Form Invariance, Topological Fluctuations and Mass Gap of Yang-Mills Theory

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Outline

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2. 3D
3. 4D
4. Topological Fluctuations
5. Yang-Mills Mass Gap Problem

How to solve Yang-Mills equation?

For $SU(N)$ gauge field in Euclidean space, can we find a systematic way to solve the Yang-Mills equation

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Can we solve it?

Form invariance condition

One of Wightman's axioms on QFT:

$$(O^{-1})_\mu{}^\nu A_\nu(O\,x) = V^{-1} A_\mu(x)\, V + V^{-1}\partial_\mu V$$

A_μ: a gauge field
O: a Lorentz transformation
V: a gauge transformation

For the flat spacetime, i.e., O has rigid parameters:
Theorem

$$\left(O^{-1}\right)_\mu{}^\nu A_\nu(O\,x) = V^{-1} A_\mu(x)\, V$$

where V has only rigid parameters.

C. H. Gu, Phys. Rept. **80**, 251 (1981).

$SU(2)$ gauge field in the 3-dimensional Euclidean space

$$A_\mu = p(\tau)\left(U^{-1}\partial_\mu U\right) , \tag{2}$$

where $\tau \equiv x_\mu x^\mu$, and U is an $SU(2)$ group element.

$$U = \exp\left[T_a \psi^a(x)\right] = \exp\left[T_a \frac{\psi^a(x)}{|\psi(x)|}|\psi(x)|\right] = \exp\left[T_a\, \omega^a{}_\mu\, \hat{n}^\mu |\psi(x)|\right]$$
$$= \exp\left[T_a\, \omega^a{}_\mu\, \hat{n}^\mu \theta(\tau)\right] , \tag{3}$$

with

$$\hat{n}^\mu \equiv x^\mu/|x| , \quad \theta(\tau) \equiv |\psi(x)| , \quad T^a = \sigma^a/2i . \tag{4}$$

We have defined a matrix $\omega^a{}_\mu$ to connect the two unit vectors $\hat{n}^\mu$ and $\psi^a(x)/|\psi(x)|$ in different spaces. The Ansatz (2) must satisfy the form invariance condition:

$$\left(O^{-1}\right)_\mu{}^\nu A_\nu(O\,x) = V^{-1} A_\mu(x)\, V ,$$

where O is a constant $SO(3)$ group element, and V is a constant $SU(2)$ group element.

Form invariance

In paper we proved that ω is restricted to be a constant $O(3)$ group element. If we assume that $\det \omega = 1$, the Ansatz (2) becomes

$$A_\mu = p(\tau)\left(U^{-1}\partial_\mu U\right) , \quad U = \exp\left[T_a\, \omega^a{}_\mu\, \hat{n}^\mu \theta(\tau)\right] , \tag{5}$$

with a constant $SO(3)$ group element ω. With a proper choice of the generators T_a, we can write the matrix ω as

$$\omega = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6}$$

and therefore

$$U = \exp\left[T_i\, \hat{n}^i \theta(\tau)\right] . \tag{7}$$

Boundary and topological charge

For the 3-dimensional Euclidean space, the appropriate topological term is the Chern-Simons term:

$$S_{CS} = \frac{ik}{4\pi} \int d^3x \, \mathrm{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) . \tag{8}$$

In general, S_{CS} takes values in $\mathbb{R}/2\pi\mathbb{Z}$. If we require that S_{CS} takes values in $2\pi\mathbb{Z}$, it will not affect the quantum Yang-Mills theory in the path integral.

$$S_{CS} = \frac{ik}{4\pi} \int d^3x \, \left(\frac{2}{3} p^3 - p^2 \right) \mathrm{Tr}(U^{-1}dU) \wedge (U^{-1}dU) \wedge (U^{-1}dU) \, , \tag{9}$$

which is essentially a Wess-Zumino term. We can define

$$S_{CS} = 2\pi i k B \, , \tag{10}$$

where B is the winding number.

Boundary and topological charge

$$B = \frac{3}{2\pi^2} \sum_{\beta} \left(\frac{2}{3} p_\beta^3 - p_\beta^2 \right) \left(\theta_\beta - \frac{1}{2} \sin 2\theta_\beta \right) \int dS_\beta \, \hat{n} \cdot (\partial_1 \hat{n} \times \partial_2 \hat{n}) \, ,$$

where β denotes the singular points, for instance $\tau = 0$ and $\tau = \infty$ in our case, and

$$\frac{1}{4\pi} \int dS \, \hat{n} \cdot (\partial_1 \hat{n} \times \partial_2 \hat{n}) = \pm 1 \, , \tag{11}$$

where the contributions from $\tau = 0$ and $\tau = \infty$ have an opposite sign due to the boundary orientation. Since B should also be an integer, the boundary values of p and θ at the singular points will be constrained.

Boundary and topological charge

We can list the possible boundary conditions

Winding number B	$p\|_{\tau=0}$	$p\|_{\tau=\infty}$	$\theta\|_{\tau=0}$	$\theta\|_{\tau=\infty}$
0	0	0	π	π
0	1/2	1/2	π	π
0	1	1	π	π
...	...	...	...	...

In sum, we have the ansatz

$$A_\mu = p(\tau)\left(U^{-1}\partial_\mu U\right)\ , \quad U = \exp\left[T_i\,\hat{n}^i\theta(\tau)\right]\ , \tag{12}$$

with the possible boundaries listed above.

Classical solutions

One can easily solve the Yang-Mills equation

$$D_\mu F_{\mu\nu} = 0\,, \tag{13}$$

where we obtain $\theta = \pi$ and

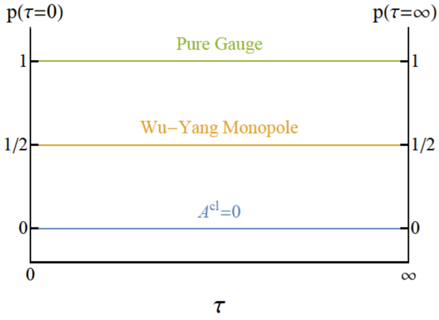


Figure: Spherically symmetric solutions to 3D Yang-Mills equation.

$SU(2)$ gauge field in the 4-dimensional Euclidean space

$$A_\mu = p(\tau, x_4)\left(U^{-1}\partial_\mu U\right) , \quad U = \exp\left[T_i\,\hat{n}^i\,\theta(\tau, x_4)\right] , \tag{14}$$

where $\tau \equiv x^\mu x_\mu$, and μ runs from 1 to 4, while i runs from 1 to 3. The functions $p(\tau, x_4)$ and $\theta(\tau, x_4)$ depend on both τ and x_4, while $\hat{n}^i$ is a unit vector depending only on x_1, x_2 and x_3:

$$\hat{n}^i \equiv \frac{x^i}{|x|} , \tag{15}$$

where $|x|^2 \equiv \sum_{i=1}^{3} x^i x_i$. The form invariance condition

$$(\Lambda^{-1})_\mu{}^\nu\, A_\nu(\Lambda\, x) = V^{-1}\, A_\mu(x)\, V$$

where Λ is an $SO(4)$ Lorentz transformation and V is an $SU(2)$ gauge transformation, both of which have parameters independent of x.

Fix θ

Let us consider a special case $\mu = 4$

$$
\begin{aligned}
&\left(\Lambda^{-1} A(\Lambda x)\right)^{a}_{\mu=4} T_a \\
=&p' \left[\hat{n}'_a(\partial_4 \theta') + \sin\theta' (\partial_4 \hat{n}'_a)\right] T_a - p'\, T_a\, \epsilon_{abc}\, (1 - \cos\theta') \frac{x^b}{|\Lambda x|} \frac{\frac{\varphi_c \sin|\varphi|}{|\varphi|}}{|\Lambda x|} \,.
\end{aligned}
$$

After a gauge transformation, it has the expression

$$
V^{-1} A^a_{\mu=4} T_a V = p \left[\frac{\psi_a \psi_b}{|\psi|^2} (1 - \cos|\psi|) + \delta_{ab} \cos|\psi| - \epsilon_{abc} \sin|\psi| \frac{\psi_c}{|\psi|} \right] T_b \hat{n}_a \partial_4 \theta \,, \tag{16}
$$

where

$$
V = \exp\left(\psi^a\, T_a\right) \,, \tag{17}
$$

p and θ

By comparing the terms $\sim \epsilon_{abc}$, one obtains

$$\psi_c = \pm\varphi_c\,, \quad -(1-\cos\theta')\frac{p'}{|\Lambda x|^2} = \pm\frac{p}{|x|}\partial_4\theta\,. \tag{18}$$

For the special case $\Lambda = 1$

$$\begin{aligned} &-(1-\cos\theta)\frac{1}{|x|^2} = \pm\frac{1}{|x|}\partial_4\theta \\ \Rightarrow\quad &\frac{1}{|x|} = \pm\partial_4\cot\left(\frac{\theta}{2}\right) \\ \Rightarrow\quad &\cot\left(\frac{\theta}{2}\right) = \pm\frac{x_4}{|x|}\pm f(|x|)\,, \end{aligned} \tag{19}$$

where f is an arbitrary smooth function. In the paper, we proved that $f = 0$. Thus,

$$\cot\left(\frac{\theta}{2}\right) = \pm\frac{x_4}{|x|} \quad \text{and} \quad p = p(\tau)\,. \tag{20}$$

Ansatz

We choose $\cot(\theta/2) = x_4/|x|$ and the form invariant Ansatz A_μ is given by

$$A_\mu = p(\tau)\left[\frac{x_4 - 2\left(T^a x_a\right)}{\sqrt{\tau}}\right]\partial_\mu\left[\frac{x_4 + 2\left(T^b x_b\right)}{\sqrt{\tau}}\right] = 2\frac{p(\tau)}{\tau}\eta_{a\mu\nu}x_\nu T^a\,, \tag{21}$$

where $\eta_{a\mu\nu}$ is the 't Hooft symbol

$$\eta_{i\mu\nu} = -\mathrm{tr}\left(M_i M_{\mu\nu}\right) = -(M_i)_{mn}(M_{\mu\nu})_{nm} = 2(M_i)_{\mu\nu} = \left(\epsilon_{i\mu\nu4} + \delta_{i\mu}\delta_{\nu4} - \delta_{i\nu}\delta_{\mu4}\right),$$
$$\bar{\eta}_{i\mu\nu} = -\mathrm{tr}\left(N_i M_{\mu\nu}\right) = -(N_i)_{mn}(M_{\mu\nu})_{nm} = 2(N_i)_{\mu\nu} = \left(\epsilon_{i\mu\nu4} - \delta_{i\mu}\delta_{\nu4} + \delta_{i\nu}\delta_{\mu4}\right). \tag{22}$$

Topological charge and boundary conditions

For the 4D Yang-Mills theory

$$k = -\frac{1}{16\pi^2}\int d^4x\,\mathrm{Tr}\left[F^{\mu\nu}(*F_{\mu\nu})\right] \tag{23}$$

is an integer-valued quantity.
Hence, the integral becomes a surface integral, and only boundaries contribute to it.

$$k = -\frac{1}{8\pi^2}\oint_{S^3_\beta} d\Omega_\mu\,\epsilon^{\mu\nu\rho\sigma}\left(\frac{2}{3}p^3 - p^2\right)\mathrm{Tr}\left[\left(U^{-1}\partial_\nu U\right)\left(U^{-1}\partial_\rho U\right)\left(U^{-1}\partial_\sigma U\right)\right]\,, \tag{24}$$

where the surface S^3_β surrounds the singular point β, and the radius of the sphere can be taken to be very small. Hence, the factor $\frac{2}{3}p^3 - p^2$ has a constant value $\left(\frac{2}{3}p^3 - p^2\right)_\beta$ in the small sphere and can be brought outside the integration.

Topological charge and boundary conditions

We use x^μ and $\xi^i(x)$ $(i = 1, 2, 3)$ to denote the spacetime coordinates and the group coordinates respectively. Using

$$\mathrm{Tr}\left[\left(U^{-1}\partial_\nu U\right)\left(U^{-1}\partial_\rho U\right)\left(U^{-1}\partial_\sigma U\right)\right] = \frac{\partial\xi^i}{\partial x^\nu}\frac{\partial\xi^j}{\partial x^\rho}\frac{\partial\xi^k}{\partial x^\sigma}\,\mathrm{Tr}\left[\left(U^{-1}\partial_i U\right)\left(U^{-1}\partial_j U\right)\left(U^{-1}\partial_k U\right)\right], \tag{25}$$

obtain

$$dk = \frac{3}{16\pi^2}\left(\frac{2}{3}p^3 - p^2\right)_\beta (\det e)\, d^3\xi\,, \tag{26}$$

where

$$U^{-1}\partial_i U = e_i^a(\xi)\, T_a\,, \tag{27}$$

and $(\det e)d^3\xi$ is the Haar measure on the group manifold. For example

$$\frac{1}{16\pi^2}\int_{S^3_{|x|\to\infty}} (\det e)\, d^3\xi = 1\,. \tag{28}$$

Ansatz

In sum, we have

$$A_\mu = 2\frac{p(\tau)}{\tau}\eta_{a\mu\nu}x_\nu T^a\,, \tag{29}$$

with the possible boundary conditions listed as

Winding Number k	$p\|_{\tau=0}$	$p\|_{\tau=\infty}$
0	0	0
0	1/2	1/2
0	1	1
1	0	1
-1	1	0
...	...	...

Table: Boundary conditions in 4D.

Classical solutions

Obtain the solution

- $p = 1/2$: Meron
- $p = \frac{\tau}{\tau+c}$: Instanton
- $p = \frac{c}{\tau+c}$: Anti-Instanton
- $p = 1$ and $p = 0$: Pure gauge and zero

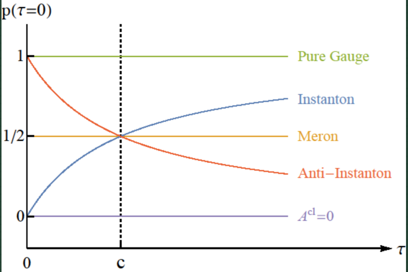


Figure: Spherically symmetric solutions to 4D Yang-Mills equation.

Classical solutions

If we adopt a new coordinate introduced by the conformal transformation

$$\zeta = \frac{1}{2}\frac{\tau - c}{\tau + c}, \tag{30}$$

then all the classical solutions can be plotted in the new coordinate

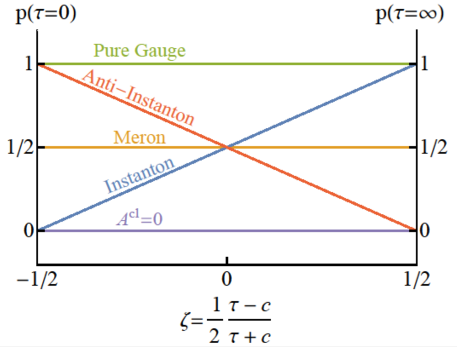


Figure: Spherically symmetric solutions to 4D Yang-Mills equation.

Discussion ...

- $$A^a_\mu = c_1(\tau)\eta_{a\mu\nu}x_\nu + c_2(\tau)\bar{\eta}_{a\mu\nu}x_\nu \tag{31}$$

Discussion ...

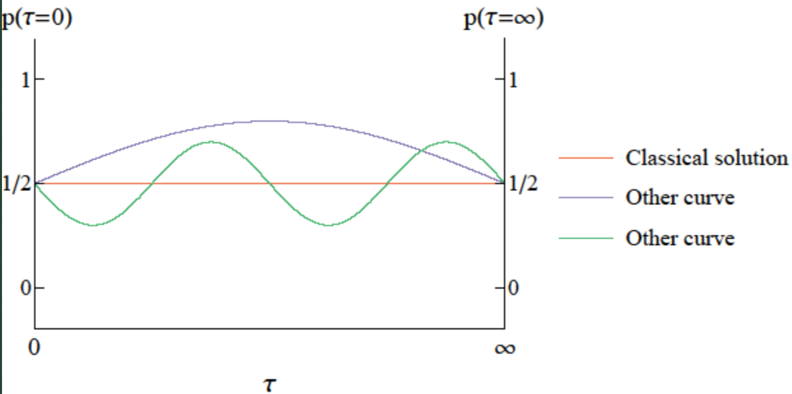
- $$A^a_\mu = c_1(\tau)\eta_{a\mu\nu}x_\nu + c_2(\tau)\bar{\eta}_{a\mu\nu}x_\nu \qquad (31)$$
- General case?
  - Subspace and sub-gauge-group space

$$(\Lambda^{-1})_\mu{}^\nu \, A_\nu(\Lambda\, x) = V^{-1}\, A_\mu(x)\, V$$

  - Lowest winding numbers

Topological Fluctuations

Take 3D Wu-Yang monopole solution as an example:



Topological modes:

Form invariance:

$$(O^{-1})_{\mu}{}^{\nu}\, A_{\nu}^{\text{top}}(O\,x) = V^{-1}\, A_{\mu}^{\text{top}}(x)\, V$$

Any configuration:

$$A_{\mu}^{\text{top}}(p_0 + \widetilde{p}, \theta_0 + \widetilde{\theta})$$

$\widetilde{p}$, $\widetilde{\theta}$: topological fluctuations

Effective lagrangian:

$$\frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a \bigg|_{\text{top}} = \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a \bigg|_{\text{cl}} + 4 \big|\partial_\tau \widetilde{\psi}\big|^2 - \frac{1}{\tau^2}\big|\widetilde{\psi}\big|^2 + \frac{1}{2\tau^2}\big|\widetilde{\psi}\big|^4 , \qquad (32)$$

Topological modes:

4D:

$$A_\mu^{\rm top}(p_0+\widetilde{p})$$

Effective lagrangian:

$$\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu}\Big|_{\rm top}=\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu}\Big|_{\rm cl}+24\left[\left(\partial_\tau\widetilde{\phi}\right)^2-\frac{1}{2\tau^2}\left(\widetilde{\phi}\right)^2+\frac{1}{\tau^2}\left(\widetilde{\phi}\right)^4\right], \tag{33}$$

Two problems?

Take 3D for example:

$$\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu}\Big|_{\rm top} = \frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu}\Big|_{\rm cl} + \frac{1}{\tau}\big|\partial_\mu\widetilde{\psi}\big|^2 - \frac{1}{\tau^2}\big|\widetilde{\psi}\big|^2 + \frac{1}{2\tau^2}\big|\widetilde{\psi}\big|^4, \qquad (34)$$

We encounter two problems:

- $1/\tau \longrightarrow$ apparent divergent
- $\widetilde{\psi} = \widetilde{\psi}(\tau)$

Topology

Topological boundary condition:

$$\widetilde{\psi}(\tau = 0) \longrightarrow 0\,, \qquad \text{and} \qquad \widetilde{\psi}(\tau = \infty) \longrightarrow 0\,. \tag{35}$$

Because it's fluctuations

$$\widetilde{\psi} \longrightarrow \quad \begin{array}{l} \text{translational invariant for most of the space} \\ \text{except} \quad \tau = 0 \quad \text{and} \quad \tau = \infty\,. \end{array} \tag{36}$$

Full theory & Effective theory

We would like to make the shift:

$$\widetilde{\psi}\left((x-x_0)^2\right) \longrightarrow \widetilde{\psi}(x^2)\,. \tag{37}$$

However, the topological fluctuations are constrained by the topological boundary conditions

$$\begin{aligned}&\int d^3x \left(\frac{1}{\tau}|\partial_\mu\widetilde{\psi}|^2 - \frac{1}{\tau^2}|\widetilde{\psi}|^2 + \frac{1}{2\tau^2}|\widetilde{\psi}|^4\right)\\ =&\left(\int_{\text{near } x_0} d^3x + \int_{\text{near } 0} d^3x + \int_{\text{else}} d^3x\right)\left(\frac{1}{\tau}|\partial_\mu\widetilde{\psi}|^2 - \frac{1}{\tau^2}|\widetilde{\psi}|^2 + \frac{1}{2\tau^2}|\widetilde{\psi}|^4\right),\end{aligned} \tag{38}$$

Full theory & Effective theory

In sum, we have

$$\begin{aligned}
&\frac{1}{g^2}\int d^3x\,\left(\frac{1}{\tau}\left|\partial_\mu\widetilde{\psi}(x-x_0)\right|^2-\frac{1}{\tau^2}\left|\widetilde{\psi}(x-x_0)\right|^2+\frac{1}{2\tau^2}\left|\widetilde{\psi}(x-x_0)\right|^4\right)\\
=&\frac{1}{g^2}\int d^3x\,\left(\frac{1}{\tau}\left|\partial_\mu\widetilde{\psi}(x)\right|^2-\frac{1}{\tau^2}\left|\widetilde{\psi}(x)\right|^2+\frac{1}{2\tau^2}\left|\widetilde{\psi}(x)\right|^4\right)\\
&-\frac{1}{g^2}\int_{\text{near } x_0} d^3x\,\left(\frac{1}{\tau}\left|\partial_\mu\widetilde{\psi}(x)\right|^2-\frac{1}{\tau^2}\left|\widetilde{\psi}(x)\right|^2+\frac{1}{2\tau^2}\left|\widetilde{\psi}(x)\right|^4\right),
\end{aligned} \tag{39}$$

The left-hand side of this equation is finite. The second term on the right-hand side is divergent and gives the difference of the integral (39) near x_0 before and after the shift (38), hence it can be viewed as a counter-term, that cancels the divergence of the first term on the right-hand side.

Effective theory

$$\begin{aligned}
\langle S\rangle_{x_0} &= \frac{1}{g^2}\int \frac{d^3x_0}{V}\int d^3x \left[\frac{1}{\tau}\left|\partial_\mu \widetilde{\psi}(\tau)\right|^2 + \frac{1}{2\tau^2}\left(|1+\widetilde{\psi}(\tau)|^2 - 1\right)^2\right] \\
&= \frac{1}{g^2}\int \frac{d^3x_0}{V}\int d^3x \left[\frac{1}{\tau}\left|\partial_\mu \widetilde{\psi}(\tau)\right|^2 + \frac{1}{2\tau^2}\left(\widetilde{\psi}^\dagger(\tau) + \widetilde{\psi}(\tau) + |\widetilde{\psi}(\tau)|^2\right)^2\right] \\
&= \frac{1}{g^2}\int \frac{d^3x_0}{V}\int d^3x \left[\frac{1}{\tau}\left|\partial_\mu \widetilde{\psi}(\widetilde{\tau})\right|^2 + \frac{1}{2\tau^2}\left(\widetilde{\psi}^\dagger(\widetilde{\tau}) + \widetilde{\psi}(\widetilde{\tau}) + |\widetilde{\psi}(\widetilde{\tau})|^2\right)^2\right] + (\text{counter}) \\
&= \frac{1}{g^2}\int_{\ell_{\rm top}^2}^{L_{\rm top}^2} \frac{2\pi\sqrt{\tau}\,d\tau}{V}\int d^3x \left[\frac{1}{\tau}\left|\partial_\mu \widetilde{\psi}(\widetilde{\tau})\right|^2 + \frac{1}{2\tau^2}\left(\widetilde{\psi}^\dagger(\widetilde{\tau}) + \widetilde{\psi}(\widetilde{\tau}) + |\widetilde{\psi}(\widetilde{\tau})|^2\right)^2\right] \\
&\approx \frac{3L_{\rm top}}{g^2L^3}\int d^3x \left[|\partial_{|x|}\widetilde{\psi}(|x|)|^2 + m_{3D}^2(|1+\widetilde{\psi}(|x|)|^2 - 1)^2\right] \\
&= \frac{3L_{\rm top}}{g^2L^3}\int d^3x \left[|\partial_{|x|}\Psi(|x|)|^2 + m_{3D}^2(|\Psi(|x|)|^2 - 1)^2\right] ,
\end{aligned} \tag{40}$$

where

$$m_{3D}^2 \equiv \frac{1}{2\ell_{\rm top}L_{\rm top}} . \tag{41}$$

Effective theory

$$\mathcal{L}_{\rm eff} \sim |\partial_{|x|}\Psi(|x|)|^2 + m_{3D}^2(|\Psi(|x|)|^2 - 1)^2 \qquad (42)$$

Boundary condition:

$$|\Psi|(|x| = 0) = |\Psi|(|x| = \infty) = 1\,. \qquad (43)$$

EOM:

$$\partial_{|x|}^2|\Psi| + \frac{2}{|x|}\,\partial_{|x|}|\Psi| - 2\,m_{3D}^2\,(|\Psi|^2 - 1)|\Psi| = 0\,. \qquad (44)$$

Effective theory

For 4D:

$$\mathcal{L}_{\text{eff}} \sim \left(\partial_{|x|} q(|x|)\right)^2 + 4\, m_{4D}^2 \left(q^2(|x|) - \frac{1}{4}\right)^2 \tag{45}$$

where

$$m_{4D}^2 \equiv \frac{2\log(L_{\text{top}}/\ell_{\text{top}})}{L_{\text{top}}^2}\,. \tag{46}$$

Boundary condition:

$$q(|x|=0) = q(|x|=\infty) = \pm\frac{1}{2}\,. \tag{47}$$

EOM:

$$2\,\Delta\, q - 16\, m_{4D}^2\, q \left(q^2 - \frac{1}{4}\right) = 0$$

with

$$\Delta\, q = \partial_{|x|}^2\, q + \frac{3}{|x|}\, \partial_{|x|} q$$

Effective theory

In general for D-dimensions:

$$\mathcal{L}_{\text{eff}} \sim \left(\partial_{|x|} u\right)^2 + \frac{1}{2}(u^2 - 1)^2 \tag{48}$$

$$\Delta u - (|u|^2 - 1)u = 0$$

$$\Delta u = \partial_{|x|}^2 u + \frac{D-1}{|x|}\,\partial_{|x|} u$$

$$\lim_{|x|\to 0} |u(x)| = \lim_{|x|\to\infty} |u(x)| = 1$$

Yang-Mills Mass Gap Problem

Mass gap Δ: ('06 Jaffe, Witten)
The Hamiltonian H has no spectrum in the interval $(0, \Delta)$ for some $\Delta > 0$.

The mass gap problem of Yang-Mills theory:
"Prove that for any compact simple gauge group G, a non-trivial quantum Yang-Mills theory exists on $\mathbb{R}^4$ and has a mass gap $\Delta > 0$."

Some Previous Attempts

- '77 Polyakov:
  (2+1)D Georgi-Glashow model has mass gap.
  In 4D YM instanton background cannot provide mass gap.
- Varying instanton profile can give glueballs mass gap at classical level.
  ('84 Diakonov, Petrov)
- With some supersymmetries,
  e.g. Seiberg-Witten theory.
- ...

Two-Point Correlation Function

Consider the operator:

$$\epsilon \equiv \frac{1}{4} F^a_{\mu\nu}\, F^a_{\mu\nu}$$

3D:

$$\langle \epsilon(-\vec{d}\,)\, \epsilon(\vec{d}\,) \rangle_{\text{Lowest State}} \approx 16\, C_1^4 \, \left(\frac{e^{-\, m_{3D}\, d}}{d} \right)^8$$

with

$$m^2_{3D} \equiv \frac{1}{2\ell_{\text{top}} L_{\text{top}}}$$

4D:

$$\langle \epsilon(-\vec{d}\,)\, \epsilon(\vec{d}\,) \rangle_{\text{Lowest State}} \approx \frac{36\, C_1^4\, \pi^2}{m^2_{4D}\, d^2} \, \left(\frac{e^{-\, m_{4D}\, d}}{d} \right)^8$$

with

$$m^2_{4D} \equiv \frac{2 \log(L_{\text{top}}/\ell_{\text{top}})}{L^2_{\text{top}}} \,. \tag{49}$$

Nonlinear Schrödinger Equation

In general for D-dimensions:

$$\Delta u - (|u|^2 - 1)u = 0$$

$$\Delta u = \partial^2_{|x|} u + \frac{D-1}{|x|}\,\partial_{|x|} u$$

$$\lim_{|x| \to 0} |u(x)| = \lim_{|x| \to \infty} |u(x)| = 1$$

Mass Gap of YM in Our Approach

Use the mass gap of 3D, 4D NLS ('15 Bao, Ruan) .
$\langle S \rangle_{x_0}$: the effective action evaluated at the first excited state on the trivial vacuum background. Assume $L \approx L_{\text{top}}$.

- For the flat space $\mathbb{R}^D$ $(D = 3, 4)$ with finite size:

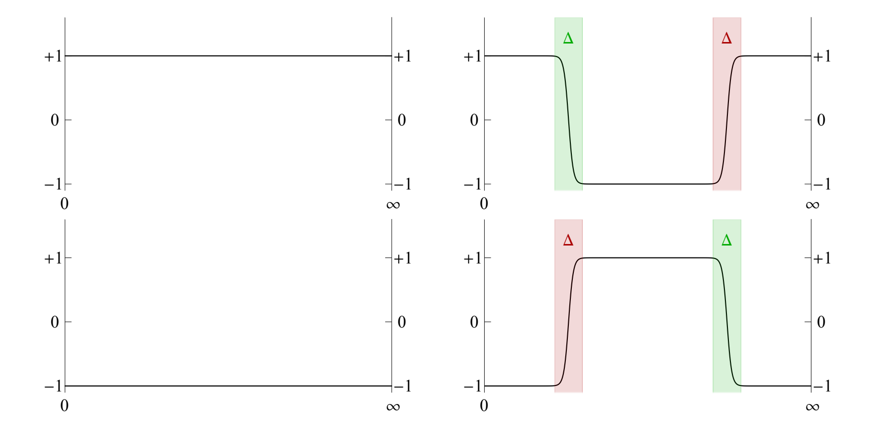
$$\langle S \rangle_{x_0} \propto \begin{cases} \frac{1}{g^2} \frac{1}{\sqrt{L_{\text{top}} \ell_{\text{top}}}} \,, & \text{for 3D}\,; \\ \frac{1}{g^2} \log\left(\frac{L_{\text{top}}}{\ell_{\text{top}}} \right) \,, & \text{for 4D}\,. \end{cases}$$

- For the sphere S^D $(D = 3, 4)$ with a radius R:

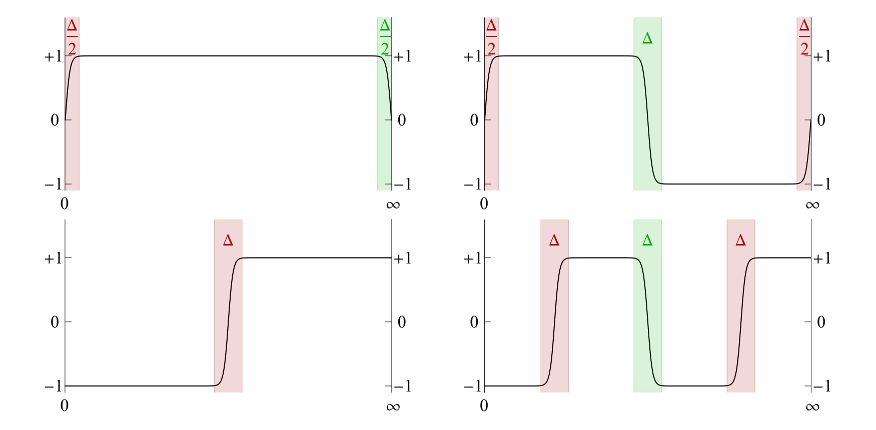
$$\langle S \rangle_{x_0} \propto \begin{cases} \frac{1}{g^2 R} \,, & \text{for 3D}\,; \\ \frac{1}{g^2} \log\left[\cot\left(\frac{\theta_0}{2} \right) \right] \,, & \text{for 4D}\,, \end{cases}$$

θ_0: physical cutoff on θ

Exact Spetrum of 1D NLS - 1



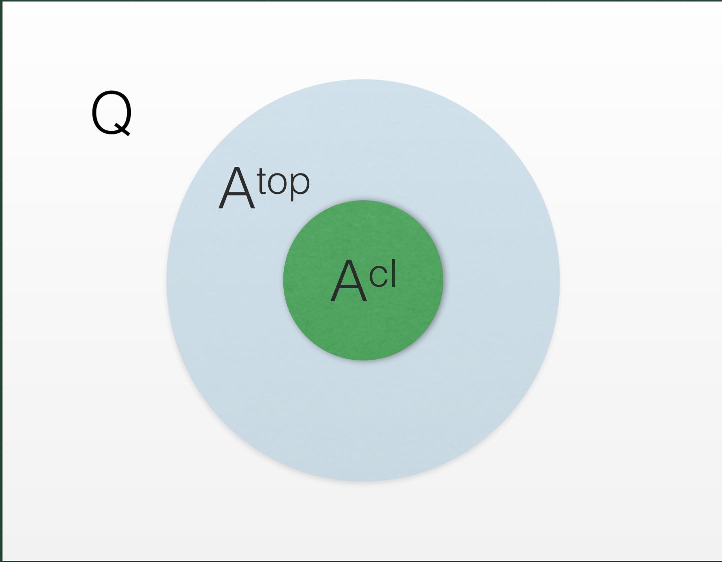
Exact Spetrum of 1D NLS - 2



Discussion ...

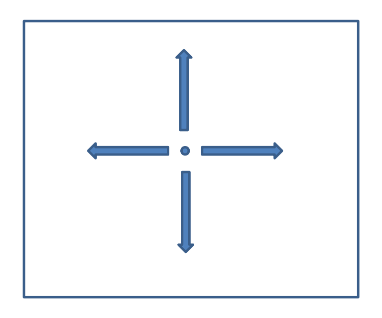
- Mass gap = quantum effects
- Low-energy physics

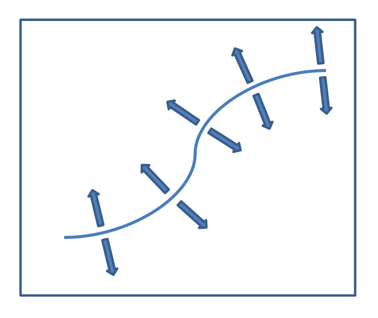
What's also in our paper



What's also in our paper

Start from different cores to probe the configuration space:





$$V = \Big(V_{\text{sol}} \cup (V_{\text{top}} \backslash V_{\text{sol}})\Big) \oplus V_{\text{top}}^{\perp}$$


Thank You!

Back up

Convention

The Lie algebra $\mathfrak{so}(4)$ has the generators given by

$$(M_{\mu\nu})_{mn} \equiv \delta_{\mu m}\delta_{\nu n} - \delta_{\mu n}\delta_{\nu m}\,, \tag{50}$$

satisfying

$$[M_{\mu\nu},\, M_{\rho\sigma}] = \delta_{\nu\rho}M_{\mu\sigma} + \delta_{\mu\sigma}M_{\nu\rho} - \delta_{\mu\rho}M_{\nu\sigma} - \delta_{\nu\sigma}M_{\mu\rho}\,. \tag{51}$$

Convention

Let us write down the generators in fundamental representations:

$$M_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \equiv J_1\,, \quad M_{14} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \equiv K_1\,,$$

$$M_{31} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \equiv J_2\,, \quad M_{24} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \equiv K_2\,,$$

$$M_{12} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \equiv J_3\,, \quad M_{34} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \equiv K_3\,. \tag{52}$$

$$[J_i,\, J_j] = -\epsilon_{ijk} J_k\,, \quad [K_i,\, K_j] = -\epsilon_{ijk} J_k\,, \quad [K_i,\, J_j] = -\epsilon_{ijk} K_k\,. \tag{53}$$

Define

$$M_i \equiv \frac{1}{2}(J_i + K_i)\,, \quad N_i \equiv \frac{1}{2}(J_i - K_i)\,. \tag{54}$$

The (anti-)commutation relations are given by

$$[M_i,\, M_j] = -\epsilon_{ijk} M_k\,, \quad [N_i,\, N_j] = -\epsilon_{ijk} N_k\,, \quad [M_i,\, N_j] = 0\,, \tag{55}$$

$$\{M_i,\, M_j\} = -\frac{1}{2}\delta_{ij}\,, \quad \{N_i,\, N_j\} = -\frac{1}{2}\delta_{ij}\,. \tag{56}$$

Fix p and θ

A general Lorentz transformation generated by M_{i4} is given by

$$\Lambda_{\mu\nu} = \left(e^{\varphi_{14}\,M_{14}+\varphi_{24}\,M_{24}+\varphi_{34}\,M_{34}}\right)_{\mu\nu}$$
$$= \delta_{\mu\nu} + \frac{\varphi_\mu}{|\varphi|}\delta_{\nu 4}\sin|\varphi| - \frac{\varphi_\nu}{|\varphi|}\delta_{\mu 4}\sin|\varphi| - 2\frac{\varphi_\mu\varphi_\nu}{|\varphi|^2}\sin^2\left(\frac{|\varphi|}{2}\right) - 2\delta_{\mu 4}\delta_{\nu 4}\sin^2\left(\frac{|\varphi|}{2}\right),$$

where $\varphi_\mu \equiv (\varphi_{i4},\, \varphi_4 = 0)$ and $|\varphi| \equiv \sqrt{(\varphi_{i4})^2}$.

$$\Lambda^{-1} A_\mu(\Lambda x) = p\left(\tau, (\Lambda x)_4\right)\,\exp\left[-T_a\,\hat{n}'_a\,\theta\left(\tau, (\Lambda x)_4\right)\right]\frac{\partial}{\partial x^\mu}\exp\left[T_b\,\hat{n}'_b\,\theta\left(\tau, (\Lambda x)_4\right)\right], \quad (57)$$

where $\tau \equiv x^\mu x_\mu$, $\hat{n}'_i \equiv \frac{(\Lambda x)_i}{|\Lambda x|}$ and $|\Lambda x| \equiv \sqrt{(\Lambda x)_i\,(\Lambda x)_i}$.

Digression: Antiferromagnet Model

$$S[\vec{n}] = s\sum_{j=1}^{N}(-1)^j S_{WZ}[\vec{n}(j)] - \frac{Js^2}{2}\int_0^T dx_0 \sum_{j=1}^{N}\left(\vec{n}(j,x_0) - \vec{n}(j+1,x_0)\right)^2$$

split $\vec{n}$ into slowly varying mode $\vec{m}(j)$ + rapidly varying mode $\vec{l}(j)$:

$$\vec{n}(j) = \vec{m}(j) + (-1)^j a_0\,\vec{l}(j)$$

$$\vec{n}^2 = \vec{m}^2 = 1\,, \quad \vec{m}\cdot\vec{l} = 0$$

In the continuum limit:

$$\mathcal{L}(\vec{m},\vec{l}) = -2a_0 Js^2\vec{l}^2 + s\vec{l}\cdot(\vec{m}\times\partial_0\vec{m}) - \frac{a_0 Js^2}{2}(\partial_1\vec{m})^2 + \frac{s}{2}\vec{m}\cdot(\partial_0\vec{m}\times\partial_1\vec{m})$$

Integrating out the rapidly varying mode $\vec{l}$:

$$\mathcal{L}(\vec{m}) = \frac{1}{2g}\left(\frac{1}{v_s}(\partial_0\vec{m})^2 - v_s(\partial_1\vec{m})^2\right) + \frac{\theta}{8\pi}\epsilon_{\mu\nu}\vec{m}\cdot(\partial_\mu\vec{m}\times\partial_\nu\vec{m})$$