3-BODY PHYSICS IN (IN)FINITE VOLUME

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I. Introduction

2-BODY EXAMPLE: σ(500)

- parameters (existence) debated for decades

→ Review Pelaez(2015)

- **dispersive techniques:** most precise resonance parameters

Colangelo et al.(2001) Garcia-Martin et al.(2011) Moussallam(2011)..

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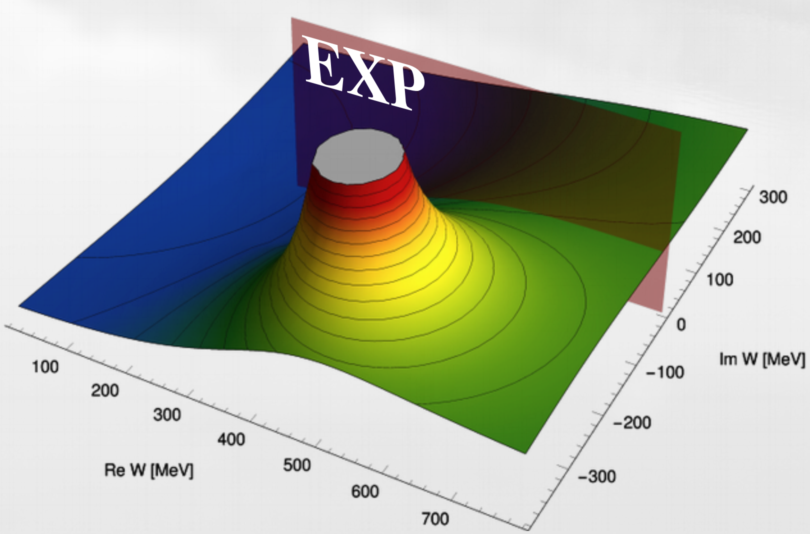

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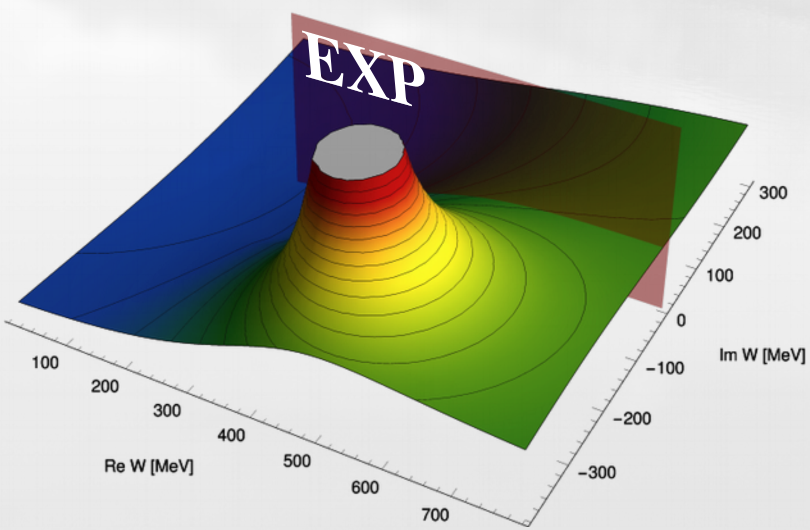

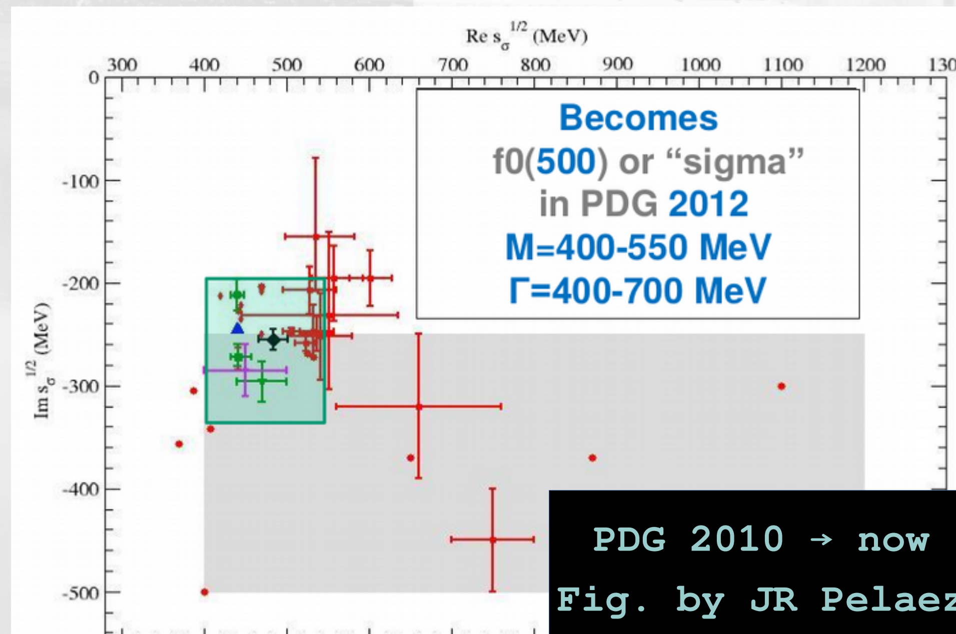

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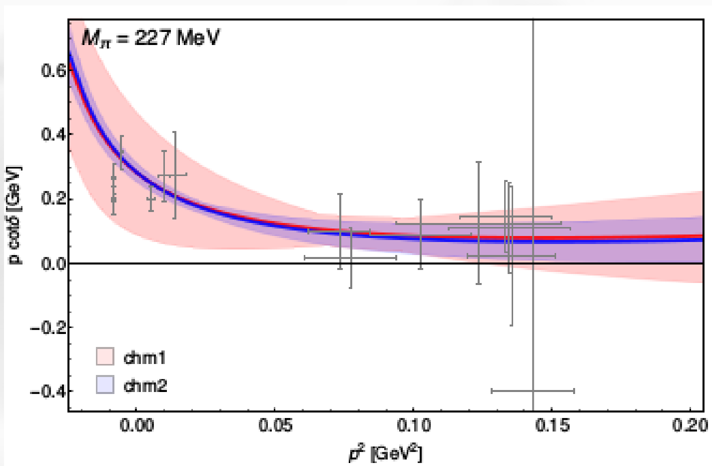

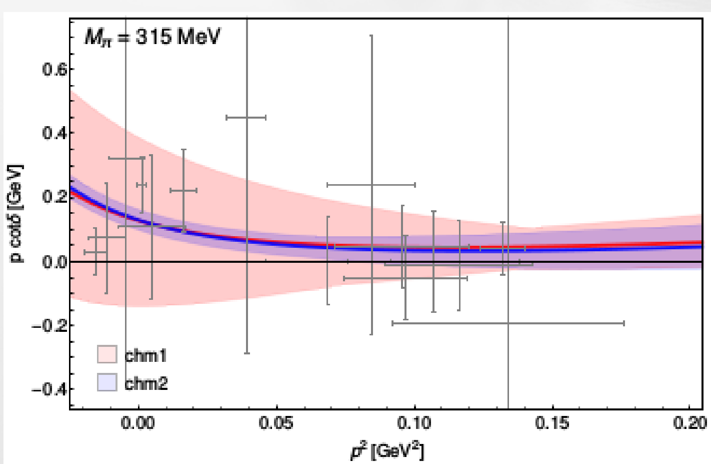

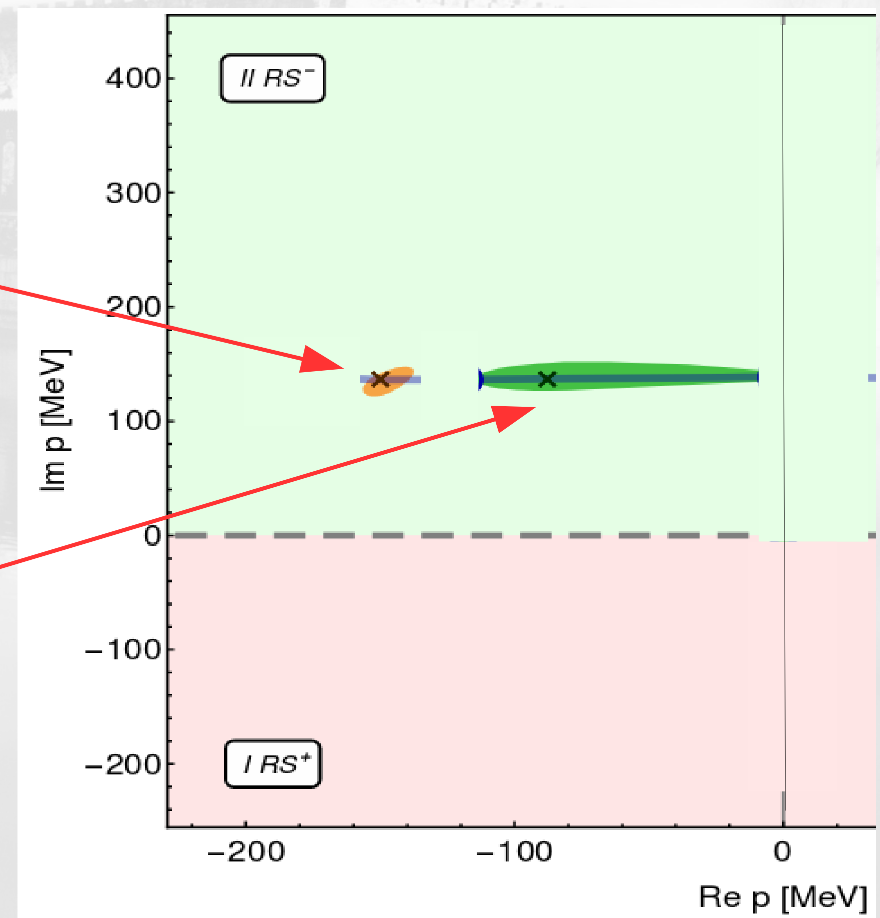

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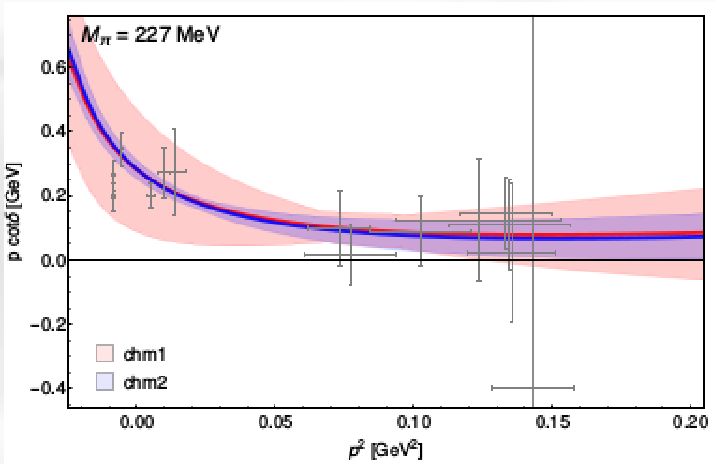
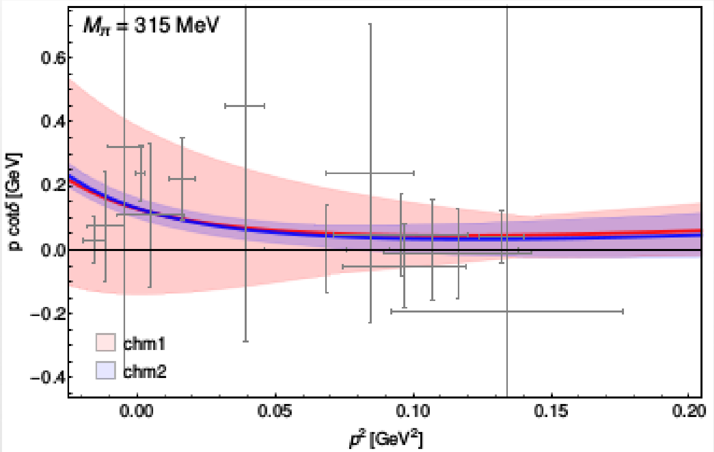
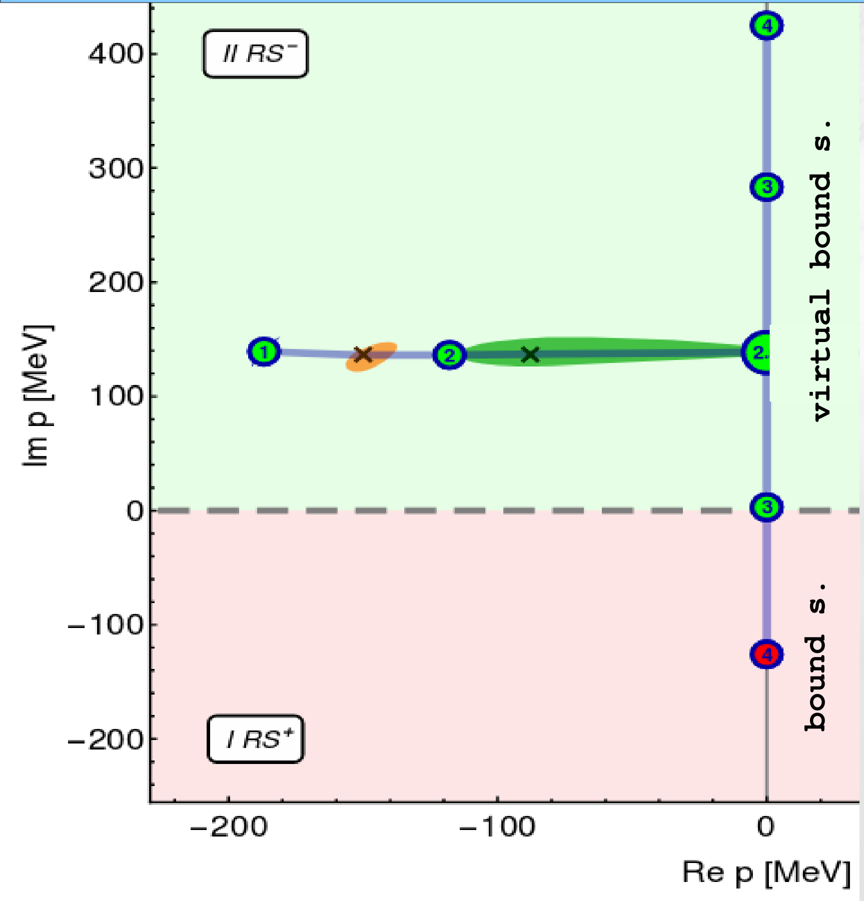




Extrapolate in m_π



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→ **Similar tools are required:**

- Scattering amplitude in infinite volume **→ I. Part**
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  - gluonic degrees of freedom
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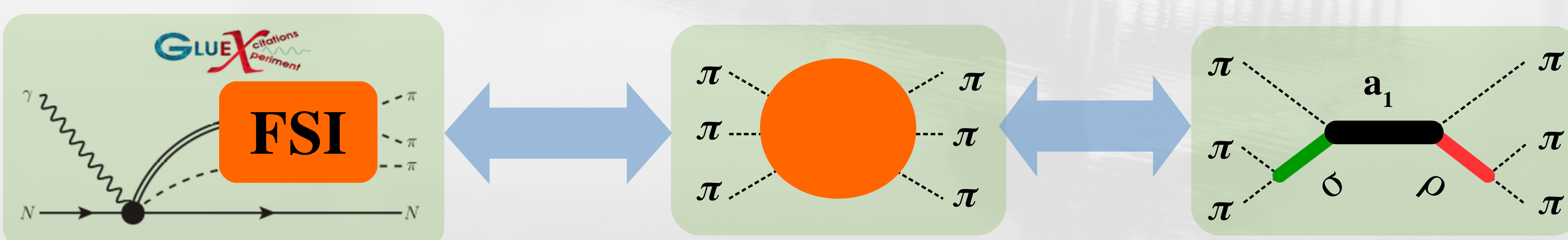


→ Need to pin down 3-hadron interaction

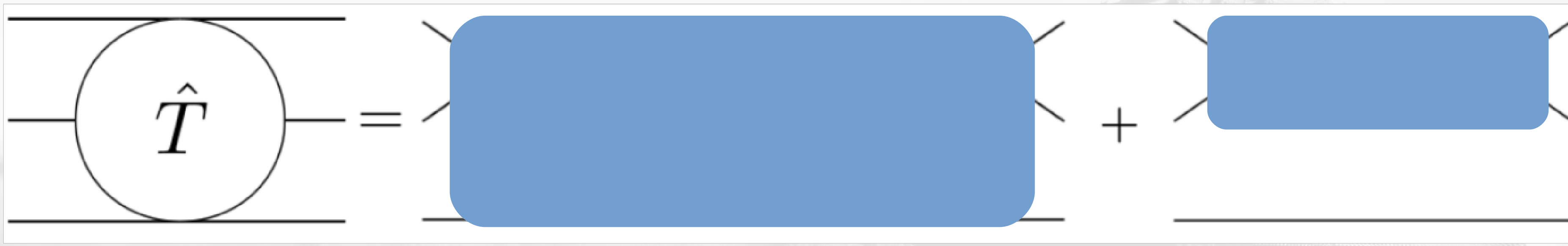
II. 3-body interactions: <u>infinite volume</u>

MM/Hu/Doring/Pilloni/Szczepaniak (2017)

Sadasivan/MM/Doring/Pilloni/Szczepaniak (in progress)

SCATTERING MATRIX (T)

- 3 asymptotic states (scalar particles of equal mass (m))
- ***Connectedness structure*** of matrix elements:



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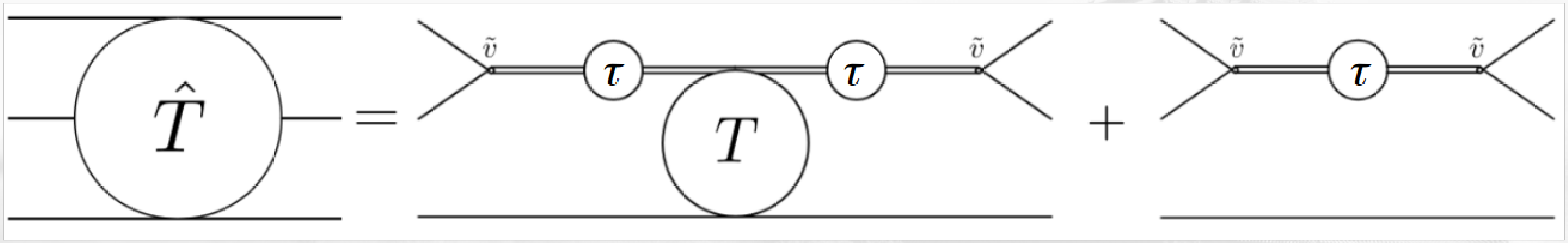

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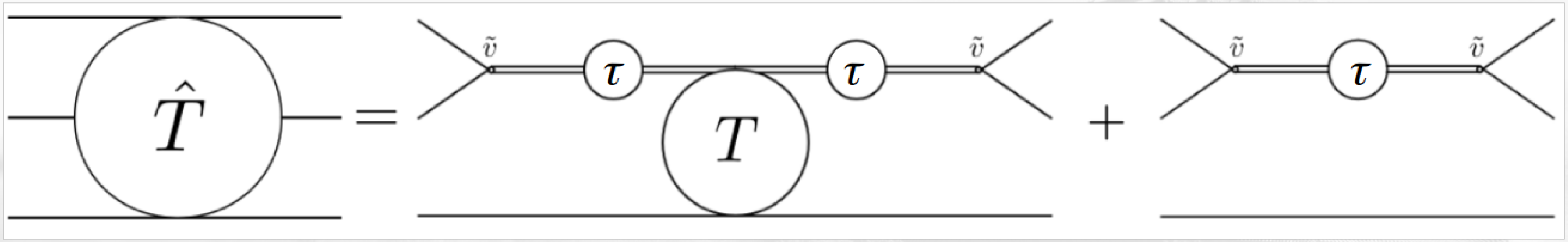

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3 unknown functions & 8 kinematic variables

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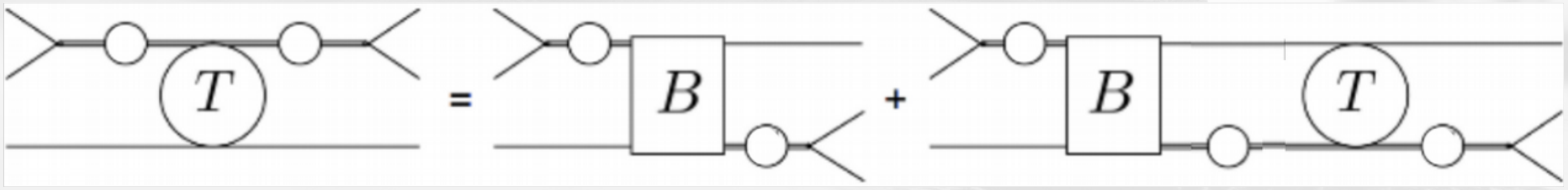

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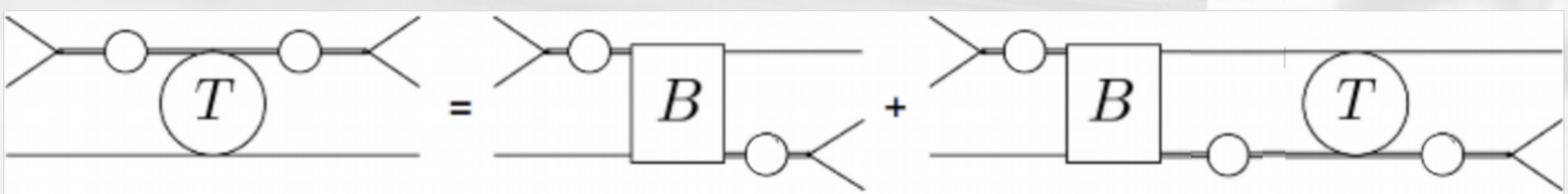

→ gives access to ***Disc[B]***

$$\mathrm{Disc}\, B(u) = 2\pi i\, \frac{\delta\left(E_Q - \sqrt{m^2 + \mathbf{Q}^2}\right)}{2\sqrt{m^2 + \mathbf{Q}^2}} v^2$$

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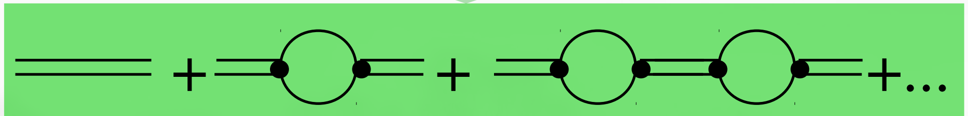

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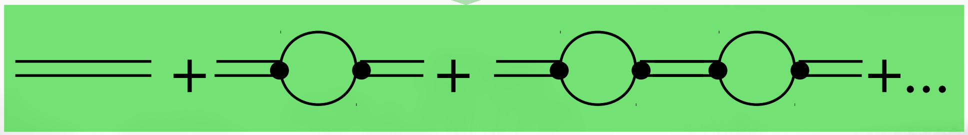

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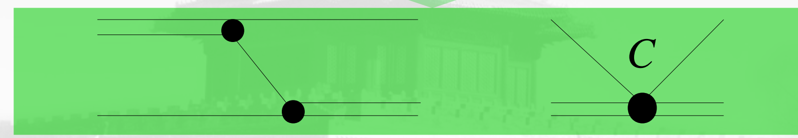

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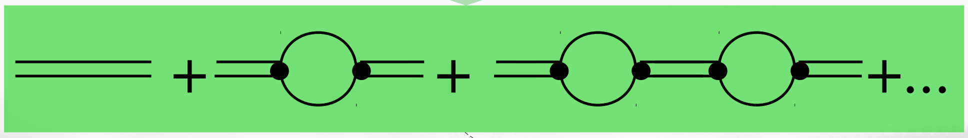

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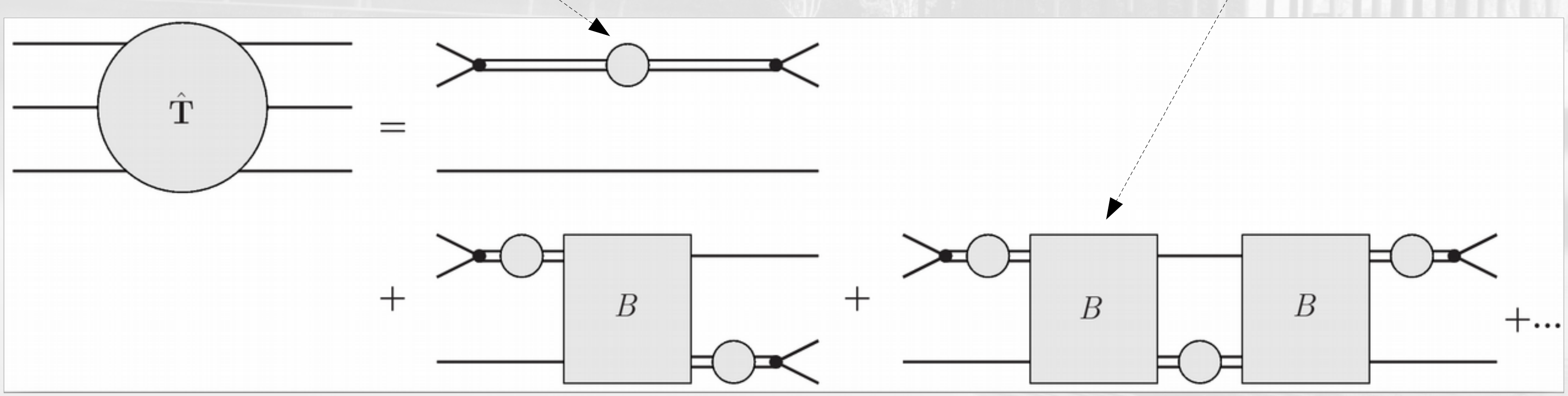
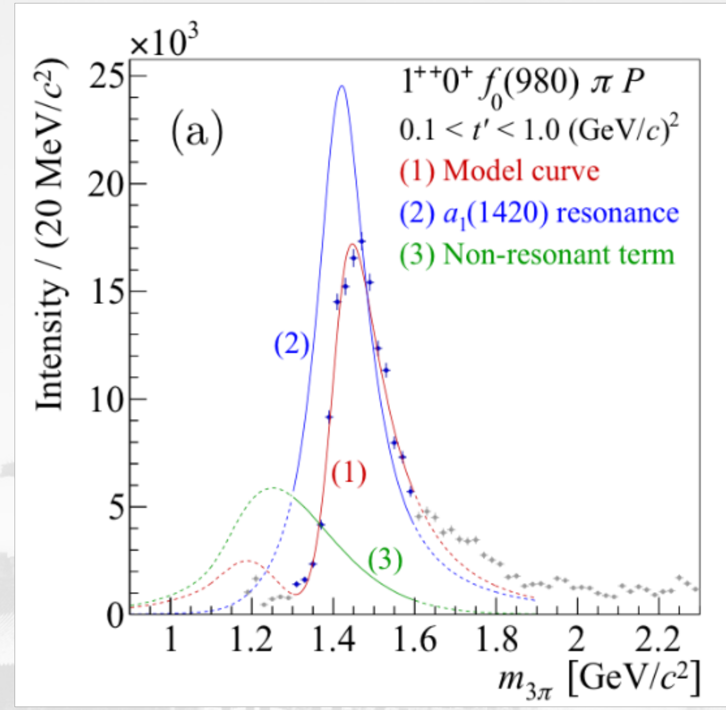


- **Manifestly unitary & covariant 3d integral equation**
- An infinite series of isobars (two-body subamplitudes) interacting with spectator:
- Unknown parameter: ***C*, *v*, *parameters of the isobar (subtraction constants)*** ← ***II. Part***

Interesting application: $a_1(1420)$

- Lineshape from COMPASS @ CERN
- in $f_0(980)\pi$ final state



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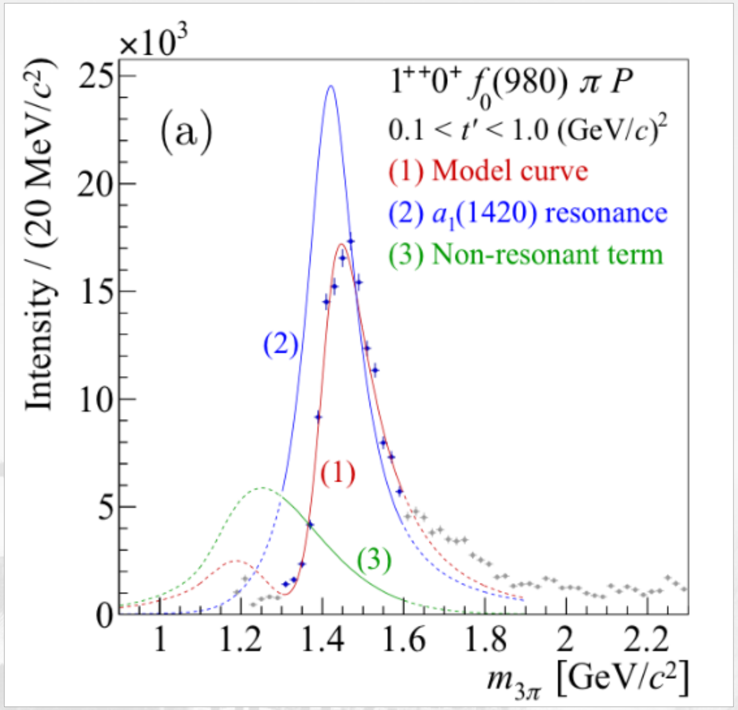

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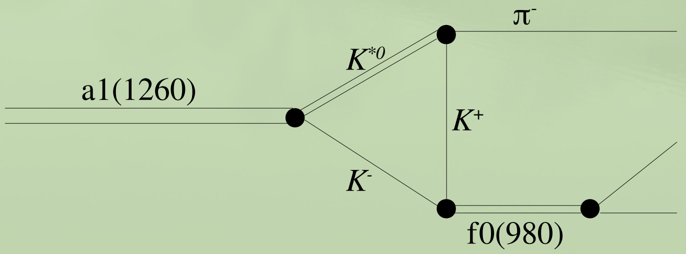
- Log-like behavior of the "triangle-diagram"

Mikhasenko/Ketzer/Sarantsev(2015) Aceti/Dai/Oset(2016)



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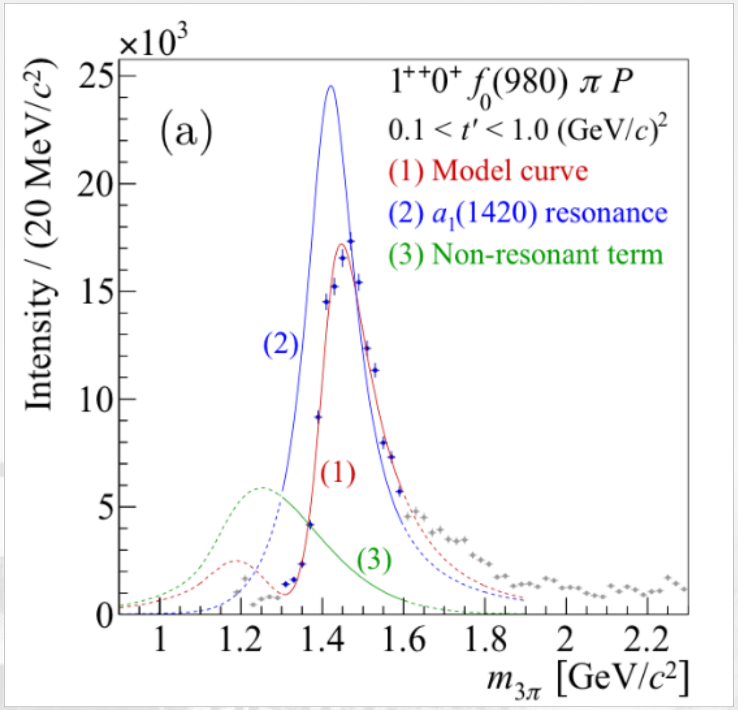

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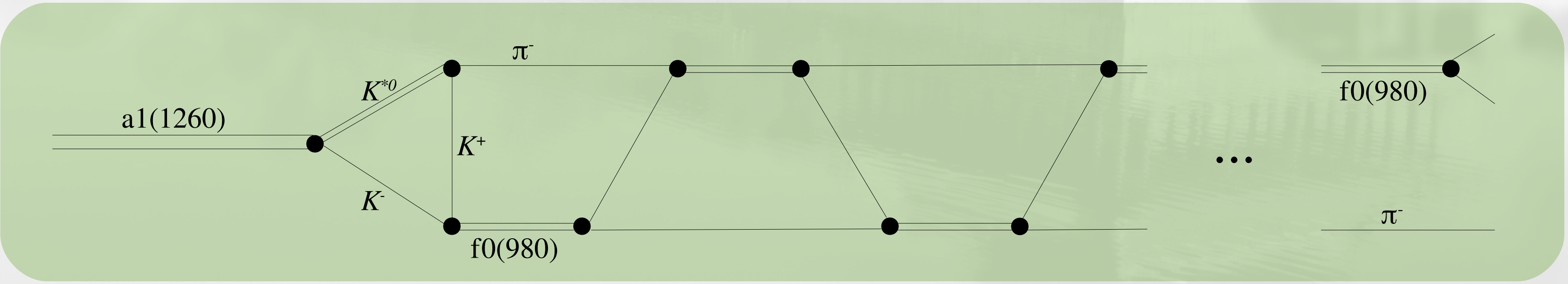

- **Q:** Does such a feature exist in full 3b-unitary FSI?

→ Check with unitary isobar approach

Sadasivan/MM/… (in progress)

III. 3-body interactions: finite volume

MM/Doring (2017)

MM/Doring (2018) → last week

LATTICE QCD

Ab-initio numerical calculations of QCD Greens functions

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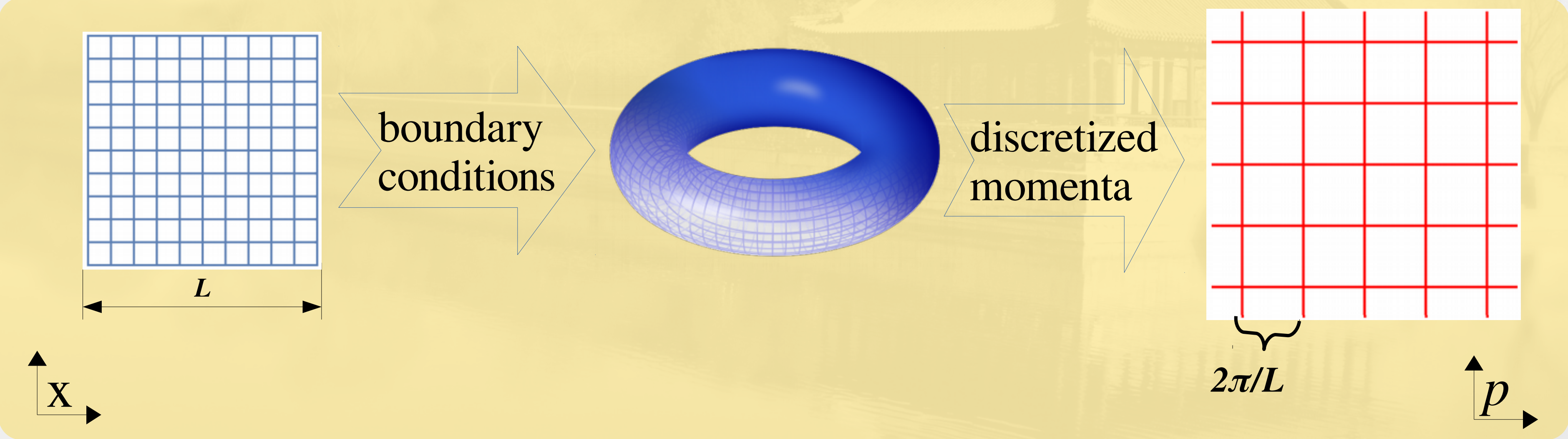

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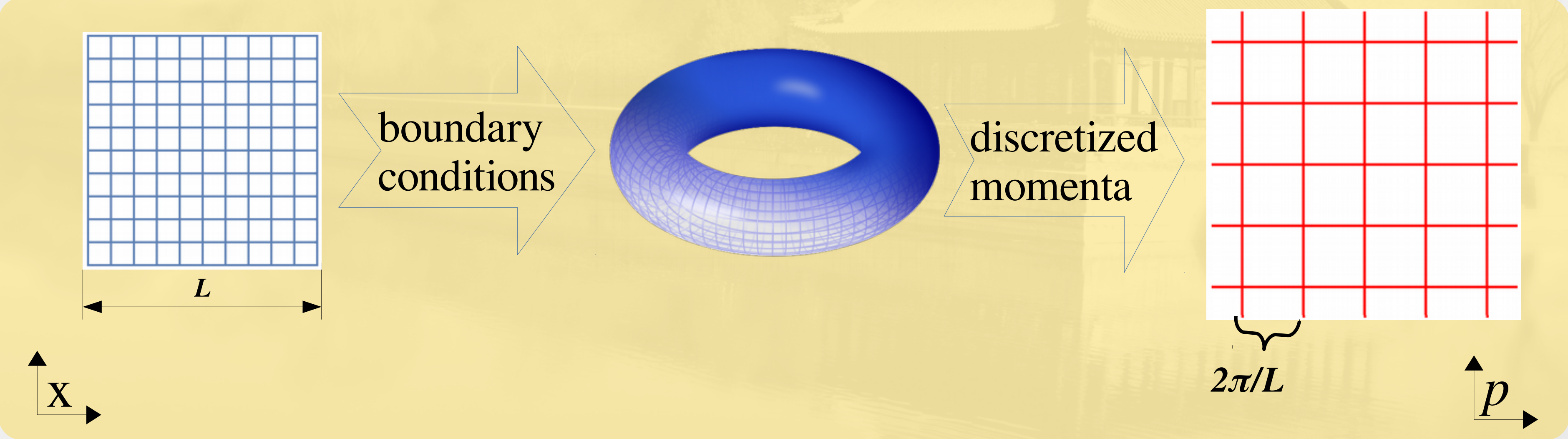

→ momenta & spectra are discretized

→ extrapolation to infinite volume ↔ **Quantization Condition**

QUANTIZATION CONDITION

2-body case

- well understood

Lüscher(1986)

- multi-channels, spin, ...

Gottlieb,Rummukainen,Feng,Li,Liu,Doring,Briceno,Rusetsky,Bernard…

3-body case

- important theoretical developments

Sharpe,Hansen,Briceno,Rusetsky,Polejaeva,Davoudi,Guo,MM,Doring…

- pilot numerical investigation

Pang/Hammer/Rusetsky/Wu(2017) MM/Doring(2017) Hansen/Briceno/Sharpe(2018)

- Finite volume spectrum of ($\pi^+\pi^+$) and ($\pi^+\pi^+\pi^+$)

  → comparison with Lattice QCD results

MM/Doring (last week)

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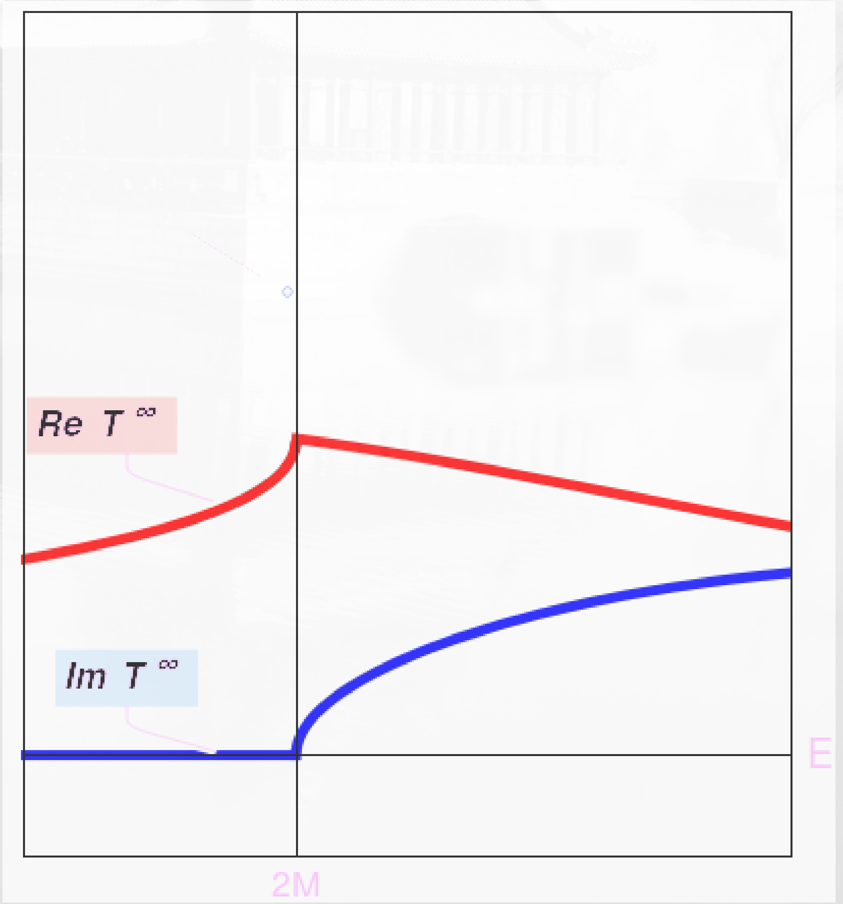
One way of thinking:

Unitarity

$$T(E) = \frac{1}{K^{-1}(E) + i\Phi(E)}$$



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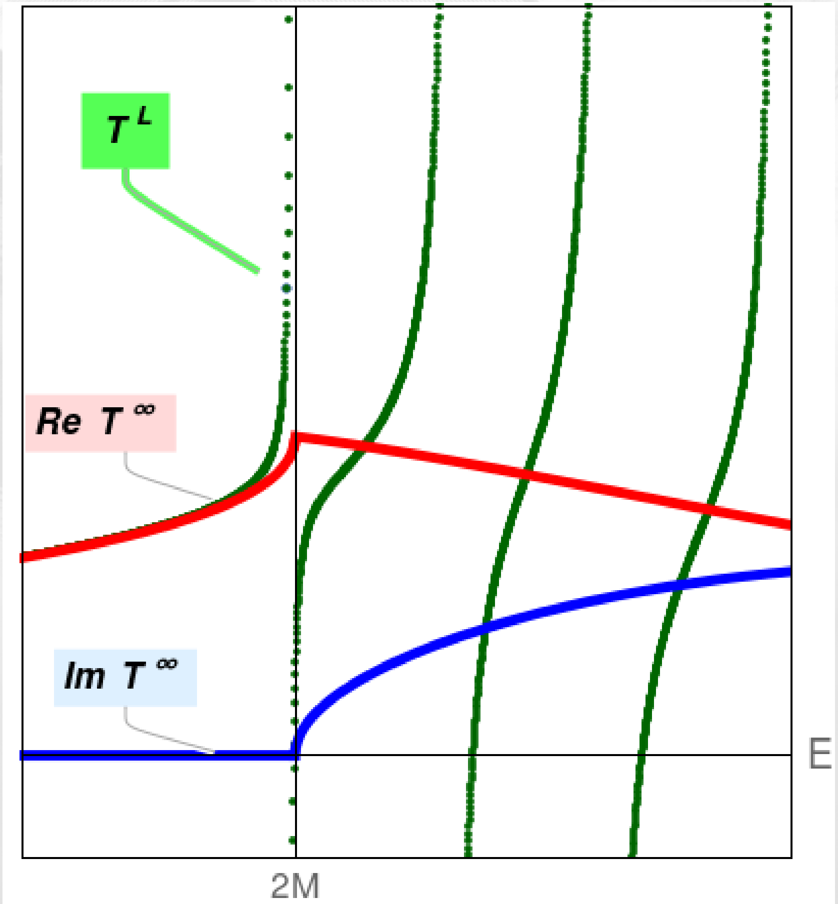

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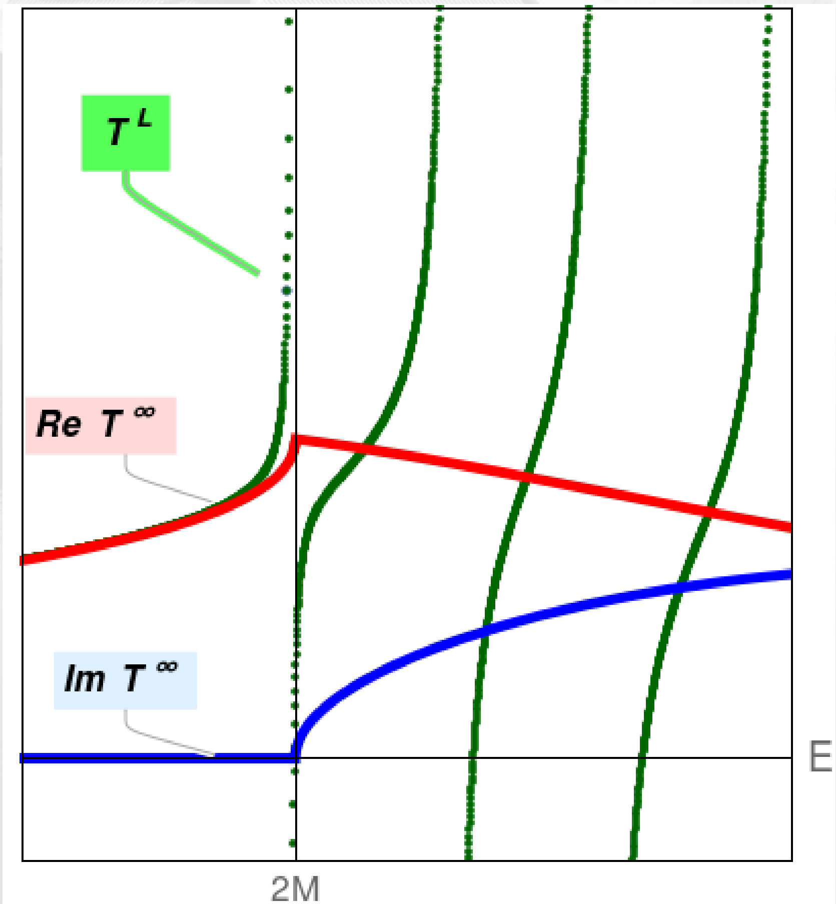
- Regular summation theorem applies for **$E<2M$**
- For **$E>2M$**: **$T(E)$** is singular
- LSZ formalism relates Greens fct. to S-matrix

  (pole positions) ↔ (energy eigenvalues)

$$K^{-1}(E^*) = -\frac{2}{\sqrt{\pi}L} Z_{00}(E^*,L)$$



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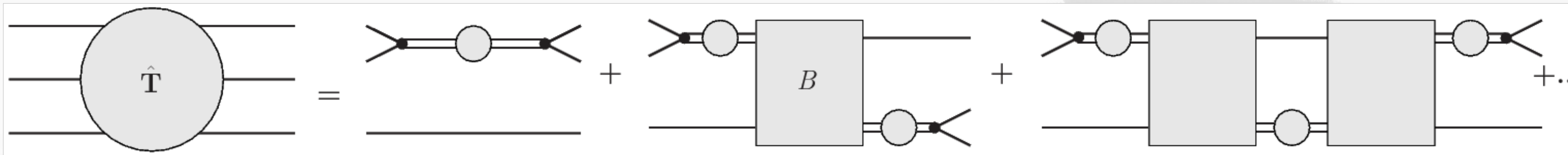

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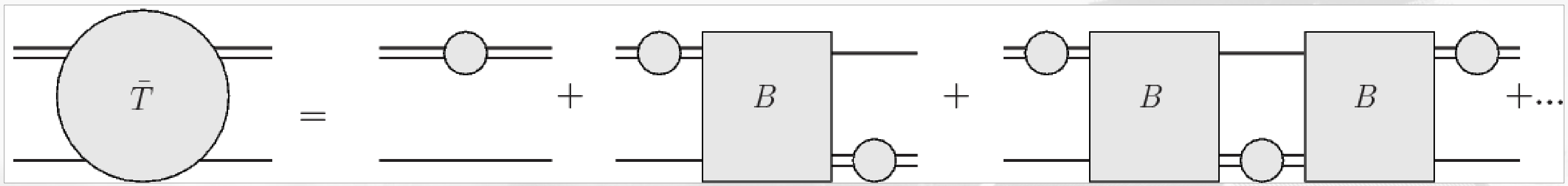
2) v is cut-free and does not contribute to poles!



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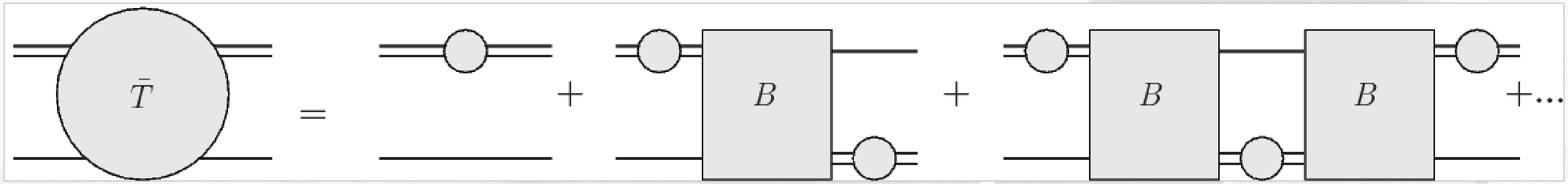


3) Discretize momenta:

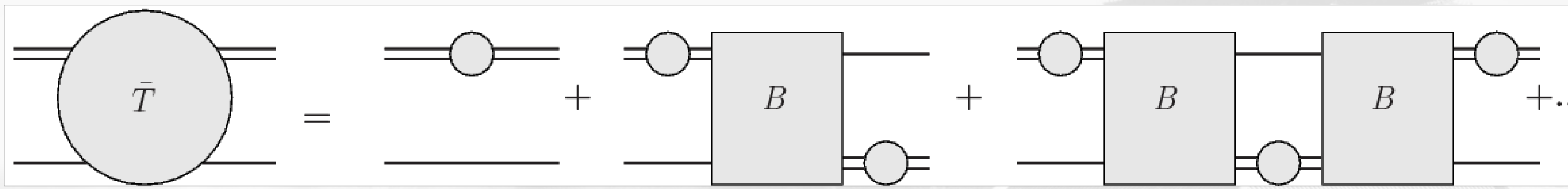
integral equation → matrix equation: $$\bar{T} = \frac{1}{B + \tau^{-1}}$$

3-BODY QUANTIZATION CONDITION

Q: Can we repeat this for 3-body case?

1) Unitary 3-body amplitude

2) ***v*** is cut-free → does not contribute to poles!



3) Discretize momenta:

integral equation → matrix equation: $$\bar{T} = \frac{1}{B + \tau^{-1}}$$

4) Project to irreducible representations of the cubic group: ***Γ***

→ Condition for poles of $\bar{T}$:

$$\mathrm{Det}\left(\mathbf{B}^{\Gamma ss'}_{uu'}(\mathbf{W^2}) + \frac{\mathbf{2E_s\,L^3}}{\vartheta(\mathbf{s})}\tau_\mathbf{s}(\mathbf{W^2})^{-1}\delta_{\mathbf{ss'}}\delta_{\mathbf{uu'}}\right) = \mathbf{0}$$

W – total energy

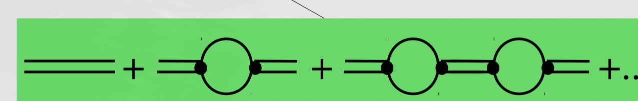
s/s' - shell index

u/u' - basis index

ϑ – multiplicity

L – lattice volume

Es – 1p. energy



REALISTIC CASE STUDY

MM/Doring
arXiv:1807.04746

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Test bed for the Quantization condition: $\pi^+\pi^+\pi^+$

- Ground state levels available from NPLQCD

Detmold et al.(2008)

L=2.5fm and ***m_π=291/352/491/591 MeV***

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  → one isobar required

  Q: Can one describe this with an “isobar”?

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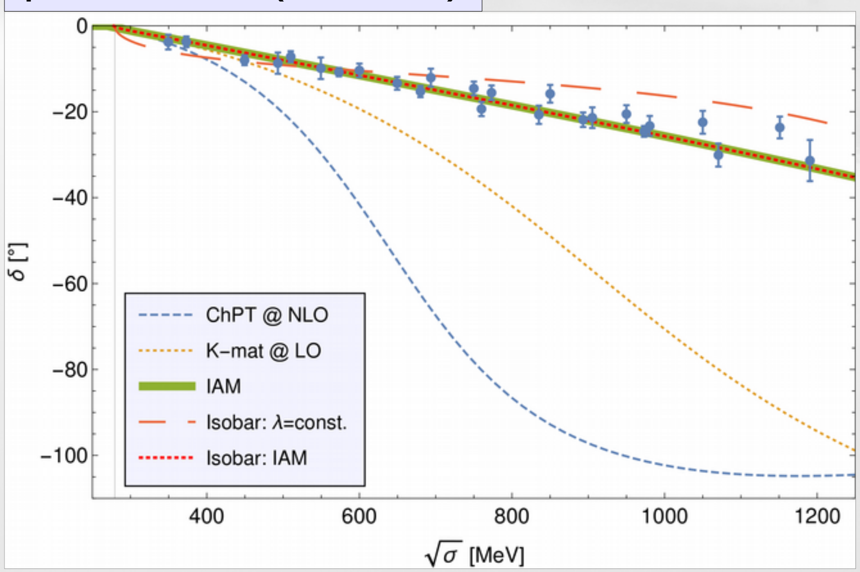
  Q: Can one describe this with an "isobar"?

  A: YES! Choose $v = v(\sigma)$

2-body
phase shift (I=2,L=0)



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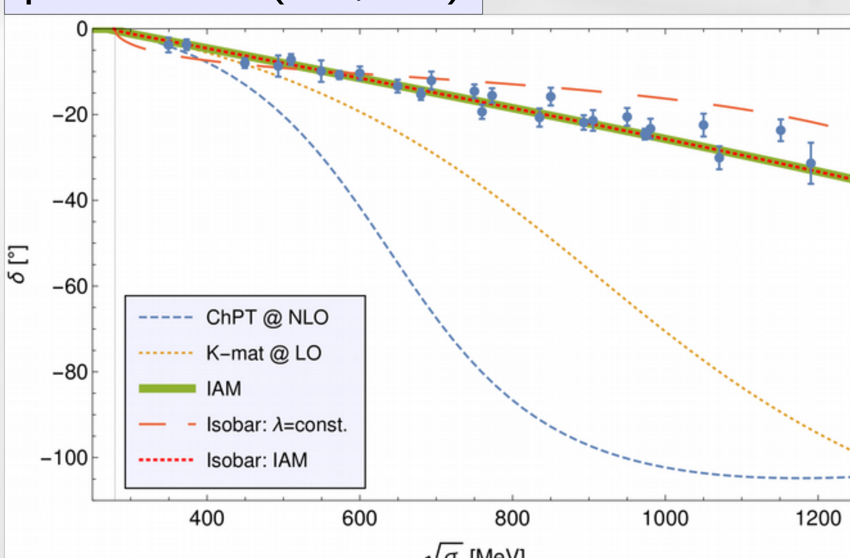
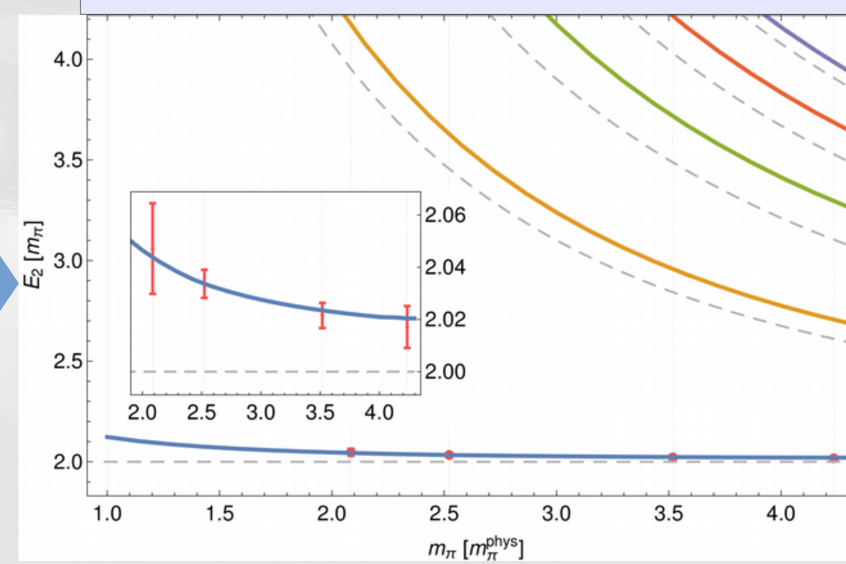


discretize

2-body finite vol. spectrum
PARAMETER FREE PREDICTION



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MM/Doring
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$$\mathrm{Det}\left(\mathbf{B}^{\Gamma ss'}_{uu'}(\mathbf{W^2}) + \frac{\mathbf{2E_s\,L^3}}{\vartheta(\mathbf{s})}\tau_\mathbf{s}(\mathbf{W^2})^{-1}\delta_{\mathbf{ss'}}\delta_{\mathbf{uu'}}\right) = \mathbf{0}$$

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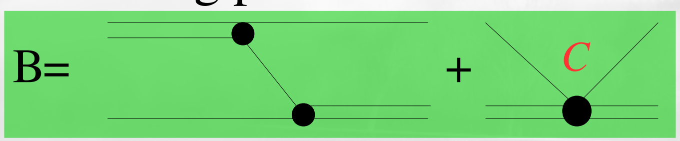
- Remaining parameter:



~ 3-body force

REALISTIC CASE STUDY

MM/Doring
arXiv:1807.04746

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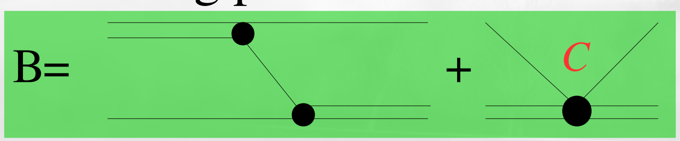
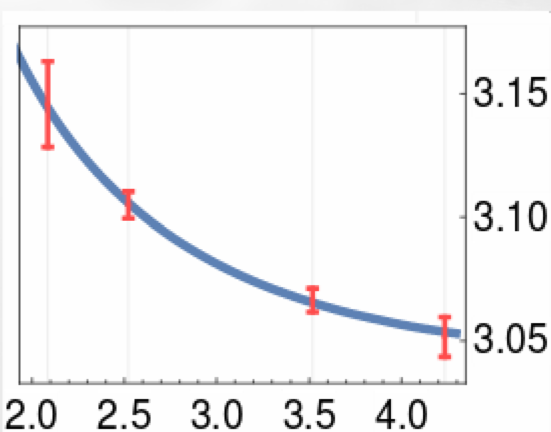


~ 3-body force

→ fit to the ground state levels by NPLQCD

→ $C=0.2\pm1.5\cdot10^{-10}$



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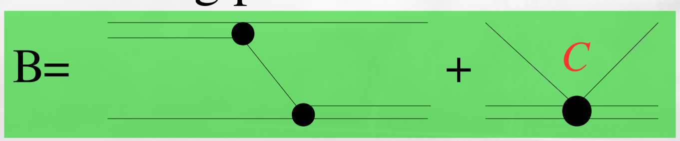
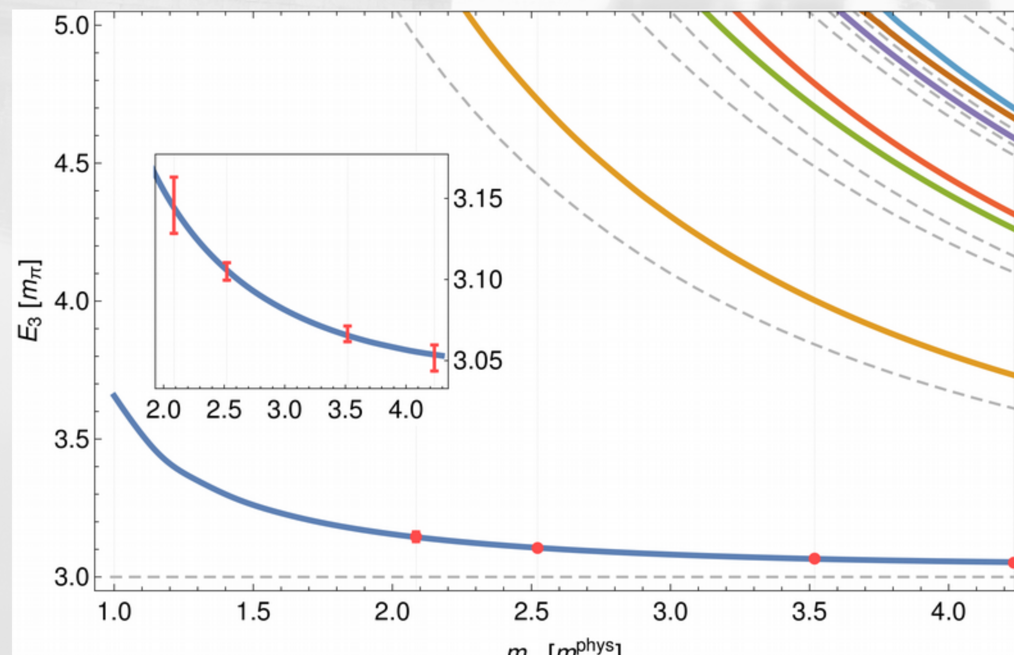


~ 3-body force

→ fit to the ground state levels by NPLQCD

→ $C=0.2\pm1.5\cdot10^{-10}$

- Exited states = prediction



3-BODY: INFINITE VOLUME

Phenomenology (exotics, etc...)

- ✔ **Unitary isobar amplitude derived**
  - **→ 2b sub-amplitudes = tower of isobars**
  - **→ 3-dim. relativistic integral equation**

Applications to ***a_1(1260) and a_1(1420)***

… in progress

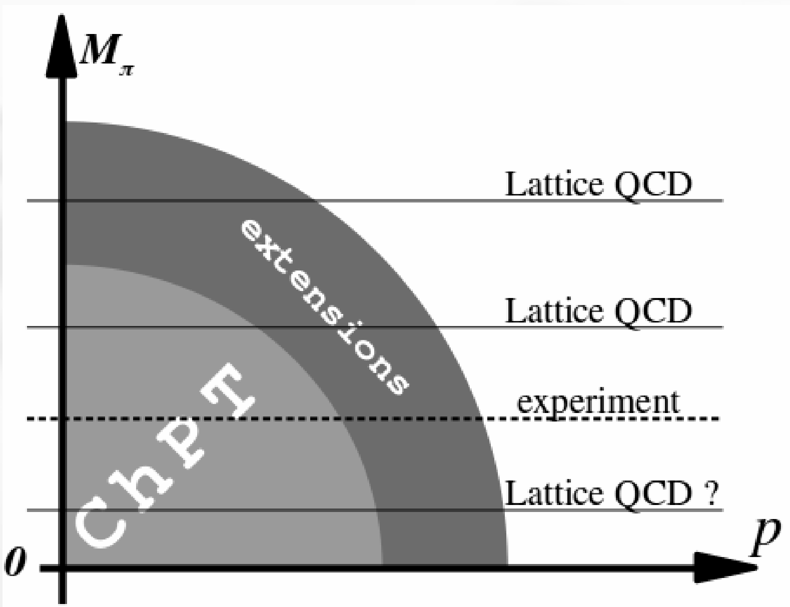
3-BODY: FINITE VOLUME

- ✔ **3-body quantization condition derived**
- ✔ **Finite volume spectrum is investigated:**
  - **2-body sepectrum predicted**
  - **3b force fitted to NPLQCD ground state results (c~0)**
  - **excited 3-body spectrum is predicted**

→ Future applications: ***N*(1440), a1(1260), …***

2-BODY EXAMPLE: σ(500)

New insights from a Lattice QCD calculations



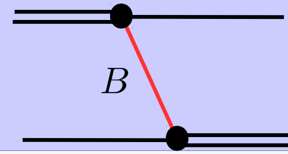
- *phase-shifts* $(N_f=3; m_\pi=236/391\ MeV)$
  Briceno et al.(2016)
- *sc. length* $(N_f=2; m_\pi=139/240/330\ MeV)$
  ETMC(2017)
- *sc. length* $(N_f=3; m_\pi=247/249/314\ MeV)$
  Fu/Chen(2018)
- *phase-shifts* $(N_f=2; m_\pi=227/315\ MeV)$
  Guo/…/MM/…(2018)

- *Mass & width around chiral limit*
  Li/Pagels(1971) Bruns/MM(2017)

Briceno et al.(2016)/ETMC(2017)Fu/Chen(2018)

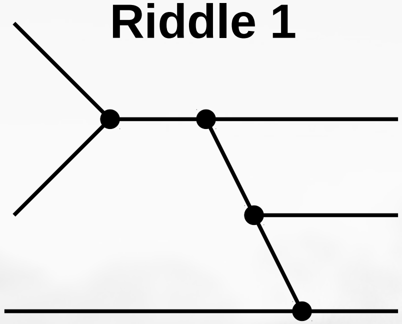
The Power of Unitarity

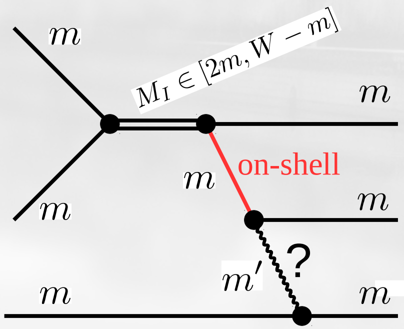
Question: Does

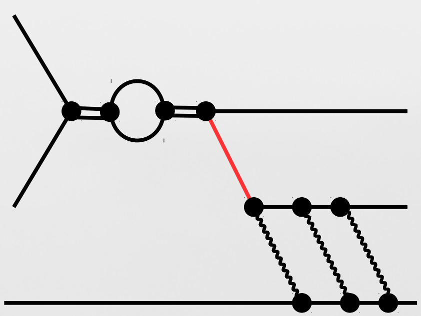


provide full imaginary part of all possible $3 \rightarrow 3$ transitions?

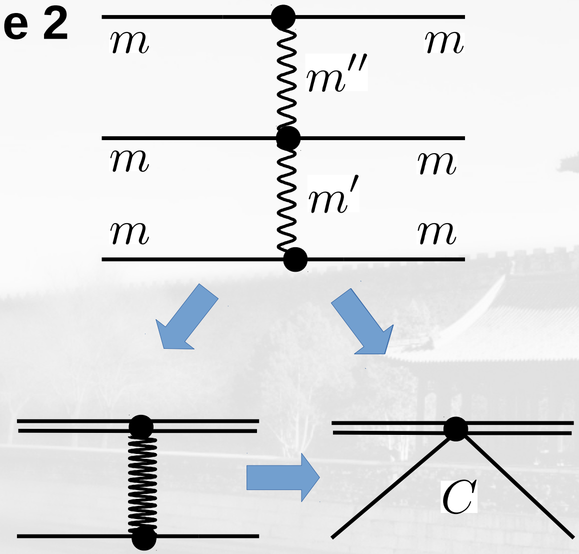
Riddle 1

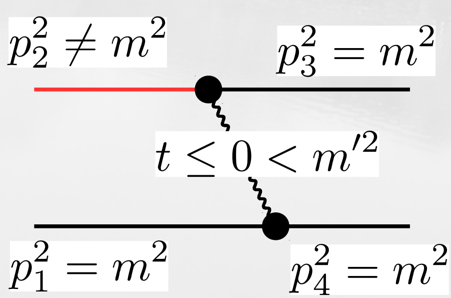


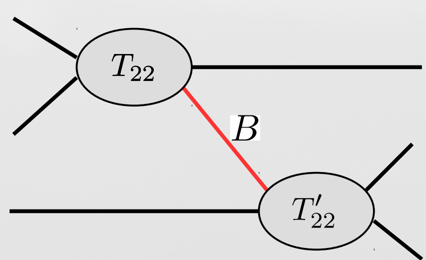




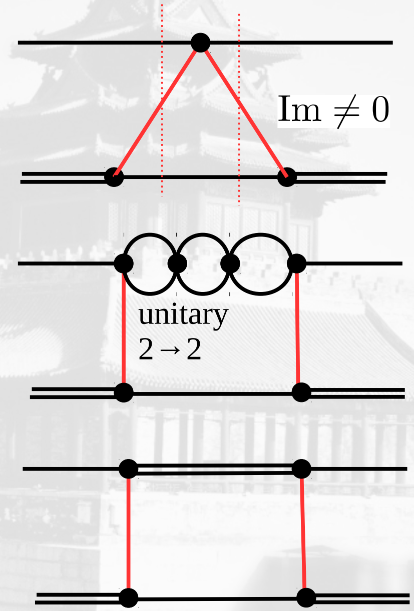
Riddle 2



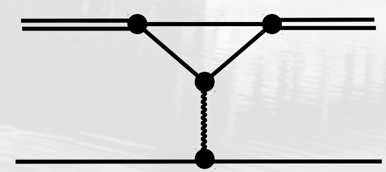




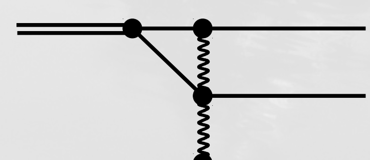
Riddle 3



Riddle 4

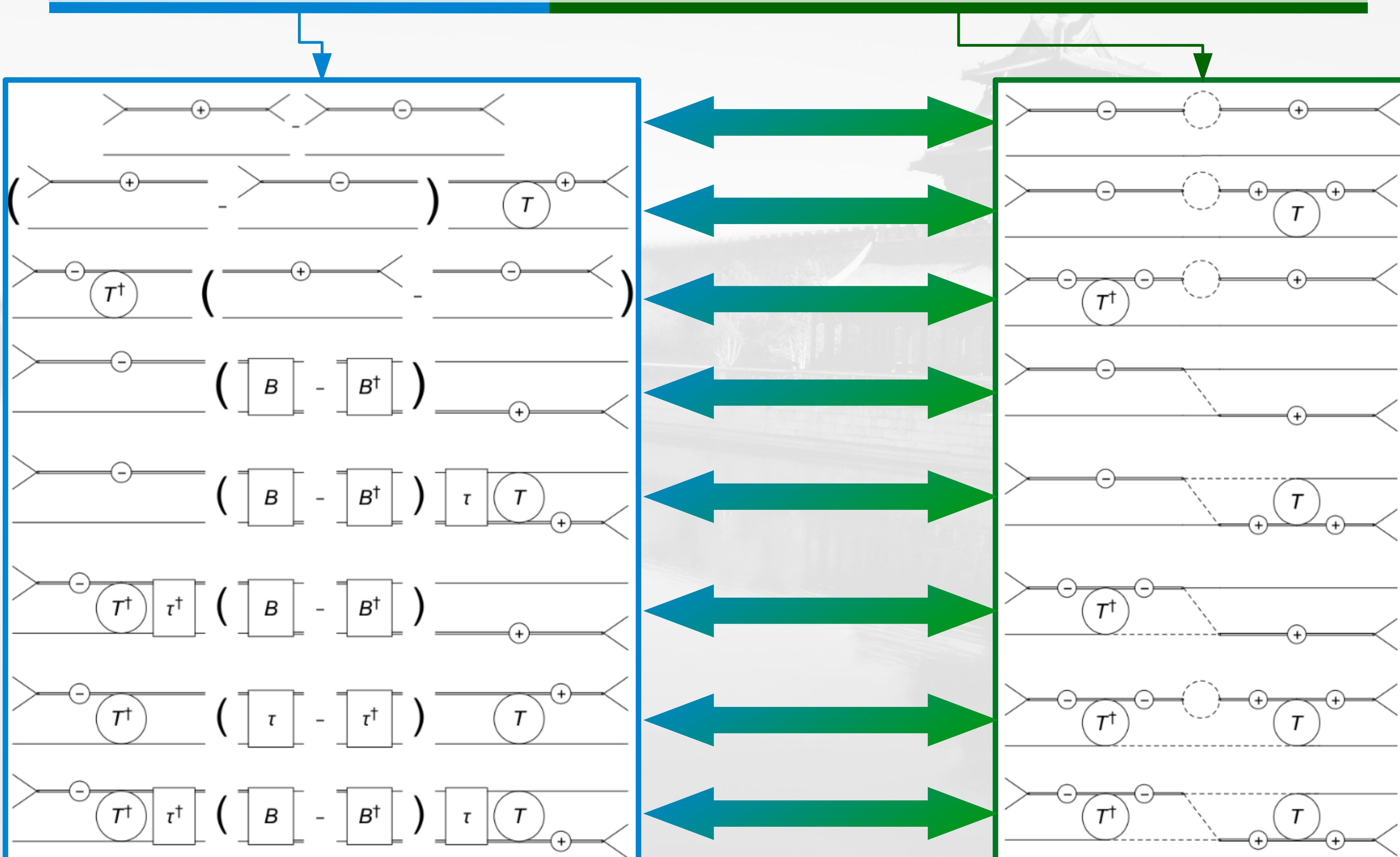


Riddle 5



3-body Unitarity (phase space integral)

$$\langle q_1, q_2, q_3|(\hat{T}-\hat{T}^\dagger)|p_1, p_2, p_3\rangle \;=\; i\int_P \langle q_1, q_2, q_3|\hat{T}^\dagger|k_1, k_2, k_3\rangle\langle k_1, k_2, k_3|\hat{T}|p_1, p_2, p_3\rangle$$



- Projection of T

$$T^{ss'}(\hat{\mathbf{p}}_j,\hat{\mathbf{p}}_{j'}) = 4\pi \sum_{\Gamma\alpha}\sum_{uu'} \chi_u^{\Gamma\alpha s}(\hat{\mathbf{p}}_j) T_{uu'}^{\Gamma ss'} \chi_{u'}^{\Gamma\alpha s'}(\hat{\mathbf{p}}_{j'})\,,$$

$$T_{uu'}^{\Gamma ss'} = \frac{4\pi}{\vartheta(s)\vartheta(s')} \sum_{j=1}^{\vartheta(s)}\sum_{j'=1}^{\vartheta(s')} \chi_u^{\Gamma\alpha s}(\hat{\mathbf{p}}_j) T^{ss'}(\hat{\mathbf{p}}_j,\hat{\mathbf{p}}_{j'}) \chi_{u'}^{\Gamma\alpha s'}(\hat{\mathbf{p}}_{j'})$$

Cancellations:

→ fin. vol. normalization of δ-distribution!

$$\bar{T}^{A_1^+}_{nm}(s) = \tau_n(s) T^{A_1^+}_{nm}(s) \tau_m(s) - 2E_n \tau_n(s) \frac{L^3}{\vartheta(n)} \delta_{nm}$$

$$T^{A_1^+}_{nm}(s) = B^{A_1^+}_{nm}(s) - \frac{1}{L^3} \sum_{x \in set_8} \vartheta(x) B^{A_1^+}_{nx}(s) \frac{\tau_x(s)}{2E_x} T^{A_1^+}_{xm}(s)$$

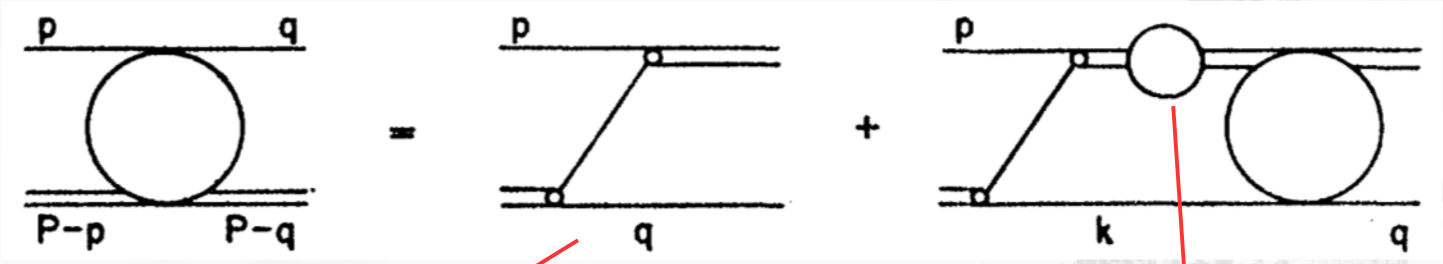
$B^{A_1^+}$ singular at $W^+ = E_m + E_n + E(\boldsymbol{q}_{nj} + \boldsymbol{p}_{mi})$

τ_m^{-1} singular at $W^{\pm\pm} = E_m \pm E((2\pi/L)\boldsymbol{y}) \pm E((2\pi/L)\boldsymbol{y} + \boldsymbol{p}_{mi})$ for $\boldsymbol{y} \in \mathbb{Z}^3$

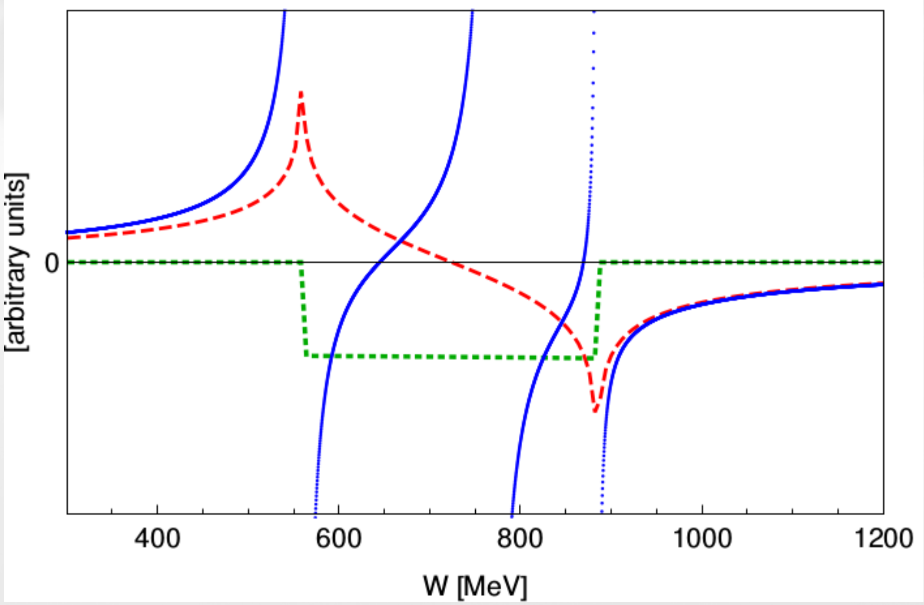
– when isobar-momenta are discretized in the 3-body cms momenta

$$\tau = \sigma(k) - M_0^2 - \frac{1}{(2\pi)^3} \int d^3\boldsymbol{\ell} \frac{\lambda^2}{2E_\ell(\sigma(k) - 4E_\ell^2 + i\epsilon)}$$

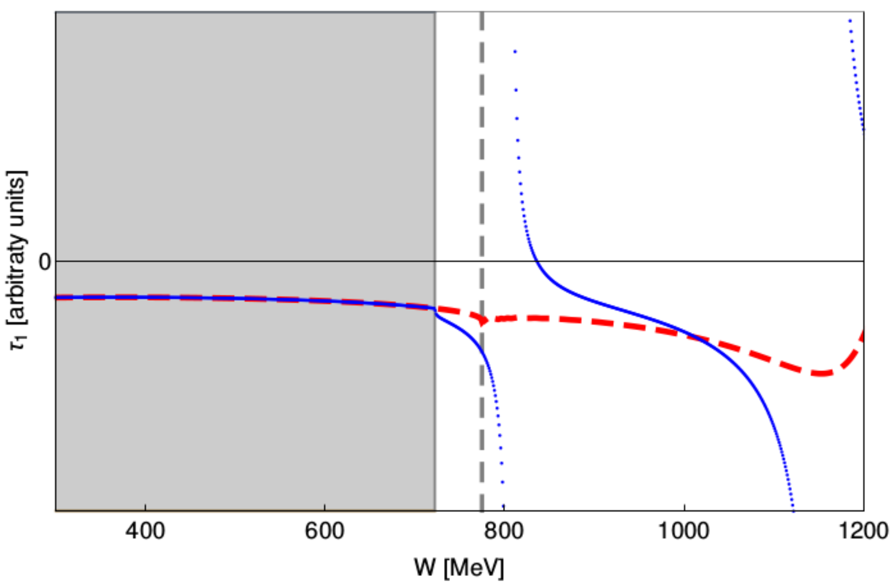
Power-law finite-volume effects dictated by three-body unitarity



S-wave infinite volume vs. A_1^+ finite volume
shell 1 to shell 1

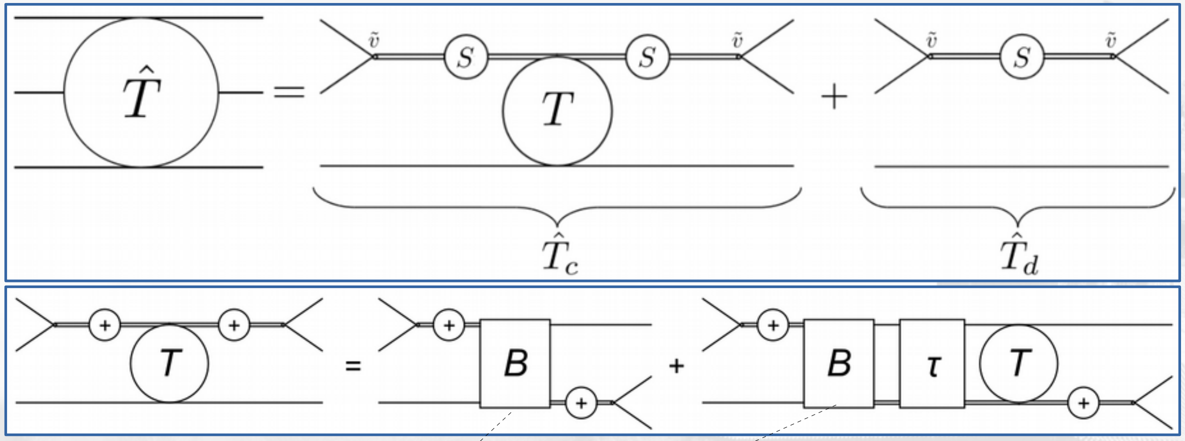


Tower of boosted 2→2 amplitudes
to implement 3-body quantization condition



SCATTERING AMPLITUDE

$3 \to 3$ scattering amplitude is a 3-dimensional integral equation



– Imaginary parts of ***B, S*** are fixed by **unitarity/matching**

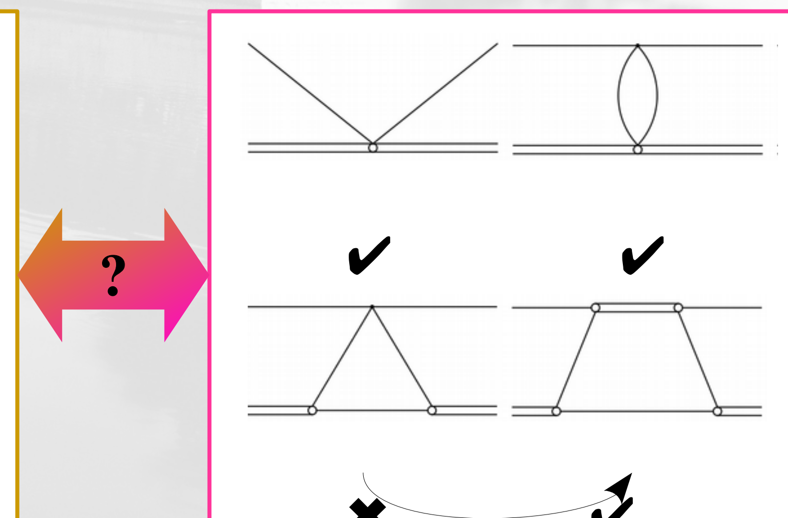
– For simplicity ***υ=λ*** (full relations available)

$$\text{Disc}\, B(u) = 2\pi i \lambda^2 \frac{\delta\left(E_Q - \sqrt{m^2 + \mathbf{Q}^2}\right)}{2\sqrt{m^2 + \mathbf{Q}^2}}$$

- un-subtracted dispersion relation

$$\langle q|B(s)|p\rangle = -\frac{\lambda^2}{2\sqrt{m^2+\mathbf{Q}^2}\left(E_Q-\sqrt{m^2+\mathbf{Q}^2}+i\epsilon\right)}$$

- one-**π** exchange in TOPT → ***R E S U L T !***



Unitarity & Matching

- 3-body Unitarity (normalization condition ↔ phase space integral)

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