *only in* problems with collinear logarithms. If these are absent, there's no hierarchy between the computed terms and the neglected ones. In any case, moving beyond LL is another story...

Multi-gluon emissions (2-leg case)

Due to the non-abelian nature of the theory, this becomes immediately cumbersome already with 2 hard legs



the squared amplitude does not coincide with the product of 2 single-emission squared amplitudes, but one finds a new piece that accounts for the correlation between the two gluons (quarks)

i.e.

individual eikonal factors

$$|M(p_1,p_2,k_1,k_2)|^2 \simeq |M(p_1,p_2)|^2 \Big\{ |M(k_1)|^2 |M(k_2)|^2 + \underbrace{|\tilde{M}(k_1,k_2)|^2}_{\text{correlated, non-abelian, contribution.}} \Big\}$$

In general, adding more gluons one finds

$$|M(p_1,p_2,k_1,\dots,k_n)|^2 \simeq |M(p_1,p_2)|^2 \Big\{ \prod_{i=1}^{n} |M(k_i)|^2 +$$

$$+ \sum_{a>b} |\tilde{M}(k_a,k_b)|^2 \prod_{\substack{i=1\\ i\neq a,b}}^{n} |M(k_i)|^2 +$$

$$+ \sum_{a>b} \sum_{\substack{c>d\\ c,d\neq a,b}} |\tilde{M}(k_a,k_b)|^2 |\tilde{M}(k_c,k_d)|^2 \prod_{\substack{i=1\\ i\neq a,b,c,d}}^{n} |M(k_i)|^2 + \dots$$

$$+ \sum_{a>b>c} |\tilde{M}(k_a,k_b,k_c)|^2 \prod_{\substack{i=1\\ i\neq a,b,c}}^{n} |M(k_i)|^2 + \dots \Big\}$$

we can extend the above decomposition to account for virtual corrections by "dressing" each real-emission matrix element with additional loop corrections

e.g.

$$M(k_i) = M^{(0)}(k_i) + M^{(1)}(k_i) + \dots$$

$$\tilde{M}(k_i, k_j) = \tilde{M}^{(0)}(k_i, k_j) + \tilde{M}^{(1)}(k_i, k_j) + \dots$$

$$\vdots$$

Since resummation is obtained by iterating a given set of squared amplitudes, we need to understand at which logarithmic order each of the building blocks of the above decomposition enters.

$\boxed{\text{This is, unfortunately, an observable-dependent statement.}}$

e.g. consider thrust

when computing $\Sigma(\tau)$ we've seen that $|M^{(0)}(k)|^2$ in its soft limit ($z \to 0$) gives

$$|M^{(0)}(k)|^2 \longrightarrow \alpha_s L^2 + \dots, \quad \text{with } L = \ln 1/\tau$$

similarly, at second order we obtain

$$|M^{(0)}(k_1)|^2 |M^{(0)}(k_2)|^2 \to \alpha_s^2 L^4 + \dots \quad (\text{same order as before when } L \sim 1/\alpha_s)$$

$$|\tilde{M}^{(0)}(k_1, k_2)|^2 + 2\mathrm{Re}\, M^{(0)}(k) M^{(1)}(k) \to \alpha_s^2 L^3 + \alpha_s^2 L^2 + \dots$$

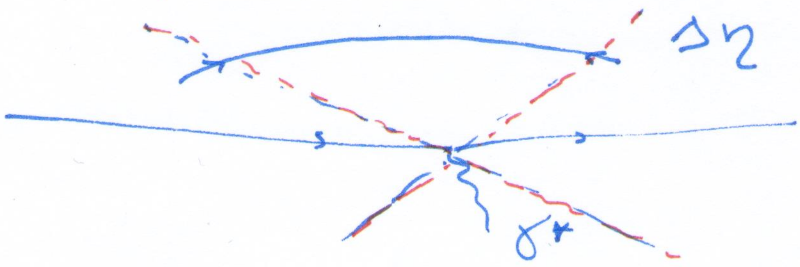
↑ of relative order $1/L$ ⇒ SUBLEADING

at higher orders: $\prod_{i=1}^{n} |M(k_i)|^2 \sim \alpha_s^n L^{2n}$; $|\tilde{M}(k_1, k_2, \dots, k_n)|^2 \sim \alpha_s^n L^{n+1}$ (including virtuals)

⇒ I have found a clear hierarchy between the correlated blocks:

→ ‖ I can iterate a fixed number of them to obtain the all-order amplitude at a given logarithmic accuracy e.g. 2 : consider a slight variant of the problem.

→ Measure Thrust only in a fixed rapidity gap with respect to the thrust axis



we then find, in the soft limit and $\tau \to 0$

$|M^{(0)}(k)|^2 \longrightarrow \underline{\alpha_s L}$

↳ single (soft) logarithm, since there is no sensitivity to the collinear divergence (fully cancels with the virtuals)

$|\tilde{M}^{(0)}(k_1, k_2)|^2 \longrightarrow \alpha_s^2 L^2 \to$ same order as above

(with $M^{(1)} M^{(0)*}$)

$|\tilde{M}^{(0)}(k_1, k_2, k_3)|^2 \longrightarrow \alpha_s^3 L^3 \to$ same order as above

(plus virtuals)

⋮

At each new order in α_s, I need to account for a brand new contribution that enters at the same logarithmic order. The solution is more <u>cumbersome</u>.

To deal with this type of problems (called non global) we have to rely on a different approximation for the squared amplitude

strong energy ordering $\longrightarrow$ Bassetto - Ciafaloni - Marchesini

(one is sensitive to the energy gap within the branchings)

(Phys. Rept. 100 (1983) 201-272)

and the evolution of soft radiation at all orders in this regime is governed by the so called

Banfi - Marchesini - Smye equation.

(JHEP 0208 (2002) 006)

[see also: Dasgupta, Salam Phys. Lett. B512 (2001) 323-330]

In these lectures I will not talk about these problems any more, and we fully focus on global observables in processes with 2 ~~Born~~ Born legs, for which the decomposition of the squared amplitude given above highlights the correct logarithmic hierarchy.

Comment on angular ordering: an important take-home message of this discussion is that the picture of angular ordering is never correct for gluon branchings (secondary branchings).

In any case (global and non-global) one ~~would~~ would be missing a contribution, coming from the full $|\tilde{M}(k_1, k_2)|^2$ that matters at the same logarithmic order.

This implies that the angular-ordered picture must be completely abandoned when talking about higher orders, as the interference structure of the soft radiation cannot be (fully) avoided. (unless azimuthal averaging is allowed).