DISPERSIVE APPROACH TO THREE BODY SCATTERING

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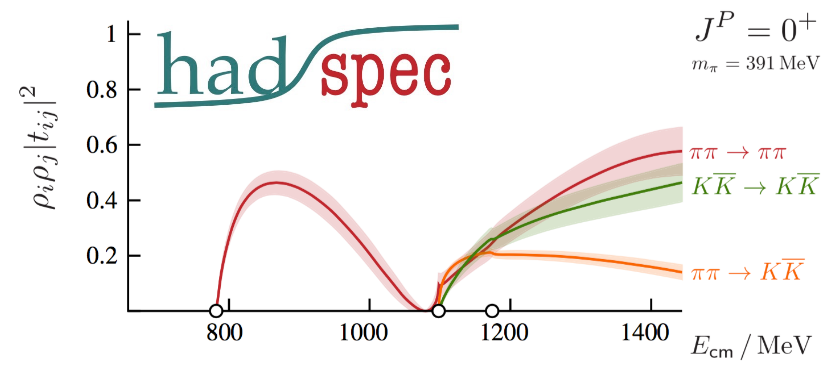
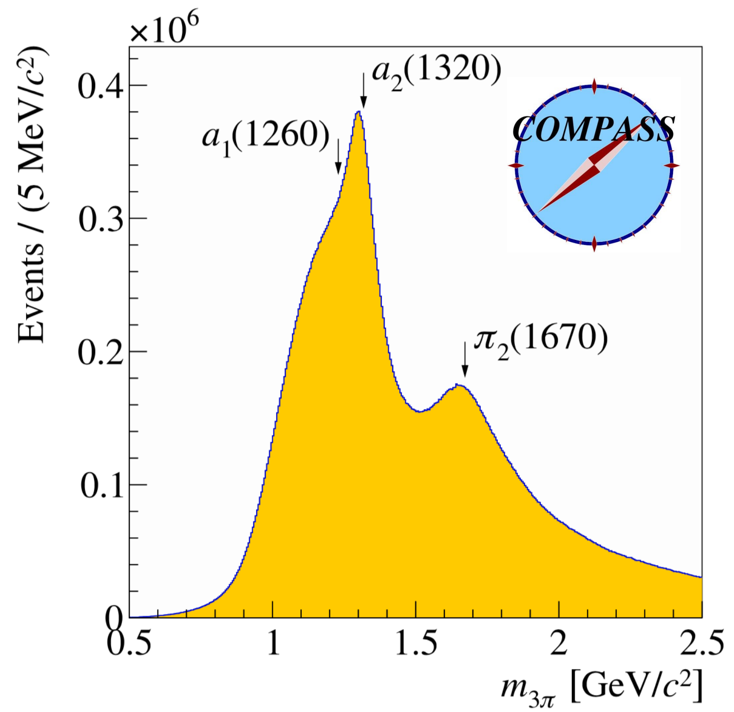
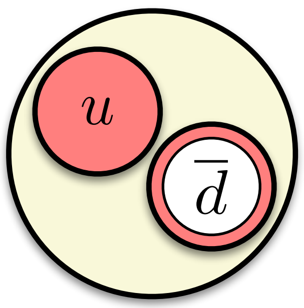
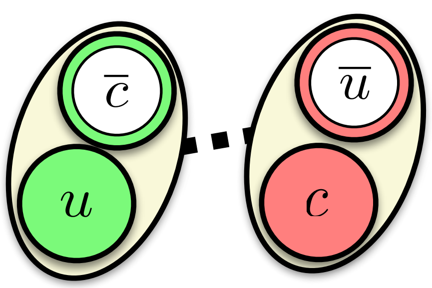
Hadrons and QCD

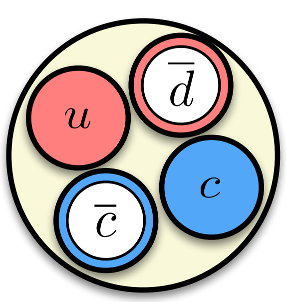
- Signatures of hadrons in data
  - Peaks and dips on **Real** energy axis
  - Patterns in mass/ spin
- Theoretical description
  - **Complex** energy plane singularities
- Interpretation
  - Constituent quarks, exotica

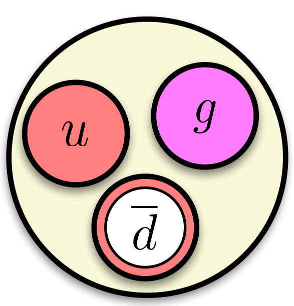


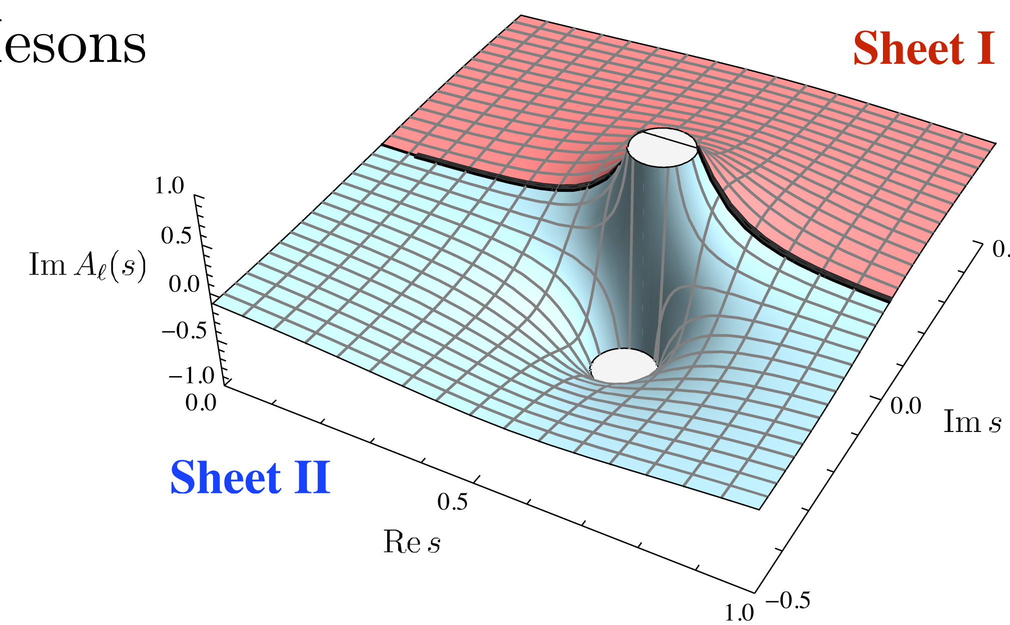



Mesons


Mesonic-Molecules

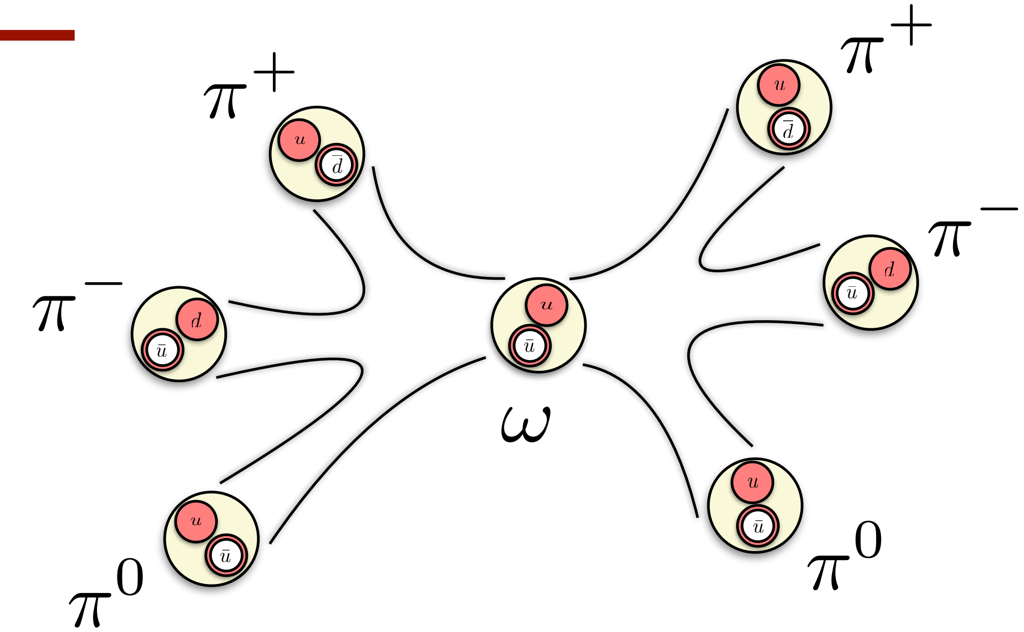

Tetraquarks


Hybrids

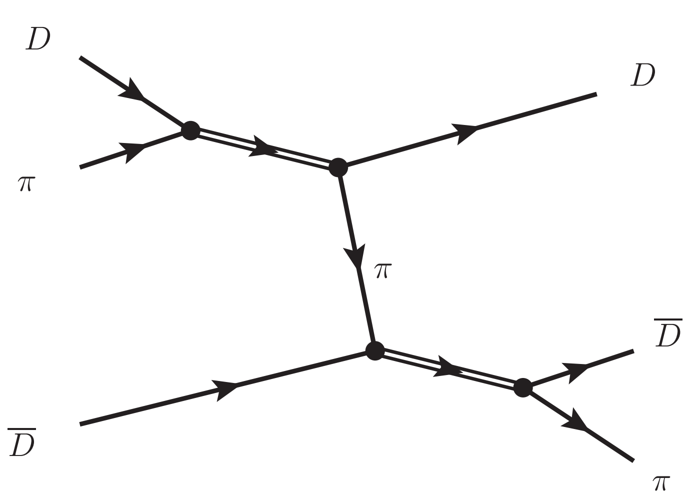


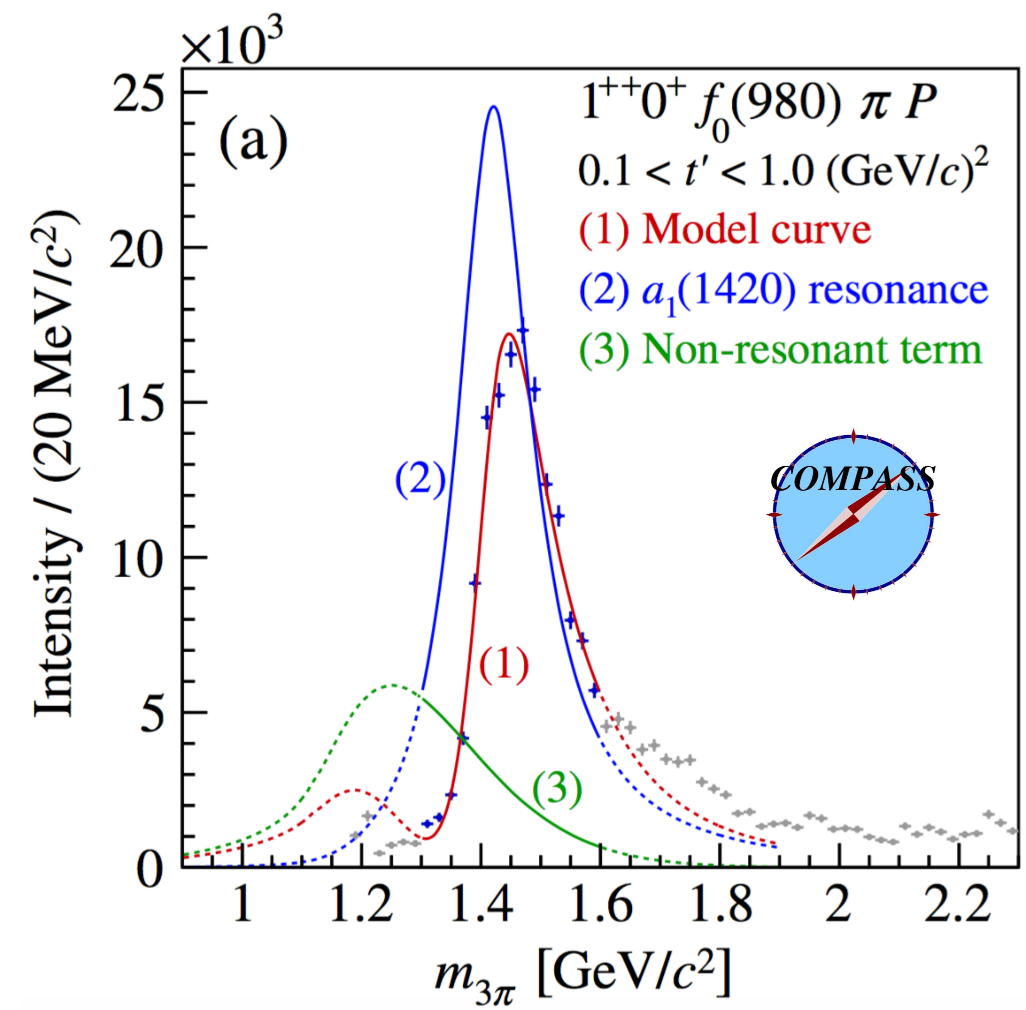
3-Body Phenomenology

- Many Resonances decay to 3-particles
  - New high-statistics experimental results
  - Advancements in Lattice Formalism
- Signatures of new resonances
  - High-Precision PWA of 3π yields $a_1(1420)$, Incompatible with quark model/ Regge expectations
- *XYZ* States
  - Lie near thresholds



$X(3872) \to D\overline{D}\pi$





Scattering Phenomenology

- Model independent constraints - ***S*-matrix theory**
  - **Unitarity**, **Analyticity**, **Crossing**, and **Poincaré Symmetry**

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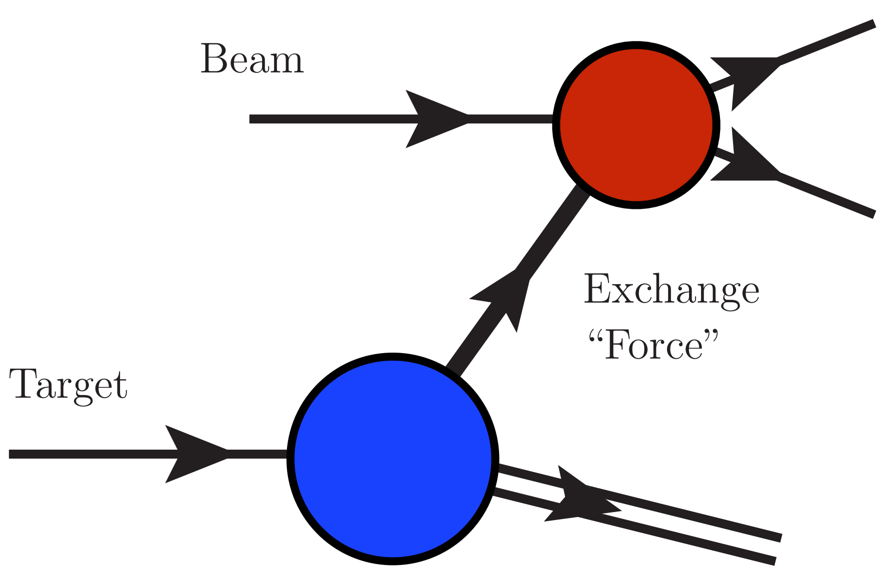
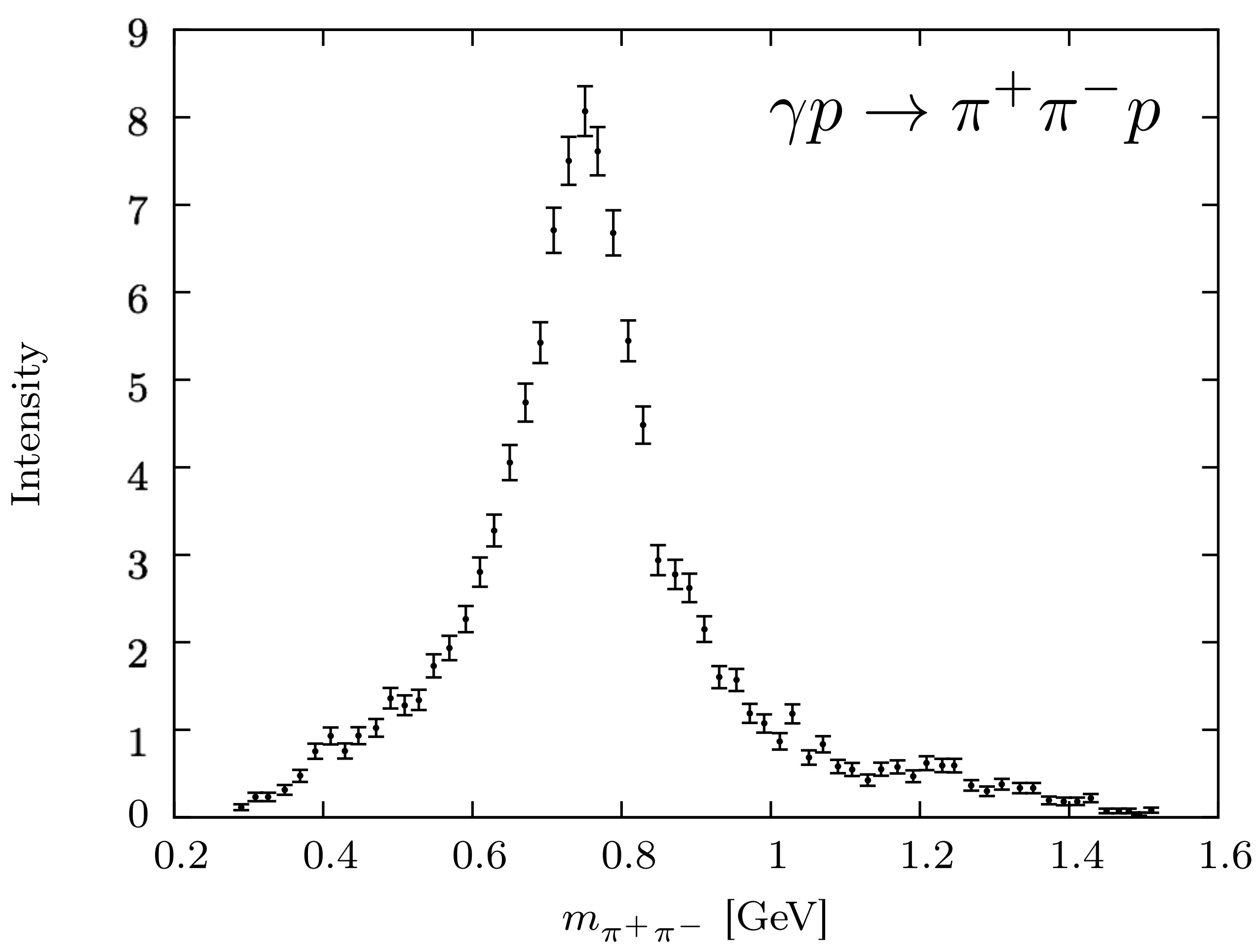
Data (Experimental/Lattice)





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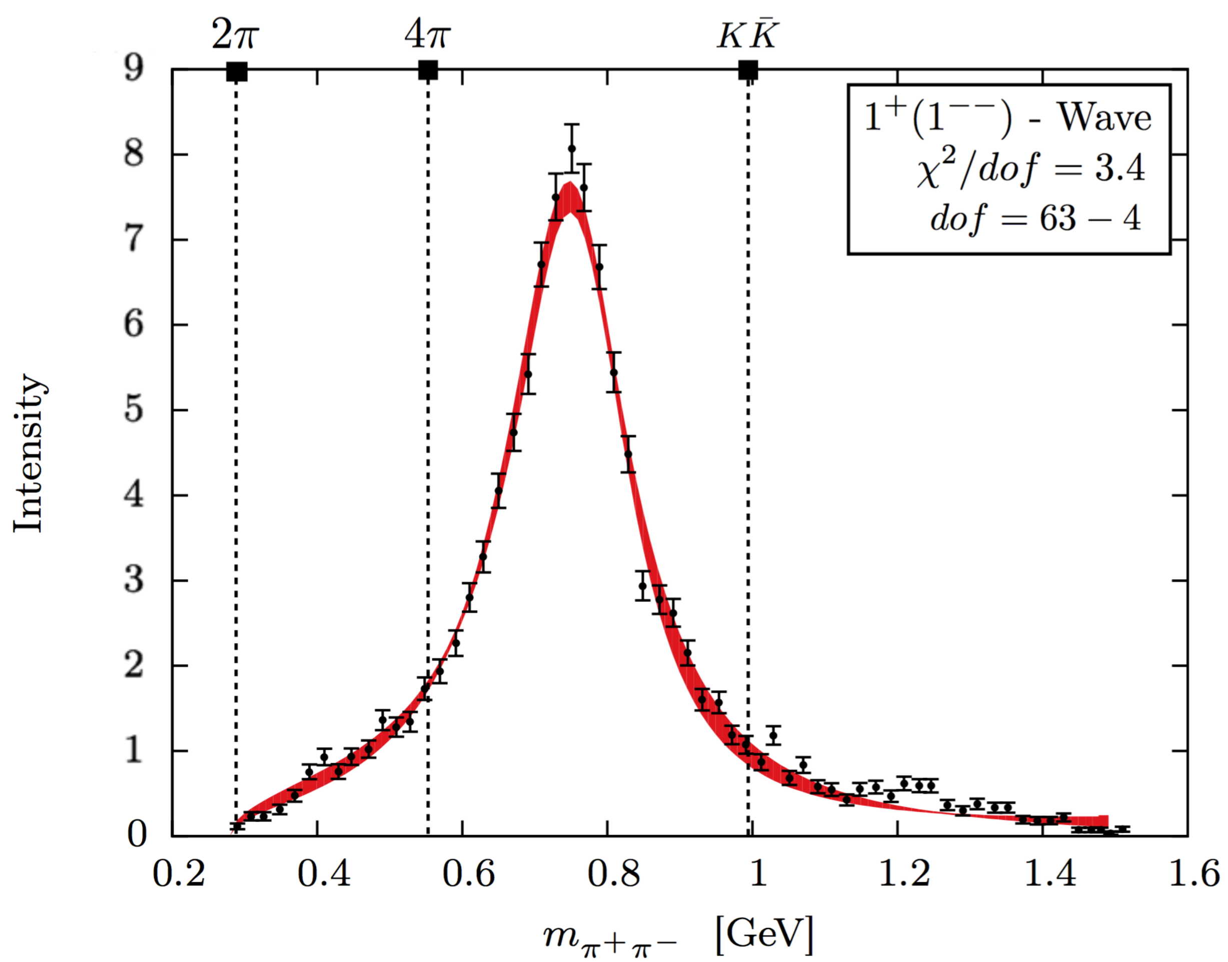
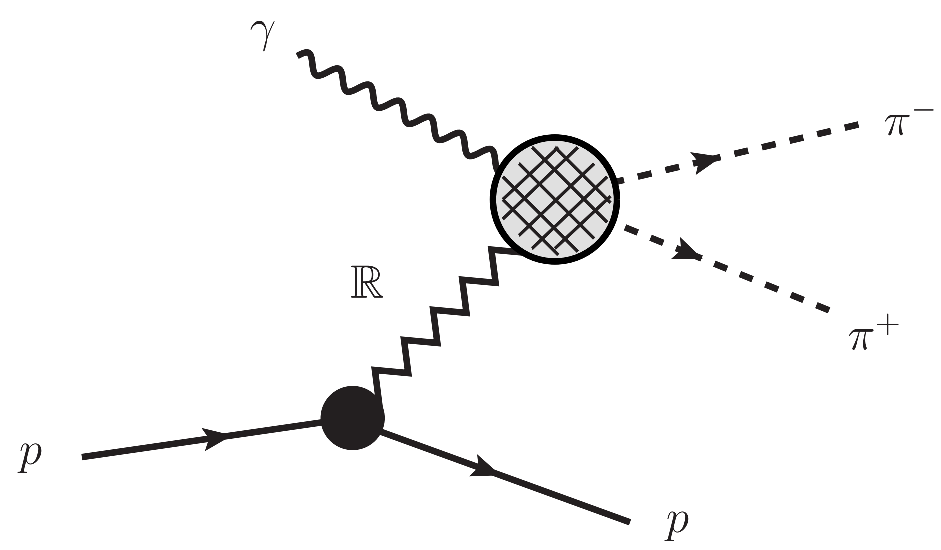
Amplitude Analysis





Amplitude Model

$$\mathcal{A}_\ell(s) = K_\ell^{-1}(s) - \frac{s}{\pi}\int_{s_{th}}^{\infty} ds' \frac{\rho(s')}{s'-s}$$

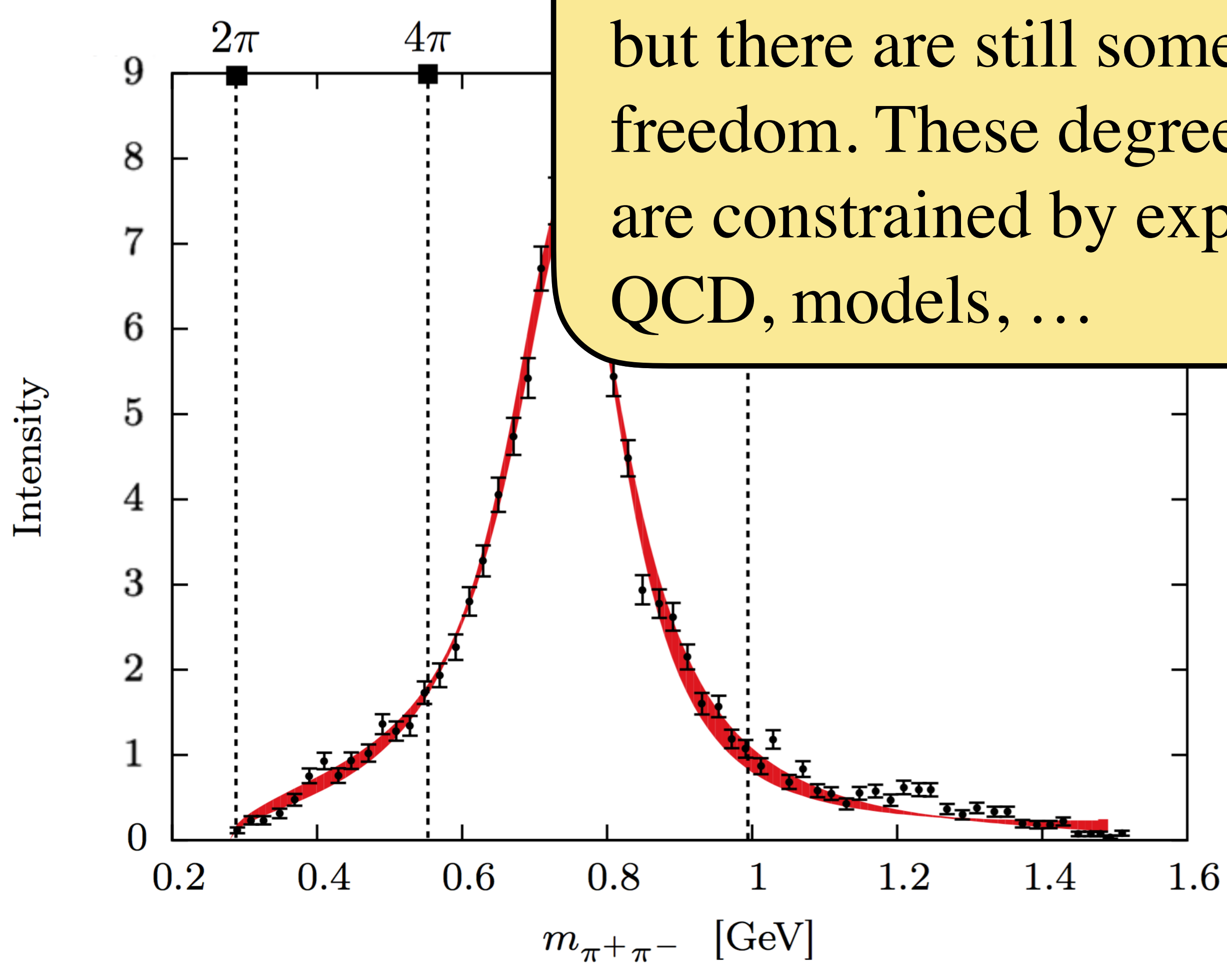
Functional form fixed from *S*-matrix constrains

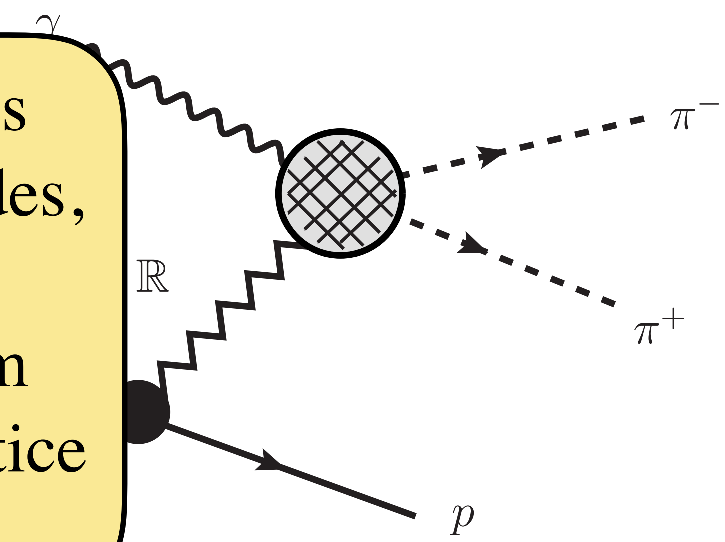
Scattering Phenomenology

- Model independent constraints - ***S*-matrix theory**
  - **Unitarity**, **Analyticity**, **Crossing**, and **Poincaré Symmetry**

Amplitude

Imposing these constraints gives us general forms of reaction amplitudes, but there are still some degrees of freedom. These degrees of freedom are constrained by experiment, lattice QCD, models, …





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Functional form fixed from *S*-matrix constrains

Scattering Phenomenology

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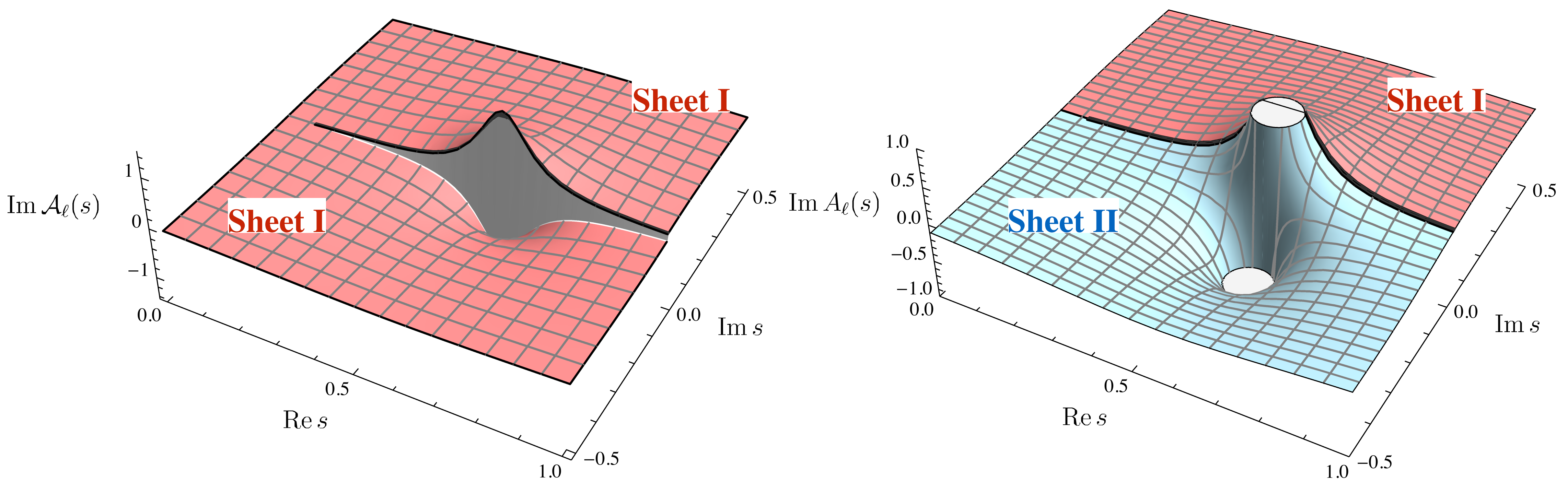
  - **Unitarity**, **Analyticity**, **Crossing**, and **Poincaré Symmetry**

Analytic Continuation

$$\mathcal{A}_\ell(s) \to \mathcal{A}_\ell^{\mathrm{II}}(s) = \frac{\mathcal{A}_\ell(s)}{1 + 2i\rho(s)\mathcal{A}_\ell(s)}$$



Scattering Phenomenology

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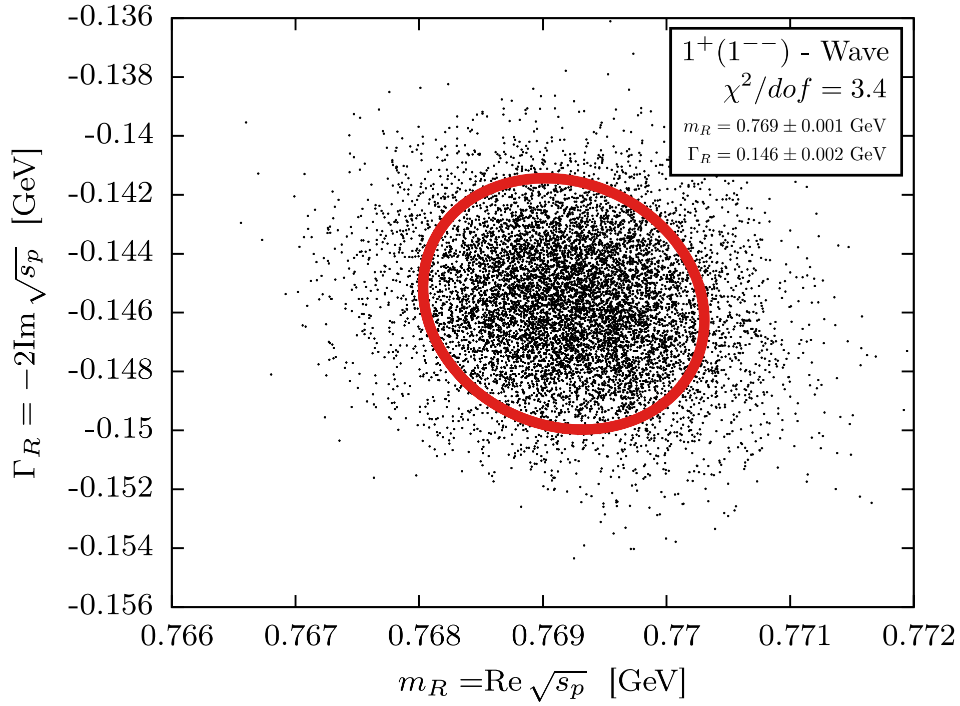
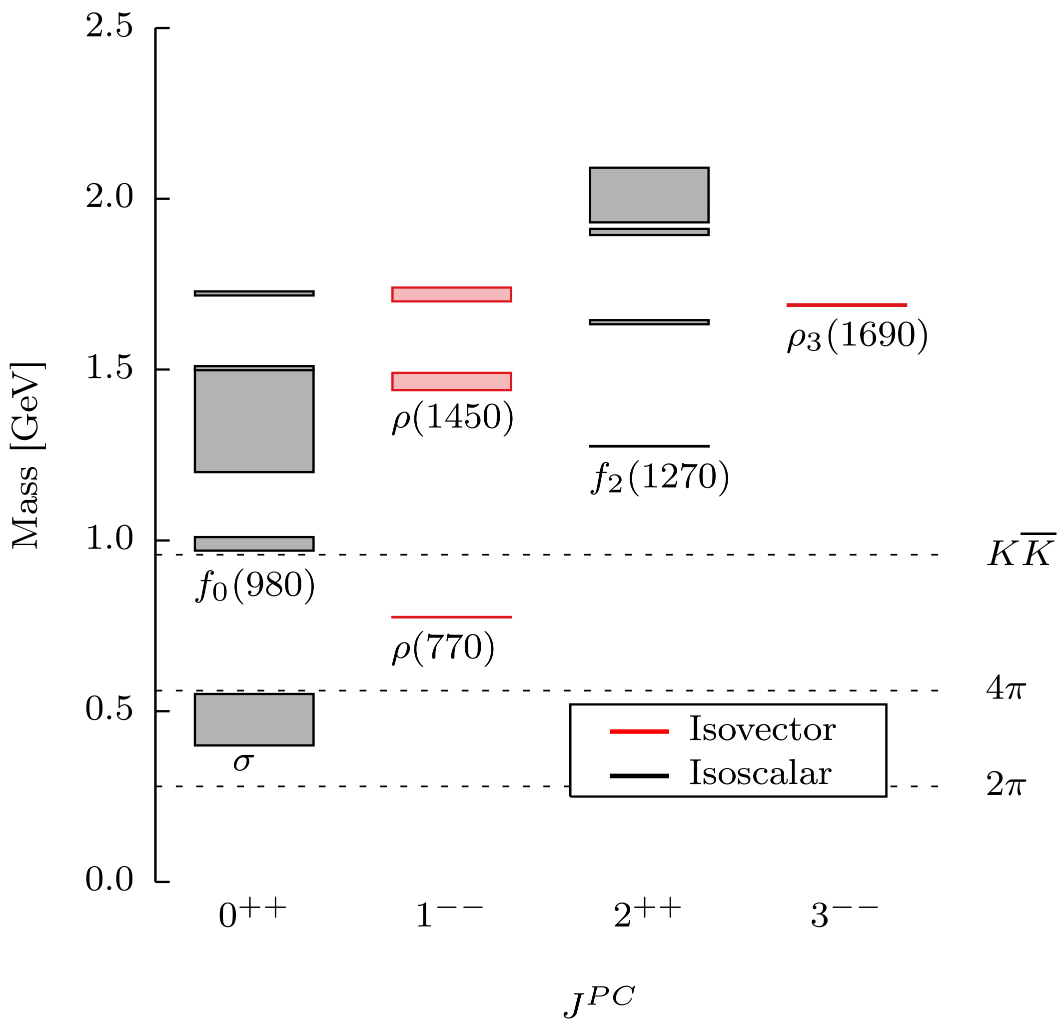
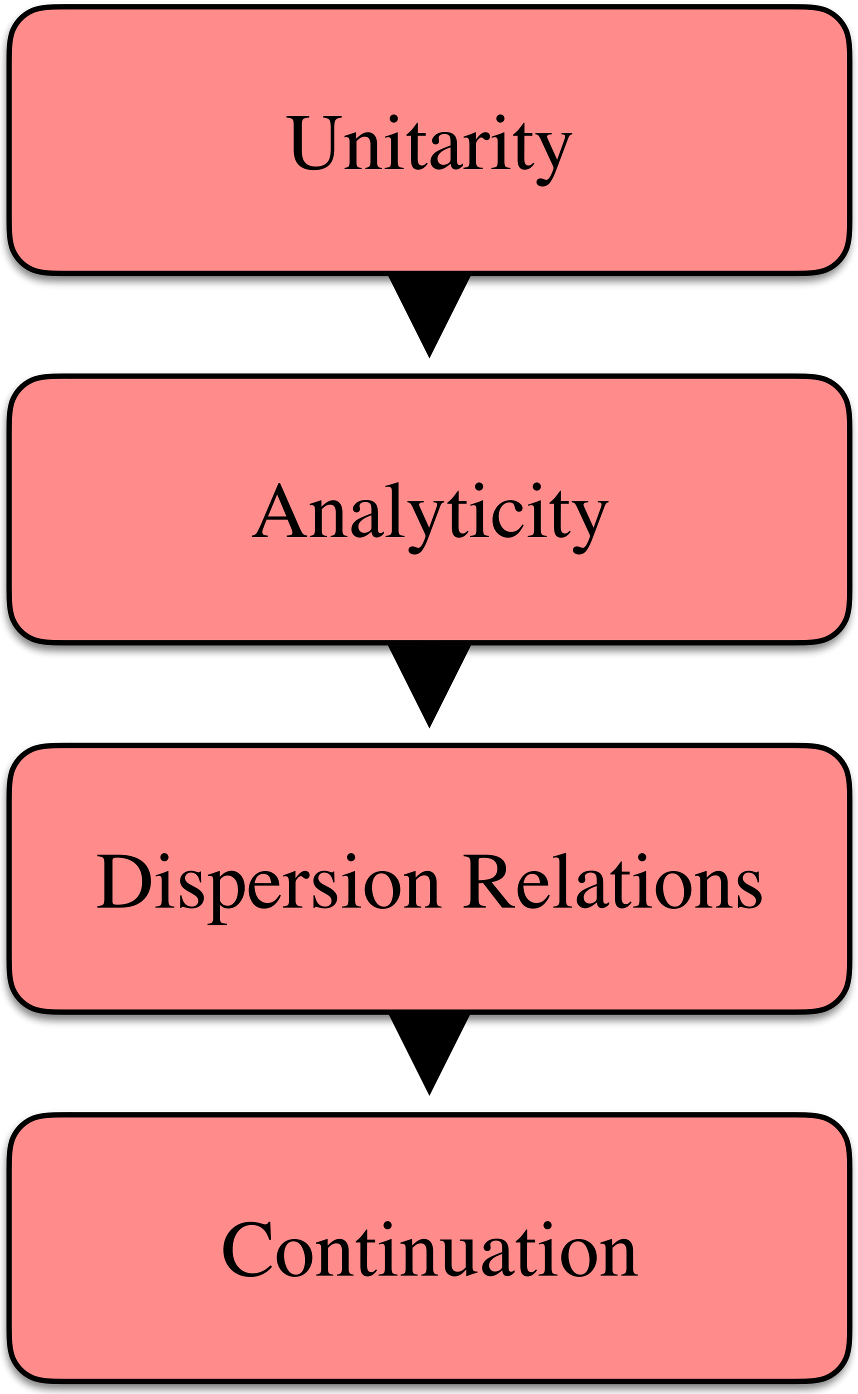
Resonance Parameters



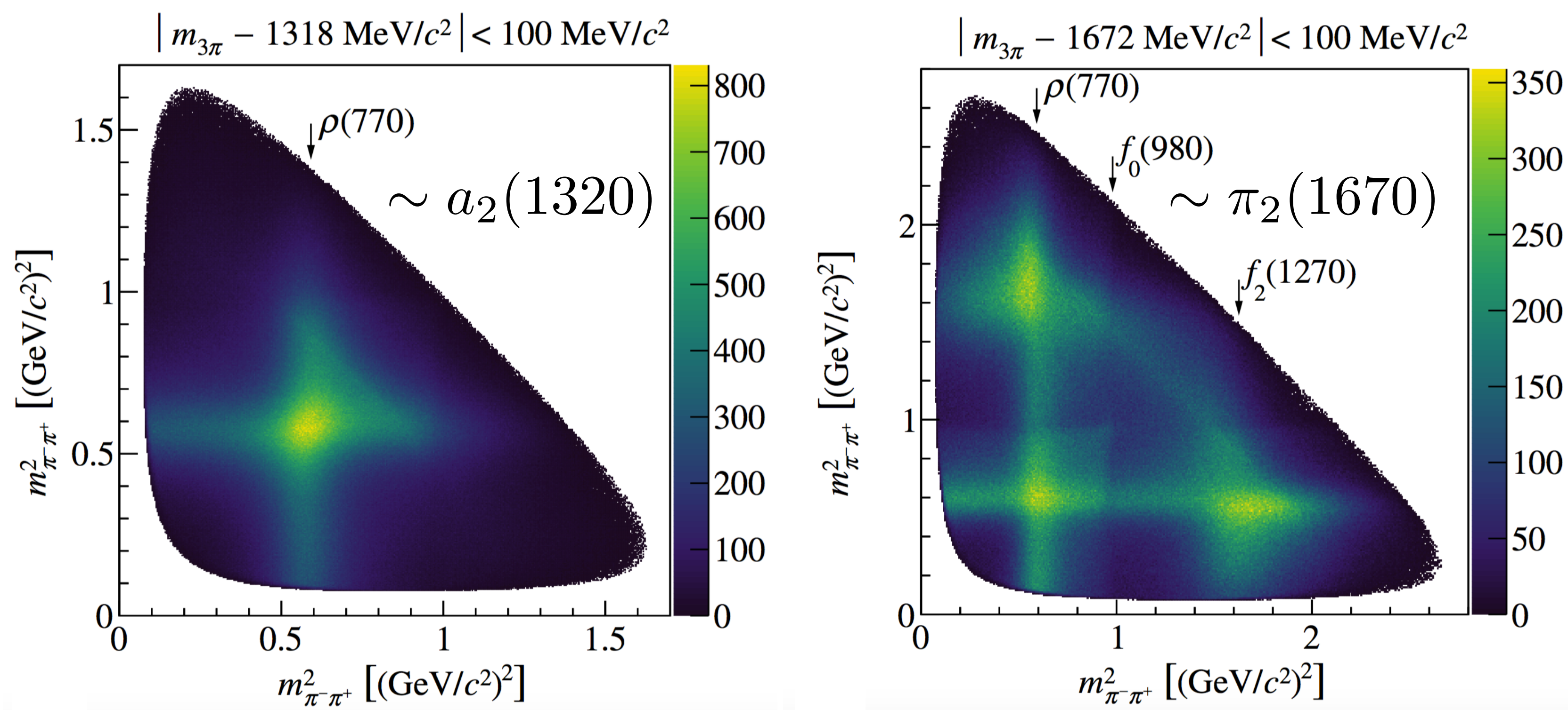
$$m_R = 769 \pm 1\,\mathrm{MeV}$$
$$\Gamma_R = 149 \pm 2\,\mathrm{MeV}$$



3→3 Scattering

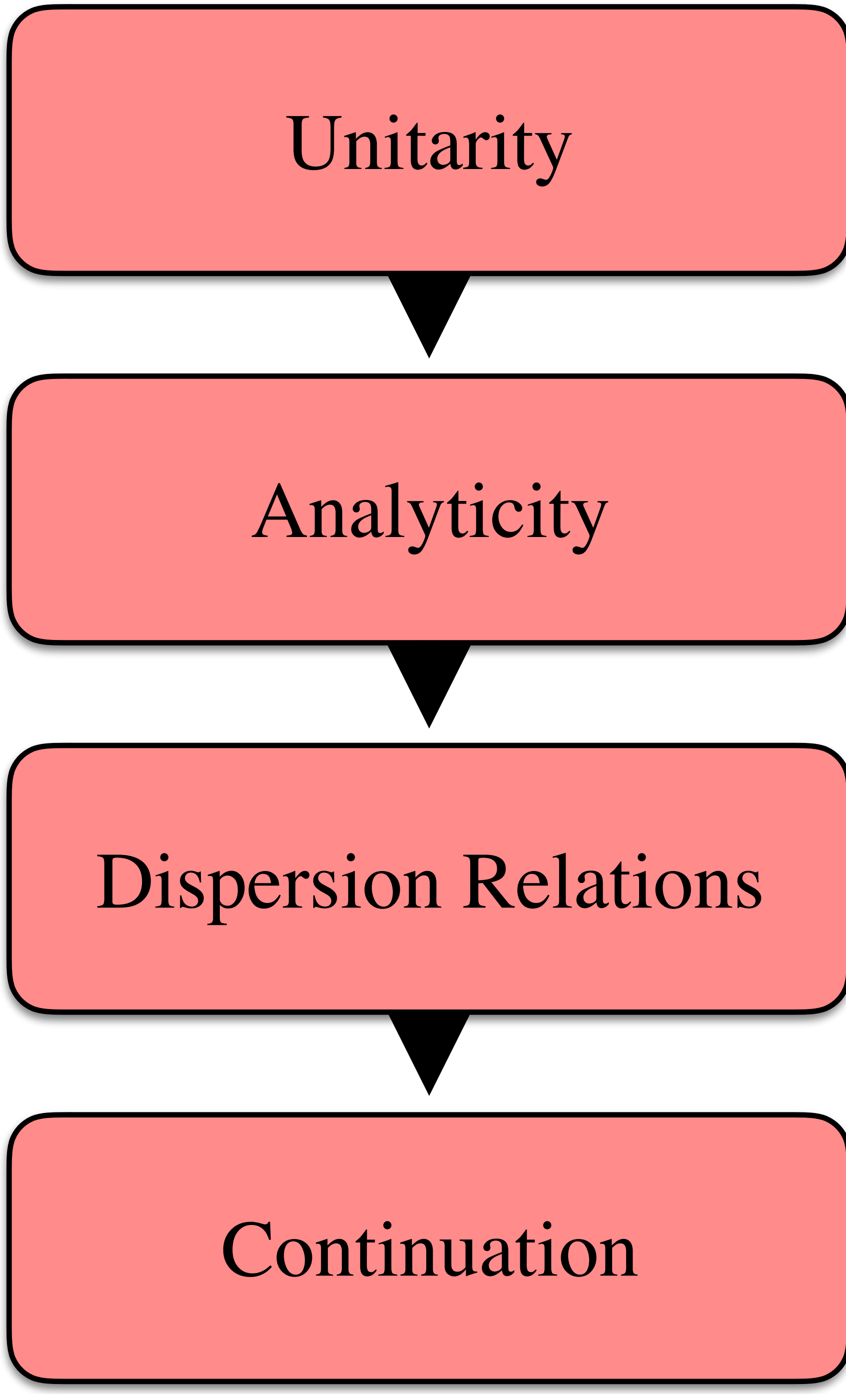


- Consider the elastic scattering *123→123*
  - Distinguishable, spinless particles
- Isobar Model
  - Interacting two-particle sub-channel (**Isobar**) interacts with third particle (**Spectator**)



C. Adolph et al. [COMPASS], Phys. Rev. D 95, no. 3, 032004 (2017)

3→3 Scattering

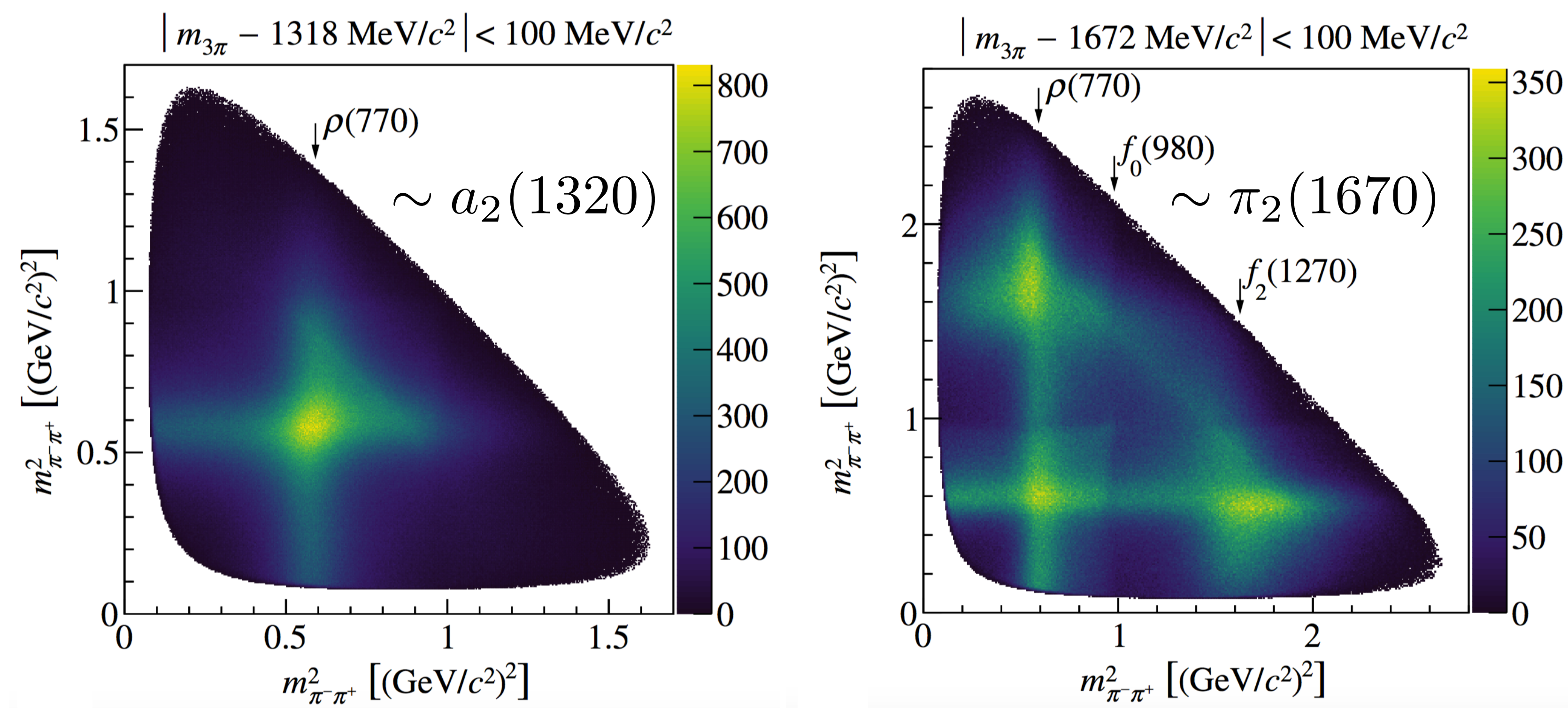


- Consider the elastic scattering *123→123*

  ...ss particles

Multiple variables →
Isobar approximation

- – Interacting two-particle sub-channel (**Isobar**) interacts with third particle (**Spectator**)



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3→3 Scattering

Unitarity

Analyticity

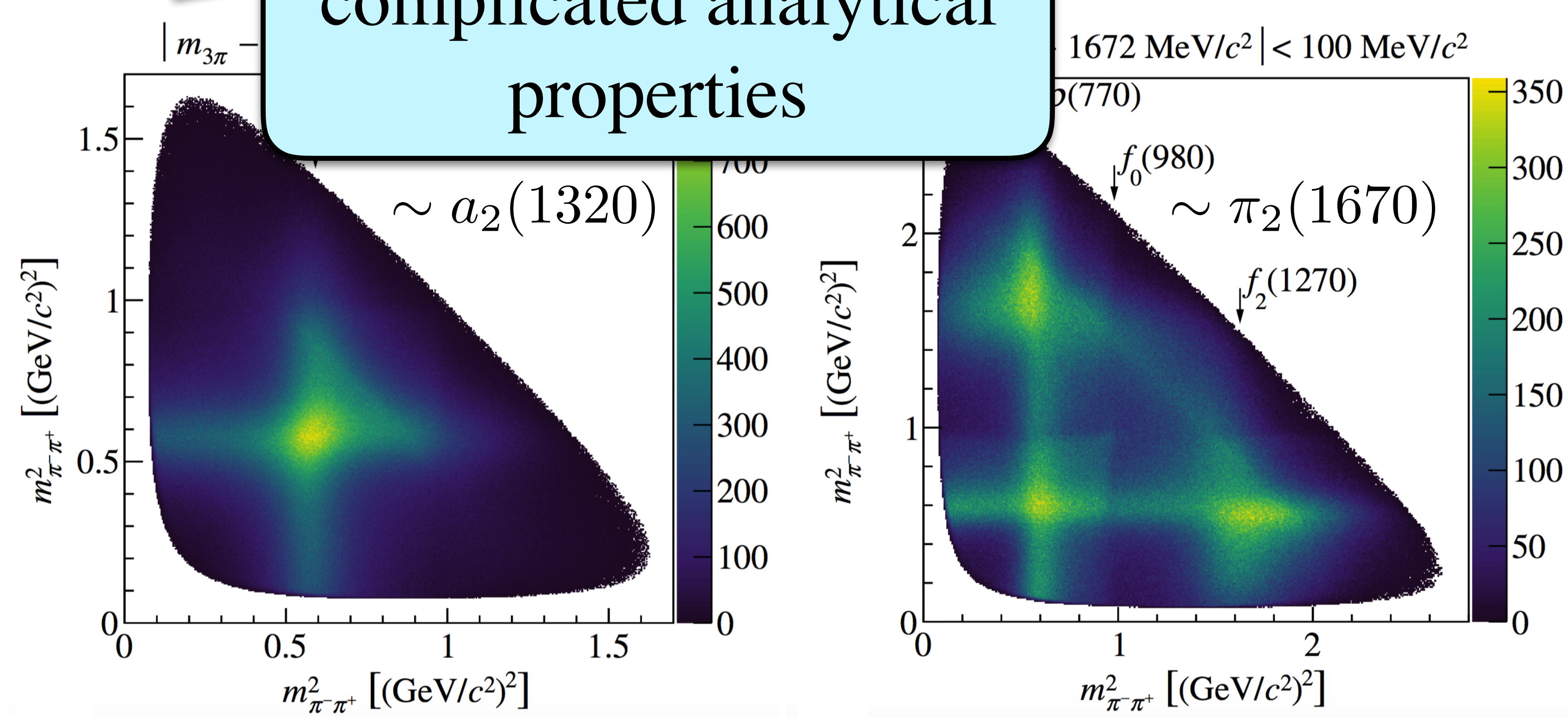
Dispersion Relations

Continuation

- Consider the elastic scattering *123→123*
  - ess particles
- Interacting two-particle sub-channel (**Isobar**) interacts with third particle (**Spectator**)

Multiple variables → Isobar approximation

Isobar/Partial Waves → complicated analytical properties



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3→3 Scattering

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Unitarity

Multiple variables → Isobar approximation

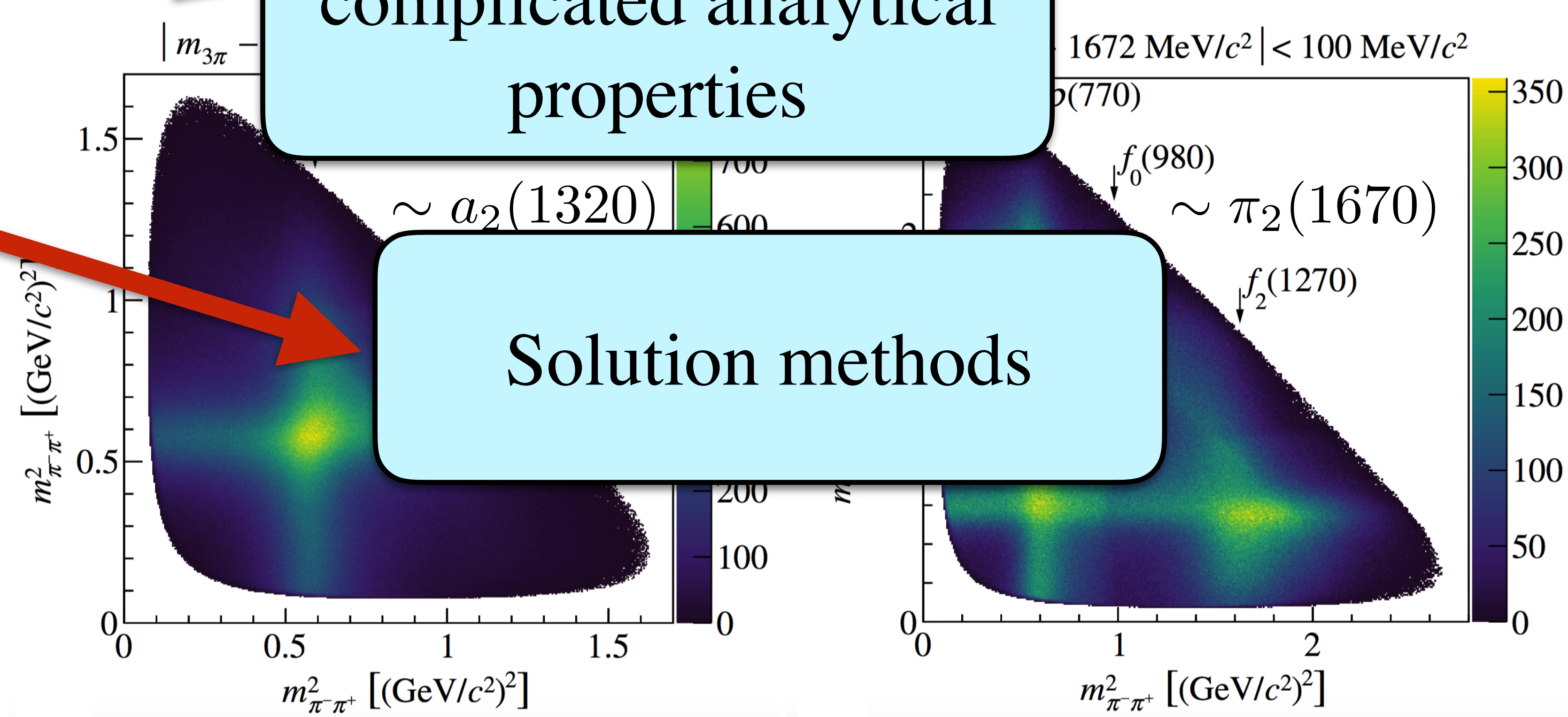
Analyticity

Isobar/Partial Waves → complicated analytical properties

Dispersion Relations

Solution methods

Continuation



C. Adolph et al. [COMPASS], Phys. Rev. D 95, no. 3, 032004 (2017)

3→3 Scattering

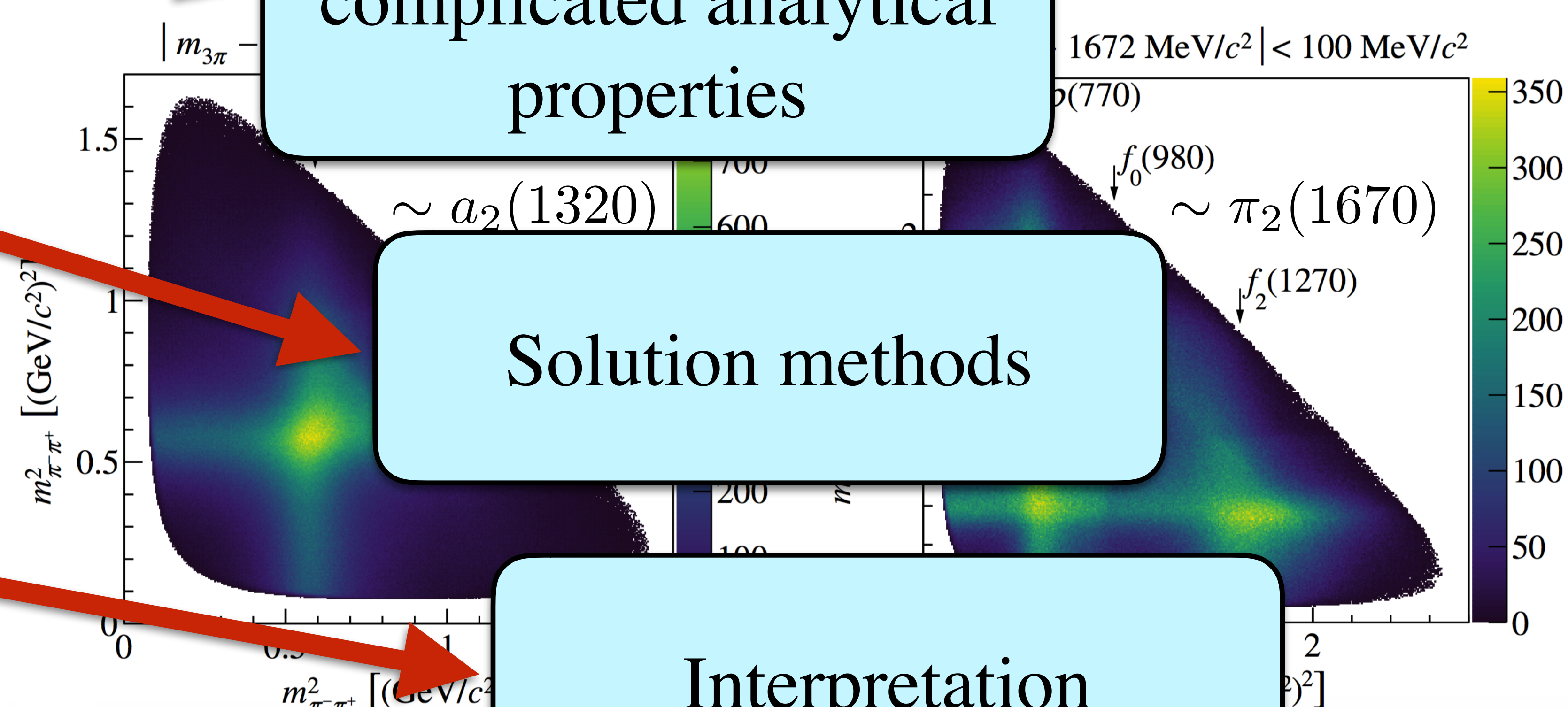
- Consider the elastic scattering *123→123*
- ...ss particles
- Interacting two-particle sub-channel (**Isobar**) int... (**Spectator**)

Unitarity → Multiple variables → Isobar approximation

Analyticity → Isobar/Partial Waves → complicated analytical properties

Dispersion Relations → Solution methods

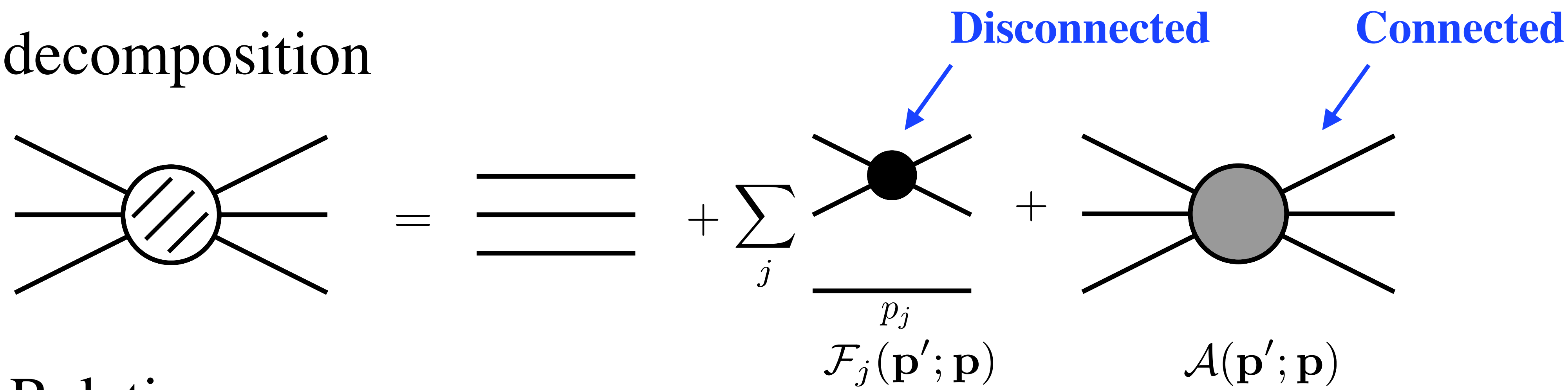
Continuation → Interpretation



Phys. Rev. D 95, no. 3, 032004 (2017)

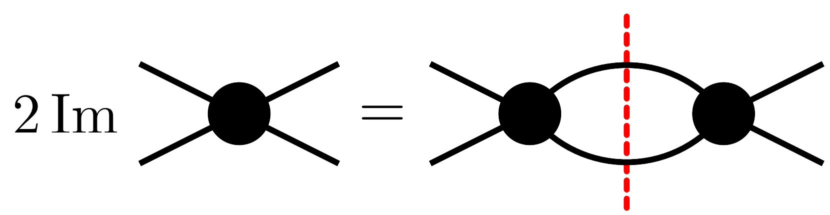
3→3 Scattering

- S-matrix decomposition

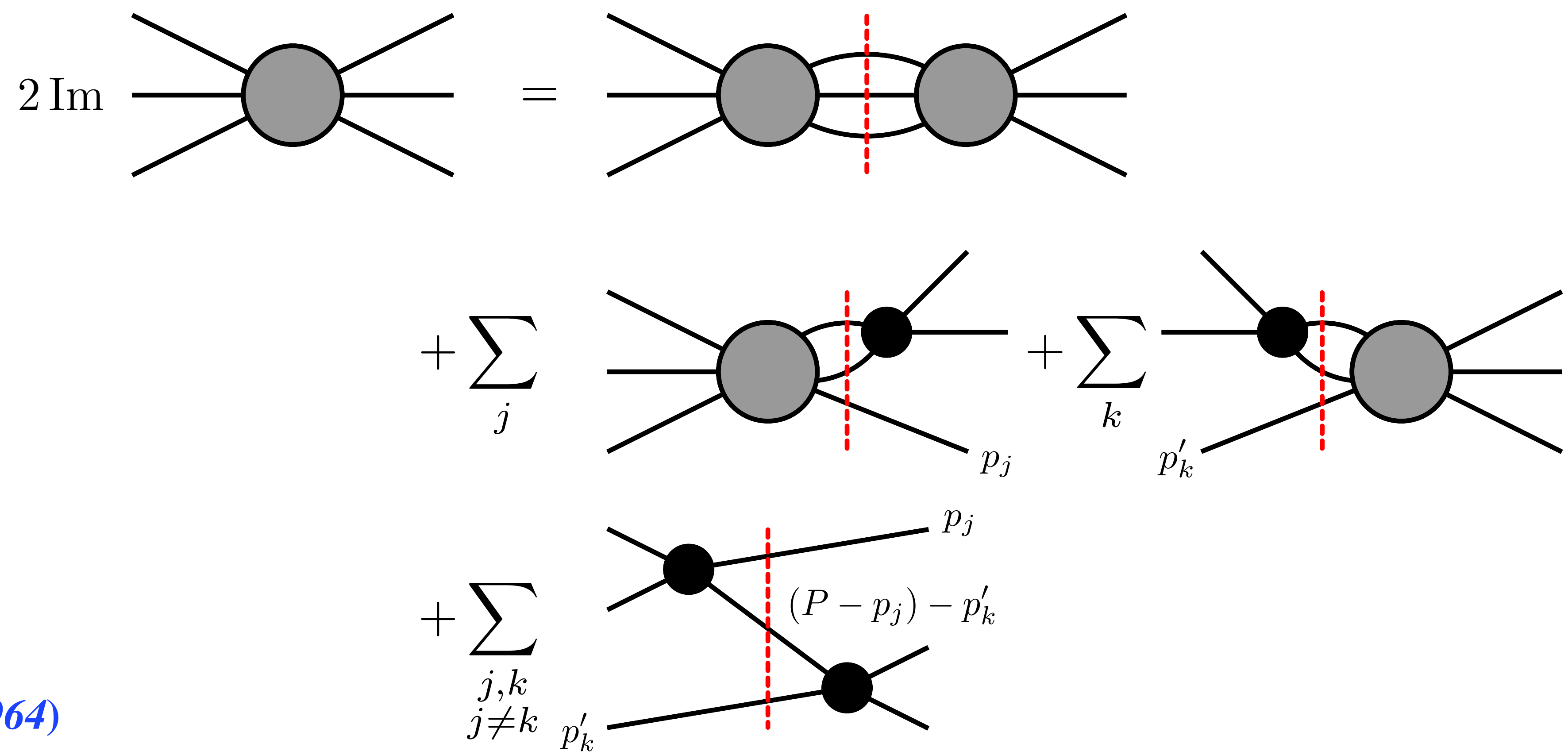


- Unitarity Relations

Disconnected



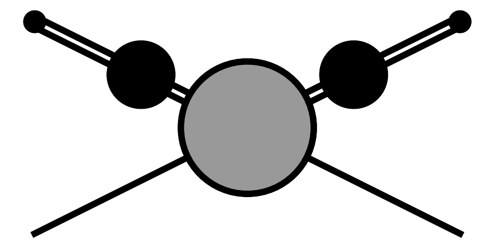
Connected



G.N. Fleming,
Phys. Rev. 135, B551 (1964)

Partial Wave Decomposition

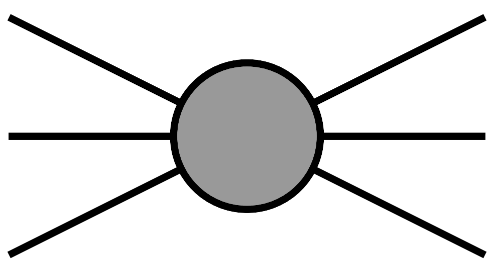
- Partial wave expansion of **3** $\to$ **3** elastic amplitude
  - Remove angular dependence $\to$ 3 Energy variables



Pair-Spectator Amplitude

$$\mathcal{A}(\sigma', s, \sigma; \widehat{\mathbf{p}}'^{\star}, \widehat{\mathbf{p}}^{\star}, \widehat{\mathbf{q}}', \widehat{\mathbf{q}}) = \sum_{J} \sum_{\ell', s'} \sum_{\ell, s} \mathcal{A}^{J}_{\ell' s'; \ell s}(\sigma', s, \sigma) \sum_{M} Z^{JM\,*}_{\ell' s'}(\widehat{\mathbf{p}}'^{\star}, \widehat{\mathbf{q}}') Z^{JM}_{\ell s}(\widehat{\mathbf{p}}^{\star}, \widehat{\mathbf{q}})$$

Full 3 $\to$ 3 Amplitude



Rotational Function

Isobar-Spectator Angle Defined in CMS Frame

$$Z^{JM}_{\ell s}(\widehat{\mathbf{p}}^{\star}, \widehat{\mathbf{q}}) = \sqrt{\frac{2\ell+1}{4\pi}} \sqrt{\frac{2s+1}{4\pi}} \sum_{\lambda=-s}^{s} \langle J\lambda | \ell 0 s \lambda \rangle \, \mathcal{D}^{(J)}_{M\lambda}(\widehat{\mathbf{p}}^{\star}) \mathcal{D}^{(s)}_{\lambda 0}(\widehat{\mathbf{q}})$$

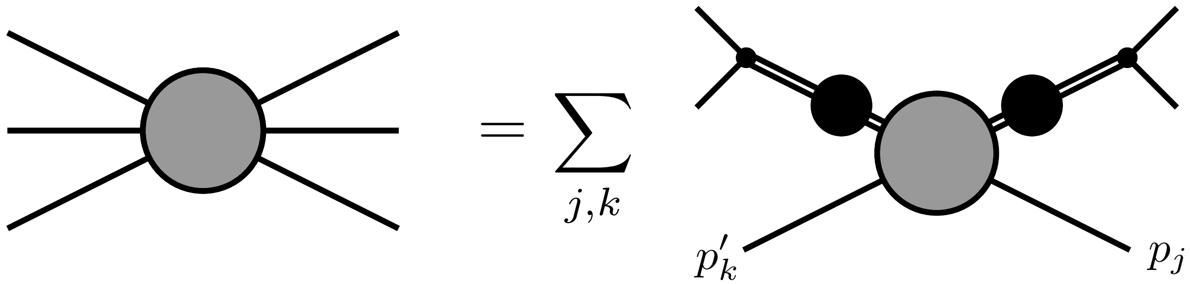
Isobar Decay Angles Defined in IRS Frame

- Isobar-Spectator Scattering angle $\Theta^{\star}$

$$\sum_{M} \mathcal{D}^{(J)\,*}_{M\lambda'}(\widehat{\mathbf{p}}'^{\star}) \mathcal{D}^{(J)}_{M\lambda}(\widehat{\mathbf{p}}^{\star}) = \mathcal{D}^{(J)}_{\lambda\lambda'}(\widehat{\mathbf{p}}'^{\star} \cdot \widehat{\mathbf{p}}^{\star}) \qquad \cos\Theta^{\star} = \widehat{\mathbf{p}}'^{\star} \cdot \widehat{\mathbf{p}}^{\star}$$

Isobar Model

- Assume sum over pair-wise interactions ⇒ **Isobar-Spectator amplitudes**
  - Partial wave expansion truncated
  - Leads to complicated sub-channel energy-dependence

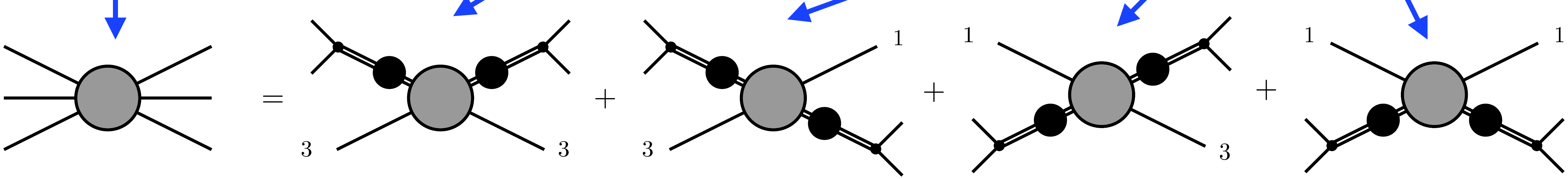


$$\mathcal{A}(\mathbf{p}';\mathbf{p}) = \sum_{j,k} \mathcal{A}_{kj}(\mathbf{p}';\mathbf{p})$$

- Partial Wave Projected Isobar-Spectator (PWIS) Amplitudes

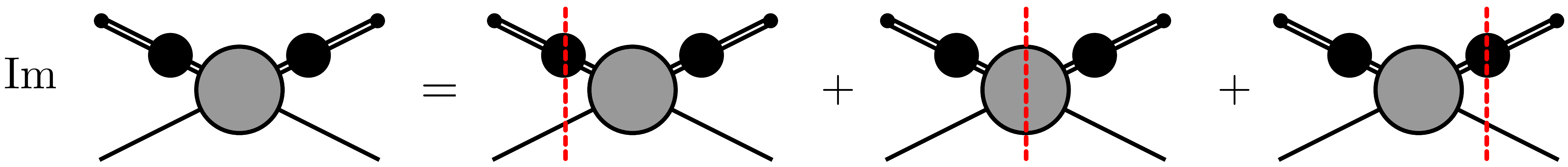
$$\widetilde{\mathcal{A}}^J_{\ell' s';\ell s}(\sigma', s, \sigma) = \widetilde{\mathcal{A}}^J_{\ell' s';\ell s}(\sigma', s, \sigma)\big|_{\mathrm{Iso}} + \sum_{\substack{j,k \\ \text{crossed}}} \left[X^J_{\ell' s';\ell s}\right]_{jk}(\sigma', s, \sigma)$$

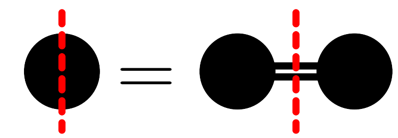
Full 3→3 PW Amplitude **PWIS Amplitude** **Recoupling PWIS Amplitudes**

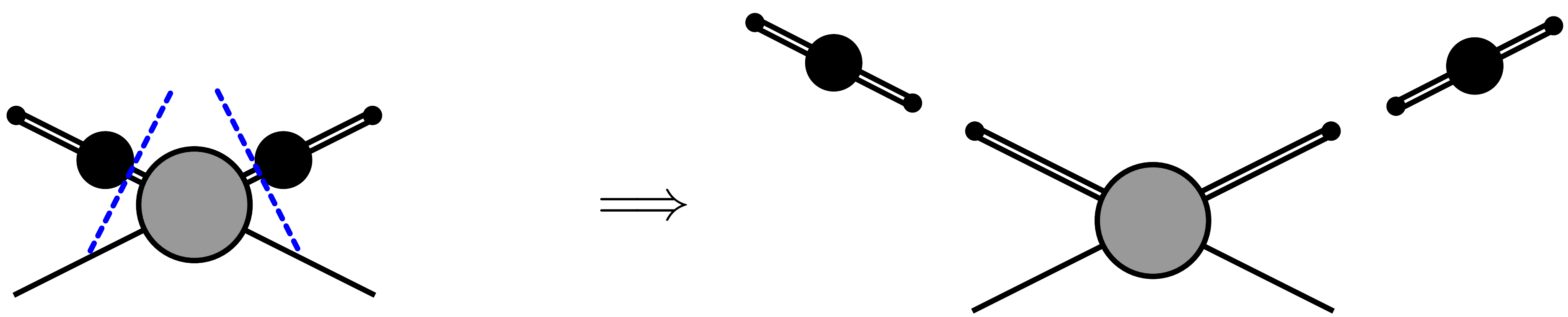


Amputation

- Imaginary part of Isobar-Specator amplitude



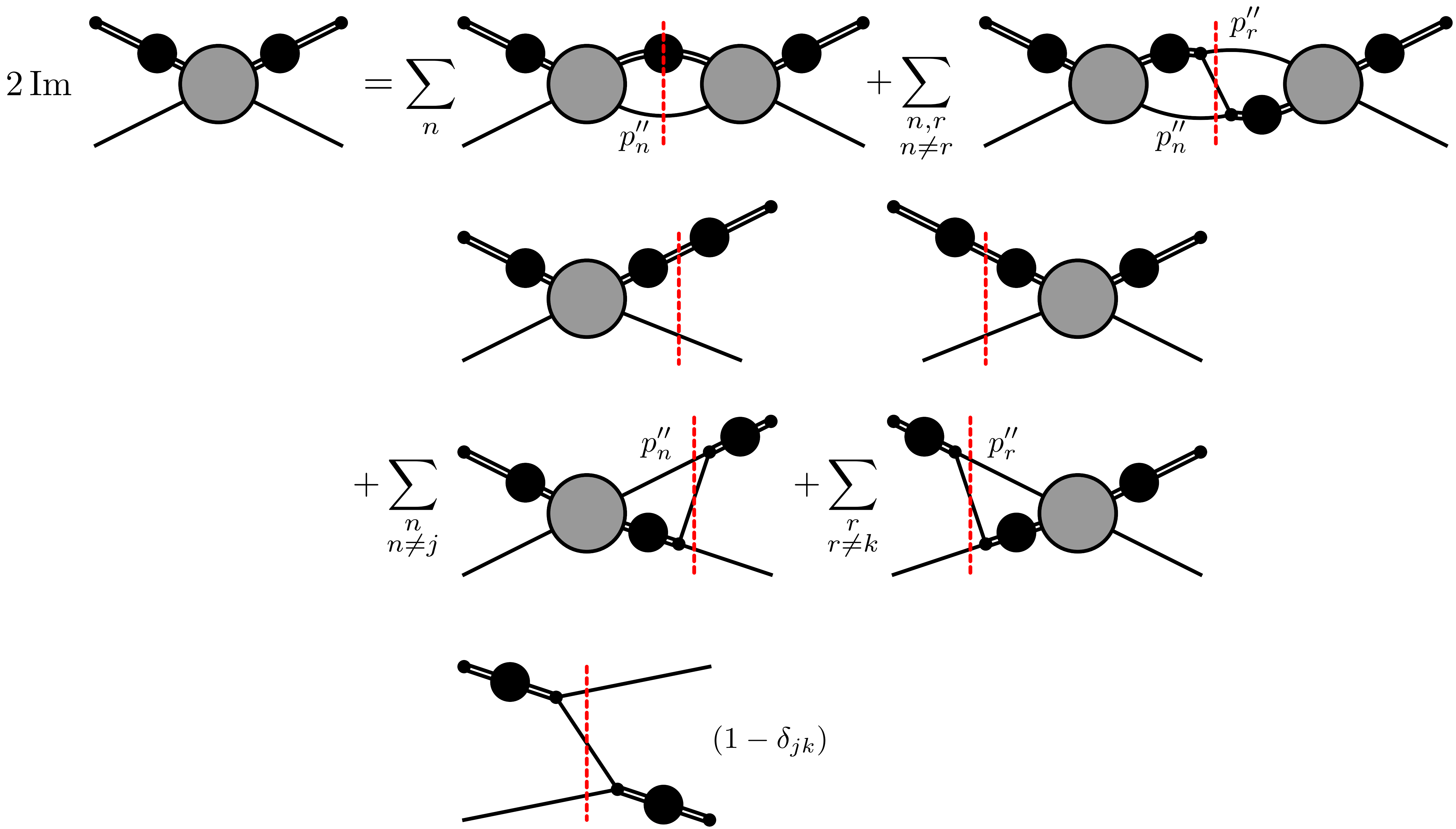
- Use unitarity for partial wave **2→2** amplitude 
  - Factorize the 2-particle sub-channel denominators



$$\mathcal{A}_{kj} = f_k(\sigma'_k)\widetilde{\mathcal{A}}_{kj}(\sigma'_k, s, \sigma_j)f_j(\sigma_j)$$

Amputated Isobar-Spectator Unitarity

- PWIS Unitarity Relations



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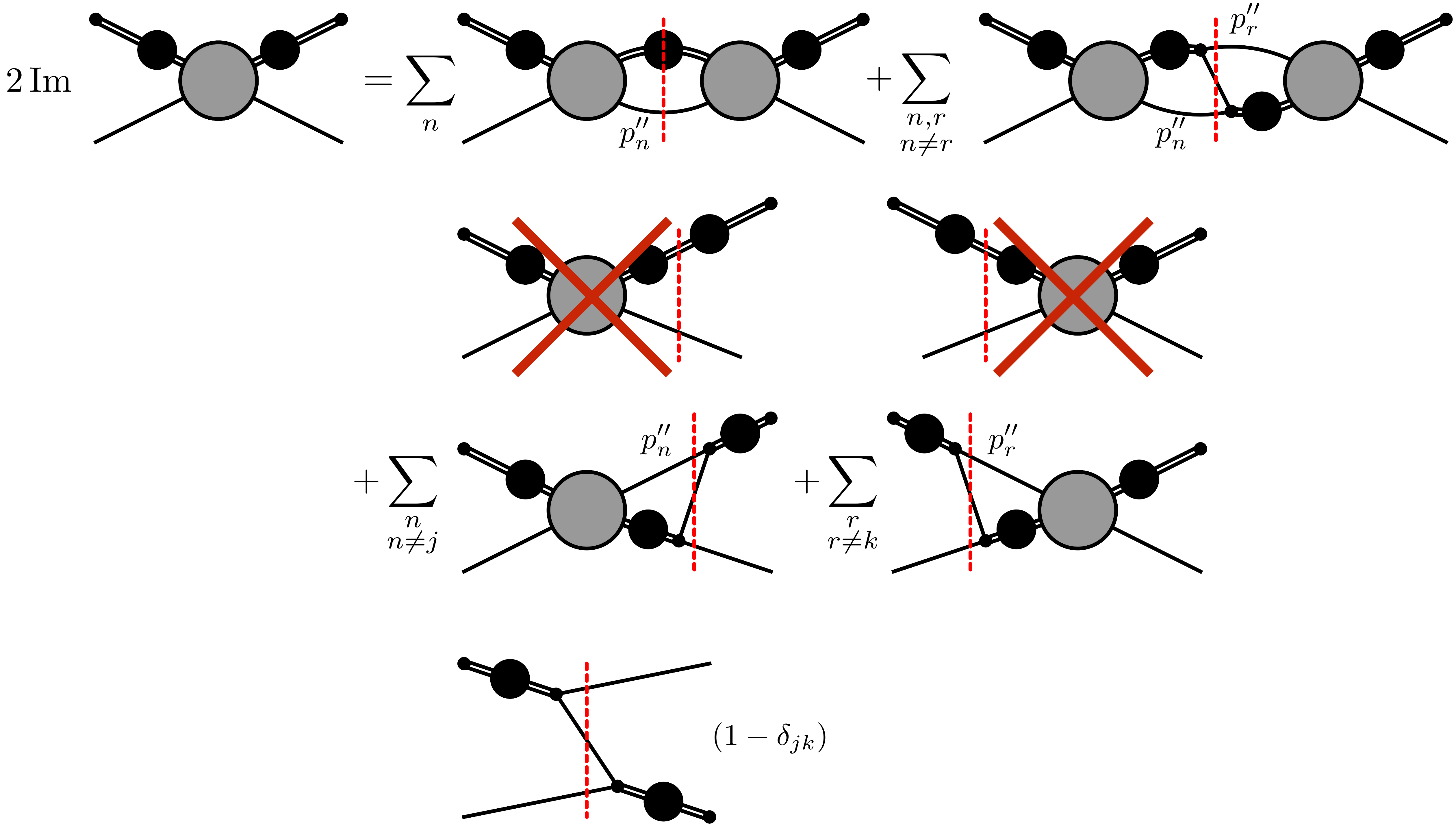

Amputated Isobar-Spectator Unitarity

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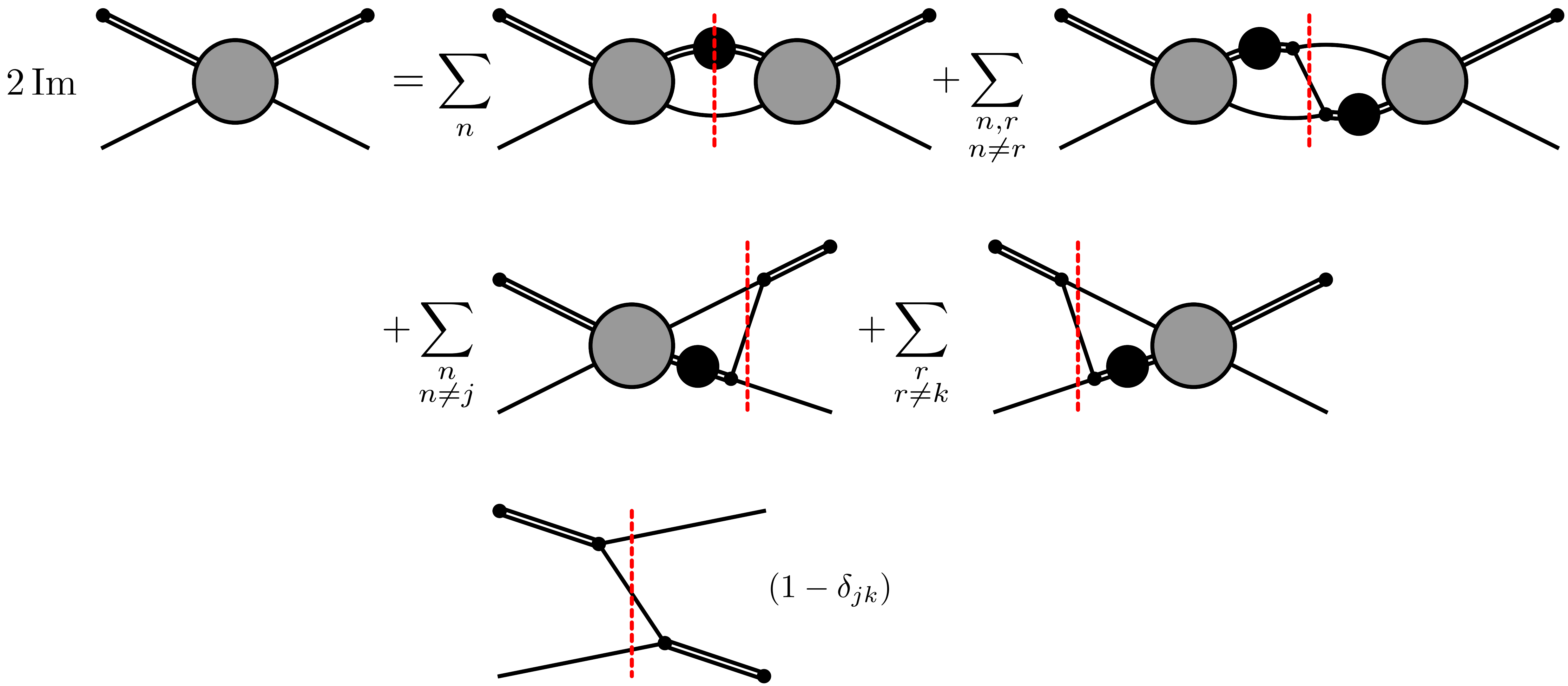
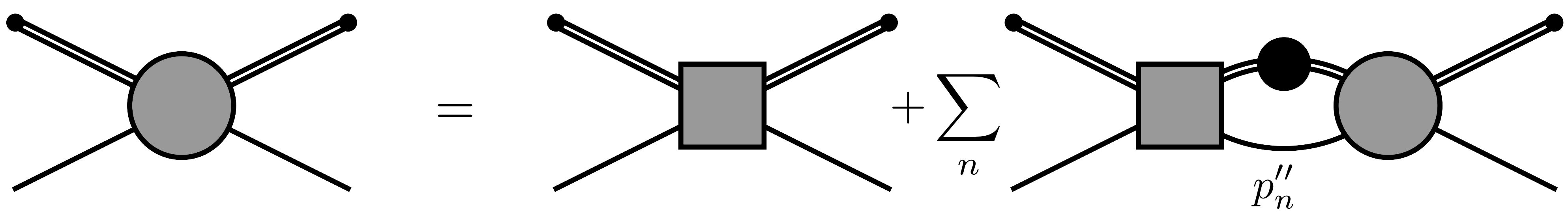


B-Matrix Parameterization

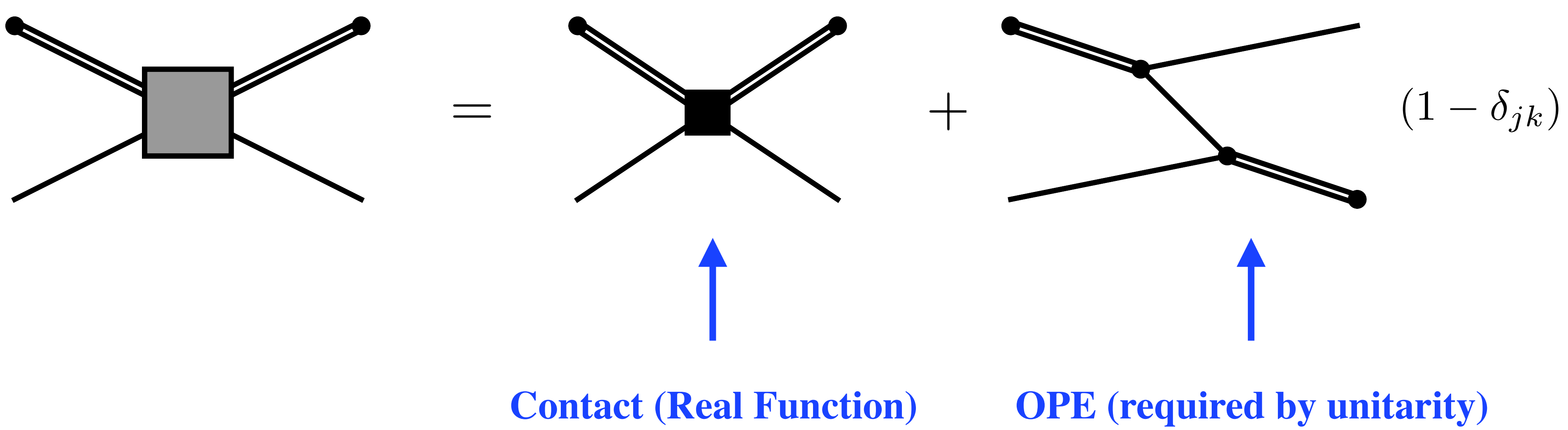
M. Mai et al. [JPAC], Eur. Phys. J. A53 (2017) no.9, 177

- Unitary parameterization of PWIS amplitude

$$\widetilde{\mathcal{A}}_{kj} = \mathcal{B}_{kj} + \sum_n \int \mathcal{B}_{kn} \tau_n \widetilde{\mathcal{A}}_{nj}$$



- B-Matrix composed of OPE and Contact

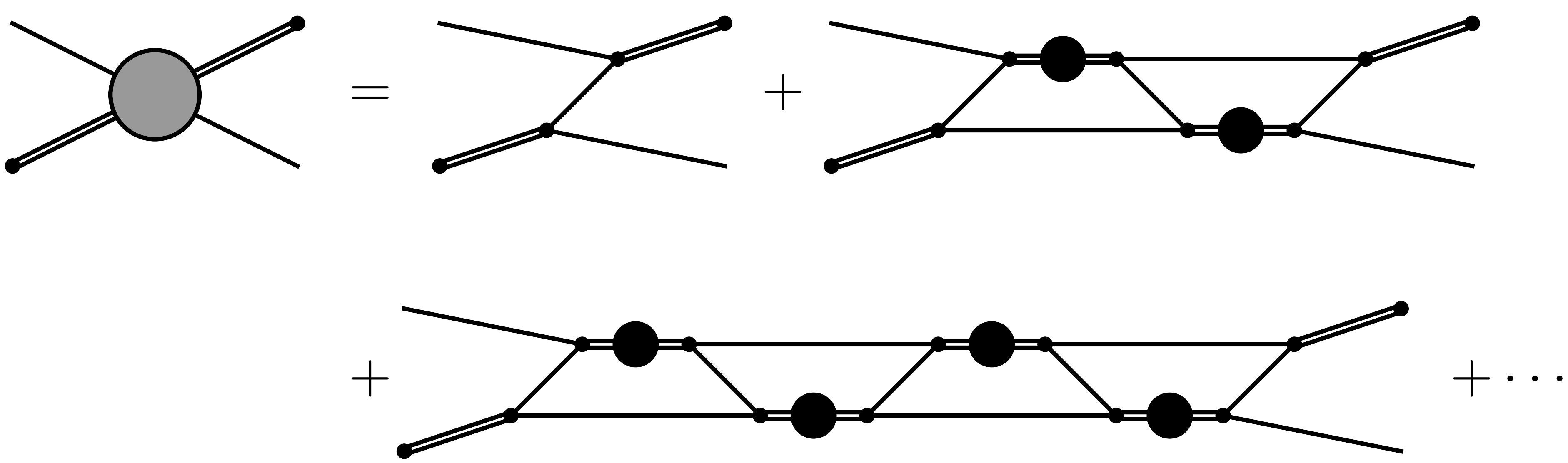


Bubbles, Triangles, and Boxes

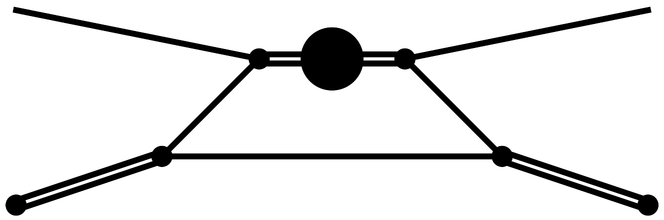
- Ladder summation of exchanges

e.g. OPE driving term

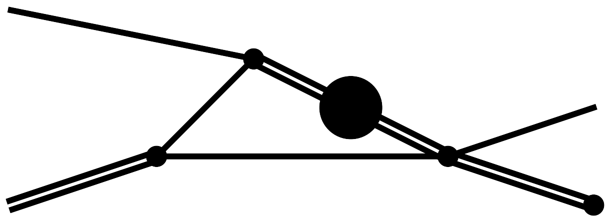
$$\widetilde{\mathcal{A}}_{13} = [\mathbb{1} - \mathcal{K}_{11} \cdot \tau_1]^{-1} \widetilde{\mathcal{E}}_{13}$$



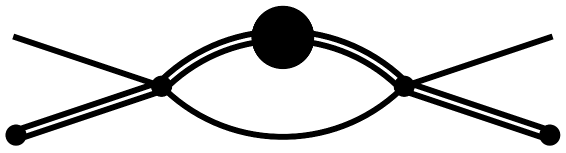
- Analytic properties of OPE, bubble, triangle, box



Box



Triangle



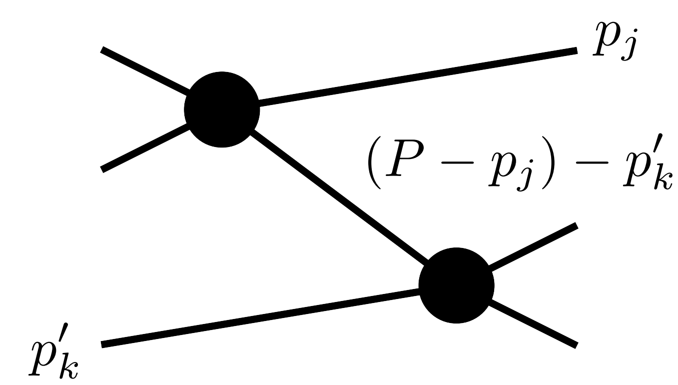
Bubble

One-Particle Exchange

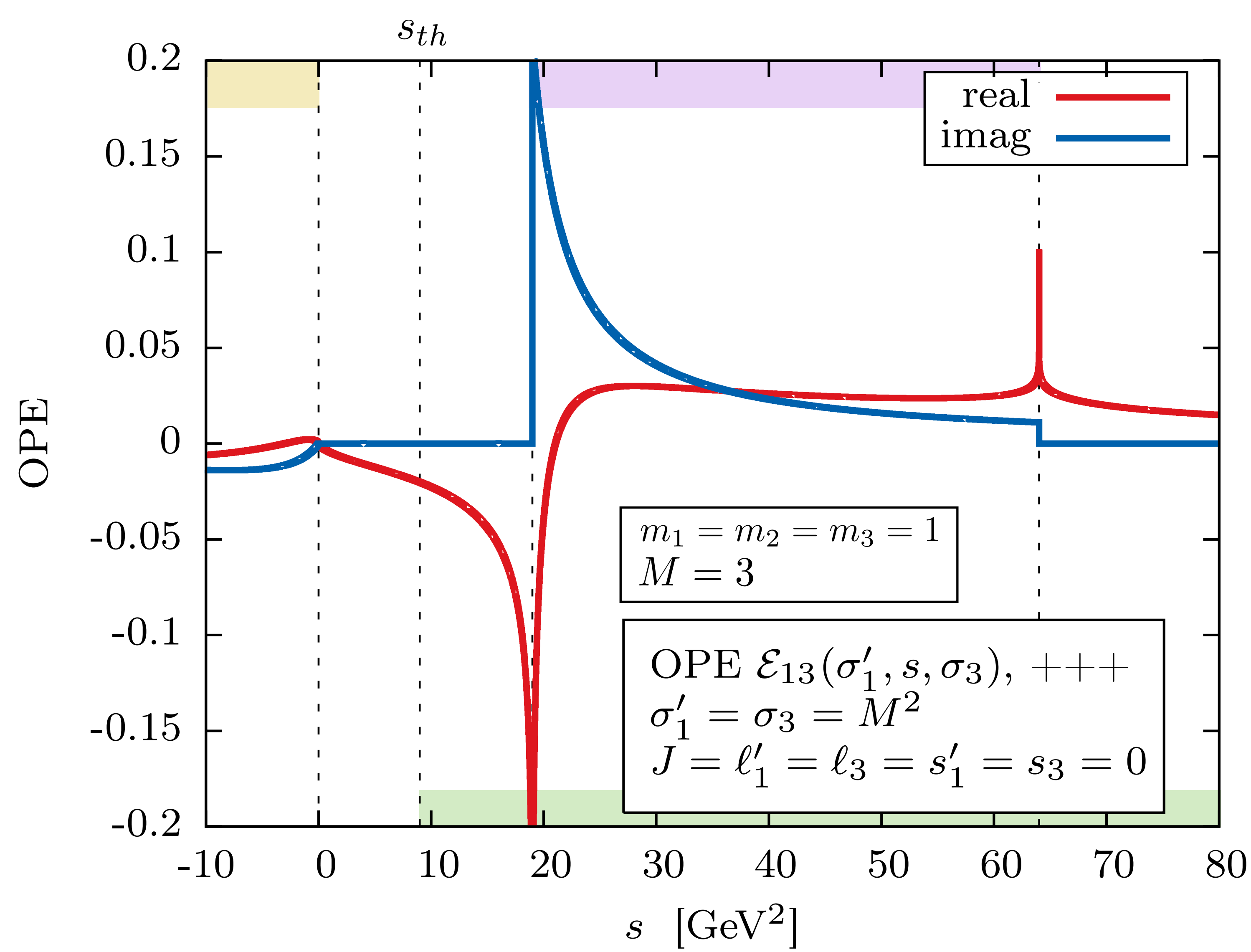
- OPE $\Rightarrow$ Driving term for ladder summation

$$\widetilde{\mathcal{E}}_{13}(\sigma_1', s, \sigma_3) = -\frac{1}{8\pi|\mathbf{p}_1'^{\star}||\mathbf{p}_3^{\star}|}\log\left(\frac{1+z_{13}^{\mu}}{1-z_{13}^{\mu}}\right)$$

$$\mathcal{E}_{kj}(\mathbf{p}';\mathbf{p}) =$$



$$z_{13}^{\mu} = \frac{\sigma_3 + m_1^2 - m_2^2 - E_3^{\star}\,\omega_1'^{\star}}{|\mathbf{p}_1'^{\star}||\mathbf{p}_3^{\star}|}$$



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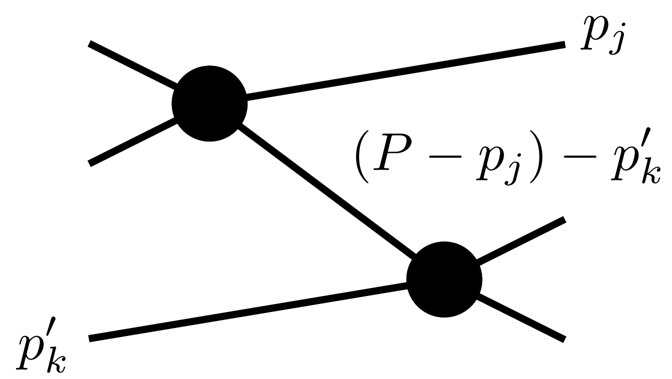
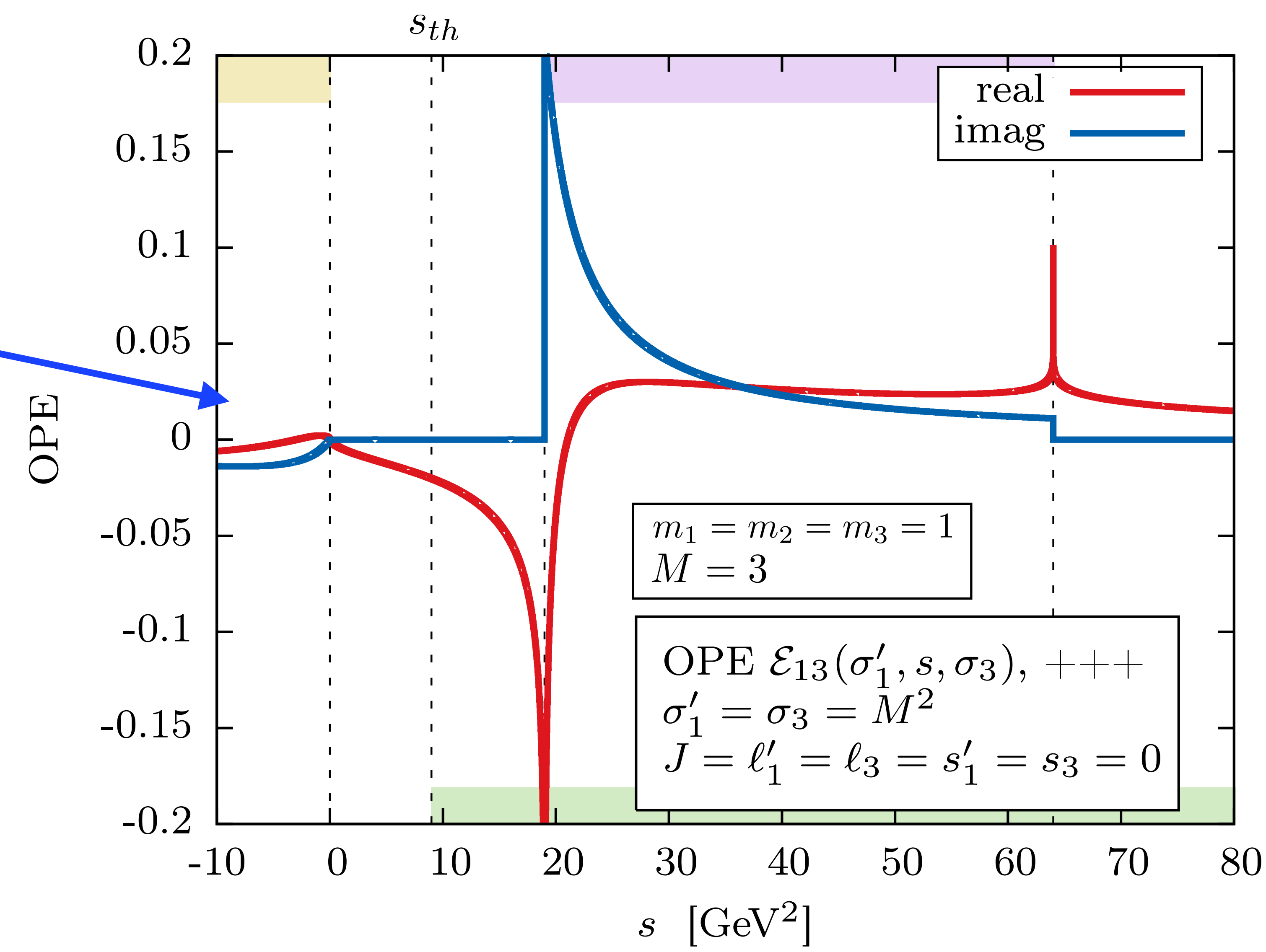


$$z_{13}^\mu = \frac{\sigma_3 + m_1^2 - m_2^2 - E_3^\star\,\omega_1'^\star}{|\mathbf{p}_1'^\star||\mathbf{p}_3^\star|}$$

Virtual Particle Exchange cut



One-Particle Exchange

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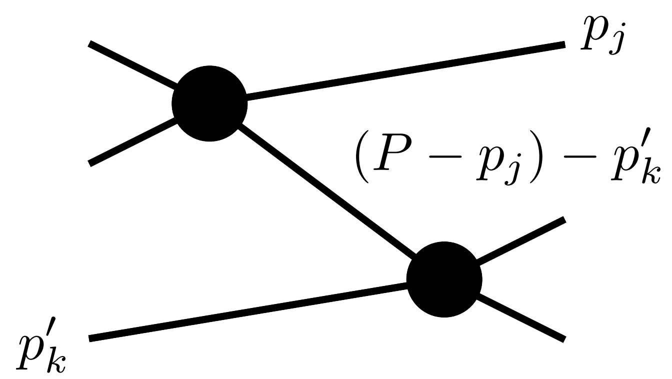
$$\widetilde{\mathcal{E}}_{13}(\sigma_1', s, \sigma_3) = -\frac{1}{8\pi|\mathbf{p}_1'^{\star}||\mathbf{p}_3^{\star}|}\log\left(\frac{1+z_{13}^{\mu}}{1-z_{13}^{\mu}}\right)$$

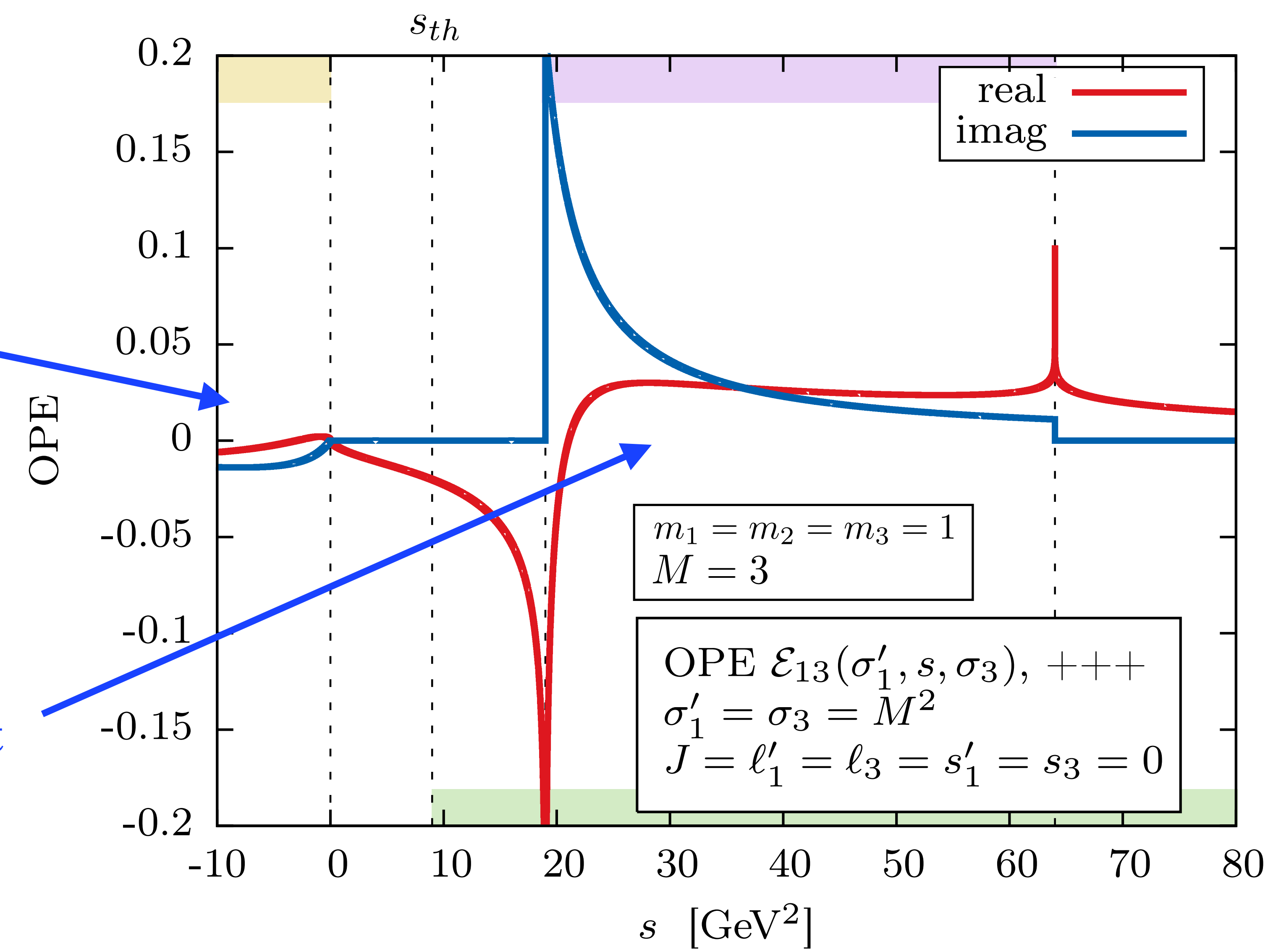
$$\mathcal{E}_{kj}(\mathbf{p}';\mathbf{p}) =$$



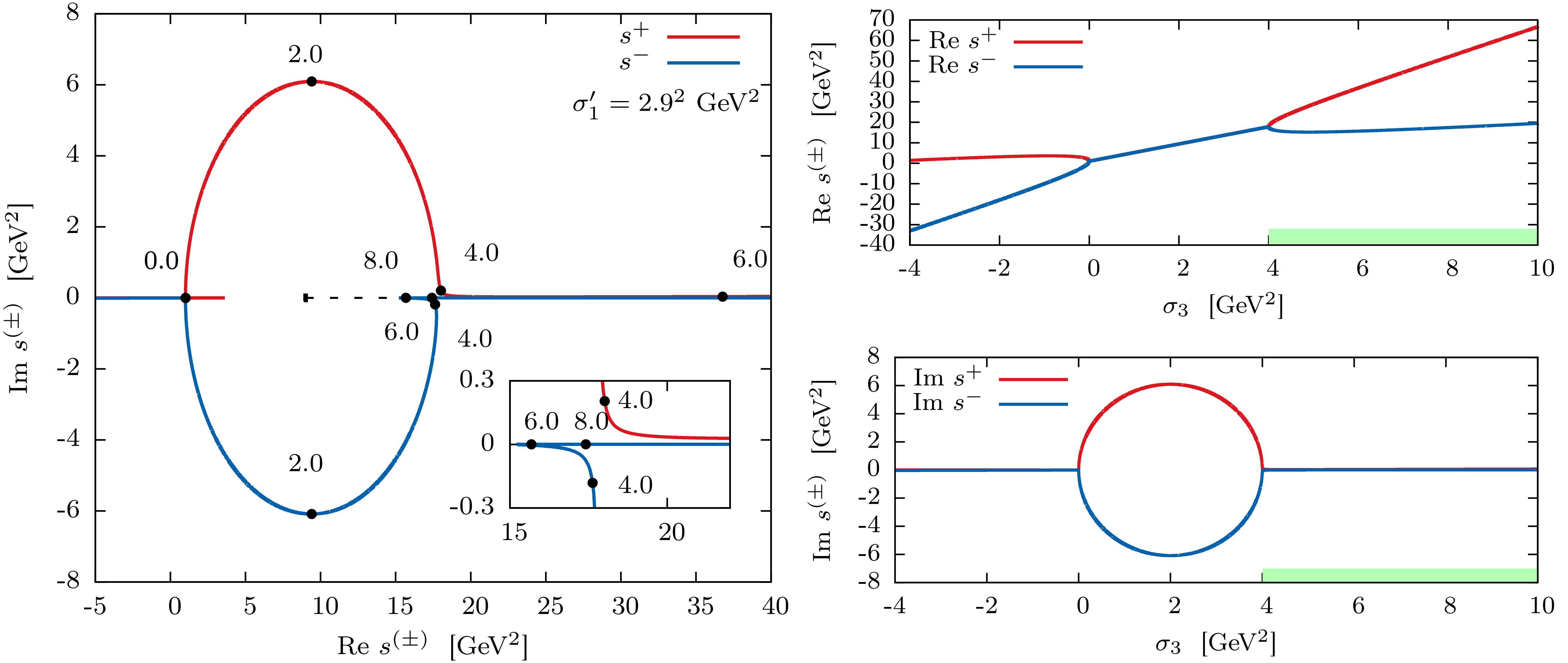
$$z_{13}^{\mu} = \frac{\sigma_3 + m_1^2 - m_2^2 - E_3^{\star}\,\omega_1'^{\star}}{|\mathbf{p}_1'^{\star}||\mathbf{p}_3^{\star}|}$$

Virtual Particle Exchange cut

Real Particle Exchange cut

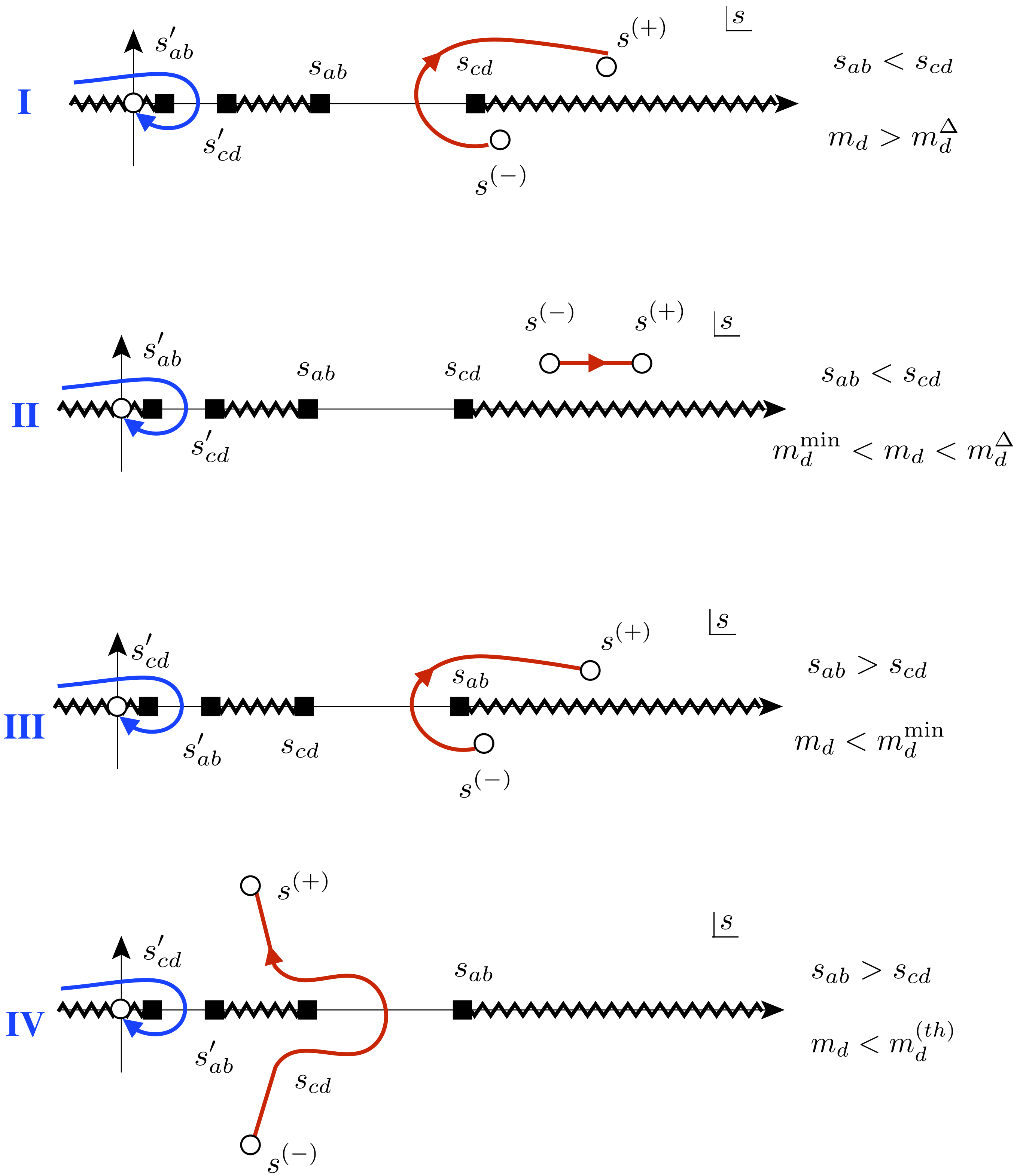


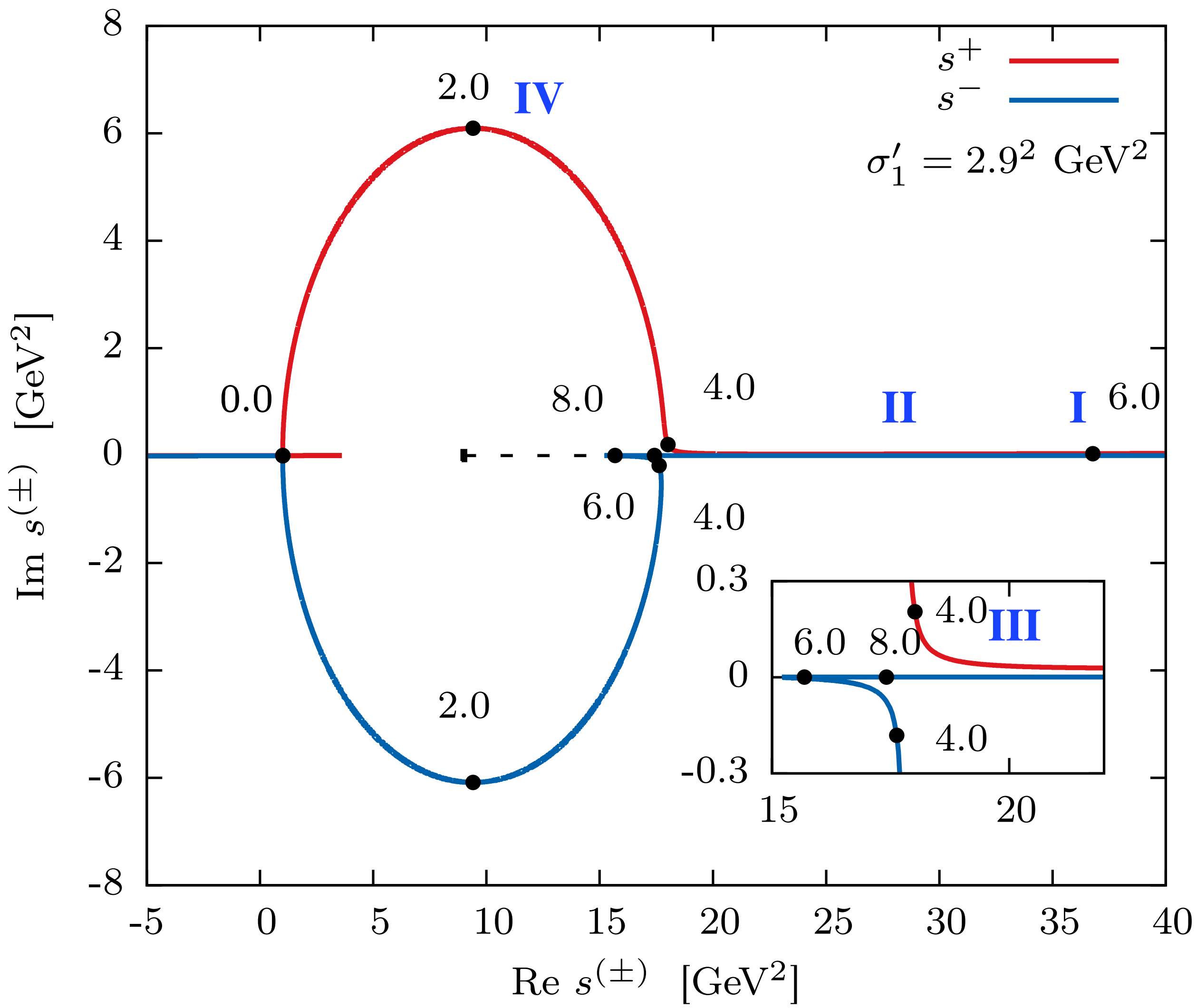
Analytic Structure



$$\widetilde{\mathcal{E}}_{13}(\sigma_1', s, \sigma_3) = \frac{1}{2\pi i}\int_{-\infty}^{0} ds' \, \frac{\Delta_s \widetilde{\mathcal{E}}_{13}(\sigma_1', s', \sigma_3)}{s' - s} + \frac{1}{2\pi i}\int_{s^{(-)}}^{s^{(+)}} ds' \, \frac{\Delta_s \widetilde{\mathcal{E}}_{13}(\sigma_1', s', \sigma_3)}{s' - s}$$

Analytic Structure



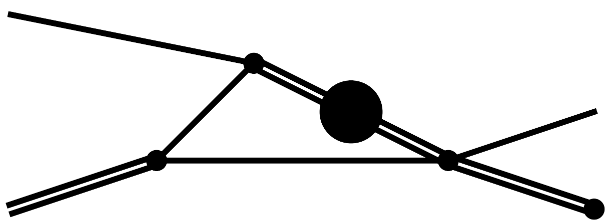


Triangle Diagram

I.J.R. Aitchison and R. Pasquier
Phys. Rev. 152, 4 (1966)

- Triangle diagram - Comparison to off-shell amplitude
  - Imaginary parts equal, difference in the real part

$\mathcal{R}(\sigma', s, \sigma) \to \text{constant}$



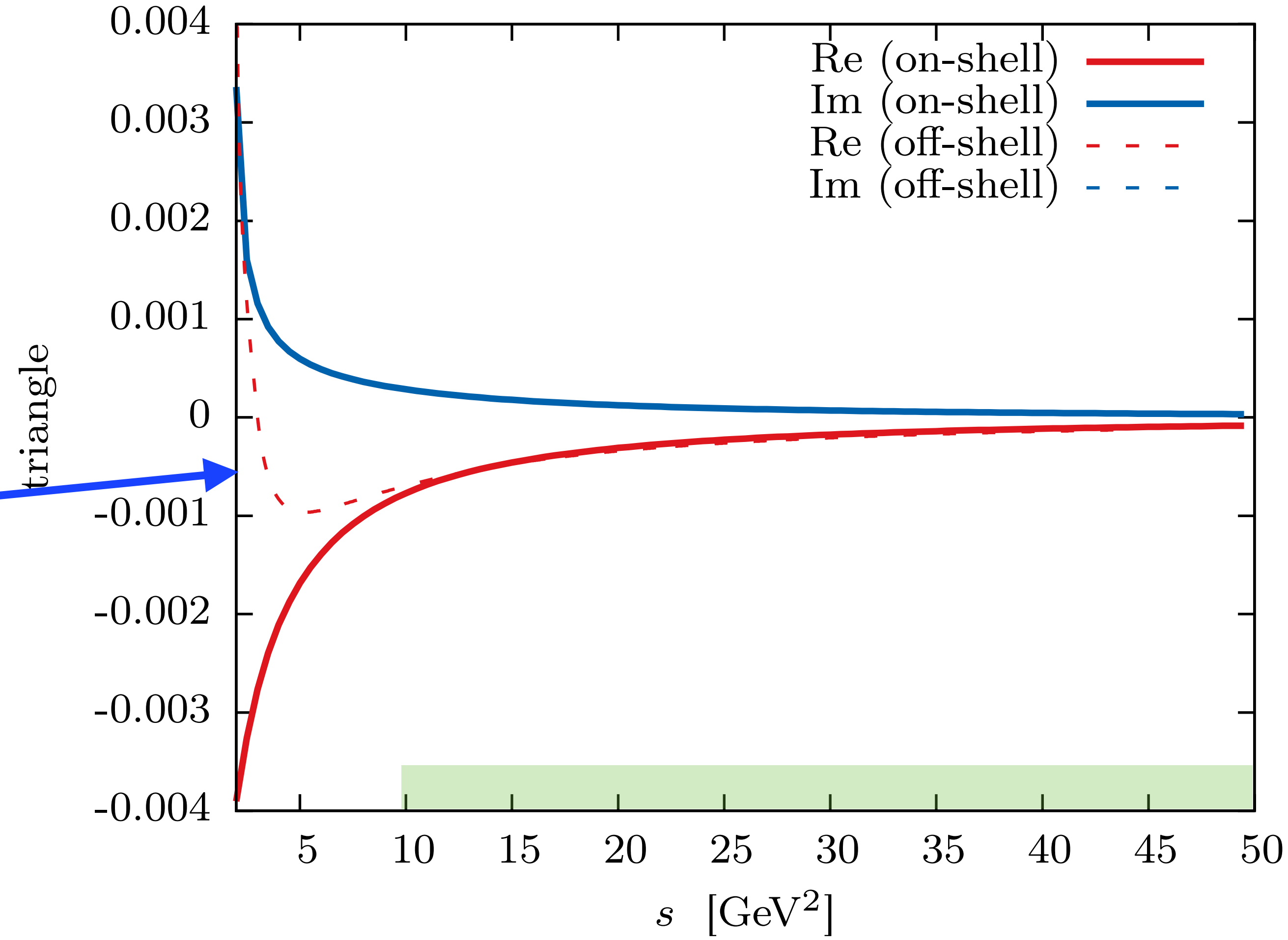
$$\widetilde{\mathcal{A}}_{\Delta}(\sigma', s, \sigma) = \int_{-\infty}^{(\sqrt{s}-m)^2} d\sigma'' \, \widetilde{\mathcal{E}}(\sigma', s, \sigma'')\tau(s, \sigma'')$$ **Off-shell triangle**

$$\widetilde{\mathcal{A}}_{\Delta}^{B}(\sigma', s, \sigma) = \int_{\sigma^{(th)}}^{(\sqrt{s}-m)^2} d\sigma'' \, \widetilde{\mathcal{E}}(\sigma', s, \sigma'')\tau(s, \sigma'')$$ **On-shell triangle**

$$\tau(s, \sigma) = \frac{1}{\pi}\rho(s, \sigma) f(\sigma)$$ **e.g. stable particle**

$$\text{Im } \widetilde{\mathcal{A}}_{\Delta}(\sigma', s, \sigma) = \text{Re } \widetilde{\mathcal{A}}_{\Delta} + i\widetilde{\mathcal{E}}(\sigma', s, M^2)\rho(s, M^2)$$

Difference is real part



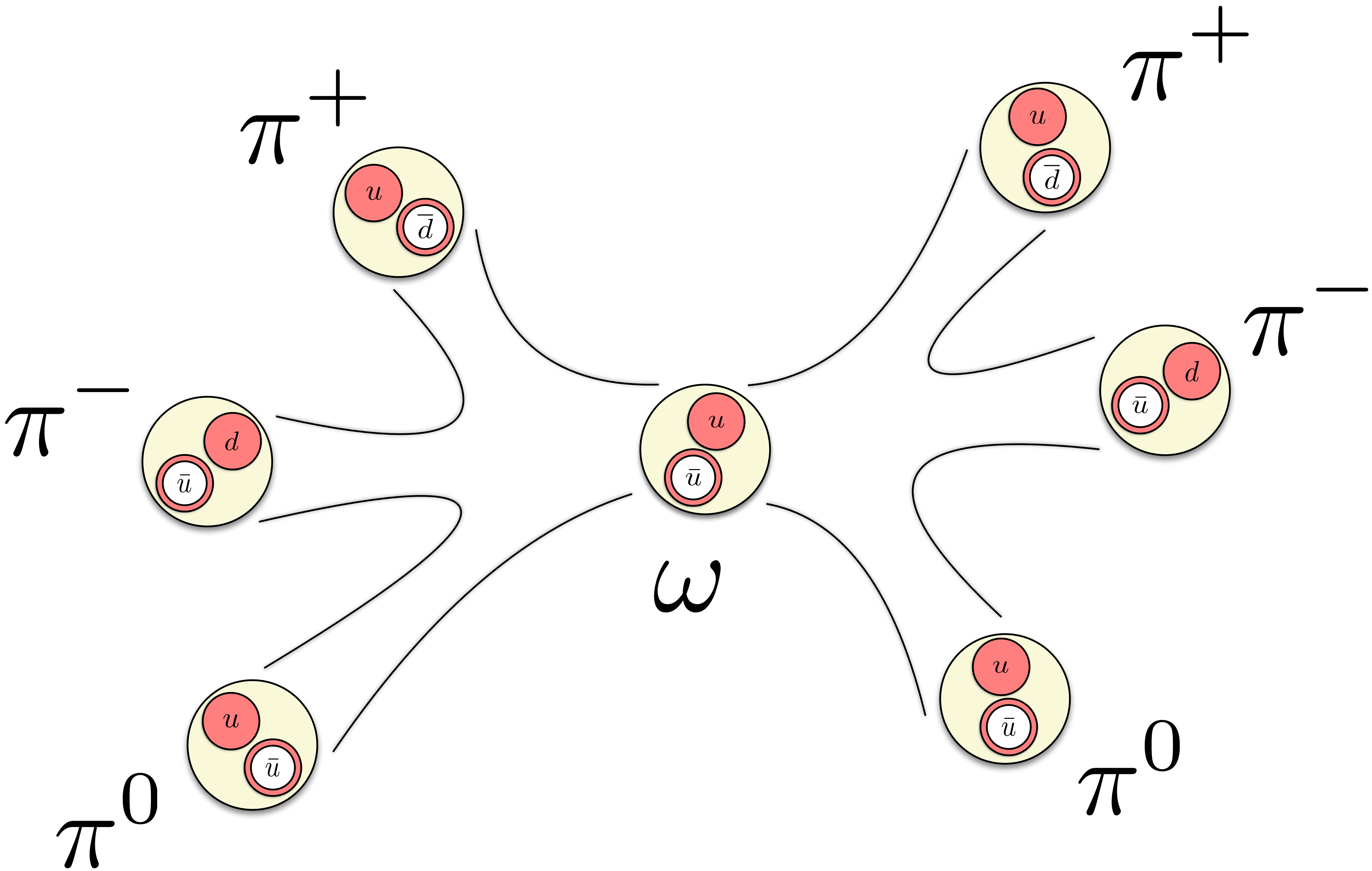
Outlook and Future Directions

- **3→3** unitarity relations within the isobar model *AJ et al* [*JPAC*] *- In Preparation*
  - Spinless external particles
  - Finite number of isobars with spin

- B-matrix parameterization
  - Satisfies unitarity
  - Numerical test underway to determine feasibility of solution
  - Studying behavior when analytically continued

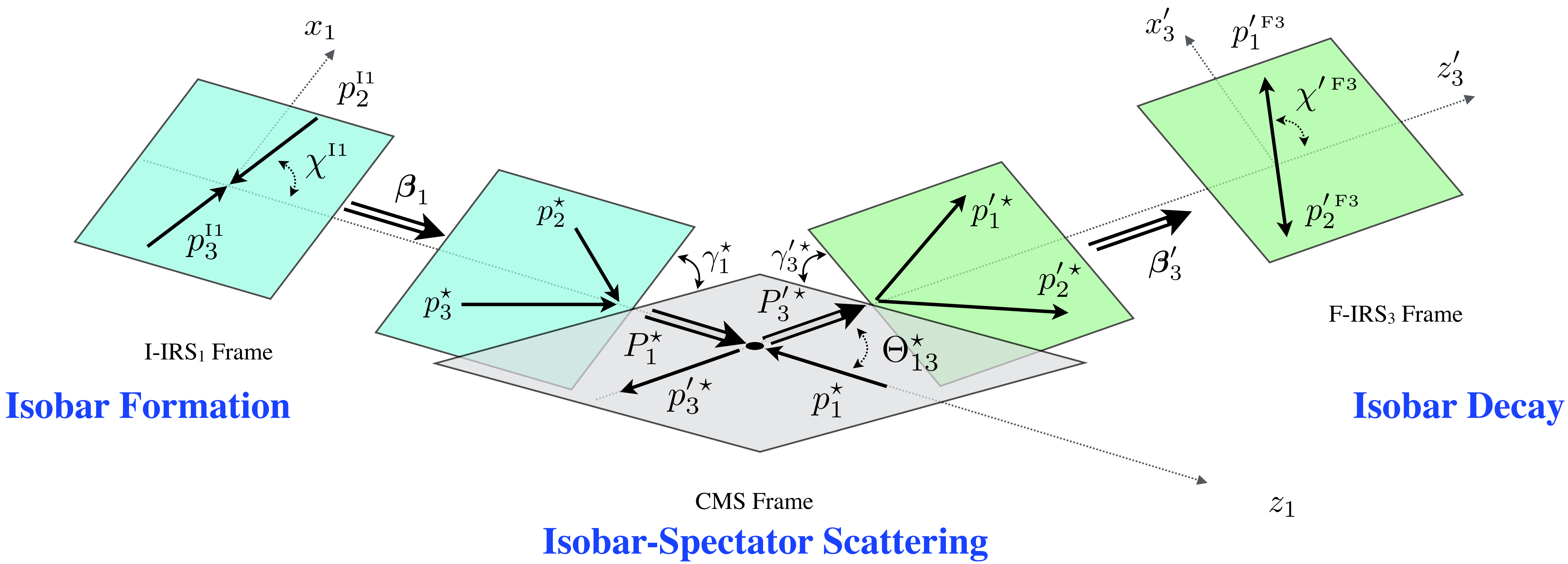
- Analytic Extensions
  - Can we use the B-matrix to write dispersive representation?
  - Continual Investigation into OPE, Box, Triangle, Bubble

Back-up



π^+
u
$\bar{d}$
π^-
d
$\bar{u}$
π^0
u
$\bar{u}$
ω
u
$\bar{u}$
π^+
u
$\bar{d}$
π^-
d
$\bar{u}$
π^0
u
$\bar{u}$

Kinematics



x_1
p_2^{I1}
χ^{I1}
p_3^{I1}
$\boldsymbol{\beta}_1$
$p_2^\star$
$p_3^\star$
$\gamma_1^\star$
$\gamma_3'^\star$
$P_3'^\star$
$P_1^\star$
$\Theta_{13}^\star$
$p_3'^\star$
$p_1^\star$
$p_1'^\star$
$p_2'^\star$
$\boldsymbol{\beta}_3'$
x_3'
$p_1'^{F3}$
z_3'
χ'^{F3}
$p_2'^{F3}$
F-IRS3 Frame
I-IRS1 Frame
Isobar Formation
Isobar Decay
CMS Frame
z_1
Isobar-Spectator Scattering

Unitarity Relations

Disconnected 2→2 Unitarity Relation

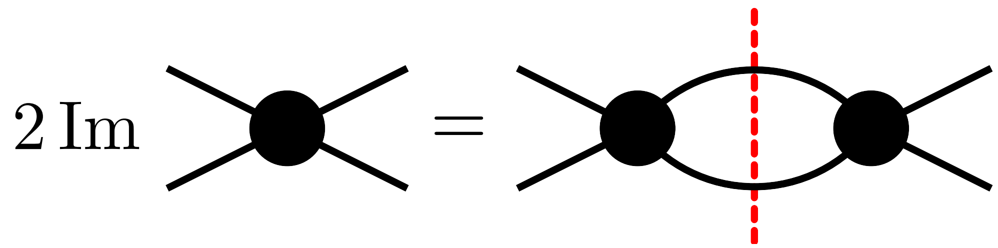
$$\mathrm{Im}\,\mathcal{F}_j(\mathbf{p}';\mathbf{p}) = \frac{|\mathbf{q}^{(j)}|}{32\pi^2\sqrt{\sigma_j}} \int d\Sigma''^{\,(j'')}\, \mathcal{F}_j^*(\mathbf{p}'';\mathbf{p}')\mathcal{F}_j(\mathbf{p}'';\mathbf{p})\Theta(\sigma_j - \sigma_j^{(th)})$$

Connected 3→3 Unitarity Relation

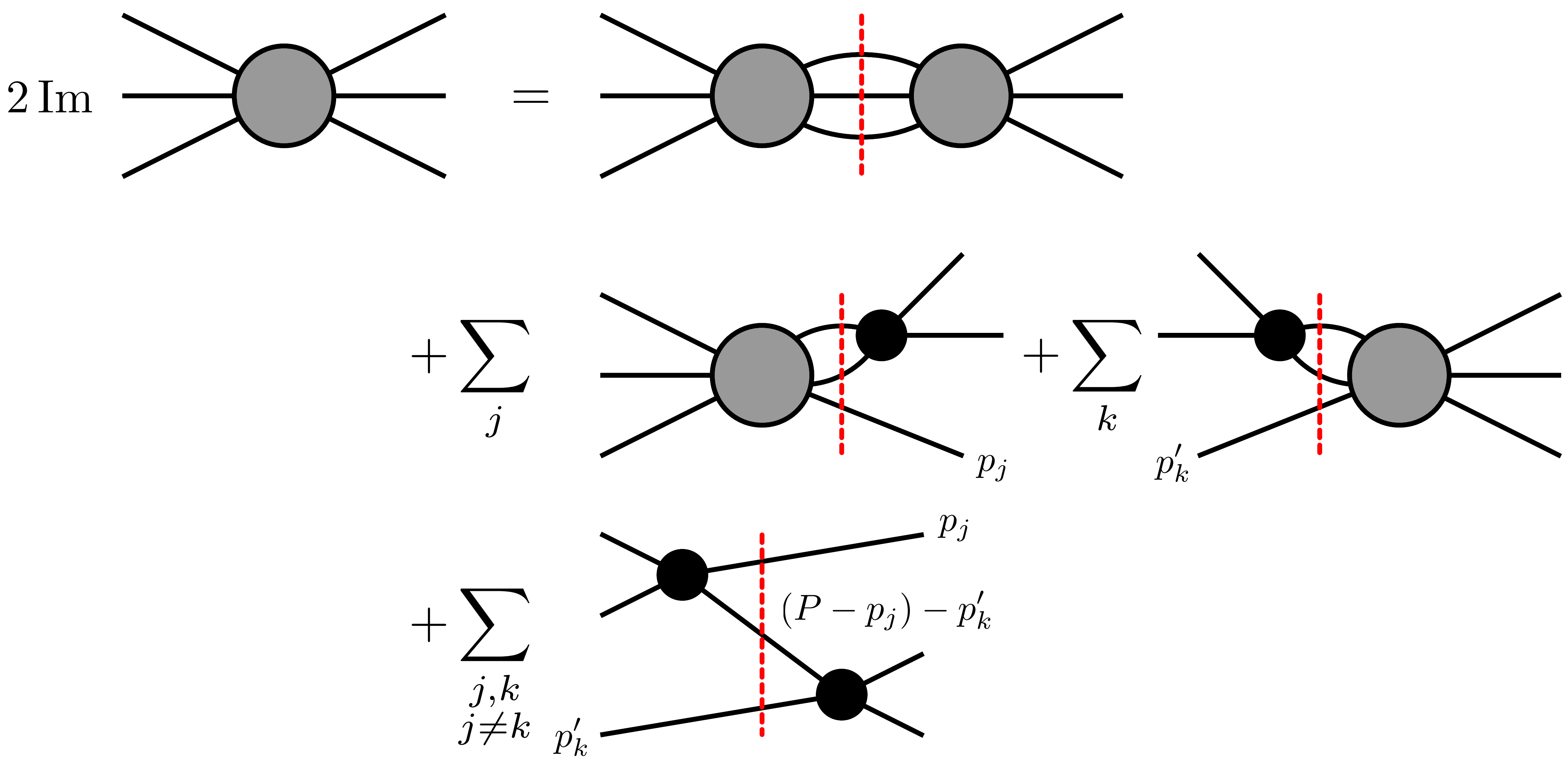
$$\begin{aligned}
\mathrm{Im}\,\mathcal{A}(\mathbf{p}';\mathbf{p}) = {} & \frac{1}{2(2\pi)^5}\int \frac{d^3\mathbf{p}_1''}{2\omega(\mathbf{p}_1'')}\frac{d^3\mathbf{p}_2''}{2\omega(\mathbf{p}_2'')}\frac{d^3\mathbf{p}_3''}{2\omega(\mathbf{p}_3'')}\,\delta^{(4)}(P''-P)\,\mathcal{A}^*(\mathbf{p}'';\mathbf{p}')\mathcal{A}(\mathbf{p}'';\mathbf{p}) \\
& + \sum_k \frac{|\mathbf{q}'^{\,(k')}|}{32\pi^2\sqrt{\sigma_k'}}\int d\Sigma''^{\,(k'')}\,\mathcal{F}_k^*(\mathbf{p}'';\mathbf{p}')\mathcal{A}(\mathbf{p}'';\mathbf{p})|_{\mathbf{p}_k''=\mathbf{p}_k'}\Theta(\sigma_k' - \sigma_k^{(th)}) \\
& + \sum_j \frac{|\mathbf{q}^{(j)}|}{32\pi^2\sqrt{\sigma_j}}\int d\Sigma''^{\,(j'')}\,\mathcal{A}^*(\mathbf{p}'';\mathbf{p}')|_{\mathbf{p}_j''=\mathbf{p}_j}\mathcal{F}_j(\mathbf{p}'';\mathbf{p})\Theta(\sigma_j - \sigma_j^{(th)}) \\
& + \sum_{\substack{j,k \\ j\neq k}} \pi\,\delta(u_{jk}-\mu^2)\,\mathcal{F}_k^*(\mathbf{p}'';\mathbf{p}')|_{\mathbf{p}_j''=\mathbf{p}_j}\mathcal{F}_j(\mathbf{p}'';\mathbf{p})|_{\mathbf{p}_k''=\mathbf{p}_k'}\,,
\end{aligned}$$

Unitarity Relations

Disconnected 2→2 Unitarity Relation



Connected 3→3 Unitarity Relation



Unitarity of B-Matrix