Numerical approach to multi-scale multi-loop integrals

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Mar 29th, 2018 @ School and Workshop on pQCD @ West Lake

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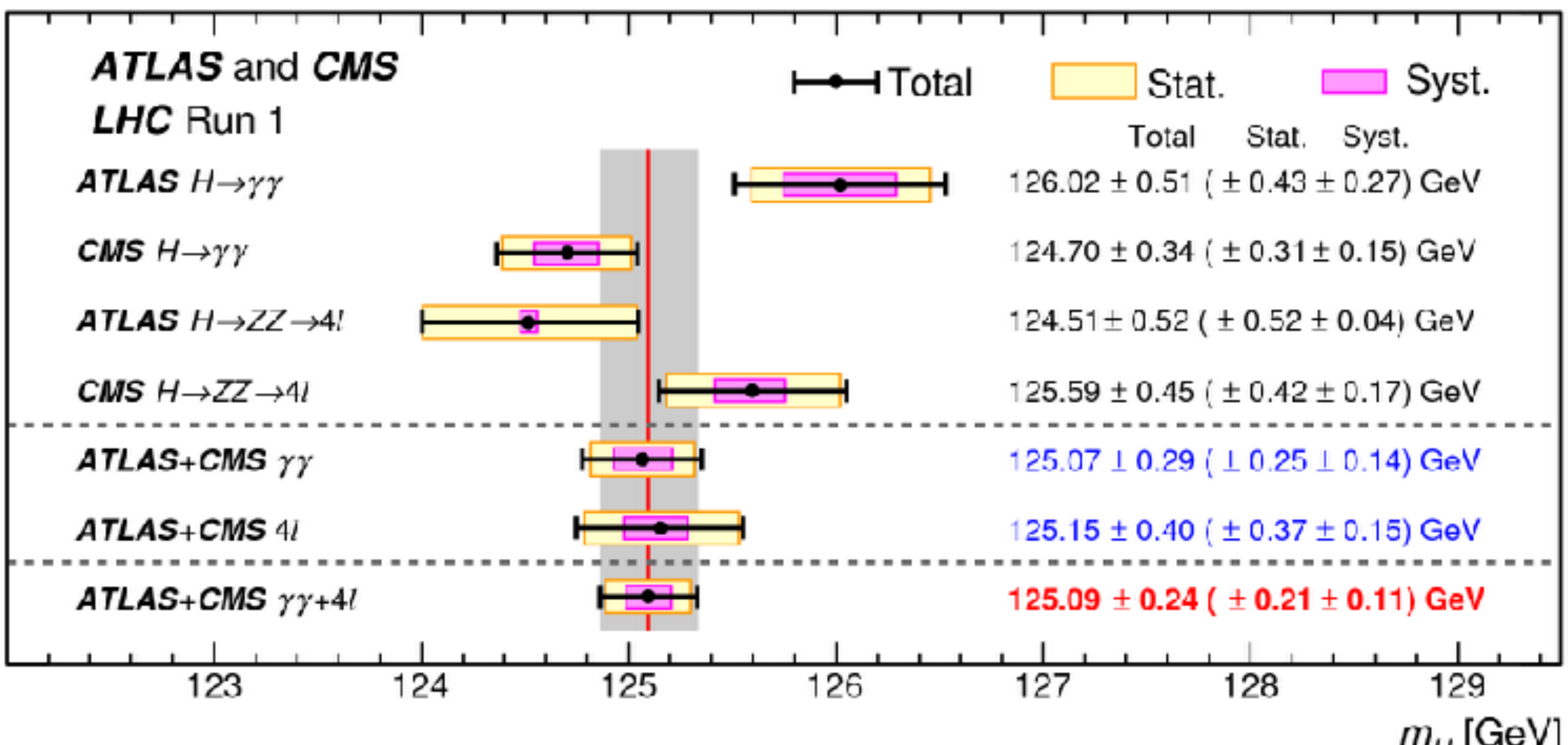
ATLAS
ALICE
CMS


ATLAS and CMS
LHC Run 1
Total
Stat.
Syst.
Total Stat. Syst.
ATLAS $H\to\gamma\gamma$
126.02 ± 0.51 (± 0.43 ± 0.27) GeV
CMS $H\to\gamma\gamma$
124.70 ± 0.34 (± 0.31 ± 0.15) GeV
ATLAS $H\to ZZ\to 4l$
124.51 ± 0.52 (± 0.52 ± 0.04) GeV
CMS $H\to ZZ\to 4l$
125.59 ± 0.45 (± 0.42 ± 0.17) GeV
ATLAS+CMS $\gamma\gamma$
125.07 ± 0.29 (± 0.25 ± 0.14) GeV
ATLAS+CMS $4l$
125.15 ± 0.40 (± 0.37 ± 0.15) GeV
ATLAS+CMS $\gamma\gamma$+$4l$
125.09 ± 0.24 (± 0.21 ± 0.11) GeV
123
124
125
126
127
128
129
m_H [GeV]

State-of-Higgs

- Anastasiou et al. JHEP 1605 (2016) 058
  N^3LO inclusive Higgs Xsection in infinite m_t (finite m_t @NLO)

- Dulat et al. arXiv:1704.08220
  NNLO differential Higgs Xsection in EFT

- Banfi et al. JHEP 1604 (2016) 049
  N^3LO+NNLL Jet vetoed Higgs Xsection in EFT

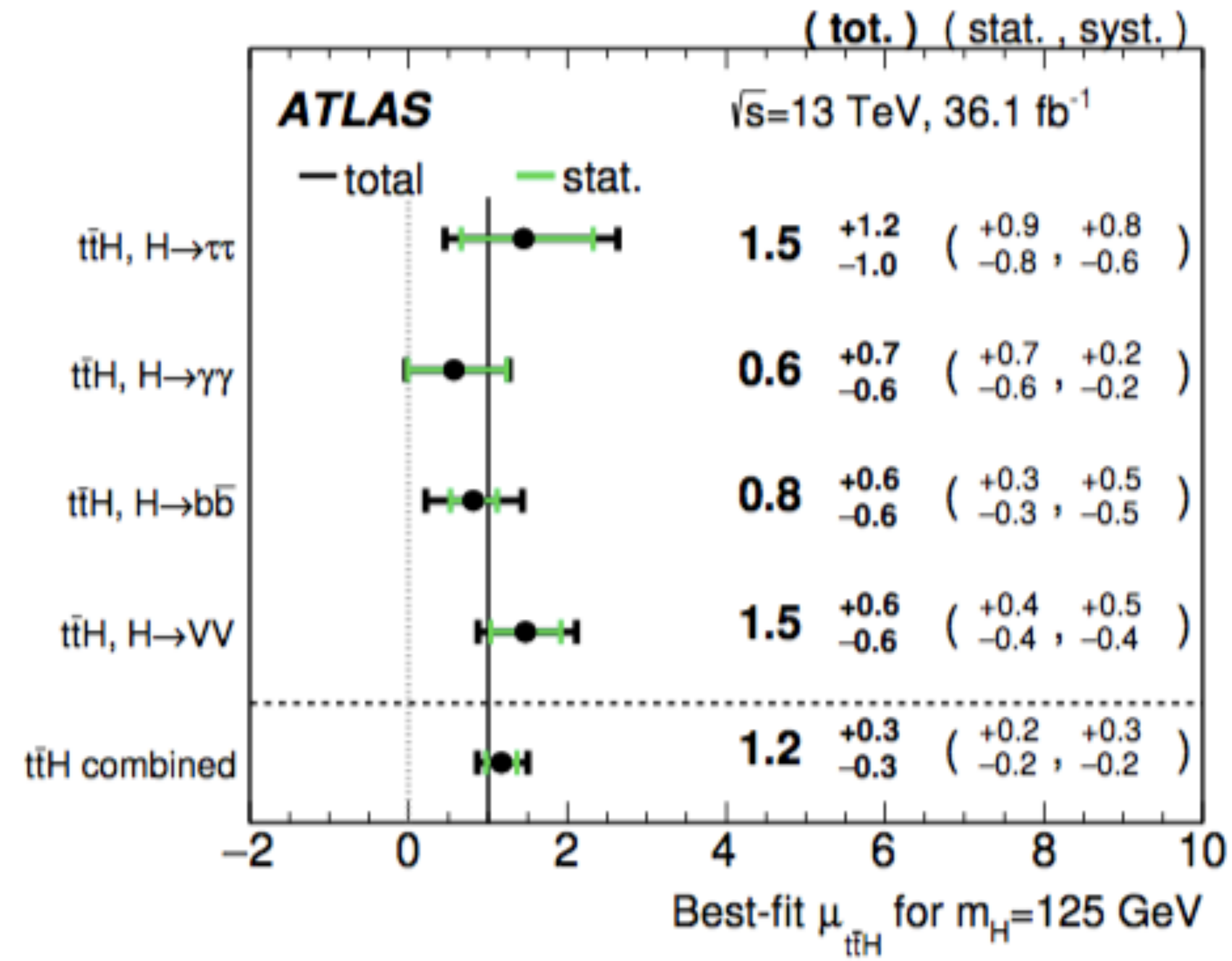
- Chen et al. JHEP 1610 (2016) 066
  NNLO Higgs+Jet in EFT (finite m_t @LO)

- Grigo et al. NPB888 (2014) 17
  NNLO Higgs pair in EFT

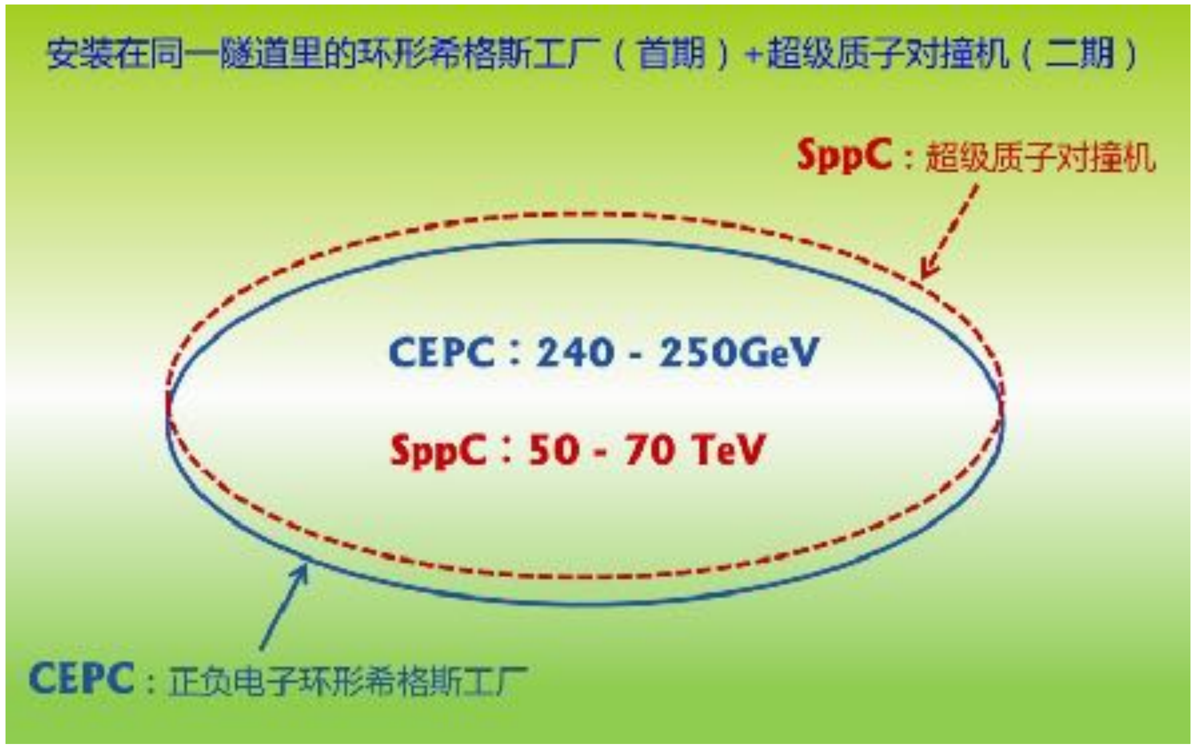
- Grigo et al. NPB900 (2015) 412
  NNLO Higgs pair in $1/m_t$ expansion

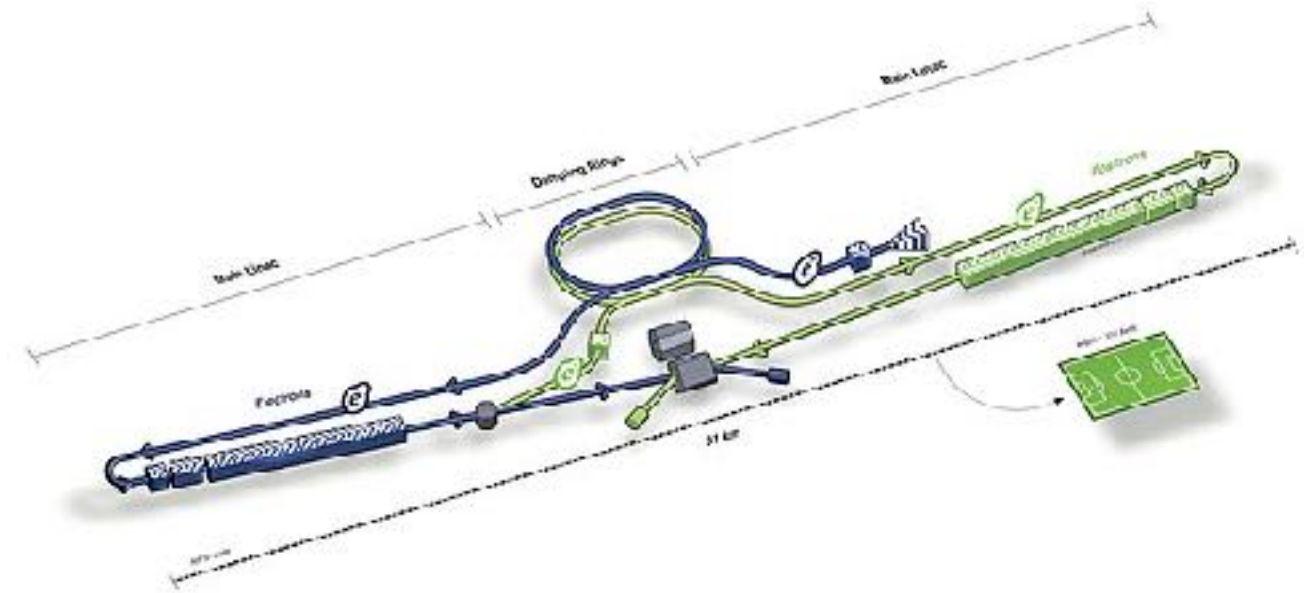
- Borowka et al. JHEP 1610 (2016) 107
  NLO Higgs pair with finite m_t

- Many other calculations……



Higgs Factory





CEPC-SPPC100TeV

FCC
hh ee he

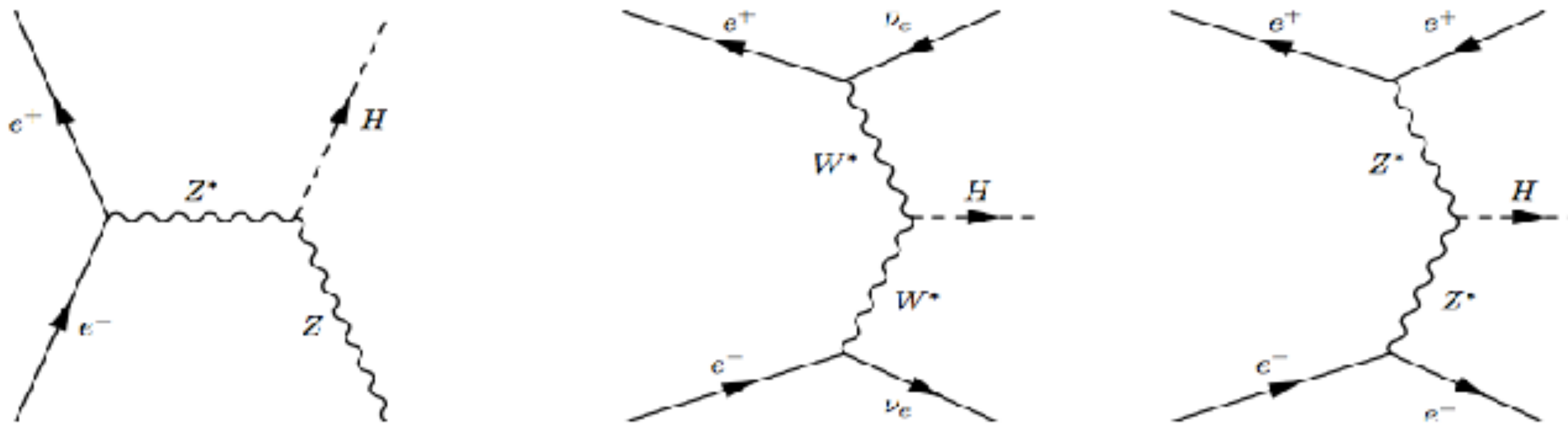
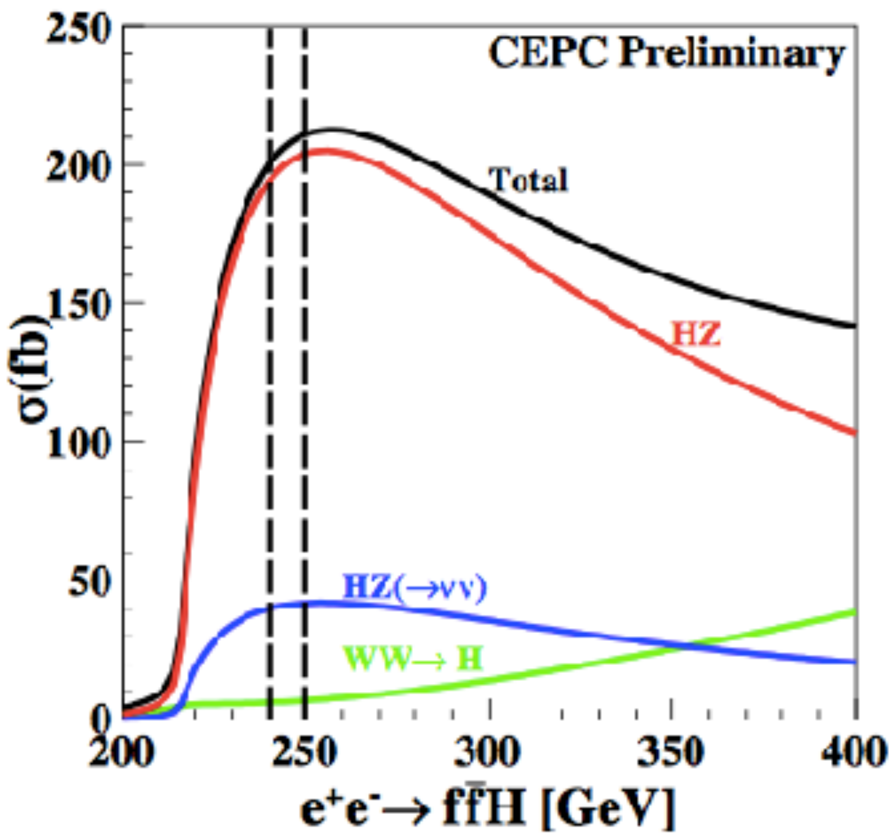
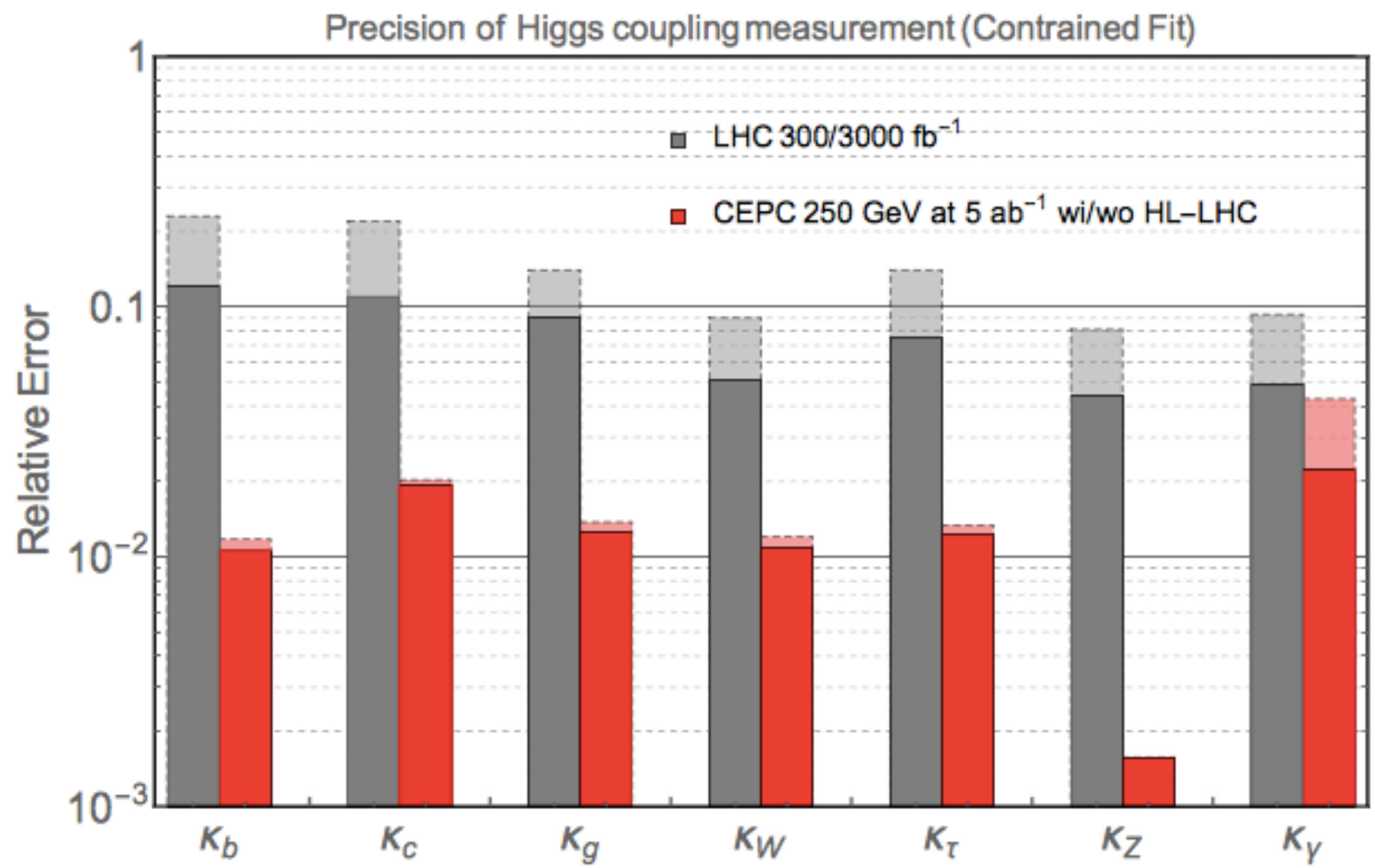


Figure 3.6 Feynman diagrams of the $e^+e^- \to ZH$, $e^+e^- \to \nu\bar{\nu}H$ and $e^+e^- \to e^+e^-H$ processes.





Needs & Obstacles

- Higher accuracy of data from LHC, HL-LHC, ILC/CEPC/FCC-ee (high order (SUSY)EW corrections?)

- Multiple scales induce problems on analytical evaluation of higher-loop.

- Analytical expressions can reveal important behaviors, but progress is getting slower. Special math may be behind, but how? when? where?

- By demand from experiments, practically more theoretical predictions can be obtained by numerical approaches.

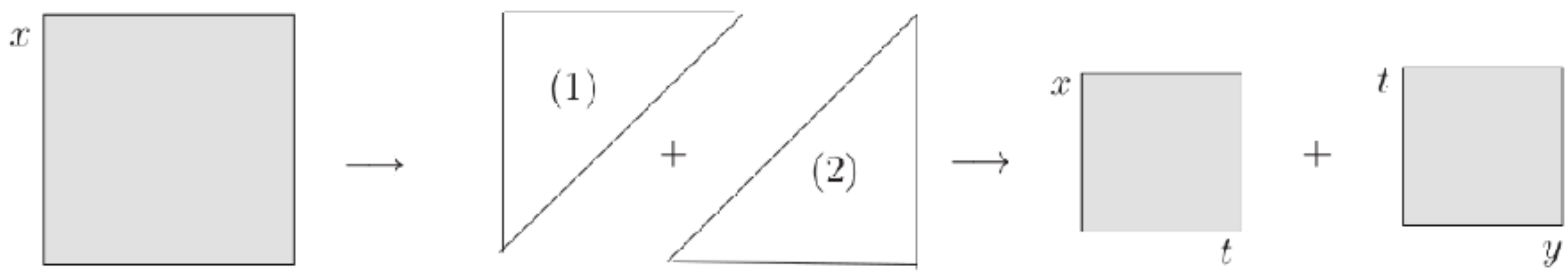
Numerical approaches for multi-scale multi-loop

- **Mellin-Barnes Representation**
  Many tools, faster, difficult on many scales.

- **Sector Decomposition**
  More general, slower, okay for many scales.

Sector Decomposition

$$I = \int_0^1 dx \int_0^1 dy\; x^{-1-\epsilon} y^{-\epsilon} (x + (1-x)y)^{-1}$$



$$I = \int_0^1 dx\; x^{-1-\epsilon} \int_0^1 dt\; t^{-\epsilon} (1 + (1-x)t)^{-1}$$

$$+ \int_0^1 dy\; y^{-1-2\epsilon} \int_0^1 dt\; t^{-1-\epsilon} (1 + (1-y)t)^{-1}$$

G.Heinrich, Int.J.Mod.Phys.A23:1457-1486,2008

$$G^{\mu_1\cdots\mu_R}_{l_1\cdots l_R} = \int \prod_{l=1}^{L} d^D\kappa_l \, \frac{k_{l_1}^{\mu_1}\cdots k_{l_R}^{\mu_R}}{\prod_{j=1}^{N} P_j^{\nu_j}\left(\{k\},\{p\},m_j^2\right)} \, ,$$

$$d^D\kappa_l = \frac{\mu^{4-D}}{i\pi^{\frac{D}{2}}}\, d^D k_l \, , \qquad P_j\left(\{k\},\{p\},m_j^2\right) = \left(q_j^2 - m_j^2 + i\delta\right) ,$$

Feynman parameterization

$$\frac{1}{\prod_{j=1}^{N} P_j^{\nu_j}} = \frac{\Gamma(N_\nu)}{\prod_{j=1}^{N}\Gamma(\nu_j)} \int_0^\infty \prod_{j=1}^{N} dx_j \, x_j^{\nu_j-1} \delta\left(1 - \sum_{i=1}^{N} x_i\right) \frac{1}{\left[\sum_{j=1}^{N} x_j P_j\right]^{N_\nu}} \, ,$$

where $N_\nu = \sum_{j=1}^{N} \nu_j$, leads to

$$G^{\mu_1\cdots\mu_R}_{l_1\cdots l_R} = \frac{\Gamma(N_\nu)}{\prod_{j=1}^{N}\Gamma(\nu_j)} \int_0^\infty \prod_{j=1}^{N} dx_j \, x_j^{\nu_j-1} \delta\left(1 - \sum_{i=1}^{N} x_i\right) \int d^D\kappa_1 \cdots d^D\kappa_L$$

$$\times \, k_{l_1}^{\mu_1}\cdots k_{l_R}^{\mu_R} \left[\sum_{i,j=1}^{L} k_i^{\mathrm{T}} M_{ij} k_j - 2\sum_{j=1}^{L} k_j^{\mathrm{T}} \cdot Q_j + J + i\delta\right]^{-N\nu} ,$$

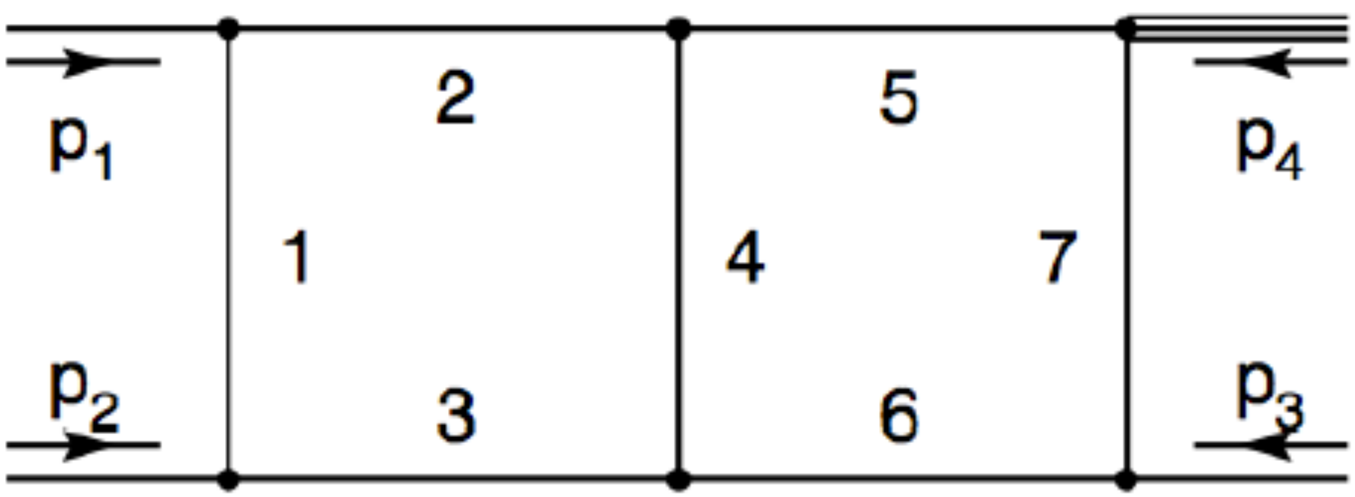
Integrate out loop momenta

$$G^{\mu_1\cdots\mu_R}_{l_1\cdots l_R} = (-1)^{N_\nu}\frac{1}{\prod_{j=1}^{N}\Gamma(\nu_j)}\int_0^\infty \prod_{j=1}^{N} dx_j\, x_j^{\nu_j-1}\delta\left(1-\sum_{l=1}^{N}x_l\right)$$
$$\times\sum_{m=0}^{[R/2]}\left(-\frac{1}{2}\right)^m \Gamma(N_\nu - m - LD/2)[(\tilde{M}^{-1}\otimes g)^{(m)}\tilde{l}^{(R-2m)}]^{\Gamma_1,\ldots,\Gamma_R}$$
$$\times\frac{\mathcal{U}^{N_\nu-(L+1)D/2-R}}{\mathcal{F}^{N_\nu-LD/2-m}}, \tag{7}$$

where

$$\mathcal{F}(\mathbf{x}) = \det(M)\left[\sum_{j,l=1}^{L} Q_j M^{-1}_{jl} Q_l - J - i\delta\right], \tag{8}$$
$$\mathcal{U}(\mathbf{x}) = \det(M)\,, \quad \tilde{M}^{-1} = \mathcal{U}M^{-1}\,, \quad \tilde{l} = \mathcal{U}v$$

U and F can be determined geometrically



$$\mathcal{U}(\mathbf{x}) = \sum_{T \in \mathcal{T}_1} \Big[\prod_{j \in \mathcal{C}(T)} x_j \Big] ,$$

$$\mathcal{F}_0(\mathbf{x}) = \sum_{\hat{T} \in \mathcal{T}_2} \Big[\prod_{j \in \mathcal{C}(\hat{T})} x_j \Big] (-s_{\hat{T}}) ,$$

$$\mathcal{F}(\mathbf{x}) = \mathcal{F}_0(\mathbf{x}) + \mathcal{U}(\mathbf{x}) \sum_{j=1}^{N} x_j m_j^2 .$$

$$\mathcal{U} = x_{123} x_{567} + x_4 x_{123567} ,$$

$$\mathcal{F} = (-s_{12})(x_2 x_3 x_{4567} + x_5 x_6 x_{1234} + x_2 x_4 x_6 + x_3 x_4 x_5)$$
$$+ (-s_{23}) x_1 x_4 x_7 + (-p_4^2) x_7 (x_2 x_4 + x_5 x_{1234}) ,$$

where $x_{ijk\dots} = x_i + x_j + x_k + \cdots$ and $s_{ij} = (p_i + p_j)^2$.

First generate primary sectors to eliminate Delta function

$$\int_0^\infty d^N x = \sum_{l=1}^{N} \int_0^\infty d^N x \prod_{\substack{j=1 \\ j\neq l}}^{N} \theta(x_l \geq x_j) \,.$$

$$x_j = \begin{cases} x_l t_j & \text{for } j < l \,, \\ x_l & \text{for } j = l \,, \\ x_l t_{j-1} & \text{for } j > l \end{cases}$$

$$G_l = \int_0^1 \prod_{j=1}^{N-1} dt_j \, \frac{\mathcal{U}_l^{N_\nu-(L+1)D/2}(\mathbf{t})}{\mathcal{F}_l^{N_\nu-LD/2}(\mathbf{t})} \,, \quad l = 1, \ldots, N \,.$$

Determine a sub-set of parameters ti

$$\mathcal{S} = \{t_{\alpha_1}, \ldots, t_{\alpha_r}\}$$

Then divide into r sub-sectors

$$\prod_{j=1}^{r} \theta(1 \geq t_{\alpha_j} \geq 0) = \sum_{k=1}^{r} \prod_{\substack{j=1 \\ j \neq k}}^{r} \theta(t_{\alpha_k} \geq t_{\alpha_j} \geq 0) \,.$$

$$t_{\alpha_j} \to \begin{cases} t_{\alpha_k} t_{\alpha_j} & \text{for } j \neq k \,, \\ t_{\alpha_k} & \text{for } j = k \,. \end{cases}$$

$$G_{lk} = \int_0^1 \left(\prod_{j=1}^{N-1} dt_j \, t_j^{a_j - b_j \epsilon} \right) \frac{\mathcal{U}_{lk}^{N_\nu - (L+1)D/2}}{\mathcal{F}_{lk}^{N_\nu - LD/2}} \,, \quad k = 1, \ldots, r \,.$$

$$\mathcal{U}_{lk_1 k_2 \ldots} = 1 + u(\mathbf{t}) \,, \quad \mathcal{F}_{lk_1 k_2 \ldots} = -s_0 + \sum_\beta (-s_\beta) f_\beta(\mathbf{t}) \,,$$

All the coefficients of divergences are finite (complicated).

Decomposition strategies

- **Hironaka's polyhedra game**
  Bogner and Weinzerl, Comput.Phys.Commun. 178 (2008) 596; A. V. Smirnov and V. A. Smirnov, JHEP 05 (2009) 004;

- **Geometric method**
  Kaneko and Ueda, Comput.Phys.Commun. 181 (2010) 1352

Iteration of certain strategy will show explicitly dimensional regulators, where poles can be extracted.

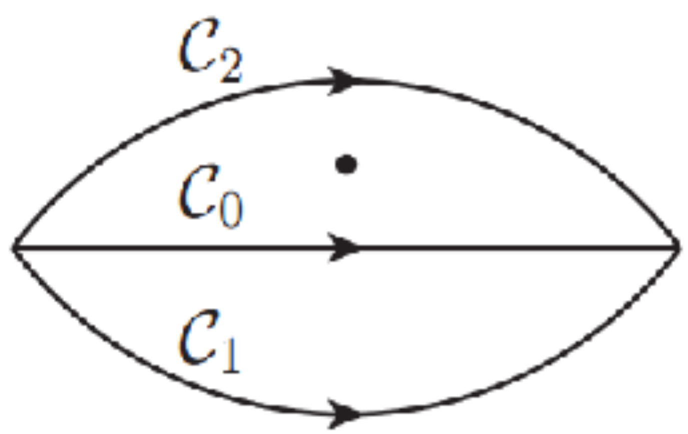
$$I_j = \int_0^1 dt_j \, t_j^{(a_j - b_j \epsilon)} \mathcal{I}(t_j, \{t_{i \neq j}\}, \epsilon) \ ,$$

$$I_j = \sum_{p=0}^{|a_j|-1} \frac{1}{a_j + p + 1 - b_j \epsilon} \frac{\mathcal{I}_j^{(p)}(0, \{t_{i \neq j}\}, \epsilon)}{p!} + \int_0^1 dt_j \, t_j^{a_j - b_j \epsilon} R(\vec{t}, \epsilon) \ .$$

$$I_j = -\frac{1}{b_j \epsilon} \mathcal{I}_j(0, \{t_{i \neq j}\}, \epsilon) + \int_0^1 dt_j \, t_j^{-1 - b_j \epsilon} \Big(\mathcal{I}(t_j, \{t_{i \neq j}\}, \epsilon) - \mathcal{I}_j(0, \{t_{i \neq j}\}, \epsilon) \Big) \ ,$$

Contour Deformation

$$I_s = C(\epsilon)\lim_{\delta\to 0}\int_0^1 \frac{\mathcal{D}(\vec{x},\epsilon)\,\mathcal{H}_s(\vec{x},\epsilon)}{[\mathcal{F}_s(\vec{x},m_i^2,s_{jk})-i\delta]^{a+b\epsilon}}$$

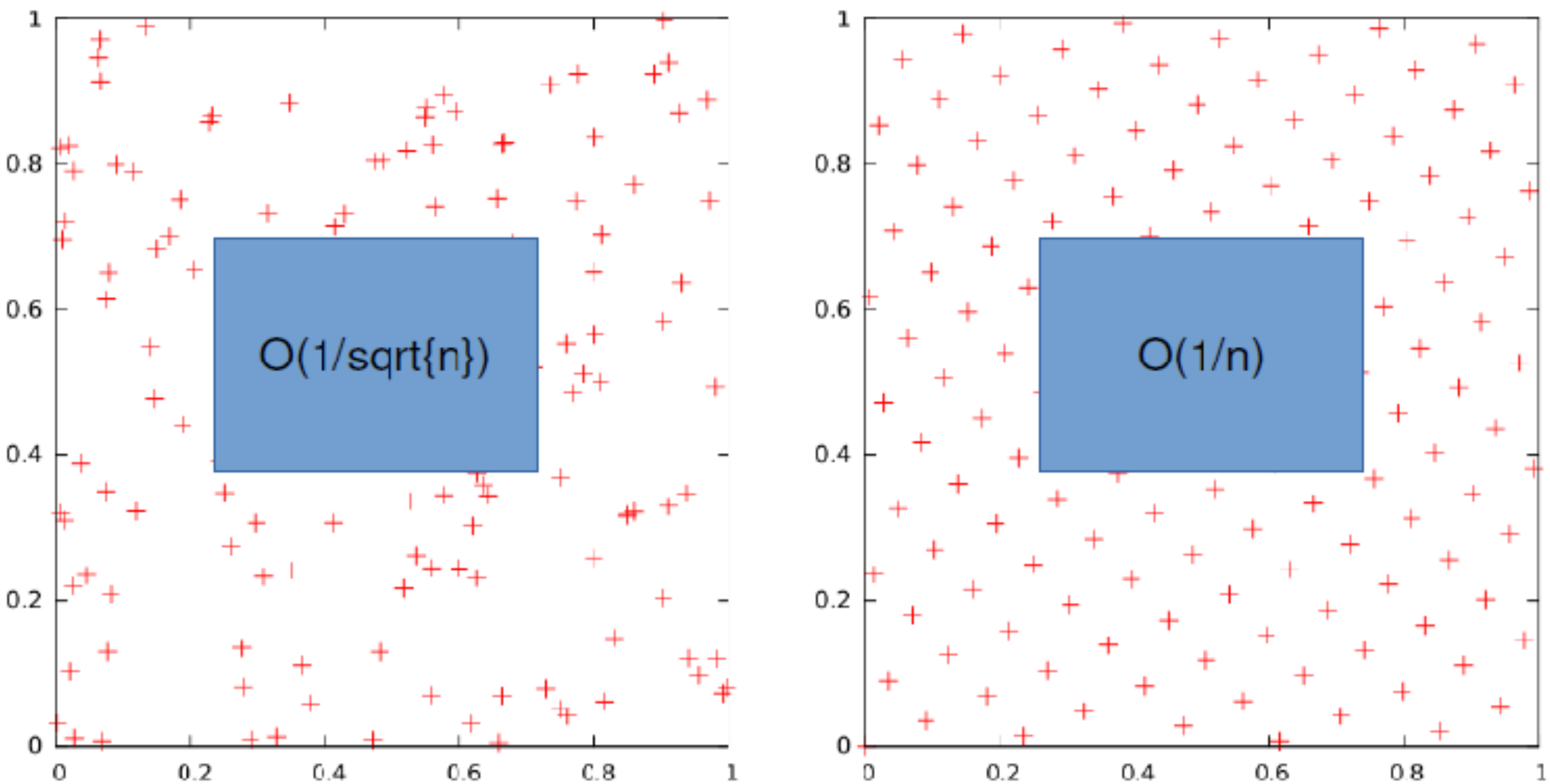


$$z_i \;=\; x_i - i\lambda x_i^{\alpha}(1-x_i)^{\beta}\frac{\partial\mathcal{F}_s}{\partial x_i}$$

$$\lim_{\delta\to 0}\int_0^1 \frac{\mathcal{D}(\vec{x},\epsilon)\,\mathcal{H}_s(\vec{x},\epsilon)}{[\mathcal{F}_s(\vec{x},m_i^2,s_{jk})-i\delta]^{a+b\epsilon}} = \int_{\mathcal{C}}\frac{\mathcal{D}(\vec{z},\epsilon)\,\mathcal{H}_s(\vec{z},\epsilon)}{[\mathcal{F}_s(\vec{z},m_i^2,s_{jk})]^{a+b\epsilon}}$$

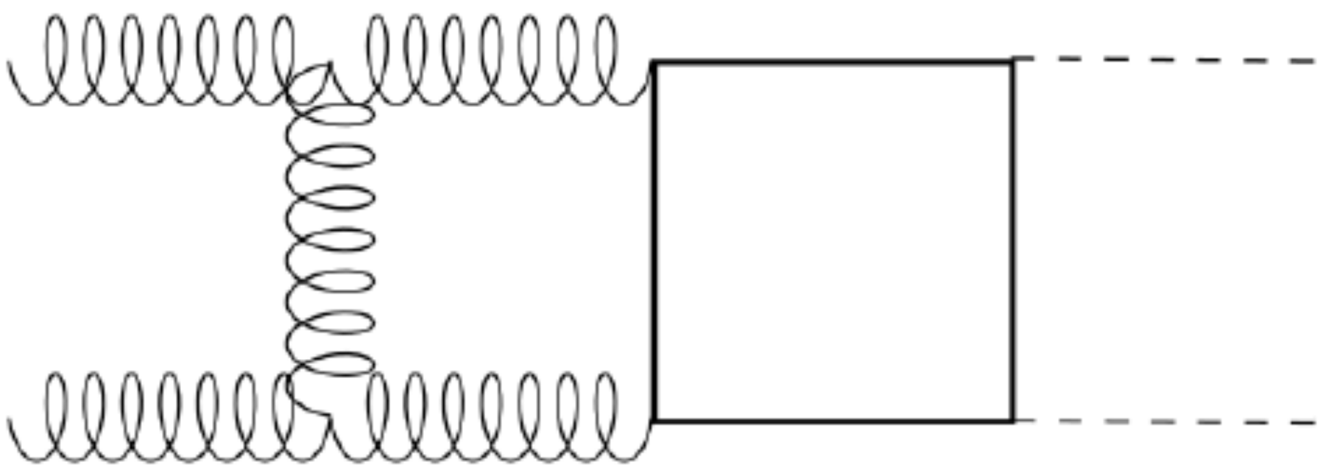
Improve via quasi-Monte-Carlo

$$I(f) = \int_0^1 d^s x f(\vec{x}) \qquad I_{estimate}(f) = \sum_{i=0}^{n-1} f(\vec{x}_i)$$



Z. Li et al., Chin.Phys. C40 (2016) no.3, 033103

Implementation on GPU

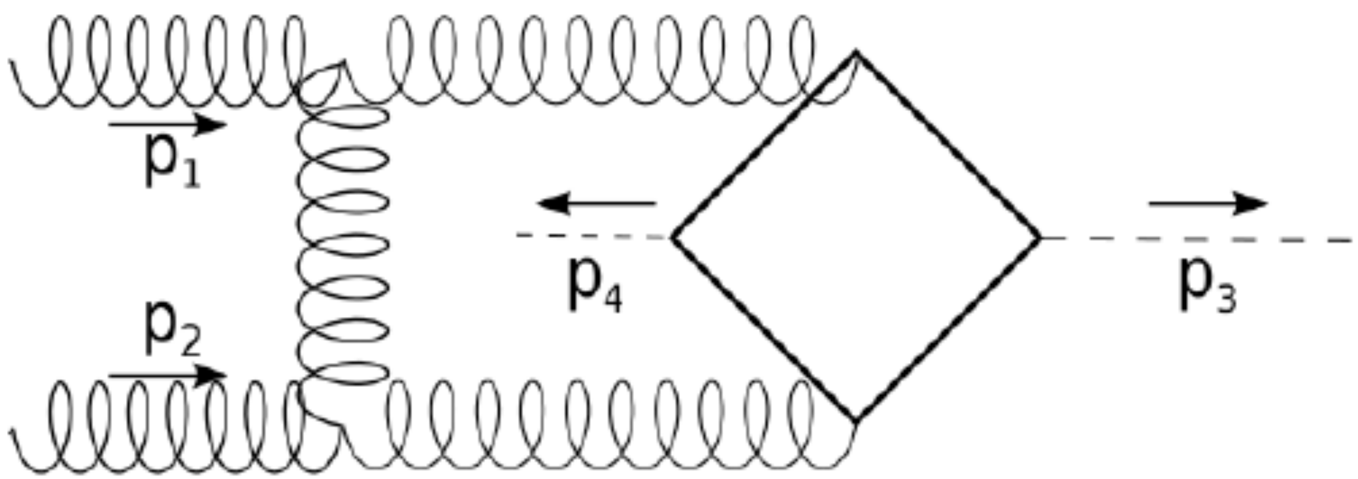


$$I_C = e^{-2\epsilon\gamma_E} s^{-3-2\epsilon} \sum_{i=0}^{i=2} \frac{P_i}{\epsilon^i}.$$

	Vegas/CPU	QMC/GPU
P_2	$-7.959 \pm 0.009 - 10.586i \pm 0.009i$	$-7.949 \pm 0.003 - 10.585i \pm 0.005i$
P_1	$3.9 \pm 0.1 - 28.1i \pm 0.1i$	$3.831 \pm 0.005 - 28.022i \pm 0.005i$
P_0	$-3.9 \pm 0.8 + 92.3i \pm 0.8i$	$-4.63 \pm 0.07 + 92.13i \pm 0.07i$
Integration Time	45540s	19s

Z. Li et al., Chin.Phys. C40 (2016) no.3, 033103

Implementation on GPU



$$I_D = e^{-2\epsilon\gamma_E} s^{-3-2\epsilon} \sum_{i=0}^{i=2} \frac{P_i}{\epsilon^i}.$$

	Vegas/CPU	QMC/GPU
P_2	$-3.848 \pm 0.004 + 0.0005i \pm 0.003i$	$-3.8482 \pm 0.0007 + 0.0004i \pm 0.0003i$
P_1	$3.81 \pm 0.03 - 6.41i \pm 0.03i$	$3.83 \pm 0.02 - 6.40i \pm 0.02i$
P_0	$77.2 \pm 0.2 + 20.1i \pm 0.2i$	$77.2 \pm 0.1 + 19.9i \pm 0.1i$
Integration Time	54290s	20s

Z. Li et al., Chin.Phys. C40 (2016) no.3, 033103

Mixed QCD-EW corrections for Higgs boson production at e+e- colliders

$\sqrt{s}$ (GeV)	σ_{LO} (fb)	σ_{NLO} (fb)	σ_{NNLO} (fb)	$\sigma_{\mathrm{NNLO}}^{\mathrm{exp.}}$ (fb)
240	256.3(9)	228.0(1)	230.9(4)	230.9(4)
250	256.3(9)	227.3(1)	230.2(4)	230.2(4)
300	193.4(7)	170.2(1)	172.4(3)	172.4(3)
350	138.2(5)	122.1(1)	123.9(2)	123.6(2)
500	61.38(22)	53.86(2)	54.24(7)	54.64(10)

TABLE I. The NNLO predictions for the total cross sections at various collider energies.

Y.Gong et al., Phys.Rev. D95 (2017) no.9, 093003

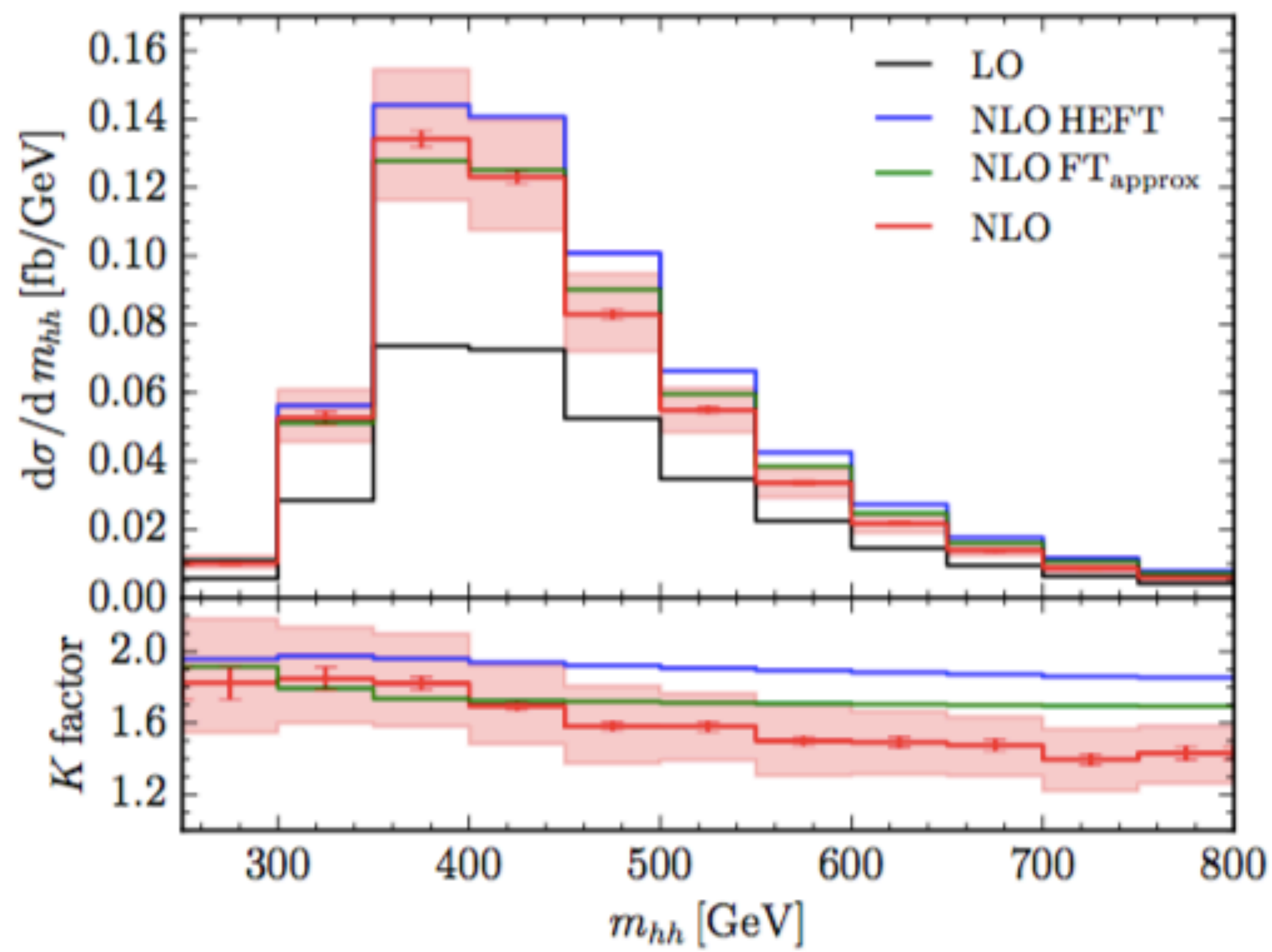
Mixed electroweak-QCD corrections to e+e−→HZ at Higgs factories

$\sqrt{s}$	schemes	$\sigma_{\rm LO}$ (fb)	$\sigma_{\rm NLO}$ (fb)	$\sigma_{\rm NNLO}$ (fb)
240	$\alpha(0)$	223.14 ± 0.47	229.78 ± 0.77	$232.21^{+0.75+0.10}_{-0.75-0.21}$
	$\alpha(M_Z)$	252.03 ± 0.60	$228.36^{+0.82}_{-0.81}$	$231.28^{+0.80+0.12}_{-0.79-0.25}$
	G_μ	239.64 ± 0.06	$232.46^{+0.07}_{-0.07}$	$233.29^{+0.07+0.03}_{-0.06-0.07}$
250	$\alpha(0)$	223.12 ± 0.47	229.20 ± 0.77	$231.63^{+0.75+0.12}_{-0.75-0.21}$
	$\alpha(M_Z)$	252.01 ± 0.60	$227.67^{+0.82}_{-0.81}$	$230.58^{+0.80+0.14}_{-0.79-0.25}$
	G_μ	239.62 ± 0.06	231.82 ± 0.07	$232.65^{+0.07+0.04}_{-0.07-0.07}$

TABLE II: The unpolarized Higgsstrahlung cross sections at $\sqrt{s} = 240(250)$ GeV in three different input schemes. To estimate the uncertainties caused by the input parameters (first entry), we take $M_W = 80.385 \pm 0.015$ GeV, $m_t = 174.2 \pm 1.4$ GeV and $\Delta\alpha^{(5)}_{\rm had}(M_Z) = 0.02764 \pm 0.00013$. We also change the strong coupling constant from $\alpha_s(M_Z)$ to $\alpha_s(\sqrt{s})$ (second entry) with its central value taken as $\alpha_s = \alpha_s(\sqrt{s}/2)$. For the conversion from the $\alpha(0)$ scheme to the $\alpha(M_Z)$ and G_μ schemes, we use $\Delta\alpha(M_Z)|_{\rm NLO} = \Delta\alpha(M_Z)|_{\rm NNLO} = 0.059$ and $\Delta r|_{\rm NLO} = 0.0293$, $\Delta r|_{\rm NNLO} = 0.0331$, respectively.

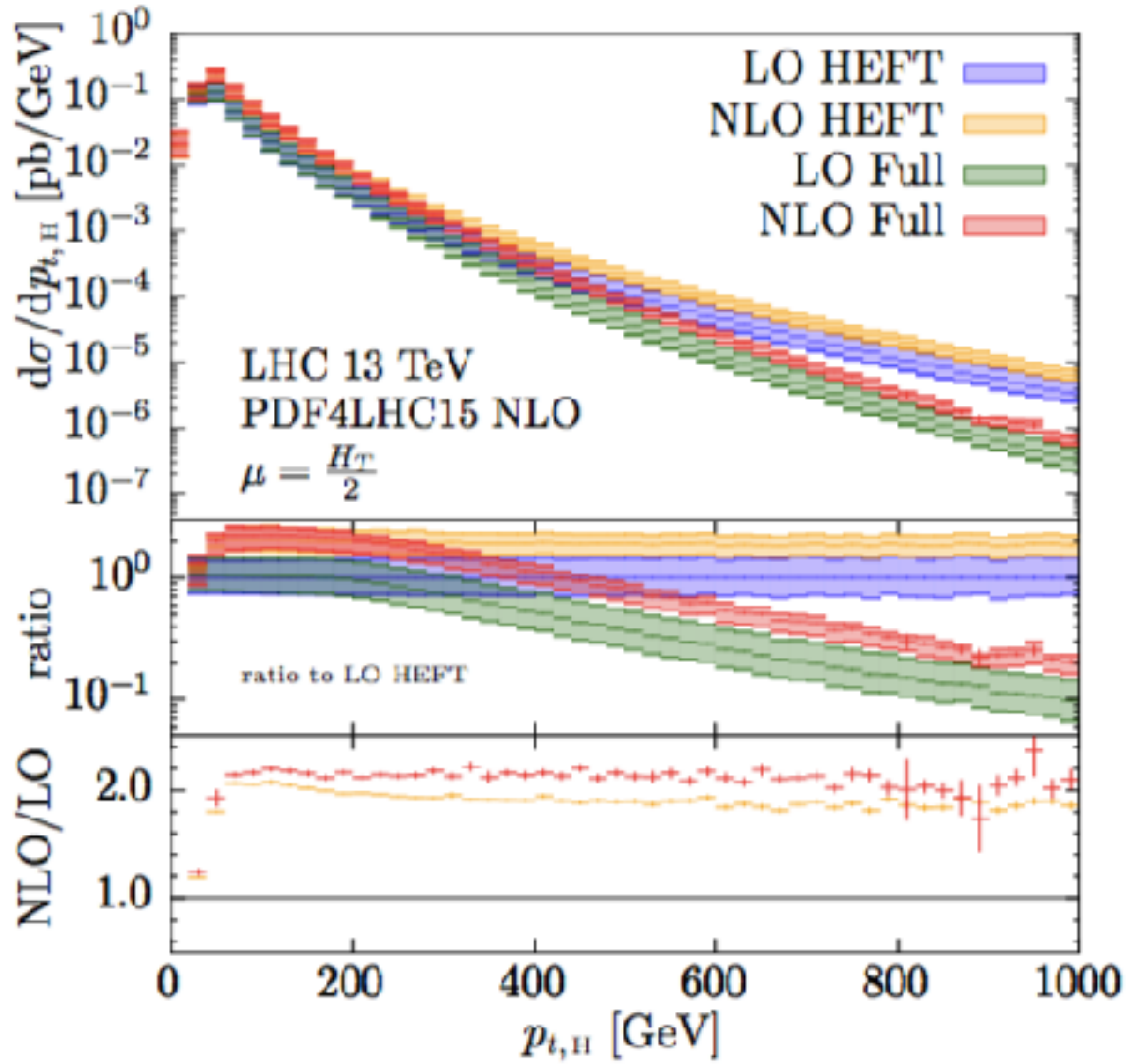
Qing-Feng Sun et al., Phys.Rev. D96 (2017) no.5, 051301

Higgs Boson Pair Production in Gluon Fusion at Next-to-Leading Order with Full Top-Quark Mass Dependence



S. Borowka et. al. Phys.Rev.Lett. 117 (2016) no.1, 012001

NLO QCD corrections to Higgs boson plus jet production with full top-quark mass dependence



S. P. Jones, et. al. arXiv:1802.00349

Prospect

- Numerical approach can give numbers for multi-scale multi-loop processes.
- Hard to identify physics structures, eg. large logs, symmetries etc..
- Many precise predictions can be expected in a few years, Higgs/single-top/ttH/dijet/new-physics?.
- Still hope breakthrough on analytical approach.

Thank you!