New developments in partial-wave analysis with light hadrons

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Era of High-Precision Data Sets

- E.g. from BESIII, COMPASS, GlueX, VES, ...
- Interesting excited light-meson states often decay into multi-body final states
  - Requires modelling of decay amplitudes
  - Often isobar model is used
- Partial-wave analyses (PWA) limited by systematic uncertainties due to model dependence
- Estimation of model dependence often difficult

Systematic uncertainties of the isobar model: (some) challenges

1. How to determine the wave set?
2. How to verify and improve the isobar parametrizations?

This talk

- Discuss ideas to study or reduce model dependence
- Use as an example PWA of $\pi^-\pi^-\pi^+$ data from COMPASS

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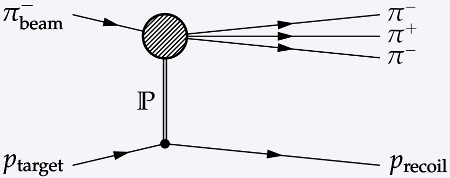
Example: Diffractive $\pi^-\pi^-\pi^+$ Production at COMPASS

C. Adolph *et al.*, PRD **95** (2017) 032004



- 191 GeV/c pion beam
- Exclusive measurement
  - Clean data sample
- 46×10^6 $\pi^-\pi^-\pi^+$ events
- Squared four-momentum transfer $0.1 < t' < 1.0\,(\text{GeV}/c)^2$
- Well-known 3π resonances appear in $m_{3\pi}$ spectrum

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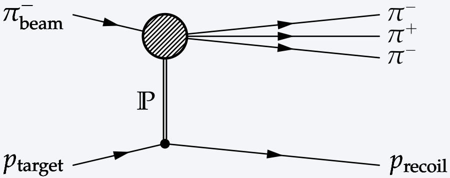

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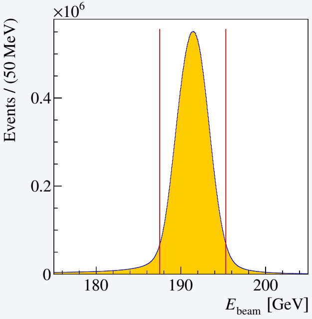

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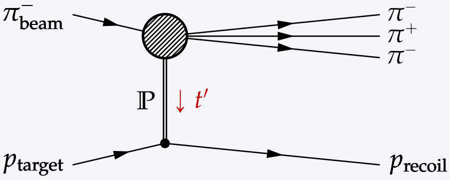

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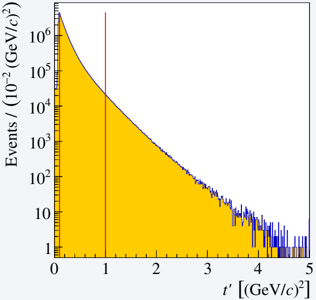

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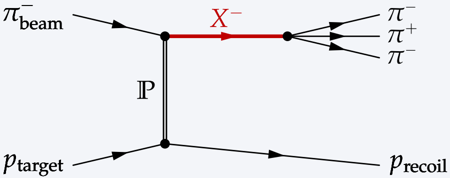

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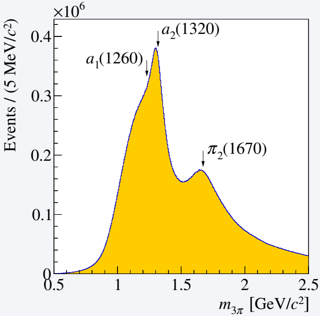

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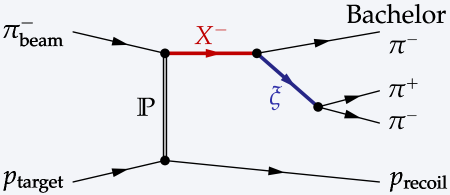
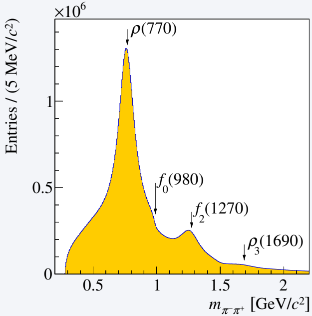
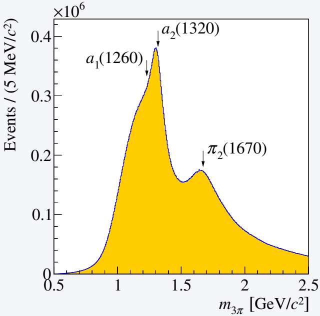
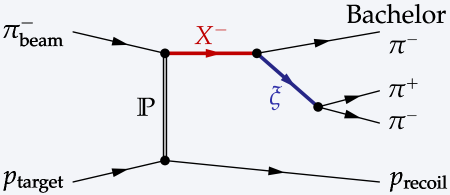
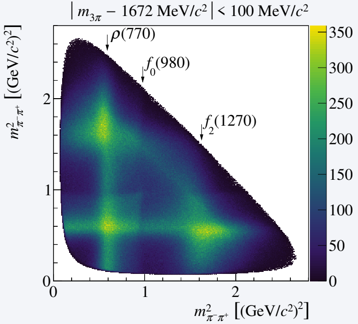


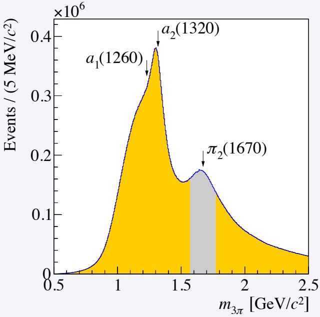




Decay of X via intermediate $\pi^-\pi^+$ resonances = "isobars"







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Partial-Wave Decomposition of $\pi^-\pi^-\pi^+$ Final State

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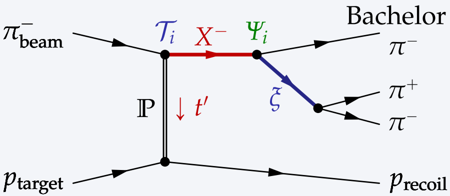

Ansatz: Factorization of production and decay

$$\mathrm{d}\sigma \propto \left| \sum_i^{\text{waves}} \mathcal{T}_i(m_{3\pi}, t')\, \Psi_i(m_{3\pi}, \tau) \right|^2 \mathrm{d}m_{3\pi}^2\ \mathrm{d}t'\ \mathrm{dLIPS}_3(m_{3\pi}, \tau)$$

- PWA model: coherent sum of partial-wave amplitudes
- Decay amplitudes $\Psi_i(m_{3\pi}, \tau)$
  - Describe 5-dimensional τ distribution of partial waves
  - Calculated using isobar model and helicity formalism
- Transition amplitudes $\mathcal{T}_i(m_{3\pi}, t') \Rightarrow$ **interesting physics**
  - Dependence on $m_{3\pi}$ and t' unknown
  - Extracted from data: PWA fits in narrow $m_{3\pi}$ and t' bins

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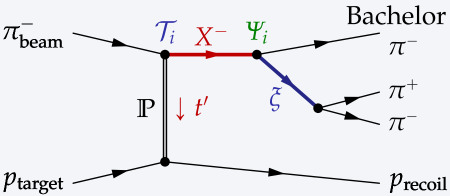

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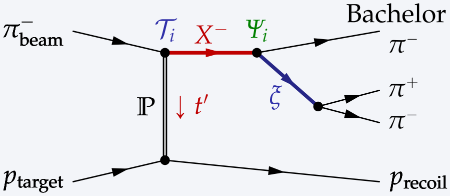

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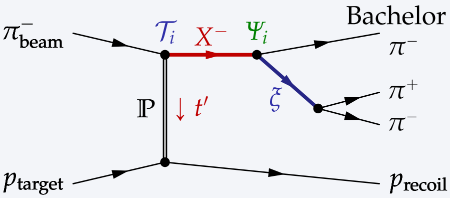

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PWA fit in given $(m_{3\pi}, t')$ bin

- Neglect $m_{3\pi}$ and t' dependence within bin

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- Model for measured τ distribution: intensity $\mathcal{I}(\tau; \{\mathcal{T}_i\})$
- Extended likelihood function

$$\mathcal{L}_{\text{ext}}(\{\mathcal{T}_i\}) = \underbrace{\frac{\overline{N}^N e^{-\overline{N}}}{N!}}_{\substack{\text{Poisson} \\ \text{distribution}}} \prod_{k=1}^{N} \underbrace{\frac{\mathcal{I}(\tau_k; \{\mathcal{T}_i\})}{\int \text{dLIPS}_3(\tau)\, \eta(\tau)\, \mathcal{I}(\tau; \{\mathcal{T}_i\})}}_{\substack{\text{Probability density} \\ \text{for event } k}}$$

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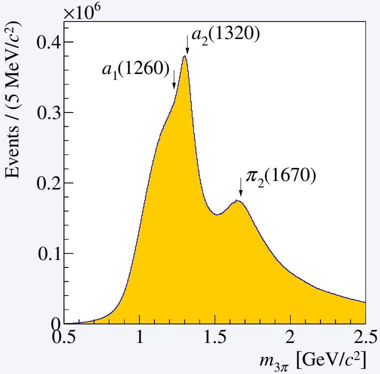
- PWA makes no assumptions about contributing 3π resonances

Partial-wave notation: $J^{PC}\, M^{\varepsilon}\, \xi\, \pi\, L$



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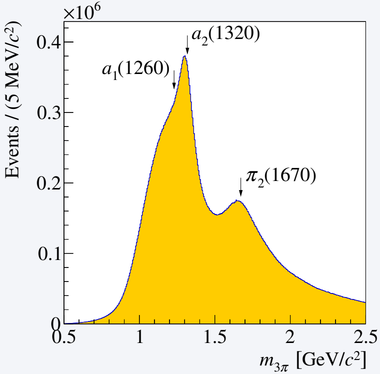

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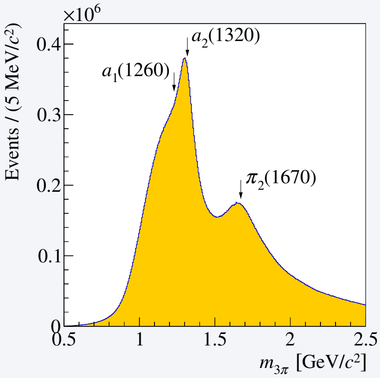

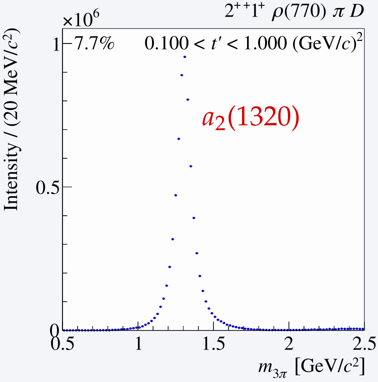

PWA of $\pi^-\pi^-\pi^+$ Final State: Major Waves

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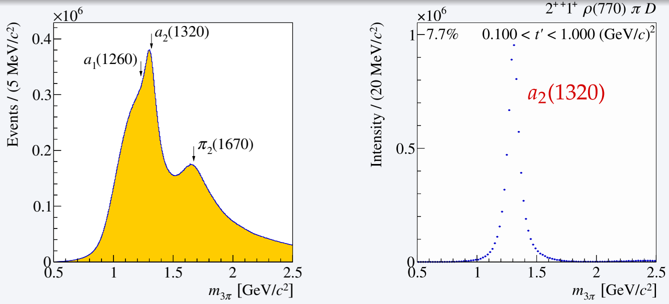

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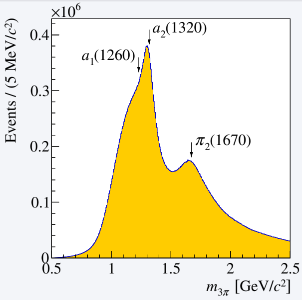

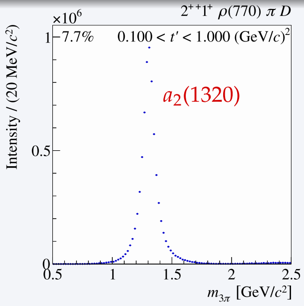

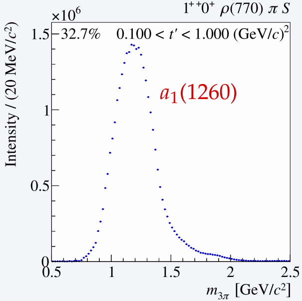

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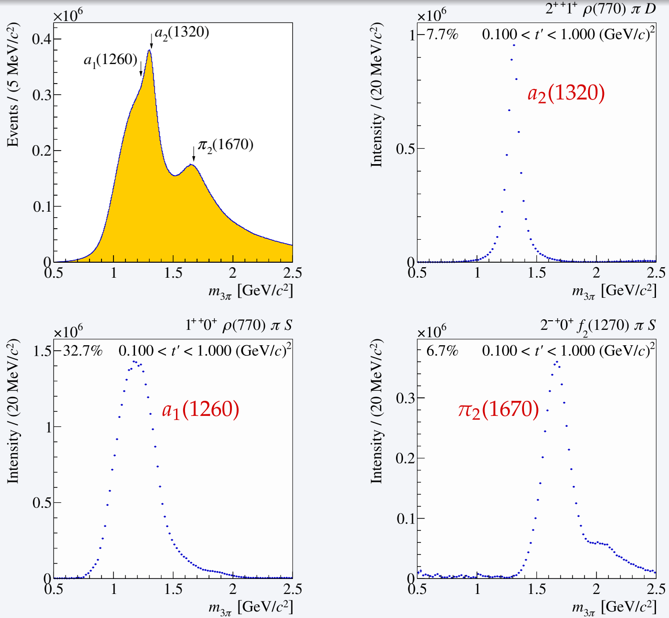

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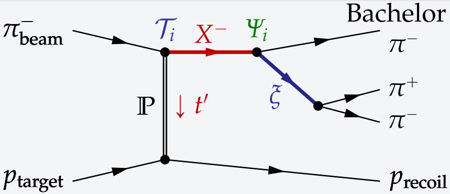

PWA model

$$\mathcal{I}(\tau) = \left| \sum_{i}^{\text{waves}} \mathcal{T}_i \Psi_i(\tau) \right|^2$$

- In principle, infinitely many waves may contribute
- But only finite data sample $\Rightarrow$ have to select wave set

Up to now, wave sets constructed "by hand"

- Often stepwise trial-and-error procedure
- Remove or add single waves and look at change of $\ln \mathcal{L}_{\text{ext}}$ or intensity of wave

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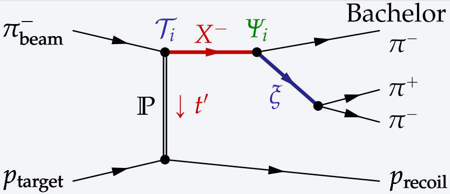

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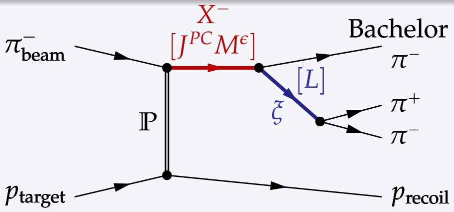

Fit model

- Included isobar resonances:

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$[\pi\pi]_S$	$= 0^{++}$
$\rho(770)$	1^{--}
$f_0(980)$	0^{++}
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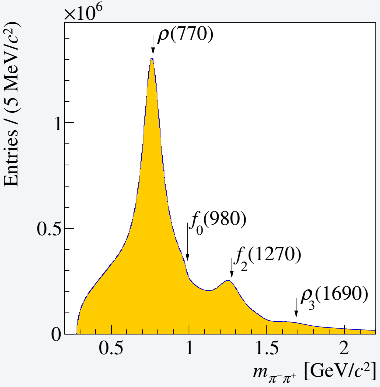
$f_0(1500)$	0^{++}
$\rho_3(1690)$	3^{--}

- Requires precise knowledge of isobar $\to \pi^-\pi^+$ amplitudes



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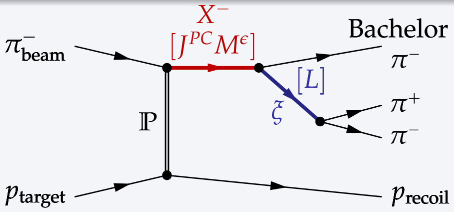

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- Requires precise knowledge of isobar $\to \pi^-\pi^+$ amplitudes
- J and L up to 6
- 87 partial waves
- Additional incoherent isotropic background wave
- Derived from 128 wave set by removing waves with relative intensities $\lesssim 10^{-4}$

How to determine the wave set?

Construction "by hand"

- Not a well-defined procedure
- Difficult to document and reproduce
- Danger of introducing observer bias
- Very time consuming procedure
- Stepwise procedures are known to be suboptimal

A. Miller, *Subset Selection in Regression*, Chapman and Hall/CRC, London (2002)

Alternative approach: add regularization terms to log-likelihood function

Guegan *et al.*, JINST **10** (2015) 09002; K. Bicker, PhD thesis, TUM (2016); O. Drotleff, Master thesis, TUM (2015); F. Kaspar, Master thesis, TUM (2017)

$$\ln \mathcal{L}(\{\mathcal{T}_i\}) = \ln \mathcal{L}_{\text{ext}}(\{\mathcal{T}_i\}) + \ln \mathcal{L}_{\text{reg}}(\{\mathcal{T}_i\})$$

- Choose $\mathcal{L}_{\text{reg}}(\{\mathcal{T}_i\})$ such that
  - leaves waves with large intensities unchanged
- Use systematically constructed set of allowed partial waves up to cut-off criterion = "wave pool"

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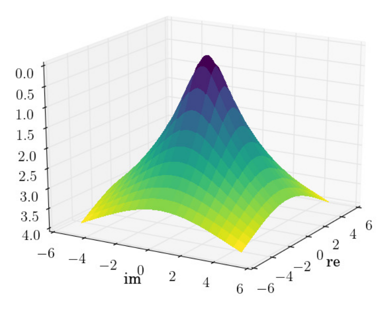
$$\mathcal{L}_{\text{reg}}(\{\mathcal{T}_i\}) = \prod_i^{\text{waves}} \frac{1}{1 + |\mathcal{T}_i|^2 / \Gamma^2}$$

$$\ln \mathcal{L}_{\text{Cauchy}} = \ln \mathcal{L}_{\text{ext}} - \sum_i^{\text{waves}} \ln \left[1 + \frac{|\mathcal{T}_i|^2}{\Gamma^2} \right]$$

- $\mathcal{L}_{\text{reg}}$ has Cauchy form in $|\mathcal{T}_i|$
- "Heavy tailed" distribution
- Pulls intensities of small waves toward zero
- Small bias for large waves

$\ln \mathcal{L}_{\text{reg}}(\{\mathcal{T}_i\})$ in $\mathcal{T}_i$ plane



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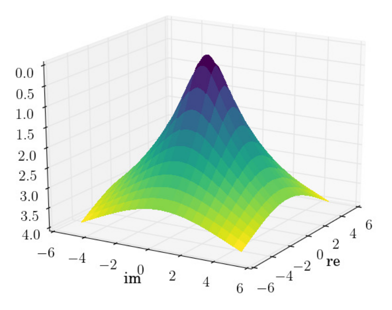

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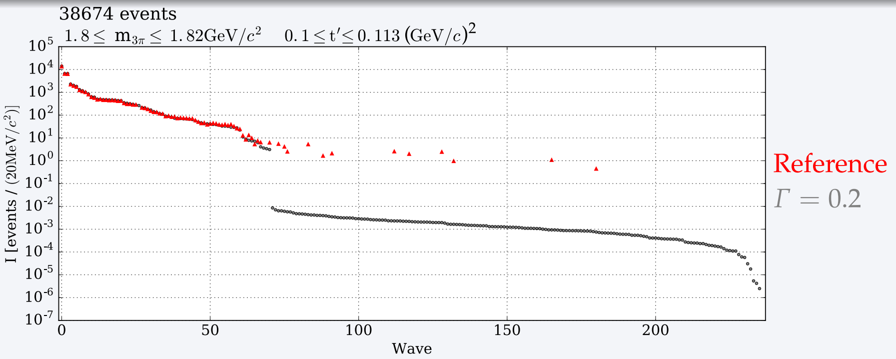

Monte Carlo data generated using 88-wave fit result from real data

- **Black:** 235 waves in wave pool ordered by intensity
- **Red:** waves from fit with 88-wave input model
- Clear drop in intensities ⇒ clear place where to cut
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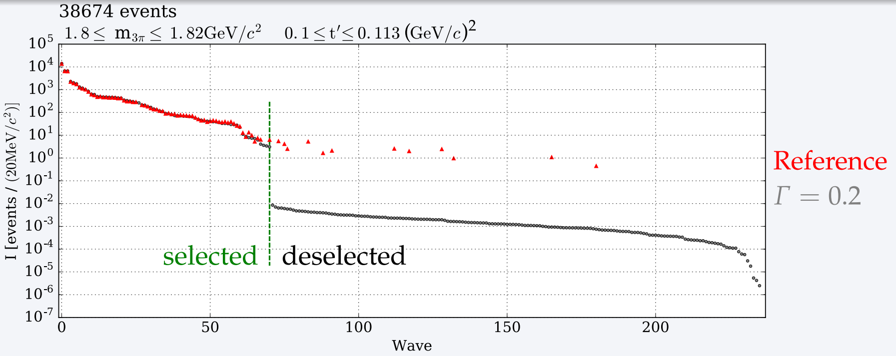

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How to determine the wave set?

LASSO Regularization Term

"Least Absolute Shrinkage and Selection Operator" Tibshirani, J. Royal Stat. Soc. B **58** (1996) 267

$$\mathcal{L}_{\text{reg}}(\{\mathcal{T}_i\}) = \prod_i^{\text{waves}} e^{-\lambda\,|\mathcal{T}_i|}$$

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- Suppresses waves with small intensities effectively
- Penalizes also waves with large intensities ⇒ potential bias

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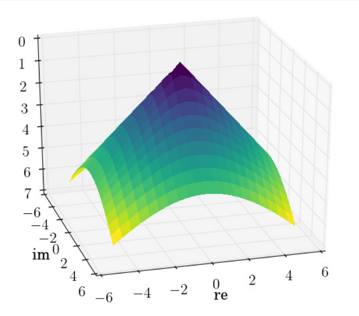

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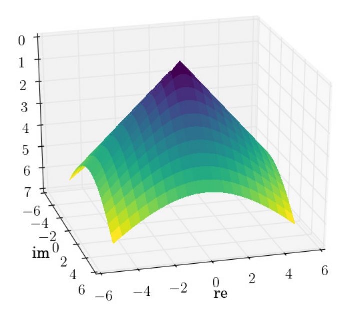

How to determine the wave set?

Regularization of likelihood function

Promising approach

- Method makes wave-set selection reproducible and bias explicit
- Choice of regularization terms is subjective
  - Applying different regularization terms ⇒ study wave-set dependence of PWA result
- Allows to systematically study dependence of PWA result on
  - Set of isobars
  - Isobar parametrizations
  - Inclusion of higher partial waves
  - …
- Studies with Monte Carlo and real data still work in progress

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Regularization of likelihood function

Challenges

- Regularization terms make likelihood function multimodal
- Cauchy and LASSO are two extreme cases of a continuum of regularization terms
  - Cauchy prior: heavy-tailed ⇒ low bias
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- Model-selection problem is ill-posed at low $m_{3\pi}$
  - Small phase space ⇒ only low-mass tails of isobars contribute
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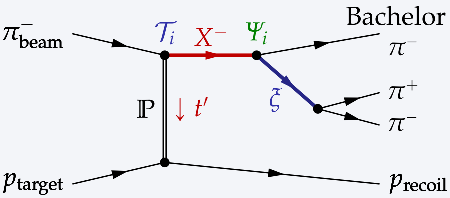
How to verify and improve the isobar parametrizations?



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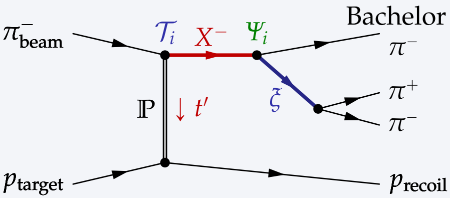

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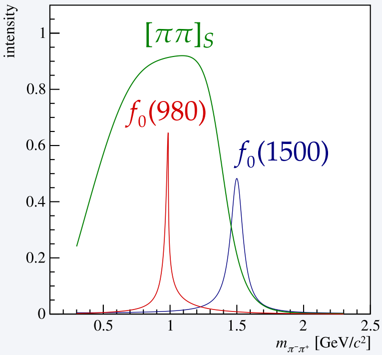
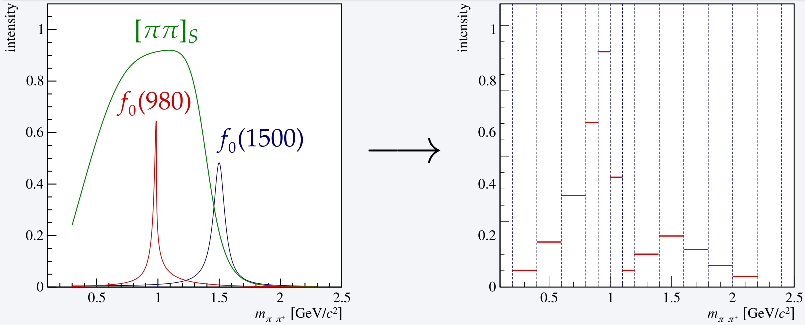


How to verify and improve the isobar parametrizations?

Novel analysis method inspired by (Q)MIPWA

E791, PRD **73** (2006) 032204

- Replace fixed $J^{PC} = 0^{++}$ isobar parametrizations by piece-wise constant amplitudes in $m_{\pi^-\pi^+}$ bins for 3π waves with $J^{PC} = 0^{-+}$, 1^{++}, and 2^{-+}
- Extract $m_{3\pi}$ dependence of total $J^{PC} = 0^{++}$ isobar amplitude from data
  - *Advantage:* reduction of model bias
  - *Caveats:* significant increase in number of fit parameters



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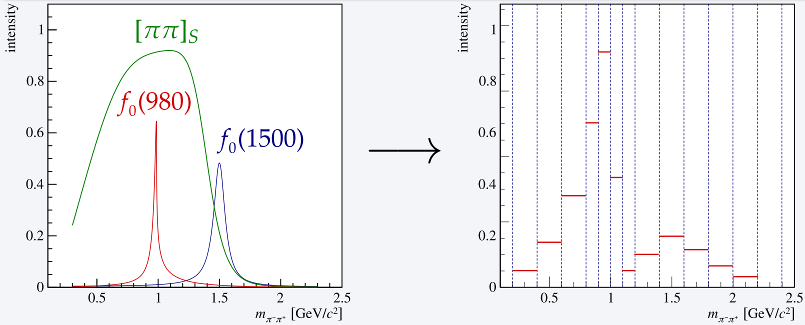
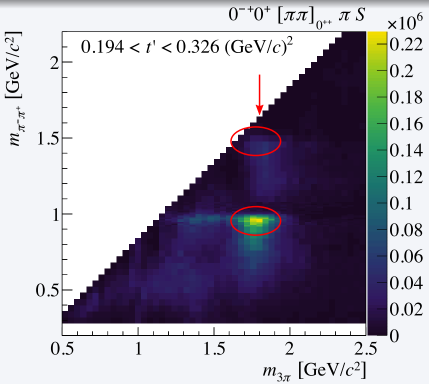
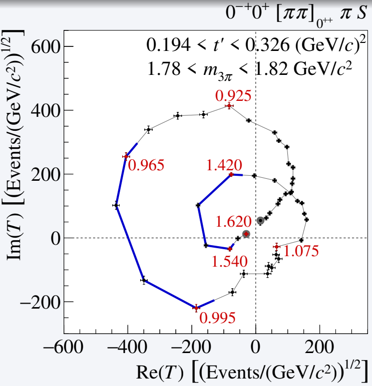


$\pi\pi$ S-Wave Amplitude in $J^{PC} = 0^{-+}$ 3π Wave

COMPASS, PRD **95** (2017) 032004





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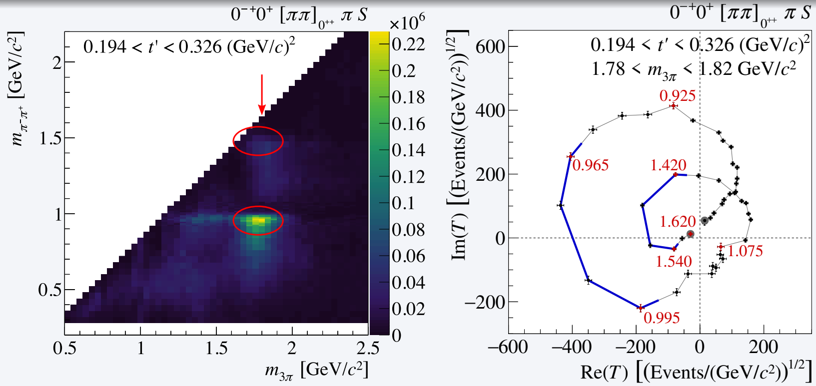

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Freed-Isobar PWA of $\pi^-\pi^-\pi^+$ Final State

PWA model with more freed waves

- Based on 88-wave model from COMPASS, PRD **95** (2017) 032004
- Free isobar amplitudes of 11 largest + 3 "interesting" waves
- 69 waves with fixed isobar parametrizations remain

Challenge

- Continuous mathematical ambiguities for some $\pi^-\pi^+$ amplitudes ("zero modes")
- Resolution requires additional constraints
  - See talk by D. Ryabchikov and F. Krinner *et al.*, PRD **97** (2018) 114008
- *Here:* amplitudes after resolution of ambiguity

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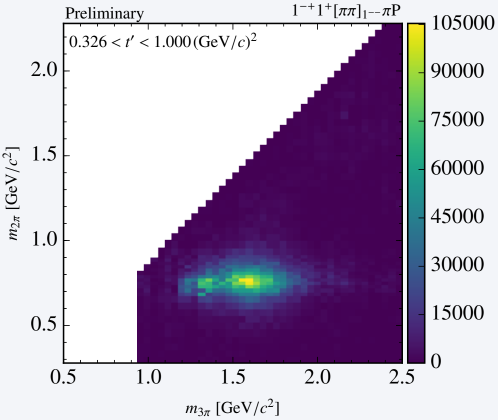

- 3π wave has spin-exotic quantum numbers
- Peak at $m_{3\pi} \approx 1.6\,\mathrm{GeV}/c^2$ correlated with $\rho(770)$
- Shape of $\rho(770)$ peak deviates slightly from Breit-Wigner

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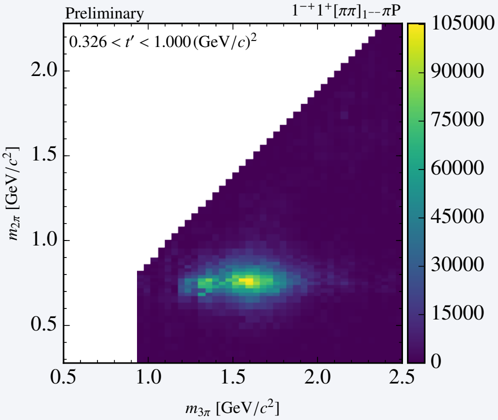

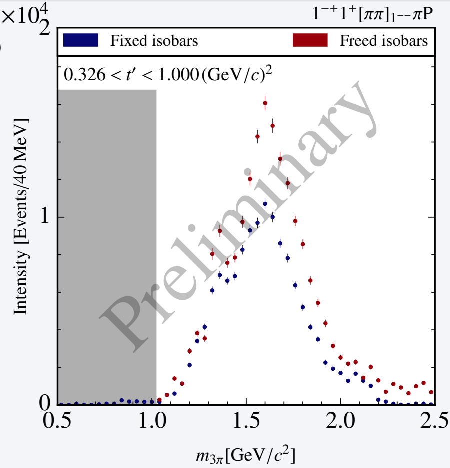

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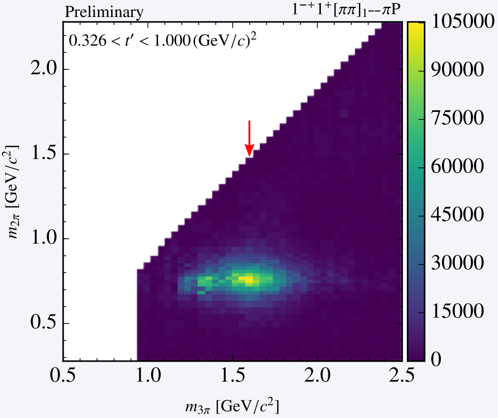

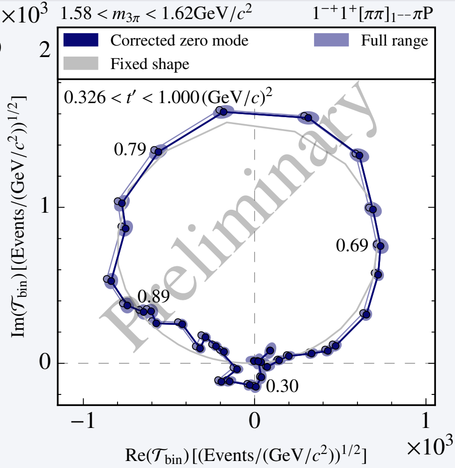

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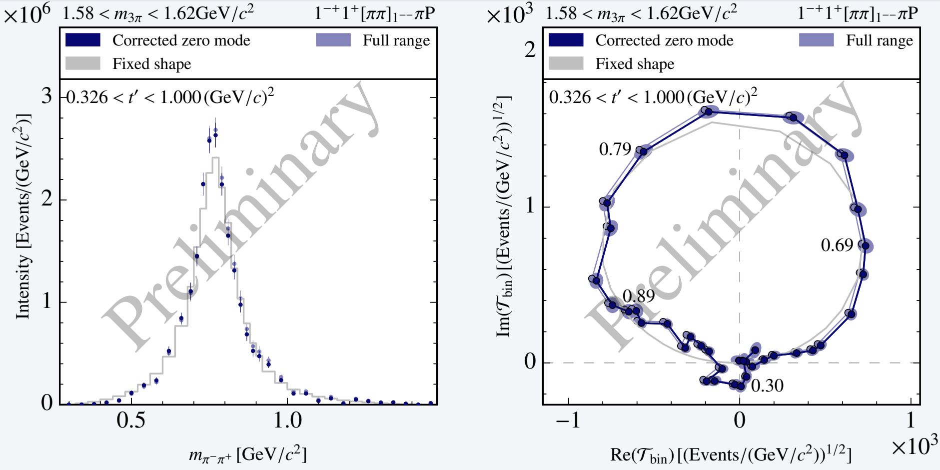

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Freed-Isobar Partial-Wave Analysis

- Greatly reduces model bias in isobar analyses
- Detailed insight into 2π vs. 3π dynamics
  - Verify/learn isobar parametrizations from data
  - Search for higher excited isobar states
  - Study of final-state-interaction effects
- Ambiguities appear when many waves are freed
  - Can be identified and resolved
- Method directly applicable to heavy-meson decays
  - (Q)MIPWA is now a tool to extract physics not just to cross check single isobar amplitudes F. Krinner *et al.*, PRD **97** (2018) 114008

Challenges

- Requires large data sets (several 10^5 events)
- Modeling of extracted isobar amplitudes
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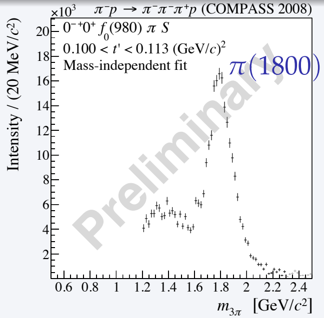
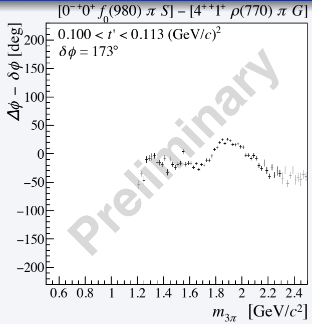
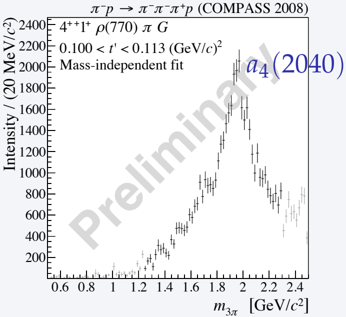
Outline

Resonance Extraction





- *Result of PWA:* partial-wave amplitudes as function of $m_{3\pi}$
- Fit several waves simultaneously by resonance model
- Exploit relative phase
- Resonances parametrized by Breit-Wigner amplitudes



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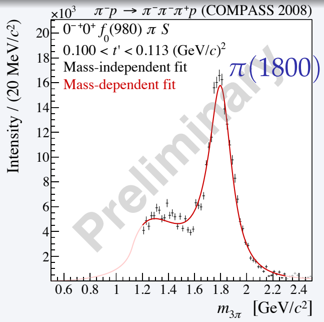
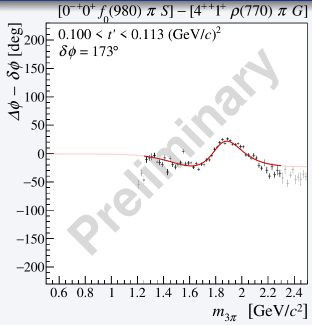
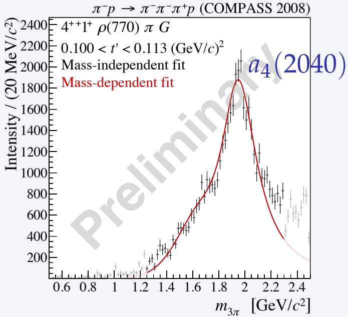




- *Result of PWA:* partial-wave amplitudes as function of $m_{3\pi}$
- Fit several waves simultaneously by resonance model
- Exploit relative phase
- Resonances parametrized by Breit-Wigner amplitudes



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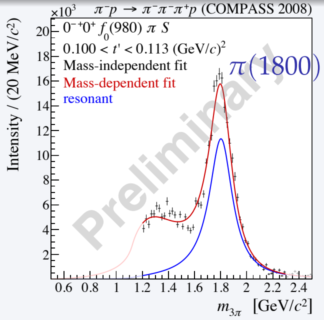

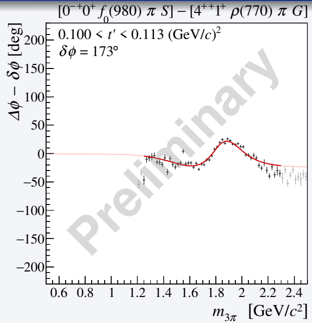

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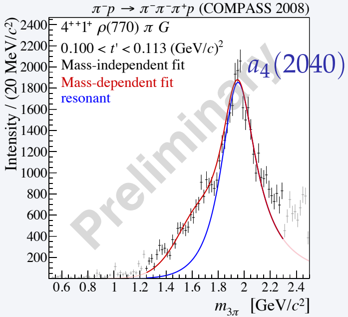

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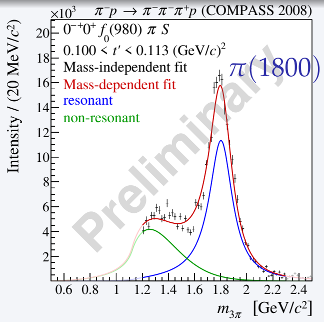

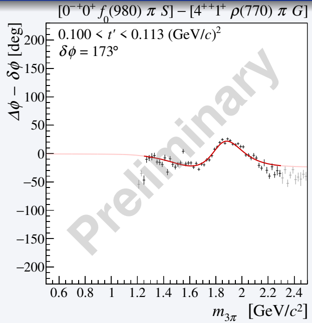

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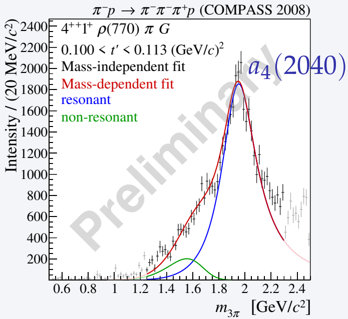
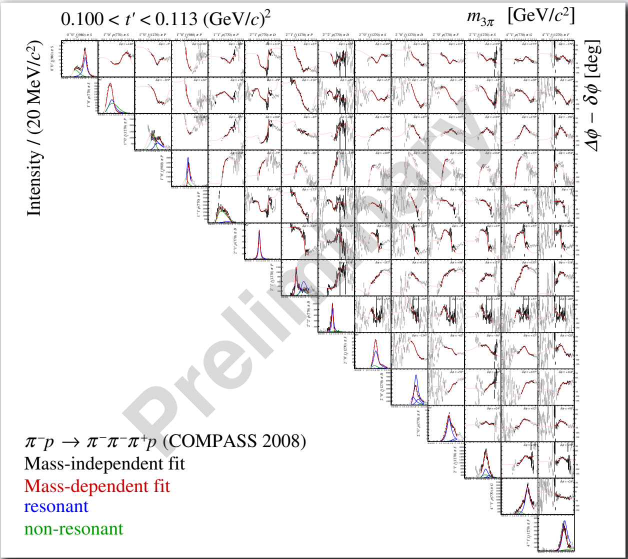


Resonance-Model Fit of $\pi^-\pi^-\pi^+$ Data



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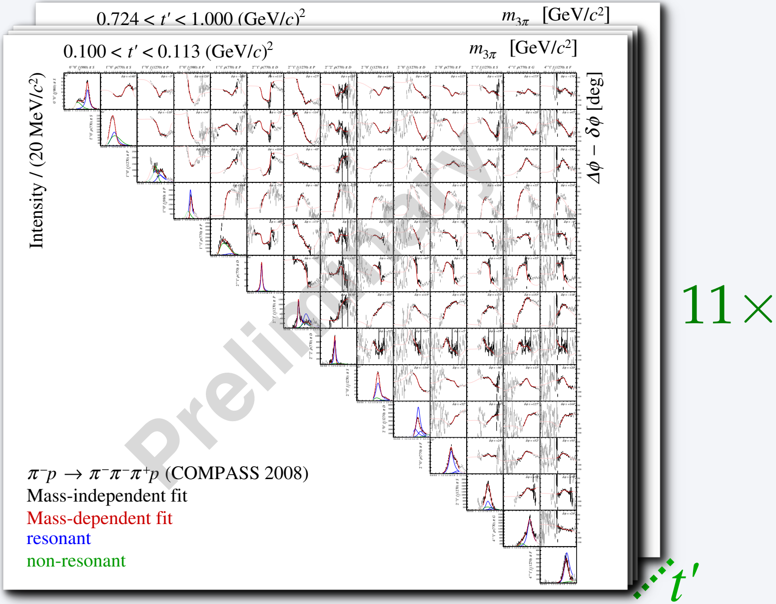
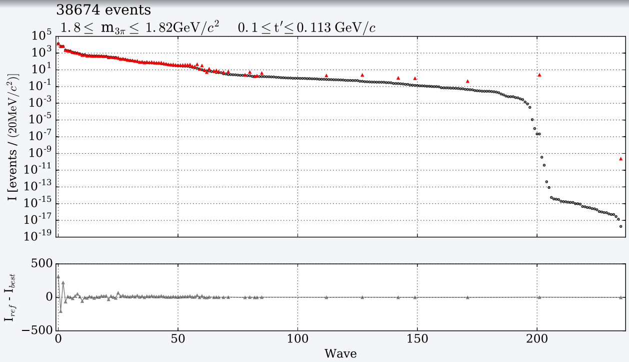


LASSO Method



Reference

$\lambda = 1$

Mass bin at $m_{3\pi} = 1.8\ \mathrm{GeV}/c^2$ (MC Data)

- Also LASSO produces clear drop in intensity distribution
  - B. Guegan *et al.* used cut on relative intensity $> 10^{-3}$ $\Rightarrow$ unnecessary
- Position depends strongly on value of λ $\Rightarrow$ dials wave-set size
- Increased bias in larger waves with increased λ
- Need criterion to tune λ

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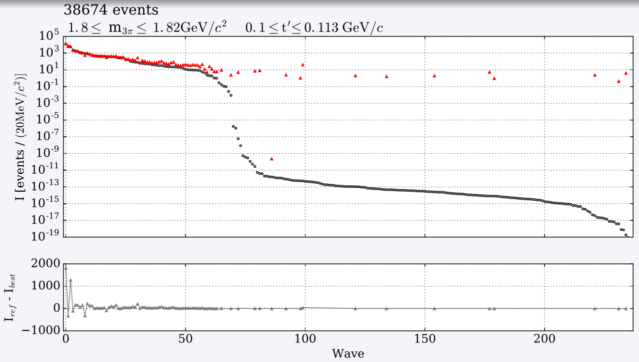

Reference

$\lambda = 5$

Mass bin at $m_{3\pi} = 1.8\,\mathrm{GeV}/c^2$ (MC Data)

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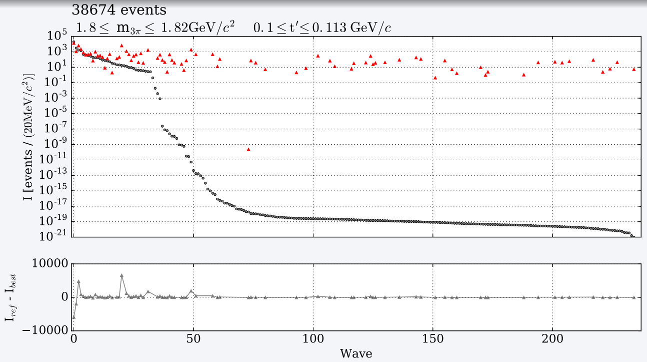
LASSO Method



Reference

$\lambda = 20$

Mass bin at $m_{3\pi} = 1.8 \,\mathrm{GeV}/c^2$ (MC Data)

- Also LASSO produces clear drop in intensity distribution
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Estimation of LASSO Parameter

B. Guegan *et al.* suggest to use information criteria (IC)

- IC are relative measure of model quality
  $\Rightarrow$ tradeoff between goodness of fit and model complexity
- Akaike IC (AIC): choose value of λ that minimizes

$$2k - 2\ln\hat{\mathcal{L}}$$

  k: number of selected parameters
  $\hat{\mathcal{L}}$: maximum likelihood value
- Bayesian IC (BIC): choose value of λ that minimizes

$$k\ln n - 2\ln\hat{\mathcal{L}}$$

  n: number of data points
- λ scans are work in progress
  - BIC seems to prefer λ round 5
  - AIC give typically larger waves sets than BIC

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