Dispersive analysis of $K^+ \to \pi^+ \ell^+ \ell^-$ decays

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(Work in progress)

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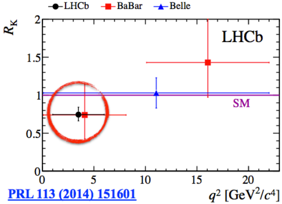
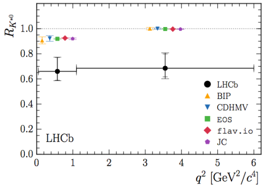
Motivation: Flavor anomalies in B decays

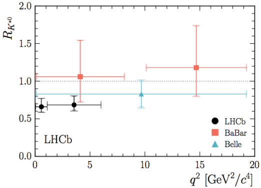
- Hints of Lepton Flavor Universality Violation (LFUV) in $B \to K^{(*)}\ell^+\ell^-$

$$R_K = 0.745^{+0.090}_{-0.074}(\text{stat}) \pm 0.036(\text{syst})\,, \quad 1 \le q^2 \le 6\,\text{GeV}^2$$

$$R_{K^{*0}} = \begin{cases} 0.66^{+0.11}_{-0.07}(\text{stat}) \pm 0.03(\text{syst}) & \text{for} \quad 0.045 < q^2 < 1.1\,\text{GeV}^2\,, \\ 0.69^{+0.11}_{-0.07}(\text{stat}) \pm 0.05(\text{syst}) & \text{for} \quad 1.1 \quad < q^2 < 6.0\,\text{GeV}^2\,, \end{cases}$$



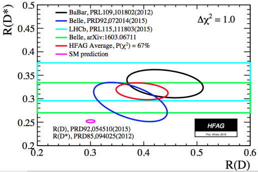




- Measured rates for $B \to D\tau\nu_\tau$ and $B \to D^*\tau\nu_\tau$ are enhanced relative to SM

$$R(D)_{\text{exp}} = \frac{BR(B \to D\tau\nu_\tau)}{BR(B \to D\ell\nu_\ell)} = 0.407(39)(24)\,,$$

$$R(D^*)_{\text{exp}} = \frac{BR(B \to D^*\tau\nu_\tau)}{BR(B \to D^*\ell\nu_\ell)} = 0.304(13)(7)\,,$$



Motivation: Flavor anomalies in B decays

- Expressed in terms of the effective $\Delta B = 1$ Hamiltonian

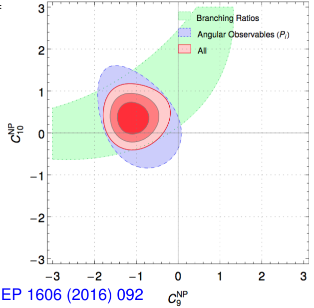
$$\mathcal{H}_{eff} = -\frac{4G_F}{\sqrt{2}} V_{tb} V_{ts}^* \sum_i C_i^B Q_i^B \,,$$

potential NP interpreted as contributions to $C_{9,10}^B$

$$Q_9^B = \frac{e^2}{32\pi^2}[\bar{s}\gamma^\mu(1-\gamma_5)b] \sum_{\ell=e,\mu} [\bar{\ell}\gamma_\mu \ell]\,,$$

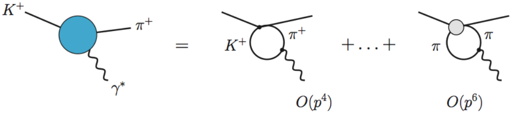
$$Q_{10}^B = \frac{e^2}{32\pi^2}[\bar{s}\gamma^\mu(1-\gamma_5)b] \sum_{\ell=} [\bar{\ell}\gamma_\mu\gamma_5 \ell]\,.$$

- An explanation of the B-anomalies typically required pulls of $C_{9,10} \sim \mathcal{O}(1)$



Descotes-Genon et.al. JHEP 1606 (2016) 092

- Examine the role of $K^+ \to \pi^+ \ell^+ \ell^-$ decays in testing B-anomalies
- Dominant contribution to $K^+ \to \pi^+ \ell^+ \ell^-$ due to $K^+ \to \pi^+ \gamma^*$



[Ecker et al. (87)] [D'Ambrosio et al. (98)]

$$V_+(z) = a_+ + b_+ z + V_+^{\pi\pi}(z)\,, \qquad z = q^2/m_K^2$$

- a_+ and b_+ are related to chiral LEC's poorly known
- Fits to E865 and NA48/2 spectra data yields:

$$a_+^{ee} = -0.587(10)\,, \qquad a_+^{\mu\mu} = -0.575(39)\,,$$

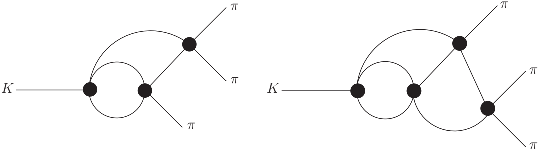
- One can show that (Crivellin et. al. Phys.Rev. D93 (2016) no.7, 074038)

$$C_9^{B,\mu\mu} - C_9^{B,ee} = -\frac{a_+^{\mu\mu} - a_+^{ee}}{\sqrt{2} V_{ts}^* V_{td}} \simeq -19 \pm 79\,,$$

Kaon probes of LFUV

- Determination of $a_+^{\mu\mu} - a_+^{ee}$ needs **one order of magnitude improvement** to probe NP explanations of B-anomalies
- Improvements of this size **possible at the NA62** experiment

  $\Rightarrow$ High-statistics: Number of decay~ 50 times larger than NA48/2
- **Proposal to experimentalists**: (re)measure $K^+ \to \pi^+\ell^+\ell^-$ sepctrum to determine a_+ at high precision
- **Proposal from theorists**: revisit the description with the advances of the field to understand better low-energy meson dynamics
  - This work: Khuri-Treiman for (P-wave) $K^+ \to \pi^+\pi^+\pi^-$ decays



Outline

1. **Review of the $K^+ \to \pi^+\ell^+\ell^-$ decays in ChPT**

2. **$K^+ \to \pi^+\ell^+\ell^-$: Dispersive approach (this work)**
    - $K^+ \to \pi^+\pi^+\pi^-$
    - Pion vector form factor

3. **Fits to N48/2 $K^+ \to \pi^+\ell^+\ell^-$ data**

4. **Outlook**

Non-leptonic weak amplitudes in the effective chiral Lagrangian

- The effective Lagrangian we are concerned with reads

$$\begin{aligned}
\mathcal{L}_{\rm eff} &= \mathcal{L}_{\rm strong} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \mathcal{L}_{\rm em} + \frac{G_F}{\sqrt{2}} V_{ud} V_{us} \left(\mathcal{L}_{\Delta \rm S=1} + \mathcal{L}^{\rm em}_{\Delta S=1} \right) \\
\mathcal{L}_{\rm strong} &= \frac{f^2}{4} \langle \partial_\mu U \partial^\mu U^\dagger \rangle + \frac{f^2}{4} \langle U^\dagger \chi + \chi^\dagger U \rangle \\
\mathcal{L}_{\Delta \rm S=1} &= g_8 \left(L_\mu L^\mu \right)_{23} + {\rm h.c.}\,, \quad L_\mu = i f^2 U \partial_u U^\dagger \\
\mathcal{L}_{\rm em} &= -e A_\mu {\rm tr}(\hat{Q} V^\mu) + \ldots\,, \quad V_\mu = \frac{1}{2} i f^2 [U, \partial_\mu U^\dagger] \\
\mathcal{L}^{\rm em}_{\Delta S=1} &= e g_8 f^2 A_\mu \{ L^\mu, \Delta \}_{23} + \ldots\,, \quad \Delta = U [\hat{Q}, U^\dagger]
\end{aligned}$$

Calculation of the $K^+ \to \pi^+\ell^+\ell^-$ decay amplitude

Ecker, Pich, de Rafael Nuclear Physics B291 (1987)

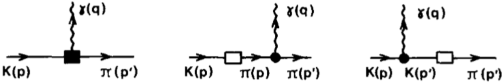
- Lowest Order $\mathcal{O}(p^2)$

■ : vertices from $\mathcal{L}^{\rm em}_{\Delta S=1}$

□ : vertices from $\mathcal{L}_{\Delta S=1}$

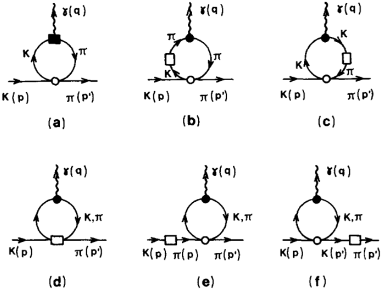
● : vertices from $\mathcal{L}_{\rm em}$

○ : vertices from $\mathcal{L}_{\rm strong}$



$$A(K^+ \to \pi^+\gamma) = \frac{G_F}{\sqrt{2}} V_{ud}V_{us} g_8 f^2 (p+p')_\mu \left(2ie + 2ip^2 \frac{i}{p^2 - M_\pi^2} ie + ie \frac{i}{(p')^2 - M_K^2} 2i(p')^2\right) = 0$$

- One-Loop $\mathcal{O}(p^4)$

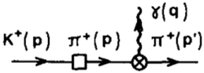


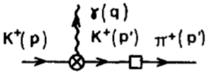
→ vanish because $\mathcal{O}(p^2)$ vertices

Counterterms $\mathcal{O}(p^4)$

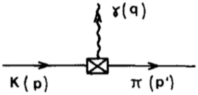
Ecker, Pich, de Rafael Nuclear Physics B291 (1987)

purely strong and e.m. origin





e.m. induced from $\mathcal{L}^{\mathrm{em},4}_{\Delta S=1}$



$$A(K^+ \to \pi^+ \gamma) = \frac{1}{3}\frac{G_F}{\sqrt{2}} V_{ud} V_{us} g_8 e \left(\mathrm{w}_1 - \mathrm{w}_2 + 3(\mathrm{w}_2 - 4L_9)\right) q^2 \epsilon^\mu (p+p')_\mu$$

- Final amplitude in terms of renormalized couplings

$$A(K^+ \to \pi^+ \gamma) = \frac{G_F V_{ud} V_{us} g_8 e}{\sqrt{2}(4\pi)^2} q^2 \hat{\phi}_+(q^2) \epsilon^\mu (p+p')_\mu$$

$$\hat{\phi}_+(q^2) = -\left[\phi_K(q^2) + \phi_\pi(q^2) + \mathrm{w}_+\right]$$

$$\mathrm{w}_+ = -\frac{1}{3}(4\pi)^2 (\mathrm{w}_1^r - \mathrm{w}_2^r + 3(\mathrm{w}_2^r - 4L_9^r)) - \frac{1}{6}\log\left(\frac{M_K^2 M_\pi^2}{\mu^4}\right)$$

State-of-the-art $\mathcal{O}(p^6)$

D'Ambrosio, Ecker, Isidori Portolés JHEP 9808 (1998) 004

- Spectrum in the dilepton invariant mass

$$\frac{d\Gamma}{dz} = \frac{G_F^2 \alpha^2 M_K^5}{12\pi(4\pi)^4} \lambda^{3/2}(1, z, r_\pi^2) \sqrt{1 - 4\frac{r_\ell^2}{z}} \left(1 + 2\frac{r_\ell^2}{z}\right) |V(z)|^2 , \quad r_P = M_P/M_K \, , \quad z = q^2/M_K^2$$

- Form factor: polynomial + $\mathcal{O}(p^6)$ unitarity correction

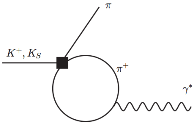
$$V(z) = a_+ + b_+ z + V^{\pi\pi}(z) \, ,$$

- Polynomial: Low Energy Constants of the ChPT framework

$$a_+ = \frac{V_{ud} V_{us} g_8}{\sqrt{2}} \left(\frac{1}{3} - \mathrm{w}_+\right) , \quad b_+ = \frac{V_{ud} V_{us} g_8}{\sqrt{2}} \frac{1}{60}$$

$$\mathrm{w}_+ = \frac{64\pi^2}{3} (N_{14}^r - N_{15}^r + 3L_9^r) + \frac{1}{3} \ln \frac{\mu^2}{M_K M_\pi}$$

- Unitarity correction



$$V_j^{\pi\pi}(z) = \overbrace{\frac{\alpha_j + \beta_j (z - z_0)/r_\pi^2}{G_F M_K^2 r_\pi^2}}^{K\to\pi\pi\pi} \overbrace{\left[\frac{4}{9} - \frac{4}{3z} + \frac{4}{3z}\left(1 - \frac{z}{4}\right) G(z)\right]}^{\text{1-loop}} \overbrace{\left[1 + \frac{z}{r_\rho^2}\right]}^{F_V(z)},$$

State-of-the-art $\mathcal{O}(p^6)$

- Polynomial dominates over the loop correction $V_+^{\pi\pi}(z)$
- $N_{14}^r - N_{15}^r$ poorly known (Bijnens et.al. EPJC 39, 347 (2005), Cappiello et.al. EPJC 78, 265 (2018))

$$L_9 = 7 \times 10^{-3}, \; N_{14} = -10.4 \times 10^{-3}, \; N_{15} = 5.95 \times 10^{-3},$$
$$a_+ = -0.236$$

$$L_9 = 5.9(4) \times 10^{-3}, \quad N_{14} = -2(28) \times 10^{-4}, \quad N_{15} = 1.65(22) \times 10^{-3},$$
$$a_+ = -1.012$$

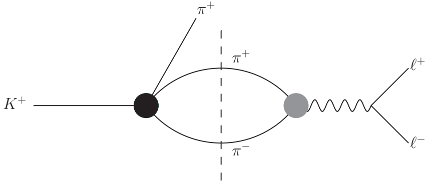
- Theoretical ideas on a_+ and b_+: $\frac{b_+}{a_+} \sim \frac{\mathcal{O}(p^6)}{\mathcal{O}(p^4)} < 1$.

Source	a_+	b_+	b_+/a_+	$BR \times 10^9$
$K^+ \to \pi^+ e^+ e^-$ E865	$-0.587(10)$	$-0.655(44)$	~ 1.20	$294(15)$
$K^+ \to \pi^+ e^+ e^-$ NA48/2	$-0.578(16)$	$-0.779(66)$	~ 1.35	$314(10)$
$K^+ \to \pi^+ \mu^+ \mu^-$ NA48/2	$-0.575(39)$	$-0.813(145)$	~ 1.41	$96.2(2.5)$
Lattice (RBC/UKQCD)	$1.6(7)$	$0.7(8)$	~ 0.4	—

- Hierarchy estimated is not correct? Higher order chiral corrections or crossed-channel contributions are not negligible?

Transition form factor $K \to \pi\ell^+\ell^-$

- Two-pion discontinuity of the $K \to \pi^+\gamma^*$ transition form factor



- Twice subtracted dispersion relation

$$
\begin{aligned}
\mathrm{disc} f_{K\pi}^{\pi\pi}(s) &= 2i\sigma(s) F_\pi^{V*}(s) \mathcal{M}_{\ell=1}^{K\to3\pi}(s)\theta(s-4m_\pi^2)\,, \\
f_{K\pi}(s) &= a + bs + \frac{s^2}{2\pi i}\int_{4m_\pi^2}^{\infty} ds' \frac{\mathrm{disc} f_{K\pi}^{\pi\pi}(s')}{(s')^2(s'-s)}\,, \\
a = a_+ &= \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\sigma(s') F_\pi^{V*}(s') \mathcal{M}_{\ell=1}^{K\to3\pi}(s')}{s'}\,, \\
b = b_+/M_K^2 &= \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\sigma(s') F_\pi^{V*}(s') \mathcal{M}_{\ell=1}^{K\to3\pi}(s')}{(s')^2}\,,
\end{aligned}
$$

$K \to 3\pi$ decay amplitude decomposition

- Assumptions: $i)$ $\Delta I = 1/2$-dominace rule; $ii)$ treat the K meson as spurious $I = 1$ triplet; $iii)$ isospin is conserved in the decay.

- Partial wave decomposition of the decay amplitude as

$$\mathcal{M}^{ijk,l}_{K\to 3\pi}(s,t,u) = \sum_{\ell=0}^{\infty}\sum_{I}(2\ell+1)P_\ell(\cos\theta)\mathcal{P}_I^{ijkl} m_\ell^I(s,t,u)\,,$$

$$s = (p_k - p_1)^2\,, \quad t = (p_k - p_2)^2\,, \quad u = (p_k - p_3)^2\,,$$

i, j, k denote the isospin of the pions, l isospin state of the interaction.

- $K \to \pi\pi\pi$ amplitude decomposition in terms of three amplitudes of fixed I and ℓ

$$\begin{aligned}\mathcal{M}(s,t,u) &= \mathcal{M}_0^0(s) + (s-u)\mathcal{M}_1^1(t) + (s-t)\mathcal{M}_1^1(u)\\ &+ \mathcal{M}_0^2(t) + \mathcal{M}_0^2(u) - \frac{2}{3}\mathcal{M}_0^2(s)\end{aligned}$$

$$\mathcal{M}_{K^+\to\pi^+\pi^+\pi^-}(s,t,u) = \mathcal{M}(t,u,s) + \mathcal{M}(u,s,t)$$

- Unitarity relation for $K \to 3\pi$ in the general form

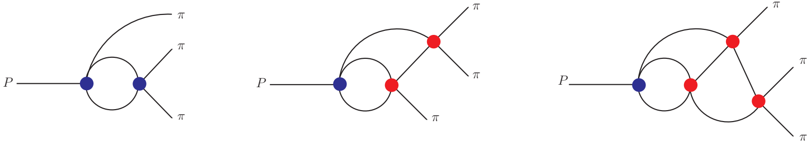
$$\mathrm{disc}\,\mathcal{M}^{ijk,l}_{K\to 3\pi}(s,t,u) = i\sum_{n'}(2\pi)^4\delta^4\left(p_1+p_2+p_3-p_{n'}\right)\mathcal{T}^{*abc\cdots,ijk}_{n'\to\pi\pi\pi}\mathcal{M}^{abc\cdots,l}_{K\to n'}$$

- Khuri-Treiman integral equations of the Omnès type:

$$\mathcal{M}^I_\ell(s) \quad = \quad \Omega^I_\ell(s)\left(P^I_\ell(s) + \frac{s^n}{\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\sin\delta^I_\ell(s')\hat{\mathcal{M}}^I_\ell(s')}{|\Omega^I_\ell(s')|(s')^n\,(s'-s)}\right),$$

- Omnès functions $\Omega^I_\ell(s)$:

$$\Omega^I_\ell(s) = \exp\left\{\frac{s}{\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\delta^I_\ell(s')}{s'(s'-s)}\right\}.$$



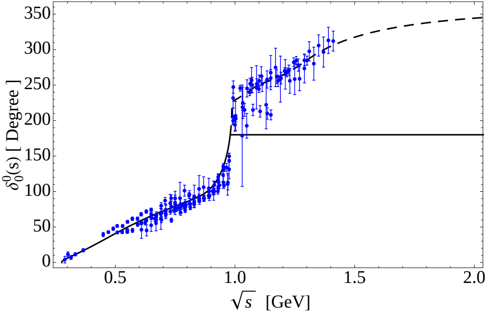
- Order of the subtraction polynomials $P^I_\ell(s)$: asymptotics of the functions $\mathcal{M}^I_\ell(s)$.
- Asymptotic behavior of $\Omega^I_\ell(s)$: assume the phase shift tends to a constant for $s \to \infty$

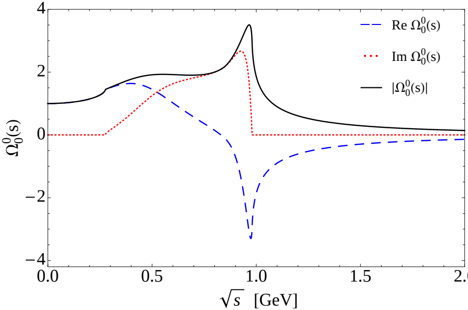
Omnès functions

Madrid-Kraków PhysRevD.83.074004

$$\Omega_0^0(s) = \exp\left\{\frac{s}{\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\delta_0^0(s)}{s'(s'-s)}\right\},$$

$$\lim_{s\to\infty} \delta_0^0(s) \to \pi\,, \quad \Omega_0^0(s) \sim \frac{1}{s}$$





Omnès functions

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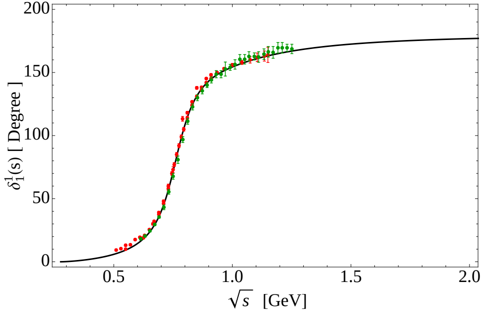
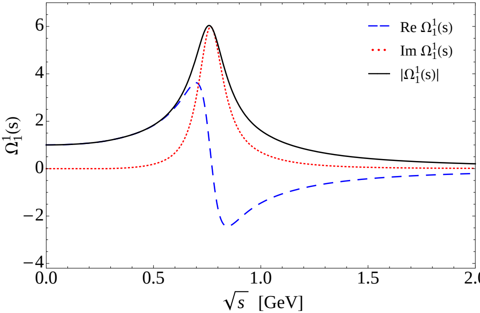
$$\Omega_1^1(s) = \exp\left\{\frac{s}{\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\delta_1^1(s)}{s'(s'-s)}\right\},$$

$$\lim_{s\to\infty} \delta_1^1(s) \to \pi\,, \quad \Omega_1^1(s) \sim \frac{1}{s}$$



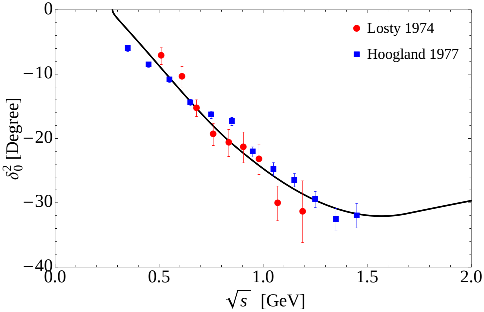


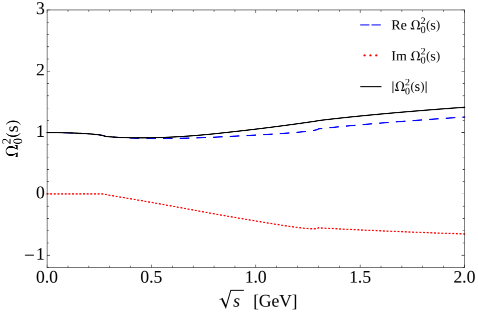
Omnès functions

Madrid-Kraków PhysRevD.83.074004

$$\Omega_0^2(s) = \exp\left\{ \frac{s}{\pi} \int_{4m_\pi^2}^{\infty} ds' \frac{\delta_0^2(s)}{s'(s'-s)} \right\},$$

$$\lim_{s\to\infty} \delta_0^2(s) \to 0\,, \quad \Omega_0^2(s) \sim \mathcal{O}(1)$$





- Assuming $\mathcal{M}(s,t,u)$ satisfies the Froissart-Martin bound: polynomial part of the amplitude grows, at most, linearly in s,t,u

$$\mathcal{M}_0^0(s) \sim \mathcal{M}_0^2(s) \sim s\,, \quad \mathcal{M}_1^1(s) \sim constant\,,$$

- the subtraction polynomial is of the form

$$P_0^0(s) = \alpha_0 + \beta_0 s + \gamma_0 s^2\,, \quad P_1^1(s) = \alpha_1 + \beta_1 s\,, \quad P_0^2(s) = \alpha_2 + \beta_2 s\,.$$

- the number of subtraction constants can be reduced because of the isospin decomposition is not unique.

$$\mathcal{M}_0^0(s) = \Omega_0^0(s)\left(\alpha_0 + \beta_0 s + \gamma_0 s^2 + \frac{s^2}{\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\sin\delta_0^0(s')\hat{\mathcal{M}}_0^0(s')}{|\Omega_0^0(s')|(s')^2\,(s'-s)}\right),$$

$$\mathcal{M}_1^1(s) = \Omega_1^1(s)\left(\beta_1 s + \frac{s}{\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\sin\delta_1^1(s')\hat{\mathcal{M}}_1^1(s')}{|\Omega_1^1(s')|s'\,(s'-s)}\right),$$

$$\mathcal{M}_0^2(s) = \Omega_0^2(s)\frac{s^2}{\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\sin\delta_0^2(s')\hat{\mathcal{M}}_0^2(s')}{|\Omega_0^2(s')|(s')^2\,(s'-s)}\,.$$

Solution to the dispersive integrals

- The solution of the integral equations is linear in the subtractions constants and can be represented by a linear combination

$$\mathcal{M}(s,t,u) = \alpha_0\mathcal{M}_{\alpha_0}(s,t,u) + \beta_0\mathcal{M}_{\beta_0}(s,t,u) + \gamma_0\mathcal{M}_{\gamma_0}(s,t,u) + \beta_1\mathcal{M}_{\beta_1}(s,t,u)\,,$$

$$\begin{aligned}
\mathcal{M}_{\alpha_0}(s,t,u) &= \mathcal{M}(s,t,u)|_{\alpha_0=1,\beta_0=0,\gamma_0=0,\beta_1=0}\,,\\
\mathcal{M}_{\beta_0}(s,t,u) &= \mathcal{M}(s,t,u)|_{\alpha_0=0,\beta_0=1,\gamma_0=0,\beta_1=0}\,,\\
\mathcal{M}_{\gamma_0}(s,t,u) &= \mathcal{M}(s,t,u)|_{\alpha_0=0,\beta_0=0,\gamma_0=1,\beta_1=0}\,,\\
\mathcal{M}_{\beta_1}(s,t,u) &= \mathcal{M}(s,t,u)|_{\alpha_0=0,\beta_0=0,\gamma_0=0,\beta_1=1}\,,
\end{aligned}$$

- each of the basis functions fulfill the reconstruction theorem

$$\begin{aligned}
\mathcal{M}_{\alpha_0}(s,t,u) &= \mathcal{M}_0^0(s)|_{\alpha_0=1,\beta_0=0,\gamma_0=0,\beta_1=0} + (s-u)\mathcal{M}_1^1(t)|_{\alpha_0=1,\beta_0=0,\gamma_0=0,\beta_1=0}\\
&+ (s-t)\mathcal{M}_1^1(u)|_{\alpha_0=1,\beta_0=0,\gamma_0=0,\beta_1=0} + \mathcal{M}_0^2(t)|_{\alpha_0=1,\beta_0=0,\gamma_0=0,\beta_1=0}\\
&+ \mathcal{M}_0^2(u)|_{\alpha_0=1,\beta_0=0,\gamma_0=0,\beta_1=0} - \frac{2}{3}\mathcal{M}_0^2(s)|_{\alpha_0=1,\beta_0=0,\gamma_0=0,\beta_1=0}\,,
\end{aligned}$$

- Perform an iteration procedure for each of the basis function separately and fix the subtraction constants after the iteration converges.

Subtraction constants

- Matching strategy adopted: Dalit-plot parameters associated to $K \to 3\pi$ agree with our dispersive representation
- These are defined by the $K \to 3\pi$ decay amplitude squared expansion

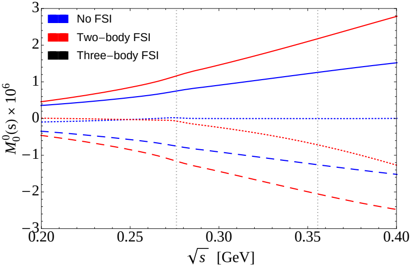
$$\left|\frac{A(s_1,s_2,s_3)}{A(s_0,s_0,s_0)}\right|^2 = 1 + gY + hY^2 + kX^2\,,$$

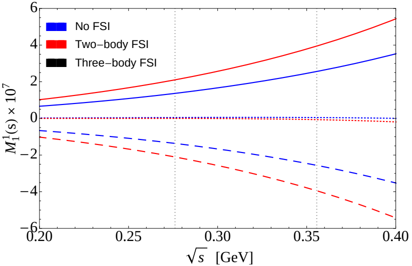
- Dalitz-plot parameters obtained in ChPT at NLO

Parameter	set 1 (Bijnens EPJC 39, 347 (2005))	set 2 (Bijnens EPJC 40, 383 (2005))
g	-0.201	-0.215
h	0.008	0.012
k	-0.0037	-0.0034
$\lvert A(s_0,s_0,s_0)\rvert^2$	$4.14 \cdot 10^{-12}$	—
$\Gamma(K \to 3\pi)$ [GeV]	—	$2.971 \cdot 10^{-18}$

- Matching approach:
  - No rescattering limit $\delta_\ell^I \to 0$ implying $\Omega_\ell^I \to 1$ and $\hat{\mathcal{M}}_\ell^I \to 0$
  - Two-body rescattering effects ($\hat{\mathcal{M}}_\ell^I = 0$)
  - Three-body effects ($\hat{\mathcal{M}}_\ell^I \neq 0$)
- (preliminary) parameters resulting from the matching:

Source	Type of matching	$\alpha_0 \times 10^7$	$\beta_0 \times 10^6$	$\gamma_0 \times 10^6$	$\beta_1 \times 10^6$
set 1	No rescattering	$1.2 + i0.7$	$-14.0 - i1.7$	$21.4 + i2.7$	-2.4
	Two-body ($\hat{\mathcal{M}}_\ell^I = 0$)	$0.3 - i1.7$	$-9.3 + i2.0$	$26.1 + i7.3$	$-1.5 + i0.1$
	Three-body ($\hat{\mathcal{M}}_\ell^I \neq 0$)				in progress
set 2	No rescattering	$0.7 + i0.4$	$-12.6 - i1.0$	$15.4 + i1.5$	-5.6
	Two-body ($\hat{\mathcal{M}}_\ell^I = 0$)	$0.01 - i1.66$	$-8.7 + i1.9$	$23.1 + i7.8$	$-3.5 + i0.1$
	Three-body ($\hat{\mathcal{M}}_\ell^I \neq 0$)				in progress





Pion vector Form Factor

- Omnès

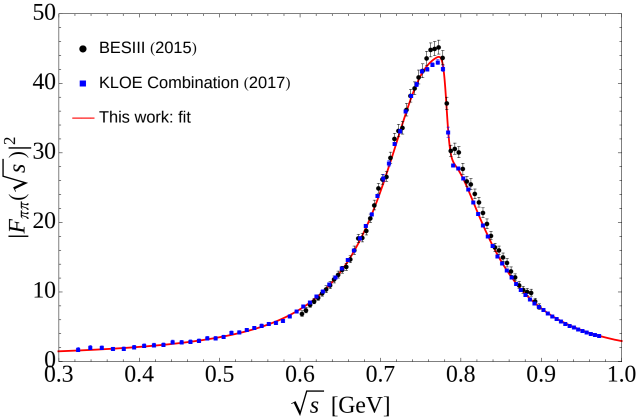
$$F_\pi^V(s) = \left(1 + \alpha_V s + \kappa \frac{s}{m_\omega^2 - s - i m_\omega \Gamma_\omega}\right) \exp\left\{\frac{s}{\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\delta_1^1(s)}{s'(s'-s)}\right\}$$

Fit to BESIII

$\alpha_V = 0.096(4)$

$\kappa = 0.018(1)$

$\chi^2_{dof} = 0.87$



Fits to $K^+ \to \pi^+\ell^+\ell^-$ experimental data (preliminary)

- Fit A: a_+ and b_+ allowed to float
- Fit B: $b_+ = 0$ to check stability of a_+
- Fit C: LFU $a_+^{ee} = a_+^{\mu\mu}$ and $b_+^{ee} = b_+^{\mu\mu}$
- Fit D: lattice predictions on a_+ and b_+ as an external restriction

$\beta_1 = -2.4$

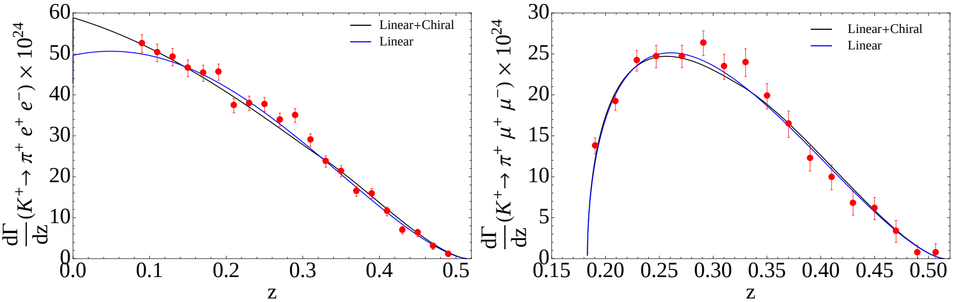
Mode	Parameter	Fit A		Fit B		Fit C		Fit D
		2-body	3-body	2-body	3-body	2-body	3-body	2-body
e^+e^-	a_+^{ee}	$-0.593(12)$		$-0.484(5)$		$-0.592(11)$		$-0.593(12)$
	b_+^{ee}	$-0.734(55)$		$= 0$	$= 0$	$-0.748(48)$		$-0.730(55)$
	$\chi^2_{\rm dof}$	1.53		6.45		1.17		2.01
$\mu^+\mu^-$	$a_+^{\mu\mu}$	$-0.603(40)$		$-0.458(9)$		$= a_+^{ee}$	$= a_+^{ee}$	$-0.608(39)$
	$b_+^{\mu\mu}$	$-0.728(146)$		$= 0$	$= 0$	$= b_+^{ee}$	$= b_+^{ee}$	$-0.707(143)$
	$\chi^2_{\rm dof}$	0.82		1.67				1.50

$\beta_1 = -1.5 + i0.1$

Mode	Parameter	Fit A		Fit B		Fit C		Fit D
		2-body	3-body	2-body	3-body	2-body	3-body	2-body
e^+e^-	a_+^{ee}	$-0.573(12)$		$-0.591(5)$		$-0.571(11)$		$-0.574(12)$
	b_+^{ee}	$-0.912(55)$		$= 0$	$= 0$	$-0.930(48)$		$-0.907(55)$
	$\chi^2_{\rm dof}$	1.34		1.40		1.07		1.87
$\mu^+\mu^-$	$a_+^{\mu\mu}$	$-0.563(40)$		$-0.603(9)$		$= a_+^{ee}$	$= a_+^{ee}$	$-0.571(39)$
	$b_+^{\mu\mu}$	$-0.974(146)$		$= 0$	$= 0$	$= b_+^{ee}$	$= b_+^{ee}$	$-0.945(144)$
	$\chi^2_{\rm dof}$	0.83		0.85				1.55

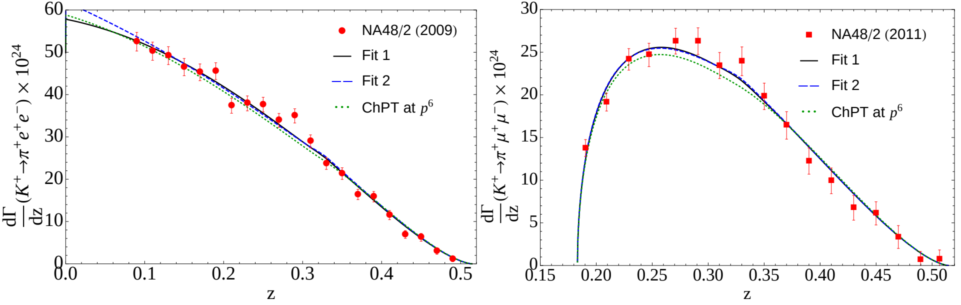
Fits to $K^+ \to \pi^+\ell^+\ell^-$ experimental data

- Call for low-energy bin points in $K^+ \to \pi^+e^+e^-$



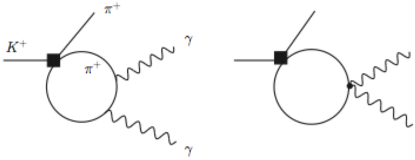
Fits to $K^+ \to \pi^+\ell^+\ell^-$ experimental data

- Call for low-energy bin points in $K^+ \to \pi^+e^+e^-$
- Region of the $\pi\pi$ threshold $(z \sim 0.32)$ slightly improved



Outlook

- $K^+ \to \pi^+ \ell^+ \ell^-$ well suited to gain information on the B-anomalies
- Dispersion relations allow to treat the most important rescattering effects properly
- The $\pi\pi$ phase shift is (almost) all what we need
- Preliminary results shows that LFU holds well...
- ...work in progress
- Other applications



Motivation

Non- Rare versus Rare Decays

BR > 10^{-5}	
Decay	**BR**
$K^+ \to \pi^+ \nu_\mu$	0.6355 (11)
$K^+ \to \pi^0 \mu^+ \nu_\mu$	0.03353 (34)
$K_L \to \pi^\pm e^\mp \nu_e$	0.4055 (12)
$K^+ \to \pi^+ \pi^0$	0.2066 (8)
$K^+ \to \pi^+ \pi^+ \pi^-$	0.0559 (4)
$K_S \to \pi^0 \pi^0$	0.3069 (5)
$K_S \to \pi^+ \pi^-$	0.6920 (5)
$K_L \to \pi^0 \pi^0 \pi^0$	0.1952 (12)
$K_L \to \pi^+ \pi^- \pi^0$	0.1254 (5)
$K^+ \to \pi^+ \pi^0 \gamma$	2.75 (15) x 10^{-4}
$K_L \to \gamma\gamma$	5.47 (4) x 10^{-4}
$K_L \to \pi^+ \pi^- \gamma$	4.15 (15) x 10^{-5}
$K_S \to \pi^+ \pi^- \gamma$	1.79 (5) x 10^{-3}

BR < 10^{-5}	
Decay	**BR x 10^5**
$K^+ \to \pi^+ \gamma\gamma$	0.1003 (56)
$K^+ \to \pi^+ e^+ e^- \gamma$	1.19 (13) x 10^{-3}
$K^+ \to \pi^+ e^+ e^-$	0.0300 (9)
$K_S \to \gamma\gamma$	0.263 (17)
$K_S \to \pi^0 \mu^+ \mu^-$	2.9 (1.5) x 10^{-4}
$K_S \to \mu^+ \mu^-$	< 9 x 10^{-4} (90% C.L.)
$K_L \to \pi^0 \gamma\gamma$	0.1274 (34)
$K_L \to e^+ e^-$	9 ($^{+6}_{-4}$) x 10^{-7}
$K_L \to \pi^+ \pi^- e^+ e^-$	0.0311 (19)
$K_L \to \mu^+ \mu^- e^+ e^-$	2.69 (27) x 10^{-4}
$K_L \to \pi^0 \mu^+ \mu^-$	< 3.8 x 10^{-5} (90% C.L.)
$K^+ \to \pi^+ \nu \bar{\nu}$	1.7 (1.1) x 10^{-5}
$K_L \to \pi^0 \nu \bar{\nu}$	< 6.7 x 10^{-3} (90% C.L.)

- Isospin decomposition of the $\pi^{+,0}\pi^{+,0}\pi^{-,+}$ state: $|I_{(12)};I_{(123)},I_3\rangle$

$$
\begin{aligned}
|0;1,1\rangle &= \frac{1}{\sqrt{3}}\left(|\pi^+\pi^-\pi^+\rangle + |\pi^-\pi^+\pi^+\rangle - |\pi^0\pi^0\pi^+\rangle\right) , \\
|1;1,1\rangle &= \frac{1}{2}\left(|\pi^+\pi^0\pi^0\rangle - |\pi^0\pi^+\pi^0\rangle - |\pi^+\pi^-\pi^+\rangle + |\pi^-\pi^+\pi^+\rangle\right) , \\
|2;1,1\rangle &= \frac{1}{2\sqrt{15}}\Big(|\pi^+\pi^-\pi^+\rangle + |\pi^-\pi^+\pi^+\rangle + 2|\pi^0\pi^0\pi^+\rangle - 3|\pi^+\pi^0\pi^0\rangle - 3|\pi^0\pi^+\pi^0\rangle + 6|\pi^+\pi^+\pi^-\rangle\Big), \\
|1;2,1\rangle &= \frac{1}{2}\left(|\pi^+\pi^0\pi^0\rangle - |\pi^0\pi^+\pi^0\rangle + |\pi^+\pi^-\pi^+\rangle - |\pi^-\pi^+\pi^+\rangle\right) , \\
|2;2,1\rangle &= \frac{1}{2\sqrt{3}}\Big(2|\pi^+\pi^+\pi^-\rangle - 2|\pi^0\pi^0\pi^+\rangle + |\pi^+\pi^0\pi^0\rangle + |\pi^0\pi^+\pi^0\rangle - |\pi^+\pi^-\pi^+\rangle - |\pi^-\pi^+\pi^+\rangle\Big), \\
|2;3,1\rangle &= \frac{1}{\sqrt{15}}\Big(2|\pi^+\pi^0\pi^0\rangle + 2|\pi^0\pi^+\pi^0\rangle + 2|\pi^0\pi^0\pi^+\rangle + |\pi^+\pi^-\pi^+\rangle + |\pi^-\pi^+\pi^+\rangle + |\pi^+\pi^+\pi^-\rangle\Big), \\
|S;1,1\rangle &= \frac{1}{\sqrt{15}}\Big(2|\pi^+\pi^-\pi^-\rangle + 2|\pi^-\pi^+\pi^+\rangle + 2|\pi^+\pi^+\pi^-\rangle - |\pi^0\pi^0\pi^+\rangle - |\pi^+\pi^0\pi^0\rangle - |\pi^0\pi^+\pi^0\rangle\Big), \\
|3\pi\rangle &= \alpha|2;3,1\rangle + \beta|S;1,1\rangle \\
R &= \frac{\Gamma(K^+\to\pi^+\pi^0\pi^0)}{\Gamma(K^+\to\pi^+\pi^+\pi^-)} \sim \left|\frac{2\alpha-\beta}{\alpha+2\beta}\right|^2 \sim 0.29 \Rightarrow I_{(123)} = 1\,\text{dominates}\,,\, \Delta I = 1/2\ \text{rule}
\end{aligned}
$$

- We define the $K \to 3\pi$ decay amplitude according to

$$\langle \pi^i(p_1)\pi^j(p_2)\pi^k(p_3)|iT|K(p_k)\rangle = i\delta^4(p_k - p_1 - p_2 - p_3)\mathcal{M}^{ijk,l}_{K\to 3\pi}(s,t,u)\,,$$

i, j, k denote the isospin of the pions, l isospin state of the interaction.

- The invariant amplitude has the isospin decomposition

$$\mathcal{M}^{ijk,l}_{K\to 3\pi}(s,t,u) = \mathcal{M}_1(s,t,u)\delta^{jk}\delta^{il} + \mathcal{M}_2(s,t,u)\delta^{jl}\delta^{ik} + \mathcal{M}_3(s,t,u)\delta^{ij}\delta^{kl}\,,$$

where the amplitudes are functions of the Mandelstam variables

$$s = (p_k - p_1)^2\,, \quad t = (p_k - p_2)^2\,, \quad u = (p_k - p_3)^2\,,$$

which fulfill the relation

$$s + t + u = m_K^2 + 3m_\pi^2 = \Sigma\,.$$

- Bose statistics: the amplitude remains invariant under the exchange of pions

$$p_2 \leftrightarrow p_3 \ (t \leftrightarrow u \text{ and } j \leftrightarrow k) \Rightarrow \mathcal{M}_1(s,t,u) = \mathcal{M}_1(s,u,t)\,,$$

$$p_1 \leftrightarrow p_2 \ (s \leftrightarrow t \text{ and } i \leftrightarrow j) \Rightarrow \mathcal{M}_2(s,t,u) = \mathcal{M}_1(t,s,u) = \mathcal{M}_1(t,u,s)\,,$$

$$p_1 \leftrightarrow p_3 \ (s \leftrightarrow u \text{ and } i \leftrightarrow k) \Rightarrow \mathcal{M}_3(s,t,u) = \mathcal{M}_1(u,t,s) = \mathcal{M}_1(u,s,t)\,,$$

- The amplitude is expressed in terms of $\mathcal{M}(s,t,u) \equiv \mathcal{M}_1(s,t,u)$

$$\mathcal{M}^{ijk,l}_{K\to 3\pi}(s,t,u) = \mathcal{M}(s,t,u)\delta^{jk}\delta^{il} + \mathcal{M}(t,u,s)\delta^{jl}\delta^{ik} + \mathcal{M}(u,s,t)\delta^{ij}\delta^{kl}$$

- In terms of the physical pions

$$\begin{aligned}
\mathcal{M}_{K^+\to\pi^+\pi^+\pi^-}(s,t,u) &= \mathcal{M}(t,u,s) + \mathcal{M}(u,s,t)\,,\\
\mathcal{M}_{K^+\to\pi^0\pi^0\pi^+}(s,t,u) &= \mathcal{M}(s,t,u)\,,\\
\mathcal{M}_{K^0\to\pi^+\pi^-\pi^0}(s,t,u) &= \mathcal{M}(s,t,u)\,,\\
\mathcal{M}_{K^0\to\pi^0\pi^0\pi^0}(s,t,u) &= \mathcal{M}(s,t,u) + \mathcal{M}(t,u,s) + \mathcal{M}(u,s,t)\,.
\end{aligned}$$

$K \to 3\pi$ kinematics

- In the CMS of particles 2 and 3 we find

$$t(s, \cos\theta_s) = \frac{1}{2}\left(\Sigma - s + \kappa_{\pi\pi}(s)\cos\theta_s\right) , \quad u(s, \cos\theta_s) = \frac{1}{2}\left(\Sigma - s - \kappa_{\pi\pi}(s)\cos\theta_s\right) ,$$

$$\cos\theta_s = \frac{t-u}{\kappa_{\pi\pi}(s)} , \quad \kappa_{\pi\pi}(s) = \lambda^{1/2}(s, m_K^2, m_\pi^2)\sigma_\pi(s) ,$$

- In the CMS of particles 1 and 3 one has

$$s(t, \cos\theta_t) = \frac{1}{2}\left(\Sigma - t + \kappa_{\pi\pi}(t)\cos\theta_t\right) , \quad u(t, \cos\theta_t) = \frac{1}{2}\left(\Sigma - t - \kappa_{\pi\pi}(t)\cos\theta_t\right) ,$$

$$\cos\theta_t = \frac{s-u}{\kappa_{\pi\pi}(t)} .$$

- In the CMS of particles 1 and 2 one finds

$$s(u, \cos\theta_u) = \frac{1}{2}\left(\Sigma - u - \kappa_{\pi\pi}(u)\cos\theta_u\right) , \quad t(u, \cos\theta_u) = \frac{1}{2}\left(\Sigma - u + \kappa_{\pi\pi}(u)\cos\theta_u\right) ,$$

$$\cos\theta_u = \frac{t-s}{\kappa_{\pi\pi}(u)} .$$

The reconstruction theorem

- Write n-times subtracted dispersion relation at fixed t

$$\mathcal{M}(s,t,u) = P^t_{n-1}(s,t,u) + \frac{s^n}{2\pi i}\int_{s_{\rm th}}^{\infty} ds' \frac{{\rm disc}\mathcal{M}(s',t,u(s'))}{(s')^n(s'-s)} + \frac{u^n}{2\pi i}\int_{u_{\rm th}}^{\infty} du' \frac{{\rm disc}\mathcal{M}(s(u'),t,u')}{(u')^n(u'-u)}\,,$$

$$s(u') = m_K^2 + 3m_\pi^2 - t - u' = s + u - u'\,,$$

$$u(s') = m_K^2 + 3m_\pi^2 - t - s' = s + u - s'\,.$$

- Express the s-and u-channel in terms of the p-w of definite isospin

$${\rm disc}\mathcal{M}(s',t,u(s')) \equiv {\rm disc}\mathcal{M}_1(s',t,u(s')) = \frac{1}{3}\left[m_0^0(s',t,u(s')) - m_0^2(s',t,u(s'))\right] = \frac{1}{3}\left[m_0^0(s') - m_0^2(s')\right]\,,$$

$${\rm disc}\mathcal{M}(s(u'),t,u') \equiv {\rm disc}\mathcal{M}_3(s(u'),t,u') = \frac{1}{2}\left[m_0^2(s(u'),t,u') - m_1^1(s(u'),t,u')\right] = \frac{1}{2}\left[m_0^2(u') - 3\cos\theta_u\, m_1^1(u')\right]\,.$$

The reconstruction theorem

- Using the expression for $\cos\theta_u = \frac{t-s}{\kappa_{\pi\pi}(u)}$

$$
\begin{aligned}
\mathcal{M}(s,t,u) &= P^t_{n-1}(s,t,u) \\
&+ \frac{1}{3}\frac{s^n}{2\pi i}\int_{s_{\rm th}}^{\infty} ds' \frac{{\rm disc}\, m^0_0(s')}{(s')^n(s'-s)} - \frac{1}{3}\frac{s^n}{2\pi i}\int_{s_{\rm th}}^{\infty} ds' \frac{{\rm disc}\, m^2_0(s')}{(s')^n(s'-s)} \\
&+ \frac{1}{2}\frac{u^n}{2\pi i}\int_{u_{\rm th}}^{\infty} du' \frac{{\rm disc}\, m^2_0(u')}{(u')^n(u'-u)} - \frac{3}{2}\frac{u^n}{2\pi i}\int_{u_{\rm th}}^{\infty} du' \frac{(t-s(u'))\,{\rm disc}\, m^1_1(u')}{k_{\pi\pi}(u')(u')^n(u'-u)}
\end{aligned}
$$

- The integral over the $\pi\pi$ P-wave can be simplified using $s(u')$

$$
\begin{aligned}
&\frac{3}{2}\frac{u^n}{2\pi i}\int_{u_{\rm th}}^{\infty} du' \frac{(t-s(u'))\,{\rm disc}\, m^1_1(u')}{k_{\pi\pi}(u')(u')^n(u'-u)} = \frac{3}{2}\frac{u^n}{2\pi i}\int_{u_{\rm th}}^{\infty} du' \frac{(t-s-u+u')\,{\rm disc}\, m^1_1(u')}{k_{\pi\pi}(u')(u')^n(u'-u)} \\
&= \frac{3}{2}\frac{u^n}{2\pi i}\int_{u_{\rm th}}^{\infty} du' \frac{{\rm disc}\, m^1_1(u')}{k_{\pi\pi}(u')(u')^n} + \frac{3}{2}(t-s)\frac{u^n}{2\pi i}\int_{u_{\rm th}}^{\infty} du' \frac{{\rm disc}\, m^1_1(u')}{k_{\pi\pi}(u')(u')^n(u'-u)}\,,
\end{aligned}
$$

The reconstruction theorem

- We are thus left with

$$\mathcal{M}(s,t,u) = P^t_{n-1}(s,t,u) + \frac{1}{3}\frac{s^n}{2\pi i}\int_{s_{\rm th}}^{\infty} ds' \frac{{\rm disc}\, m_0^0(s')}{(s')^n(s'-s)} - \frac{1}{3}\frac{s^n}{2\pi i}\int_{s_{\rm th}}^{\infty} ds' \frac{{\rm disc}\, m_0^2(s')}{(s')^n(s'-s)}$$
$$+ \frac{1}{2}\frac{u^n}{2\pi i}\int_{u_{\rm th}}^{\infty} du' \frac{{\rm disc}\, m_0^2(u')}{(u')^n(u'-u)} - \frac{3}{2}(t-s)\frac{u^n}{2\pi i}\int_{u_{\rm th}}^{\infty} du' \frac{{\rm disc}\, m_1^1(u')}{k_{\pi\pi}(u')(u')^n(u'-u)}$$

- Similarly, one can perform the same exercise at fixed u.

$$\begin{aligned}{\rm disc}\mathcal{M}(s(t'),t',u) \equiv {\rm disc}\mathcal{M}_2(s(t'),t',u) &= \frac{1}{2}\left[m_0^2(s(t'),t',u) + m_1^1(s(t'),t',u)\right] \\ &= \frac{1}{2}\left[m_0^2(t') + 3\cos\theta_t\, m_1^1(t')\right],\end{aligned}$$

$$\mathcal{M}(s,t,u) = P^u_{n-1}(s,t,u) + \frac{1}{3}\frac{s^n}{2\pi i}\int_{s_{\rm th}}^{\infty} ds' \frac{{\rm disc}\, m_0^0(s')}{(s')^n(s'-s)} - \frac{1}{3}\frac{s^n}{2\pi i}\int_{s_{\rm th}}^{\infty} ds' \frac{{\rm disc}\, m_0^2(s')}{(s')^n(s'-s)}$$
$$+ \frac{1}{2}\frac{t^n}{2\pi i}\int_{t_{\rm th}}^{\infty} dt' \frac{{\rm disc}\, m_0^2(t')}{(t')^n(t'-t)} - \frac{3}{2}(u-s)\frac{t^n}{2\pi i}\int_{t_{\rm th}}^{\infty} dt' \frac{{\rm disc}\, m_1^1(t')}{k_{\pi\pi}(t')(t')^n(t'-t)},$$

- Inserting the reconstruction theorem on both sides we obtain

$$\mathrm{disc}\,\mathcal{M}_0^0(s) = 2i\Big\{\mathcal{M}_0^0(s) + \hat{\mathcal{M}}_0^0(s)\Big\}\sin\delta_0^0(s)e^{-i\delta_0^0(s)}\,,$$

$$\mathrm{disc}\,\mathcal{M}_1^1(s) = 2i\Big\{\mathcal{M}_1^1(s) + \hat{\mathcal{M}}_1^1(s)\Big\}\sin\delta_1^1(s)e^{-i\delta_1^1(s)}\,,$$

$$\mathrm{disc}\,\mathcal{M}_0^2(s) = 2i\Big\{\mathcal{M}_0^2(s) + \hat{\mathcal{M}}_0^2(s)\Big\}\sin\delta_0^2(s)e^{-i\delta_0^2(s)}\,,$$

- The inhomogeneities $\hat{\mathcal{M}}_\ell^I$ are given by

$$\hat{\mathcal{M}}_0^0(s) = \frac{2}{3}\langle\mathcal{M}_0^0\rangle + 2(s-\Sigma)\langle\mathcal{M}_1^1\rangle + \frac{2}{3}\kappa_{\pi\pi}(s)\langle z\mathcal{M}_1^1\rangle + \frac{20}{9}\langle\mathcal{M}_0^2\rangle\,,$$

$$\hat{\mathcal{M}}_1^1(s) = \frac{1}{\kappa_{\pi\pi}(s)}\Big\{3\langle z\mathcal{M}_0^0\rangle + \frac{9}{2}(s-\Sigma)\langle z\mathcal{M}_1^1\rangle + \frac{3}{2}\kappa_{\pi\pi}(s)\langle z^2\mathcal{M}_1^1\rangle - 5\langle z\mathcal{M}_0^2\rangle\Big\}\,,$$

$$\hat{\mathcal{M}}_0^2(s) = \langle\mathcal{M}_0^0\rangle - \frac{3}{2}(s-\Sigma)\langle\mathcal{M}_1^1\rangle - \frac{1}{2}\kappa_{\pi\pi}(s)\langle z\mathcal{M}_1^1\rangle + \frac{1}{3}\langle\mathcal{M}_0^2\rangle\,,$$

- where have used the relation

$$\int d\Omega_s z^n \mathcal{M}_\ell^I(t_s') = (-1)^n \int d\Omega_s z^n \mathcal{M}_\ell^I(u_s')\,,$$

$$\langle z^n\mathcal{M}_\ell^I\rangle = \frac{1}{2}\int_{-1}^{1} dz z^n \mathcal{M}_\ell^I\left(\frac{\Sigma - s + \kappa_{\pi\pi}(s)z}{2}\right)\,.$$