Infrared subtractions in perturbative quantum chromodynamics

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School and Workshop on pQCD
Zhejiang University Yuquan Campus
Hangzhou, China, 26-30 March 2018

Lecture 2: Infrared subtractions at next-to-leading order

Recap

We considered the behavior of QCD amplitudes describing the radiation of a **soft** or **collinear** parton. We saw that:

- The leading singular behavior of QCD amplitudes in the soft and collinear limits gives rise to non-converging integrals in the phase space of the radiated partons.
- In the soft limit, QCD amplitudes factorize into:
  - an amplitude describing the hard scattering, and
  - an eikonal function which is a function of the color charges and momenta of the soft and hard partons.
- In the collinear limit, the QCD amplitude factorizes into:
  - an amplitude describing the hard scattering with one of the momenta rescaled to account for the collinear emission, and
  - a splitting function which is a function of the color charges and energies of the collinear partons, and which gives the probability of the collinear emission.
- The divergent behavior may be **regulated** by performing the phase space integral in $d = 4 - 2\epsilon$ dimensions, resulting in poles in $1/\epsilon$.
- These poles are **guaranteed** to cancel against infrared poles from the loop integration in the virtual corrections, leaving a finite result.

Integrate over full phase space?

One option would be to just integrate analytically over the full phase space of the radiated partons, obtain the poles in $1/\epsilon$, and cancel these against the virtual poles. However, this would mean that we lose **all** the information about the kinematics of the radiated parton.

This is acceptable in a few cases (e.g. in Higgs production) where we are most interested in the *inclusive cross section*.

However, for the vast majority of interesting processes that are studied, this is **not** acceptable:

- Partons give rise to hadronic energy flows called *jets*.
- The properties of jets frequently provide essential information when analyzing an experimental signature.
- May want to apply *cuts* on the jet observables.
- Since jets are usually defined through an *iterative algorithm*, analytic integration over the phase space is not appropriate.

Thus, in general, it is not feasible to simply integrate over the full phase space of the radiated parton. We have to find a different strategy to extract the poles in the real radiation.

Unresolved phase space

We cannot integrate over the full phase space of radiated partons since these partons have observable effects that we want to preserve.

However, soft or collinear radiated partons should **not** have an experimental signature – they are *unresolved*.

We can define a very large class of observables called *infrared-safe* observables which are **unchanged** by the presence of unresolved (i.e. soft or collinear) partons. (Almost) All observables that are routinely studied and reported at the LHC are infrared-safe.

Therefore we can integrate over the phase space of the *unresolved partons* without losing information about the final state.

The objective is to separate the resolved and unresolved phase spaces of the radiated partons.

Notation (1)

We write the partonic cross section for a general LO *color singlet* production

$$q(p_1)\bar{q}(p_2) \to V + g(p_3),$$

as

$$\mathrm{d}\sigma^{\mathrm{LO}} = \frac{1}{2s} F_{LM}(1,2),$$

where

- V is any color singlet final state, e.g. e^+e^-, W^+W^-, ZH, etc,
- s is the partonic center-of-mass energy,
- the factor $\frac{1}{2s}$ is the flux factor,
- the function $F_{LM}(1,2)$ is

$$F_{LM}(1,2) = \mathrm{dLips}_V \, |\mathcal{M}(1,2,V)|^2 \, \mathcal{F}_{\mathrm{kin}}(1,2,V),$$

  with

  - dLips_V is the Lorentz-invariant phase space for the colorless particles V, including the momentum-conserving delta function.
  - $\mathcal{M}(1,2,V)$ is the matrix element for the process $q_1\bar{q}_2 \to V$.
  - $\mathcal{F}_{\mathrm{kin}}$ defines an infrared-safe observable.

Notation (2)

The real emission correction $q(p_1)\bar{q}(p_2) \to V + g(p_3)$ is, in this notation,

$$\mathrm{d}\sigma^{\mathrm{R}} = \frac{1}{2s} \int [\mathrm{d}p_3] F_{LM}(1,2,3),$$

where

- $[\mathrm{d}p_3]$ is the d-dimensional phase space measure of the emitted gluon,

$$[\mathrm{d}p_3] = \frac{\mathrm{d}^{d-1} p_3}{(2\pi)^{d-1} 2E_3} \theta(E_{\mathrm{max}} - E_3),$$

  with E_{max} an energy parameter that can be fixed by energy conservation requirements;
- the function $F_{LM}(1,2,3)$ is defined analously to the LO case

$$F_{LM}(1,2,3) = \mathrm{dLips}_V \, |\mathcal{M}(1,2,3,V)|^2 \, \mathcal{F}_{\mathrm{kin}}(1,2,3,V),$$

  with $\mathcal{M}(1,2,3,V)$ the matrix element for the process $q(p_1)\bar{q}(p_2) \to V + g(p_3)$.

Basic idea of subtraction (1)

Suppose we have a function $\mathcal{S}$ with the following two properties:

- It reproduces the leading singular behavior of $F_{LM}(1,2,3)$ in all soft and collinear limits, and
- It can be integrated analytically in the d-dimensional phase space of the emitted parton.

Then we can write the real-emission cross section as

$$2s \times \mathrm{d}\sigma^{\mathrm{R}} = \int [\mathrm{d}p_3] F_{LM}(1,2,3) = \int [\mathrm{d}p_3]\big(F_{LM}(1,2,3) - \mathcal{S}\big) + \int [\mathrm{d}p_3]\mathcal{S}$$

- In the first term, the leading singular behavior in $F_{LM}(1,2,3)$ which gives rise to the non-convergent integrals over the phase space $[\mathrm{d}p_3]$ is removed by the *subtraction term* $\mathcal{S}$.
- Therefore the integral converges and we can integrate it in **four dimensional phase space** without fear.
  In practice, this is done numerically using a **Monte Carlo (MC) integration**, which allows us to simultaneously obtain the differential distribution for any infrared-safe observable, including observables of the jet.

Basic idea of subtraction (2)

Suppose we have a function $\mathcal{S}$ with the following two properties:

- It reproduces the leading singular behavior of $F_{LM}(1,2,3)$ in all soft and collinear limits, and
- It can be integrated analytically in the d-dimensional phase space of the emitted parton.

$$2s \times d\sigma^{R} = \int [dp_3] F_{LM}(1,2,3) = \int [dp_3] \big(F_{LM}(1,2,3) - \mathcal{S} \big) + \int [dp_3] \mathcal{S}$$

- The second term contains only the **unresolved** partons. We can therefore integrate over the phase space $[dp_3]$ *analytically* without affecting the jet observables, since these partons are unobservable.
- By performing this integration in d-dimensions, we **extract** poles in $1/\epsilon$ which describe the singular behavior of $\mathcal{S}$, and thus of the amplitude.
- The factorization of the amplitude-squared in the soft and collinear limits implies that these poles in $1/\epsilon$ will multiply a function F_{LM} of **lower particle multiplicity.**

Subtraction scheme for color singlet production at NLO

- The actual form of the subtraction $\mathcal{S}$ is irrelevant, provided it has the two properties mentioned before.
- Different subtraction schemes make use of different forms of $\mathcal{S}$, but these are formally equivalent.
- We will study a subtraction scheme known as FKS subtraction, developed by S. Frixione, Z. Kunszt, and A. Signer. [a]
- In this scheme, the subtraction function $\mathcal{S}$ is constructed directly from the limits.
- This is one of the most widely-used subtraction schemes for NLO calculations; the other being the **Catani-Seymour dipole method**.[b]
- The existence of these schemes enabled a large number of NLO calculations to be performed over the last two decades, and were essential for the NLO automation effort of the last 6-7 years.

[a] S. Frixione, Z. Kunszt and A. Signer, Nucl. Phys. B **485** (1997), 291; Erratum: [Nucl. Phys. B **510** (1998), 503]; S. Frixione, Nucl. Phys. **B507** (1997), 295

[b] S. Catani and M. H. Seymour, Nucl. Phys. B **485**, 291 (1997), Erratum: [Nucl. Phys. B **510**, 503 (1998)].

Notation (3)

- I define the unit vector $\vec{n}_i$ that describes the direction of the momentum of the i-th particle in $(d-1)$-dimensional space, i.e. $p_i = E_i(1, \vec{n}_i)$.
- I also define $\rho_{ij} = 1 - \vec{n}_i \cdot \vec{n}_j = 1 - \cos\theta_{ij}$ where θ_{ij} is the angle between partons i and j.
- Thus $0 \leq \rho_{ij} \leq 2$.
- Note that this means we can always write

$$p_i \cdot p_j = E_i(1, \vec{n}_i) \cdot E_j(1, \vec{n}_j) = E_i E_j (1 - \vec{n}_i \cdot \vec{n}_j) = E_i E_j \rho_{ij}$$

I define the soft and collinear operators S_i and C_{ij} as follows:

$$S_i A = \lim_{E_i \to 0} A, \quad C_{ij} A = \lim_{\rho_{ij} \to 0} A.$$

Furthermore, I will indicate the integration over the phase space of emitted partons with angle brackets, so that

$$\mathrm{d}\sigma^{\mathrm{R}} = \frac{1}{2s} \int [\mathrm{d}p_3] F_{LM}(1,2,3) \equiv \langle F_{LM}(1,2,3) \rangle.$$

Overall picture of the subtraction scheme

Now I write

$$\begin{aligned}\langle F_{LM}(1,2,3)\rangle &= \langle (I-S_3)F_{LM}(1,2,3)\rangle + \langle S_3 F_{LM}(1,2,3)\rangle \\ &= \langle (I-C_{31}-C_{32})(I-S_3)F_{LM}(1,2,3)\rangle \\ &\quad +\langle (C_{31}+C_{32})(I-S_3)F_{LM}(1,2,3)\rangle + \langle S_3 F_{LM}(1,2,3)\rangle .\end{aligned}$$

- The first term has no soft or collinear singularities, because the leading singular behavior of $F_{LM}(1,2,3)$ is removed by the operators S_3, C_{31}, and C_{32}.
- Therefore the integral in the first term is convergent. This is the *subtracted real emission term* which can be integrated in four dimensions (typically using Monte Carlo integration).
- The subtraction is performed in a **nested manner**: we first remove the soft singularities, then the collinear ones. We will discuss this further towards the end of this lecture.
- The second and third terms involve integrals over various limits of $F_{LM}(1,2,3)$, which will give rise to $1/\epsilon$ poles once we integrate in d-dimensions.
  We will look at these in greater detail now.

Integrated soft limit (1)

We first consider the term $\langle S_3 F_{LM}(1,2,3)\rangle$.

We saw in Lecture 1 that in the soft limit, the amplitude-squared factorizes into the amplitude-squared for the hard process without the soft gluon, multiplied by an eikonal factor:

$$S_3 F_{LM}(1,2,3) = \mathrm{Eik}(1,2,3) F_{LM}(1,2).$$

- The momentum of the soft gluon p_3 enters only through the eikonal factor, and not through F_{LM}.
- Thus only the eikonal function needs to be integrated over the d-dimensional phase space $[\mathrm{d}p_3]$.
- We can perform this integration *without knowing anything about the functional form of* $F_{LM}(1,2)$.
- So this is **completely general** for the process $q(p_1)\bar{q}(p_2) \to V + g(p_3)$ with p_3 soft and V a *color singlet*.

Integrated soft limit (2)

The eikonal function is

$$\mathrm{Eik}(1,2,3) = g_s^2 \mu_0^{2\epsilon} 2C_F \frac{p_1 \cdot p_2}{(p_1 \cdot p_3)(p_2 \cdot p_3)} = g_s^2 \mu_0^{2\epsilon} 2C_F \frac{E_1 E_2 \rho_{12}}{(E_1 E_3 \rho_{13})(E_2 E_3 \rho_{23})}$$
$$= g_s^2 \mu_0^{2\epsilon} 2C_F \frac{1}{E_3^2} \frac{\rho_{12}}{\rho_{13}\rho_{23}},$$

using $p_i \cdot p_j = E_i E_j \rho_{ij}$. We work in the frame where the partons 1 and 2 collide head-on, so $\theta_{12} = \pi \Rightarrow \rho_{12} = 1 - \cos\theta_{12} = 2 = \rho_{13} + \rho_{23}$. Then

$$\mathrm{Eik}(1,2,3) = g_s^2 \mu_0^{2\epsilon} 2C_F \frac{1}{E_3^2} \frac{2}{\rho_{13}(2-\rho_{13})} = g_s^2 \mu_0^{2\epsilon} C_F \frac{1}{E_3^2} \frac{1}{\eta_{13}(1-\eta_{13})}$$

where I have introduced $\eta_{ij} = \rho_{ij}/2$.

Integrated soft limit (2)

Recall from last lecture that the d-dimensional phase space measure is (including the θ function that we included earlier)

$$[\mathrm{d}p_3] = \frac{E_3^{d-3}\mathrm{d}E_3\,\mathrm{d}\cos\theta_{13}(1-\cos^2\theta_{13})^{d/2-2}\,\mathrm{d}\Omega_3^{(d-2)}}{2(2\pi)^{d-1}}\theta(E_{\max}-E_3)$$

where we are using the azimuthal angle θ_{13}, i.e. the angle of $\vec{p}_3$ with respect to $\vec{p}_1$.

Using $\eta_{13} = (1-\cos\theta_{13})/2$ we can rewrite the integration over $\cos\theta_{13}$ as

$$\int_{-1}^{+1}\mathrm{d}\cos\theta_{13}(1-\cos^2\theta_{13})^{d/2-2} = -\int_{+1}^{0}\frac{2\mathrm{d}\eta_{13}}{[4\eta_{13}(1-\eta_{13})]^{\epsilon}} = \int_{0}^{1}\frac{2\mathrm{d}\eta_{13}}{[4\eta_{13}(1-\eta_{13})]^{\epsilon}},$$

as we did in the previous lecture.

Then the integral over the phase space is

$$\int[\mathrm{d}p_3] = \int_0^{E_{\max}}\mathrm{d}E_3 E_3^{1-2\epsilon}\int_0^1\frac{2\mathrm{d}\eta_{13}}{[4\eta_{13}(1-\eta_{13})]^{\epsilon}}\int\frac{\mathrm{d}\Omega_3^{(d-2)}}{2(2\pi)^{d-1}}$$

Integrated soft limit (3)

Putting these results together, the integral of the eikonal function over the phase space of the soft gluon is

$$\int [\mathrm{d}p_3]\mathrm{Eik}(1,2,3) = C_F g_s^2 \mu_0^{2\epsilon} \int_0^{E_{\max}} \mathrm{d}E_3 E_3^{-1-2\epsilon} \int_0^1 \frac{2\mathrm{d}\eta_{13}}{[4\eta_{13}(1-\eta_{13})]^\epsilon} \frac{1}{\eta_{13}(1-\eta_{13})} \times \int \frac{\mathrm{d}\Omega_3^{(d-2)}}{2(2\pi)^{d-1}}$$

$$= C_F \left[-\frac{1}{2\epsilon} E_{\max}^{-2\epsilon} \right] \left[2^{1-2\epsilon} B(-\epsilon, -\epsilon) \right] \left[\frac{g_s^2}{8\pi^2} \frac{\mu_0^{2\epsilon} (4\pi)^\epsilon}{\Gamma(1-\epsilon)} \right],$$

where $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ is the Euler Beta function, and I have used the solid angle

$$\Omega^{(d-2)} = \frac{2\pi^{d/2-1}}{\Gamma(d/2-1)} \Rightarrow \int \frac{\mathrm{d}\Omega_3^{(d-2)}}{2(2\pi)^{d-1}} = \frac{2^{-3+2\epsilon}\pi^{-2+\epsilon}}{\Gamma(1-\epsilon)} = \frac{1}{8\pi^2} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}$$

The parameter $E_{\max}$ is essential in order for the energy integral **to be bounded from above.** This parameter should be sufficiently large, but otherwise is arbitrary. In practice, we can always take $E_{\max} = \sqrt{s}/2$.

Integrated soft limit (4)

Using $x\Gamma(x) = \Gamma(1+x)$ I can write

$$B(-\epsilon, -\epsilon) = \frac{\Gamma^2(-\epsilon)}{\Gamma(-2\epsilon)} = \frac{-2\epsilon}{\epsilon^2}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} = -\frac{2}{\epsilon}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}.$$

I also define the shorthand

$$[\alpha_s] = \frac{g_s^2}{8\pi^2}\frac{\mu_0^{2\epsilon}(4\pi)^\epsilon}{\Gamma(1-\epsilon)}.$$

Putting this together

$$\int [\mathrm{d}p_3]\mathrm{Eik}(1,2,3) = 2C_F[\alpha_s]\frac{1}{\epsilon^2}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}(2E_{\max})^{-2\epsilon}$$

$$\Rightarrow \langle S_3 F_{LM}(1,2,3)\rangle = 2C_F[\alpha_s]\frac{1}{\epsilon^2}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}(2E_{\max})^{-2\epsilon}F_{LM}(1,2).$$

Note that this term has poles in $1/\epsilon^2$ and lower.

Integrated soft-collinear limit (1)

We now consider the integrated **collinear** limit

$$\langle (C_{31} + C_{32})(I - S_3) F_{LM}(1,2,3) \rangle,$$

starting with the integrated **soft-collinear limit** $C_{31}S_3$. We have

$$\begin{aligned} C_{31}S_3F_{LM}(1,2,3) &= C_{31}\left[\mathrm{Eik}(1,2,3)\right] F_{LM}(1,2) \\ &= C_{31}\left[g_s^2\mu_0^{2\epsilon} C_F \frac{1}{E_3^2}\frac{1}{\eta_{13}(1-\eta_{13})}\right] F_{LM}(1,2) \\ &= g_s^2\mu_0^{2\epsilon} C_F \frac{1}{E_3^2}\frac{1}{\eta_{13}} F_{LM}(1,2), \end{aligned}$$

having used $C_{31} \Leftrightarrow \eta_{31} \to 0$.

The C_{32} limit is the same with the exchange $\eta_{13} \to \eta_{23} = 1 - \eta_{13}$

$$C_{32}S_3F_{LM}(1,2,3) = g_s^2\mu_0^{2\epsilon} C_F \frac{1}{E_3^2}\frac{1}{1-\eta_{13}} F_{LM}(1,2).$$

Cancellation of soft and soft-collinear limits

This means

$$(I - C_{31} - C_{32}) S_3 F_{LM}(1,2,3) = (I - C_{31} - C_{32}) \mathrm{Eik}(1,2,3) F_{LM}(1,2)$$
$$= g_s^2 \mu_0^{2\epsilon} C_F \frac{1}{E_3^2} \left[\frac{1}{\eta_{13}(1-\eta_{13})} - \frac{1}{\eta_{13}} - \frac{1}{1-\eta_{13}} \right] = 0,$$

i.e. the **soft** and **soft-collinear** poles cancel *prior to* integration over the phase space!

Therefore we do not need to consider the soft limits at all, only the collinear limits.

This is a special case for color singlet production.

In general, the partons in the hard process are not back-to-back, in which case $\eta_{31} + \eta_{32} \neq 1$ and the soft and soft-collinear pieces **do not cancel.**

Integrated collinear limits (1)

We now look at the integrated collinear limit $\langle C_{31} F_{LM}(1,2,3)\rangle$.

Recall from the first lecture that

$$\begin{aligned}
C_{31} F_{LM}(1,2,3) &= 2 g_s^2 \mu_0^{2\epsilon} \frac{1}{(p_1 - p_3)^2} P_{qq}\left(\frac{E_1}{E_1 - E_3}\right) F_{LM}(1-3,2) \\
&= -g_s^2 \mu_0^{2\epsilon} \frac{1}{E_1 E_3 \rho_{13}} P_{qq}\left(\frac{E_1}{E_1 - E_3}\right) F_{LM}(1-3,2) \\
&= -g_s^2 \mu_0^{2\epsilon} \frac{1}{E_3^2 \rho_{13}} (1-z) P_{qq}(1/z) F_{LM}(z\cdot 1,2),
\end{aligned}$$

where I have defined $z = 1 - E_3/E_1$. We can also use

$$P_{qq}(1/z) = C_F \left(\frac{1+1/z^2}{1-1/z} - \epsilon(1-1/z)\right) = -C_F \frac{1}{z}\left(\frac{1+z^2}{1-z} - \epsilon(1-z)\right) = -\frac{P_{qq}(z)}{z},$$

so that

$$C_{31} F_{LM}(1,2,3) = g_s^2 \mu_0^{2\epsilon} \frac{1}{E_3^2 \rho_{13}} (1-z) P_{qq}(z) \frac{F_{LM}(z\cdot 1,2)}{z}.$$

Aside: Commutation of soft and collinear limits

Using $z = 1 - E_3/E_1$, the soft limit $E_3 \to 0 \Leftrightarrow z \to 1$. Therefore

$$S_3 C_{31} F_{LM}(1,2,3) = \lim_{z \to 1} \left[g_s^2 \mu_0^{2\epsilon} \frac{1}{E_3^2 \rho_{13}} (1-z) P_{qq}(z) \frac{F_{LM}(z \cdot 1, 2)}{z} \right].$$

Using

$$(1-z) P_{qq}(z) = C_F (1 + z^2 + \epsilon (1-z)^2) \Rightarrow \lim_{z \to 1} [(1-z) P_{qq}(z)] = 2 C_F.$$

So

$$S_3 C_{31} F_{LM}(1,2,3) = g_s^2 \mu_0^{2\epsilon} \frac{2 C_F}{E_3^2 \rho_{13}} = g_s^2 C_F \frac{1}{E_3^2} \frac{1}{\eta_{13}},$$

which is identical to what we obtained when we computed the limits the other way around, i.e. $C_{31} S_3 F_{LM}(1,2,3)$.

This tells us that the soft and collinear limits **commute**, at least at NLO.
In some way, this justifies our method of removing the soft singularities first, and then the collinear singularities. However, there is something more profound going on here, which we will discuss in the next lecture.

Integrated collinear limits (2)

Integrating over the phase space of the collinear gluon gives

$$\langle C_{31} F_{LM}(1,2,3)\rangle = g_s^2 \mu_0^{2\epsilon} \int \frac{\mathrm{d}\Omega_3^{(d-2)}}{2(2\pi)^{d-1}} \int_0^1 \frac{2\mathrm{d}\eta_{13}}{[4\eta_{13}(1-\eta_{13})]^{\epsilon}} \frac{1}{2\eta_{13}} \times \mathcal{E},$$

where $\mathcal{E}$ is the energy integral (using $E_3 = (1-z)E_1$):

$$\begin{aligned}\mathcal{E} &= \int_0^{E_{\max}} \mathrm{d}E_3 E_3^{-1-2\epsilon} (1-z) P_{qq}(z) \frac{F_{LM}(z\cdot 1, 2)}{z} \\ &= -E_1^{-2\epsilon} \int_1^{z_{\min}} \mathrm{d}z (1-z)^{-1-2\epsilon} (1-z) P_{qq}(z) \frac{F_{LM}(z\cdot 1, 2)}{z} \\ &= E_1^{-2\epsilon} \int_{z_{\min}}^1 \frac{\mathrm{d}z}{(1-z)^{2\epsilon}} P_{qq}(z) \frac{F_{LM}(z\cdot 1, 2)}{z},\end{aligned}$$

where $z_{\min} = 1 - E_{\max}/E_1$.

Integrated collinear limits (3)

Recall that $F_{LM}(z \cdot 1, 2)$ includes a **momentum-conserving delta-function**:

$$\delta(zE_1 + E_2 - E_V).$$

This implies that for $z < z_{\min}$ there is insufficient energy to produce V, and the delta-function gives 0. Thus that we can replace $z_{\min}$ with 0 in the integral over z.

Using also

$$g_s^2 \mu_0^{2\epsilon} \int \frac{\mathrm{d}\Omega_3^{(d-2)}}{2(2\pi)^{d-1}} = [\alpha_s] \qquad \int_0^1 \frac{2\mathrm{d}\eta_{13}}{[4\eta_{13}(1-\eta_{13})]^{\epsilon}} \frac{1}{2\eta_{13}} = -\frac{4^{-\epsilon}}{\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)},$$

we have

$$\langle C_{31} F_{LM}(1,2,3) \rangle = -[\alpha_s] \frac{1}{\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} s^{-\epsilon} \int_0^1 \frac{\mathrm{d}z}{(1-z)^{2\epsilon}} P_{qq}(z) \frac{F_{LM}(z \cdot 1, 2)}{z}$$

where I have used $(2E_1)^{-2\epsilon} = s^{-\epsilon}$ in the center-of-mass frame .

Extracting soft singularities from integrated collinear limits (1)

Recall

$$P_{qq}(z) = C_F \left(\frac{1+z^2}{1-z} + \epsilon(1-z) \right)$$

then it is clear that the integral

$$\int_0^1 \frac{\mathrm{d}z}{(1-z)^{2\epsilon}} P_{qq}(z) \frac{F_{LM}(z \cdot 1, 2)}{z}$$

does not converge as $z \to 1$ – i.e. in the **soft** limit $E_3 \to 0$.

In principle this is not a problem: the factor of $(1-z)^{-2\epsilon}$ regulates the integral, and we can perform the integration expecting to see terms $\sim 1/\epsilon$.

However, we will extract the singularity from the splitting function before integrating. To do so, we write $P_{qq}(z)$ as the sum a function which is *singular* as $z \to 1$ and a function which is *regular* as $z \to 1$:

$$P_{qq}(z) = \frac{2C_F}{1-z} + P_{qq}^{\mathrm{reg}}(z), \qquad P_{qq}^{\mathrm{reg}}(z) = -C_F(1 + z + \epsilon(1-z)).$$

Extracting soft singularities from integrated collinear limits (2)

We can then write, for some generic function $G(z)$,

$$\int_0^1 \frac{\mathrm{d}z}{(1-z)^{2\epsilon}} P_{qq}(z) G(z) = \int_0^1 \mathrm{d}z \left[\frac{2C_F}{(1-z)^{1+2\epsilon}} + \frac{P_{qq}^{\mathrm{reg}}(z)}{(1-z)^{2\epsilon}} \right] G(z).$$

The second term is integrable and the **integrand** can be expanded in powers in ϵ.

In the first term, I **subtract** the function $G(z=1)$ and **add it back** and **integrate** to obtain

$$\int_0^1 \mathrm{d}z \frac{2C_F}{(1-z)^{1+2\epsilon}} G(z) = \int_0^1 \mathrm{d}z \left[\frac{2C_F}{(1-z)^{1+2\epsilon}} \Big(G(z) - G(1) \Big) + G(1) \frac{2C_F}{(1-z)^{1+2\epsilon}} \right]$$

$$= -\frac{C_F}{\epsilon} G(1) + \int_0^1 \mathrm{d}z \frac{2C_F}{(1-z)^{1+2\epsilon}} \Big(G(z) - G(1) \Big).$$

The second term is integrable, as the singularity at $z=1$ is explicitly removed by the subtraction of $G(1)$.

Extracting soft singularities from integrated collinear limits (3)

We can expand the **integrand** of the second term in ϵ:

$$\int_0^1 \mathrm{d}z \frac{2C_F}{(1-z)^{1+2\epsilon}} \Big(G(z) - G(1) \Big) = 2C_F \int_0^1 \mathrm{d}z \sum_{n=0}^{\infty} \frac{(-2\epsilon)^n}{n!} \frac{\log^n(1-z)}{1-z} \Big(G(z) - G(1) \Big)$$

$$= 2C_F \sum_{n=0}^{\infty} \frac{(-1)^n (2\epsilon)^n}{n!} \int_0^1 \mathrm{d}z \mathcal{D}_n(z) G(z),$$

where $\mathcal{D}_n(z)$ are plus-distributions, defined as

$$\int_0^1 \mathrm{d}z \mathcal{D}_n(z) G(z) = \int_0^1 \mathrm{d}z \frac{\log^n(1-z)}{1-z} (G(z) - G(1)).$$

In our case, $G(z) = F_{LM}(z \cdot 1, 2)/z$, so $G(1) = F_{LM}(1,2)$.

Integrated collinear limits (4)

Putting everything together we have

$$\int_0^1 \frac{\mathrm{d}z}{(1-z)^{2\epsilon}} P_{qq}(z) \frac{F_{LM}(z\cdot 1,2)}{z} = -\frac{C_F}{\epsilon} F_{LM}(1,2) + \int_0^1 \frac{\mathrm{d}z}{(1-z)^{2\epsilon}} P_{qq}^{\mathrm{reg}}(z) \frac{F_{LM}(z\cdot 1,2)}{z}$$

$$+2C_F \sum_{n=0}^{\infty} \frac{(-1)^n (2\epsilon)^n}{n!} \int_0^1 \mathrm{d}z \mathcal{D}_n(z) \frac{F_{LM}(z\cdot 1,2)}{z}.$$

Recalling that this is multiplied by $\sim 1/\epsilon$, we need to expand the integrands to $\mathcal{O}(\epsilon)$:

$$\int_0^1 \frac{\mathrm{d}z}{(1-z)^{2\epsilon}} P_{qq}(z) \frac{F_{LM}(z\cdot 1,2)}{z} = -\frac{C_F}{\epsilon} F_{LM}(1,2)$$

$$+C_F \int_0^1 \mathrm{d}z \Big\{ 2\mathcal{D}_0(z) - (1+z) + \epsilon\big[2(1+z)\log(1-z) - (1-z) - 4\mathcal{D}_1(z)\big] \Big\}$$

$$\times F_{LM}(z\cdot 1,2)/z.$$

Integrated collinear limits (5)

We will write

$$\int_0^1 \frac{\mathrm{d}z}{(1-z)^{2\epsilon}} P_{qq}(z) \frac{F_{LM}(z\cdot 1,2)}{z} = -C_F\left(\frac{1}{\epsilon}+\frac{3}{2}\right) F_{LM}(1,2) + \int_0^1 \mathrm{d}z \hat{P}_{qq,R}(z) \frac{F_{LM}(z\cdot 1,2)}{z},$$

with

$$\hat{P}_{qq,R}(z) = \hat{P}^{(0)}_{qq}(z) + \epsilon \hat{P}^{(\epsilon)}_{qq}(z),$$

and

$$\hat{P}^{(0)}_{qq}(z) = C_F\left(2\mathcal{D}_0(z) - (1+z) + \frac{3}{2}\delta(1-z)\right),$$

$$\hat{P}^{(\epsilon)}_{qq}(z) = C_F\Big(2(1+z)\log(1-z) - (1-z) - 4\mathcal{D}_1(z)\Big),$$

where the $+\frac{3}{2}\delta(1-z)$ accounts for the extra $-\frac{3C_F}{2}F_{LM}(1,2)$ term.

Combined integrated subtraction terms

Inserting this expression into the integrated collinear limit gives

$$\langle C_{31} F_{LM}(1,2,3)\rangle = -[\alpha_s]\frac{1}{\epsilon}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}s^{-\epsilon}\Bigg[-C_F\left(\frac{1}{\epsilon}+\frac{3}{2}\right)F_{LM}(1,2) + \int_0^1 \mathrm{d}z\,\hat{P}_{qq,R}(z)\,\frac{F_{LM}(z\cdot 1,2)}{z}\Bigg].$$

The treatment of the limit C_{32} proceeds in an analogous fashion yielding a result that is identical apart from $F_{LM}(z\cdot 1,2) \to F_{LM}(1,z\cdot 2)$.

We have now considered all **singular limits** in the real emission correction, and **integrated over the unresolved phase space** to obtain poles in $1/\epsilon$.

The real emission correction

Combining everything, the real emission correction can be written as

$$\langle F_{LM}(1,2,3)\rangle = \langle \hat{\mathcal{O}}_{\rm NLO} F_{LM}(1,2,3)\rangle + 2[\alpha_s]\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} s^{-\epsilon}\left(\frac{C_F}{\epsilon^2}+\frac{3C_F}{2\epsilon}\right) F_{LM}(1,2)$$

$$-[\alpha_s]\frac{1}{\epsilon}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} s^{-\epsilon}\int_0^1 {\rm d}z \hat{P}_{qq,R}(z)\left[\frac{F_{LM}(z\cdot 1,2)}{z}+\frac{F_{LM}(1,z\cdot 2)}{z}\right],$$

where

$$\hat{\mathcal{O}}_{\rm NLO} = (I - C_{31} - C_{32})(I - S_3).$$

Some comments:

- The first term, as before, has all singularities **explicitly subtracted**, and may be numerically integrated in four dimensions.
- The leading pole $\mathcal{O}(1/\epsilon^2)$ multiplies the pure LO process $F_{LM}(1,2)$.
- The subleading poles $\mathcal{O}(1/\epsilon)$ multiply the LO process with *reduced kinematics* on either the incoming quark or the incoming antiquark, $F_{LM}(z\cdot 1,2)$ and $F_{LM}(1,z\cdot 2)$.

Cancelling poles

We have now done what we aimed to do: we have written the real emission correction, which involves a **non-convergent integral** over the phase space of the emitted gluon, as the sum of a **convergent integral** and terms whose **divergent** behavior is made explicit as **poles** in $1/\epsilon$. All that remains is to cancel these poles.

For this, we need to know the IR poles in the virtual correction.
The computation of virtual corrections is beyond the scope of these lectures (covered by Lorenzo Tancredi and Simon Badger).

However, much as the infrared limits of real emission amplitudes behave in a universal manner, so too are the **infrared divergences of one-loop amplitudes universal.**

Using the infrared limits of the real emission amplitudes and the fact that the infrared poles are guaranteed to cancel, we can derive an expression for the universal infrared divergences of one-loop amplitudes *without computing the amplitude.* [a]

This is a tremendously important and powerful result.

[a] S. Catani, Phys. Lett. B **427** (1998), 161

Virtual poles (1)

The infrared poles of a one-loop amplitude $\mathcal{A}_1$ for a given process can be written in terms of the corresponding tree-level amplitude for the process $\mathcal{A}_0$:

$$\mathcal{A}_1 = \frac{\alpha_s(\mu^2)}{2\pi} I_1(\epsilon)\mathcal{A}_0 + \mathcal{A}_1^{\text{fin}},$$

where $I_1(\epsilon)$ is also a function of the momenta and colors of the LO partons, and $\mathcal{A}_1^{\text{fin}}$ is the finite part of the amplitude. For color singlet production $q\bar{q} \to V$

$$I_1(\epsilon) = -\frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\left(\frac{C_F}{\epsilon^2} + \frac{3C_F}{2\epsilon}\right)\left(\frac{\mu^2}{-s-i0}\right)^{\epsilon}.$$

The virtual correction is given by the interference between the one-loop and tree-level amplitudes, $2\text{Re}[\mathcal{A}_1\mathcal{A}_0^*]$. Thus

$$2s\mathrm{d}\sigma^{\mathrm{V}} = -2\frac{\alpha_s(\mu^2)}{2\pi}\frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\left(\frac{C_F}{\epsilon^2} + \frac{3C_F}{2\epsilon}\right)\left(\frac{\mu^2}{s}\right)^{\epsilon}\cos(\pi\epsilon)F_{LM}(1,2) + F_{LV}^{\text{fin}}(1,2),$$

where I have analytically continued the factor $(-s-i0)^{-\epsilon}$ and taken the real part to get the factor $\cos(\pi\epsilon)$.

Virtual poles (2)

$$2s\mathrm{d}\sigma^{\mathrm{V}} = -2\frac{\alpha_s(\mu^2)}{2\pi}\frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\left(\frac{C_F}{\epsilon^2}+\frac{3C_F}{2\epsilon}\right)\left(\frac{\mu^2}{s}\right)^{\epsilon}\cos(\pi\epsilon)F_{LM}(1,2)+F_{LV}^{\mathrm{fin}}(1,2).$$

In this expression,

$$F_{LV}^{\mathrm{fin}}(1,2) = \frac{\alpha_s(\mu^2)}{2\pi}2\mathrm{Re}[\mathcal{A}_0\mathcal{A}_1^{\mathrm{fin}*}]$$

is the finite part of the virtual corrections, which we won't concern ourselves with any further.

Virtual poles (3)

This result is given in terms of the *renormalized* strong coupling $\alpha_s(\mu^2)$, whereas our expression for the real emission corrections was in terms of *unrenormalized* $[\alpha_s]$. We therefore *unrenormalize*

$$\frac{\alpha_s(\mu^2)}{2\pi}\mu^{2\epsilon} = \mu_0^{2\epsilon}(4\pi)^{\epsilon}e^{-\epsilon\gamma_E}\frac{\alpha_{s,b}}{2\pi} = \frac{g_s^2}{8\pi^2}\frac{\mu_0^{2\epsilon}(4\pi)^{\epsilon}}{\Gamma(1-\epsilon)}e^{-\epsilon\gamma_E}\Gamma(1-\epsilon) = [\alpha_s]e^{-\epsilon\gamma_E}\Gamma(1-\epsilon).$$

The virtual correction is then

$$2s\mathrm{d}\sigma^{\mathrm{V}} = -2[\alpha_s]\cos(\pi\epsilon)\left(\frac{C_F}{\epsilon^2}+\frac{3C_F}{2\epsilon}\right)s^{-\epsilon}F_{LM}(1,2) + F_{LV}^{\mathrm{fin}}(1,2).$$

Cancelling poles (1)

Summing the real and the virtual corrections

$$
\begin{aligned}
2s\mathrm{d}\sigma^{\mathrm{R+V}} =& \langle \hat{O}_{\mathrm{NLO}} F_{LM}(1,2,3)\rangle + F_{LV}^{\mathrm{fin}}(1,2) \\
&+2[\alpha_s]s^{-\epsilon}\left(\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} - \cos(\pi\epsilon)\right)\left(\frac{C_F}{\epsilon^2}+\frac{3C_F}{2\epsilon}\right)F_{LM}(1,2) \\
&-[\alpha_s]\frac{1}{\epsilon}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}s^{-\epsilon}\int_0^1 \mathrm{d}z\hat{P}_{qq,R}(z)\left[\frac{F_{LM}(z\cdot 1,2)}{z}+\frac{F_{LM}(1,z\cdot 2)}{z}\right].
\end{aligned}
$$

Expanding the second line in ϵ, we see that **all the poles proportional to $F_{LM}(1,2)$ cancel,** leaving behind a finite contribution proportional to $F_{LM}(1,2)$:

$$
\begin{aligned}
2s\mathrm{d}\sigma^{\mathrm{R+V}} =& \langle \hat{O}_{\mathrm{NLO}} F_{LM}(1,2,3)\rangle + F_{LV}^{\mathrm{fin}}(1,2) + \frac{2\pi^2}{3}C_F[\alpha_s]s^{-\epsilon}F_{LM}(1,2) \\
&-[\alpha_s]\frac{1}{\epsilon}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}s^{-\epsilon}\int_0^1 \mathrm{d}z\hat{P}_{qq,R}(z)\left[\frac{F_{LM}(z\cdot 1,2)}{z}+\frac{F_{LM}(1,z\cdot 2)}{z}\right].
\end{aligned}
$$

Cancelling poles (2)

However, we are still left with $1/\epsilon$ poles multiplying the matrix elements with reduced kinematics $F_{LM}(z \cdot 1, 2)$ and $F_{LM}(1, z \cdot 2)$.

This is not surprising since these poles **cannot** be cancelled by the virtual corrections: the virtual corrections have *LO-like* kinematics, and will never lead to these *reduced* kinematic configurations.

The best we could have hoped for is that the virtual corrections would cancel all the poles proportional to $F_{LM}(1, 2)$, which is indeed what happened.

So what about the remaining poles??

Pdf renormalization (1)

So far, we have been considering a *partonic* cross section.

This is **not observable**: to relate it to hadronic observables, we need to **integrate** over *parton distribution functions (pdfs)*.

We can then *absorb* the divergences into the pdf, by writing the physical (observable) pdf as the sum of a bare (unobservable) pdf plus counterterms which remove the divergences.

This is analogous to the ultraviolet renormalization of the couplings and masses in QED and QCD, where the observable coupling or mass is written as a sum of a bare coupling or mass plus counterterms that remove UV divergences.

Pdf renormalization (2)

We can perform the pdf renormalization perturbatively in α_s

$$f_i^{\text{bare}}(x) = \left[\delta_{ij}\delta(1-y_1) + \frac{\alpha_s(\mu^2)}{2\pi}\frac{1}{\epsilon}\hat{P}_{ij}^{(0)}(y_1) + \mathcal{O}(\alpha_s^2)\right] \otimes f_j^{\text{ren}}(y_2),$$

where

- f_i^{bare} and $f_i^{\text{ren.}}$ are, respectively, the bare and renormalized pdfs for the parton i,
- the convolution $\otimes$ is defined as

$$[f \otimes g](x) = \int_0^1 \int_0^1 \mathrm{d}y_1 \mathrm{d}y_2 f(y_1) g(y_2) \delta(x - y_1 y_2)$$

- $\hat{P}_{ij}^{(0)}$ is the leading-order *Altarelli-Parisi kernel* for partons i and j.

For the quark-antiquark channel, the LO Altarelli-Parisi kernel is

$$\hat{P}_{qq}^{(0)} = C_F \left(2\mathcal{D}_0(z) - (1+z) + \frac{3}{2}\delta(1-z)\right),$$

which is the same function that multiplies the $F_{LM}(z \cdot 1, 2)$ and $F_{LM}(z \cdot 1, 2)$ at $\mathcal{O}(1/\epsilon)$ in $\mathrm{d}\sigma^{\mathrm{R+V}}$. We are on the right track!

Pdf renormalization (3)

The pdf renormalization

$$f_i^{\text{bare}}(x) = \left[\delta_{ij}\delta(1-y_1) + \frac{\alpha_s(\mu^2)}{2\pi}\frac{1}{\epsilon}\hat{P}_{ij}^{(0)}(y_1) + \mathcal{O}(\alpha_s^2)\right] \otimes f_j^{\text{ren}}(y_2),$$

implies that renormalizing the pdfs gives non-trivial contributions at the *next order* in α_s (and higher).
Therefore when I renormalize the pdfs, the **LO cross section** generates terms that contribute at NLO.

The pdf-unrenomalized *hadronic* LO cross section is

$$\int_0^1 \mathrm{d}x_1 \mathrm{d}x_2 f_q^{\text{bare}}(x_1) f_q^{\text{bare}}(x_1) \frac{1}{x_1 x_2 P_1 P_2} F_{LM}(x_1 P_1, x_2 P_2)$$

where P_1 and P_2 are the momenta of the incoming hadrons, and $\frac{1}{x_1 x_2 P_1 P_2}$ is the flux factor $1/(2s)$.

Pdf renormalization (4)

Let's just consider the parts of the hadronic LO cross sections which depend on x_1:

$$\int_0^1 \mathrm{d}x_1 f_q^{\mathrm{bare}}(x_1)\frac{1}{x_1}F_{LM}(x_1 P_1,\ldots).$$

Renormalizing this introduces a counterterm at $\mathcal{O}(\alpha_s)$

$$\begin{aligned}
\mathrm{d}\sigma_{\mathrm{had.}}^{\mathrm{CV},1} &= \int_0^1 \mathrm{d}x_1 \frac{\alpha_s(\mu^2)}{2\pi}\frac{1}{\epsilon}\left[\hat{P}_{qq}^{(0)}\otimes f_q^{\mathrm{ren}}\right](x_1)\times\frac{1}{x_1}F_{LM}(x_1 P_1,\ldots) \\
&= \int_0^1 \mathrm{d}x_1 \mathrm{d}y_1 \mathrm{d}y_2 \delta(x_1 - y_1 y_2)\frac{\alpha_s(\mu^2)}{2\pi}\frac{1}{\epsilon}\hat{P}_{qq}^{(0)}(y_1) f_q^{\mathrm{ren}}(y_2)\frac{1}{x_1}F_{LM}(x_1 P_1,\ldots), \\
&= \int_0^1 \mathrm{d}y_1 \mathrm{d}y_2 \frac{\alpha_s(\mu^2)}{2\pi}\frac{1}{\epsilon}\hat{P}_{qq}^{(0)}(y_1) f_q^{\mathrm{ren}}(y_2)\frac{1}{y_1 y_2}F_{LM}(y_1 y_2 P_1,\ldots)
\end{aligned}$$

where I have just used the definition of the convolution $\otimes$ in the second line.

Pdf renormalization (5)

Renaming the dummy variables $y_1 \to z$ and $y_2 \to x_1$ we have

$$d\sigma^{\mathrm{CV},1}_{\mathrm{had.}} = \int_0^1 dx_1 f_q^{\mathrm{ren}}(x_1) \frac{1}{x_1} \left[\int_0^1 dz \frac{\alpha_s(\mu^2)}{2\pi} \frac{1}{\epsilon} \hat{P}^{(0)}_{qq}(z) \right] \frac{F_{LM}(zx_1 P_1, \ldots)}{z}.$$

Thus renormalizing the pdf $f_1(x_1)$ introduces a counterterm at the *partonic* level

$$d\sigma^{\mathrm{CV},1} = \frac{\alpha_s(\mu^2)}{2\pi} \frac{1}{\epsilon} \int_0^1 dz \hat{P}^{(0)}_{qq}(z) \frac{F_{LM}(z \cdot 1, 2)}{z}$$

Renormalizing the pdf $f_q(x_2)$ creates a similar factor with $F_{LM}(z \cdot 1, 2) \to F_{LM}(1, z \cdot 2)$, so the full *pdf renormalization counterterm* is

$$d\sigma^{\mathrm{CV}} = \frac{\alpha_s(\mu^2)}{2\pi} \frac{1}{\epsilon} \int_0^1 dz \hat{P}^{(0)}_{qq}(z) \left[\frac{F_{LM}(z \cdot 1, 2)}{z} + \frac{F_{LM}(1, z \cdot 2)}{z} \right].$$

Cancelling poles (3)

Combining the *real* and *virtual* corrections with the *pdf renormalization counterterms*, we have

$$
\begin{aligned}
2s\mathrm{d}\sigma^{\mathrm{NLO}} =& \langle \hat{O}_{\mathrm{NLO}} F_{LM}(1,2,3)\rangle + F_{LV}^{\mathrm{fin}}(1,2) + \frac{2\pi^2}{3} C_F [\alpha_s] s^{-\epsilon} F_{LM}(1,2) \\
&+ \frac{1}{\epsilon} \int_0^1 \mathrm{d}z \left[\frac{\alpha_s(\mu^2)}{2\pi} \hat{P}_{qq}^{(0)}(z) - [\alpha_s] \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} s^{-\epsilon} \hat{P}_{qq}^{(0)}(z) \right] \\
&\times \left[\frac{F_{LM}(z\cdot 1,2)}{z} + \frac{F_{LM}(1,z\cdot 2)}{z} \right] \\
&- [\alpha_s] \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} s^{-\epsilon} \int_0^1 \mathrm{d}z \hat{P}_{qq}^{(\epsilon)}(z) \left[\frac{F_{LM}(z\cdot 1,2)}{z} + \frac{F_{LM}(1,z\cdot 2)}{z} \right].
\end{aligned}
$$

where I have written $\hat{P}_{qq,R}(z) = \hat{P}_{qq}^{(0)}(z) + \epsilon \hat{P}_{qq}^{(\epsilon)}(z)$.

The $1/\epsilon$ poles in the second line cancel, leaving a finite remainder.

NLO correction

After cancelling the poles we can take $\epsilon \to 0$. Then gathering the finite remainders and renormalizing the strong coupling, we find the NLO correction is

$$
\begin{aligned}
2s\mathrm{d}\sigma^{\mathrm{NLO}} =& + \langle \hat{O}_{\mathrm{NLO}} F_{LM}(1,2,3) \rangle + F_{LV}^{\mathrm{fin}}(1,2) + \frac{\alpha_s(\mu^2)}{2\pi} \frac{2\pi^2}{3} C_F F_{LM}(1,2) \\
& + \frac{\alpha_s(\mu^2)}{2\pi} \int_0^1 \mathrm{d}z \left[\hat{P}_{qq}^{(0)}(z) \log\left(\frac{s}{\mu^2}\right) - \hat{P}_{qq}^{(\epsilon)}(z) \right] \\
& \times \left[\frac{F_{LM}(z \cdot 1, 2)}{z} + \frac{F_{LM}(1, z \cdot 2)}{z} \right].
\end{aligned}
$$

The log term in the second line is due to the different energy scalings that enter with the LO Altarelli-Parisi kernel from the real corrections and the pdf renormalization. Apart from in $\alpha_s(\mu^2)$, it is the **only** place where the renormalization scale μ enters.

Recap and preview

- The poles have been **extracted and cancelled** and the limit $\epsilon \to 0$ taken, and we obtained a finite expression for the NLO corrections.
- Even though we needed a d-dimensional calculation to get to the above expression, we can now calculate every term in it in **four space-time dimensions.**
- Since we did not integrate over the *resolved* (i.e. observable) phase space of the radiated parton, the expression for the NLO correction is **fully differential** – i.e. we can use it to get a prediction for any infrared-safe observable that we construct out of the complete final state momenta.
- We considered the specific case of real emission corrections to color singlet production in the partonic channel $q\bar{q} \to V + g$, but the subtraction scheme has been **completely generalized for arbitrary production processes** at a hadron or lepton collider.

In the next lecture we will discuss the use of subtraction schemes for *NNLO* corrections. This is an area of active research. I will describe some of the new obstacles that appear at this order, and outline a few methods that are being developed.