SCET, an introduction

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Introduction - motivation

Huada School on QCD 2016: QCD in the EIC Era

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Central China Normal University
Asia/Chongqing timezone

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Invited Lecturers

- Jian-Ping Chen: Physics in Electron-ion Collider Experiments
- $\boxed{\text{Christopher Lee: Soft Collinear Effective Theory in QCD}}$
- Jianwei Qiu: QCD in collisions with polarized beams
- C.-P.Yuan: Factroization and resummation in QCD
- Bowen Xiao: Small-x and gluon saturation

Introduction - background

Consider the problems that we deal with in text books (e.g. Peskin)

- Pair annihilation: $e^+e^- \to \mu^+\mu^-$ or $q\bar{q}$
- Compton scattering :$e^-\gamma \to e^-\gamma$
- masses of the particles are usually taken to be zero($m \approx 0$) to simplify the cross-section calculation.

Now look at some laboratory energy scales:

- LHC: $6.5\,TeV_{pp}$
- RHIC: $255\,GeV_{pp}$
- HERA, BEPC-II ...
- energy scales could range

Introduction - scales

Some typical processes:

- DIS, Drell-Yan...
- Higgs production, Top decay...

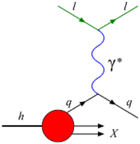
Particles involved:

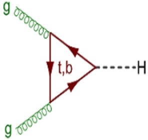
- e, u, $d < 10MeV$
- c, $\tau \sim 1GeV$
- t, $W^{\pm}$, Z, $H \sim 100GeV$

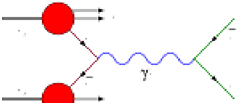
Energy scale involved:

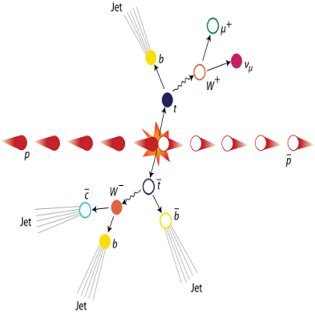
- center-of-mass energy
- collinear momentum
- transverse momentum

QFT is just the beginning...









Introduction - framework

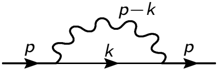
Consider a simple process:

- e.g. DIS $l(k) + A(p) \rightarrow l(k') + X$
- Large momentum transfer $Q^2 = -q^2 = (k - k')^2$
- Note the Infrared nature of QCD.
  Non-perturbative dynamics of the proton
- factorization (collinear): $k = (xP^+, 0, 0_T)$
- cross-section(σ) = hard$(\hat{H}) \otimes$ PDF$(\hat{\Phi})$
- PDF evolution using data fitting CTEQ$_{DGLAP}$
- hard kernel can be computed perturbatively

How do we deal with divergences?
- one-loop example:
  self-energy, vacuum polarization, vertex correction

Example: Self-Energy

Consider one-loop electron self-energy:



$$= \frac{i(\not{p} + m_0)}{p^2 - m_0^2}\left[-i\Sigma_2(p)\right]\frac{i(\not{p} + m_0)}{p^2 - m_0^2}$$

where we have (with dimensional regularization):

$$\Sigma_2(p) = -ie^2\mu^{2\epsilon}\int \frac{d^n k}{(2\pi)^n}\frac{\gamma^\mu(\not{k} + m_0)\gamma_\mu}{[k^2 - m_0^2 + i\epsilon][(p-k)^2 + i\epsilon]}$$

We then use Feynman parameterization to integrate out k:

$$\Sigma_2(p) = \frac{\alpha}{2\pi}(\epsilon - 1)\not{p}\left[\frac{-p^2}{4\pi\mu^2}\right]^{-\epsilon}\Gamma(\epsilon)B(2-\epsilon, 1-\epsilon)$$

divergent term comes from Γ and B function.
But why? Where does it come from? Soft? Collinear?
How will this divergence contribute at high order? Heavy mass?
details:[24]

method by regions

Let's consider one-loop self-energy with two different particle masses at zero external momentum, the integral:

$$I = \int_0^\infty dk \frac{k}{(k^2+m^2)(k^2+M^2)} = \frac{\ln \frac{M}{m}}{M^2 - m^2}$$

expanding in the large mass hierarchy limit ($m^2 \ll M^2$):

$$I = \frac{\ln \frac{M}{m}}{M^2} \left(1 + \frac{m^2}{M^2} + \frac{m^4}{M^4} + \cdots \right)$$

Is it analytic? What else can we do to this integral?
Expand the denominator without causing IR divergence?

method by regions

Let's introduce a cut-off Λ to the interval.

$$I = I_{1,\Lambda} + I_{2,\Lambda} = \int_0^\Lambda dk \frac{k}{(k^2+m^2)(k^2+M^2)} + \int_\Lambda^\infty dk \frac{k}{(k^2+m^2)(k^2+M^2)}$$

where the scale Λ is chosen to be $m \ll \Lambda \ll M$. Then

$$I_{1,\Lambda} = \int_0^\Lambda dk \frac{k}{(k^2+m^2)M^2}\left(1 - \frac{k^2}{M^2} + \frac{k^4}{M^4} + \cdots\right) = \frac{-\ln(\frac{m}{\Lambda})}{M^2} - \frac{\Lambda^2}{2M^4} + \mathcal{O}\left(\frac{\Lambda^4}{M^6}, \frac{m^2}{M^4}\ln\left(\frac{\Lambda}{m}\right)\right)$$

$$I_{2,\Lambda} = \int_\Lambda^\infty dk \frac{k}{k^2(k^2+M^2)}\left(1 - \frac{m^2}{k^2} + \frac{m^4}{k^4} + \cdots\right) = \frac{-\ln(\frac{\Lambda}{M})}{M^2} + \frac{\Lambda^2}{2M^4} + \mathcal{O}\left(\frac{\Lambda^4}{M^6}\ln\left(\frac{\Lambda}{m}\right)\right)$$

where I_1 represents the *low-energy* region, and I_2 represents the *high-energy* region. Combining the integrals, we have:

$$I = -\frac{1}{M^2}\ln\left(\frac{m}{M}\right) + \mathcal{O}\left(\frac{m^2}{M^4}\ln\left(\frac{M}{m}\right)\right)$$

which gives the same result as before. But where is Λ?
Introduction of cut-off scale is not necessary (gauge-symmetry).

dimensional regularization

Let's introduce regulator $k^{-\epsilon}$, where $\epsilon \to 0$.

$$I_1 = \int_0^\infty dk\ k^{-\epsilon} \frac{k}{(k^2+m^2)M^2}\left(1 - \frac{k^2}{M^2} + \frac{k^4}{M^4} + \cdots\right) = \frac{1}{M^2}\left(\frac{1}{\epsilon} - \ln m + \mathcal{O}(\epsilon)\right)$$

$$I_2 = \int_0^\infty dk\ k^{-\epsilon} \frac{k}{k^2(k^2+M^2)}\left(1 - \frac{m^2}{k^2} + \frac{m^4}{k^4} + \cdots\right) = \frac{1}{M^2}\left(-\frac{1}{\epsilon} + \ln M + \mathcal{O}(\epsilon)\right)$$

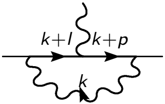
Note that both integrals integrates the entire interval. Overlap?
Let $R_1 = I_1 - I_{1,\Lambda}$, $R_2 = I_2 - I_{2,\Lambda}$. Then the overlap region (0-bin):

$$\begin{aligned}
R &= \int_\Lambda^\infty dk\ k^{-\epsilon} \frac{k}{(k^2+m^2)M^2}\left(1 - \frac{k^2}{M^2} + \cdots\right) + \int_0^\Lambda dk\ k^{-\epsilon} \frac{k}{k^2(k^2+M^2)}\left(1 - \frac{m^2}{k^2} + \cdots\right) \\
&= \int_\Lambda^\infty dk\ k^{-\epsilon} \frac{k}{k^2M^2}\left(1 - \frac{m^2}{k^2} - \frac{k^2}{M^2} + \cdots\right) + \int_0^\Lambda dk\ k^{-\epsilon} \frac{k}{k^2M^2}\left(1 - \frac{m^2}{k^2} - \frac{k^2}{M^2} + \cdots\right) \\
&= \int_0^\infty dk\ k^{-\epsilon} \frac{k}{k^2M^2}\left(1 - \frac{m^2}{k^2} - \frac{k^2}{M^2} + \cdots\right) = 0
\end{aligned}$$

where we have expanded the denominators into their respective limits.
Regions are *localized*.

vertex correction

Consider the one-loop vertex correction (neglect spin, mass)



$$= i\pi^{-d/2}\mu^{4-d}\int d^d k \frac{1}{(k^2+i0)[(k+l)^2+i0][(k+p)^2+i0]}$$

where $d = 4-2\epsilon$ is the dimensional regulator, μ the t'Hooft dimension scale. One can define the following:

$$L^2 \equiv -l^2 - i0, \quad P^2 \equiv -p^2 - i0, \quad Q^2 \equiv -(l-p)^2 - i0$$

Setting $L^2 \sim P^2 \ll Q^2$, we have small invariant mass, but large energy. One can also define the light-like reference vector:

$$n_\mu = (1,0,0,+1) \;\; and \;\; \bar{n}_\mu = (1,0,0,-1)$$

with $n^2 = \bar{n}^2 = 0$, $n \cdot \bar{n} = 2$. Then $p^\mu = [n \cdot p, \;\; \bar{n} \cdot p, \;\; p_\perp^\mu]$

Regions

Working in the minus-metric, the largest scale is Q, and we define λ to be the vanishing limit:

$$\lambda^2 \sim \frac{P^2}{Q^2} \sim \frac{L^2}{Q^2}$$

We then have $p^\mu \approx Qn^\mu/2 \sim (\lambda^2, 1, \lambda)Q$, $l^\mu \approx Q\bar{n}^\mu/2 \sim (1, \lambda^2, \lambda)Q$.
Note that the scaling is not unique.

Regions

- Hard (h): $k^\mu \sim (1, 1, 1)Q$
- Collinear to p (c): $k^\mu \sim (1, \lambda^2, \lambda)Q$
- Collinear to l ($\bar{c}$): $k^\mu \sim (\lambda^2, 1, \lambda)Q$
- Soft (s): $k^\mu \sim (\lambda^2, \lambda^2, \lambda^2)Q$

We can expand the denominator according to the scales above during integration.

Hard

The hard region $k^\mu \sim (1,1,1)Q$ correspond to:

$$(k+l)^2 = k^2 + 2k_- \cdot l_+ + \mathcal{O}(\lambda)$$
$$(k+p)^2 = k^2 + 2k_+ \cdot p_- + \mathcal{O}(\lambda)$$

Then

$$\begin{aligned}
I_h &= i\pi^{-d/2}\mu^{4-d}\int d^d k \frac{1}{(k^2+i0)(k^2+2k_-\cdot l_+ + i0)(k^2+2k_+\cdot p_- + i0)} \\
&= \frac{\Gamma(1+\epsilon)}{2l_+\cdot p_-}\frac{\Gamma^2(-\epsilon)}{\Gamma(1-2\epsilon)}\left(\frac{\mu^2}{2l_+\cdot p_-}\right)^{\epsilon} \\
&= \frac{\Gamma(1+\epsilon)}{Q^2}\left(\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\ln\frac{\mu^2}{Q^2}+\frac{1}{2}\ln^2\frac{\mu^2}{Q^2}-\frac{\pi^2}{6}\right)+\mathcal{O}(\epsilon)
\end{aligned}$$

details:[25]

where we have used Feynman parametrization on the denominators.

details:[26]

Collinear

Region collinear to p:

$$(k+l)^2 = 2k_- \cdot l_+ + \mathcal{O}(\lambda^2), \quad (k+p)^2 = \mathcal{O}(\lambda^2)$$

Then the integral:

$$\begin{aligned}
I_c &= i\pi^{-d/2}\mu^{4-d} \int d^d k \frac{1}{(k^2+i0)(2k_- \cdot l_+ + i0)[(k+p)^2+i0]} \\
&= -\frac{\Gamma(1+\epsilon)}{2l_+ \cdot p_-} \frac{\Gamma^2(-\epsilon)}{\Gamma(1-2\epsilon)} \left(\frac{\mu^2}{P^2}\right)^{\epsilon} \\
&= \frac{\Gamma(1+\epsilon)}{Q^2} \left(-\frac{1}{\epsilon^2} - \frac{1}{\epsilon} \ln \frac{\mu^2}{P^2} - \frac{1}{2} \ln^2 \frac{\mu^2}{P^2} + \frac{\pi^2}{6}\right) + \mathcal{O}(\epsilon)
\end{aligned}$$

Similarly, region collinear to l:

$$(k+l)^2 = \mathcal{O}(\lambda^2), \quad (k+p)^2 = 2k_+ \cdot p_- + \mathcal{O}(\lambda^2)$$

Then the integral:

$$\begin{aligned}
I_{\bar{c}} &= i\pi^{-d/2}\mu^{4-d} \int d^d k \frac{1}{(k^2+i0)[(k+l)^2+i0](2k_+ \cdot p_- + i0)} \\
&= -\frac{\Gamma(1+\epsilon)}{2l_+ \cdot p_-} \frac{\Gamma^2(-\epsilon)}{\Gamma(1-2\epsilon)} \left(\frac{\mu^2}{L^2}\right)^{\epsilon} \\
&= \frac{\Gamma(1+\epsilon)}{Q^2} \left(-\frac{1}{\epsilon^2} - \frac{1}{\epsilon} \ln \frac{\mu^2}{L^2} - \frac{1}{2} \ln^2 \frac{\mu^2}{L^2} + \frac{\pi^2}{6}\right) + \mathcal{O}(\epsilon)
\end{aligned}$$

Soft

Soft Region $k^\mu \sim (\lambda^2, \lambda^2, \lambda^2)Q$:

$$
\begin{aligned}
I_s &= i\pi^{-d/2}\mu^{4-d}\int d^d k \frac{1}{(k^2+i0)[(2k_-\cdot l_+ + l^2 + i0)(2k_+\cdot p_- + p^2 + i0)} \\
&= -\frac{\Gamma(1+\epsilon)}{2l_+\cdot p_-}\Gamma(\epsilon)\Gamma(-\epsilon)\left(\frac{2l_+\cdot p_-\mu^2}{L^2P^2}\right)^\epsilon \\
&= \frac{\Gamma(1+\epsilon)}{Q^2}\left(\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\ln\frac{\mu^2Q^2}{L^2P^2}+\frac{1}{2}\ln^2\frac{\mu^2Q^2}{L^2P^2}+\frac{\pi^2}{6}\right)+\mathcal{O}(\epsilon)
\end{aligned}
$$

We can check that the overlapping region R vanishes due to scaleless integral in dimensional regularization.

$$
\begin{aligned}
I &= I_h + I_c + I_{\bar{c}} + I_s \\
&= \frac{1}{Q^2}\left(\ln\frac{Q^2}{L^2}\ln\frac{Q^2}{P^2}+\frac{\pi^2}{3}+\mathcal{O}(\lambda)\right)
\end{aligned}
$$

Note that the IR divergence from hard region cancels with UV divergence from soft and collinear region.

general idea

We see that the dynamics of different scales are localized.
- e.g. we don't need to know the vacuum polarization of bottom quarks in Hydrogen binding energy. Since $E_0 = \frac{1}{2} m_e \alpha^2 \left[1 + \mathcal{O}\left(\frac{m_e}{m_b}\right) \right]$

The two types of Effective Field Theory:

Top-down:

- Physics at high energy scales are known.
- wants to investigate the physics at low energy
- integrate out particles at respective ranges, build Lagrangian
- e.g. NRQCD, SCET...

Bottom-up:

- Physics at low energy scales are known.
- wants to investigate the physics at higher energy
- build Lagrangian and assume degrees of freedom (gauge symmetry...)
- e.g. Standard model, Einstein (quantum) gravity...

field decomposition

Consider di-jet production ($SCET_I$), with hard scale Q, then the components of the each collinear jet scales as

$$p_c^\mu \sim (\frac{\Delta^2}{Q}, Q, \Delta) \sim Q(\lambda^2, 1, \lambda)$$

with $\Lambda_{QCD} \ll \Delta \ll Q$ collinear. And the component for soft (ultra-soft) jets scales as

$$p_s^\mu \sim (\frac{\Delta^2}{Q}, \frac{\Delta^2}{Q}, \frac{\Delta^2}{Q}) \sim Q(\lambda^2, \lambda^2, \lambda^2)$$

We can decompose gluon and quark fields into

$$A^\mu(x) \to A_c^\mu(x) + A_s^\mu(x), \quad \psi(x) \to \psi_c(x) + \psi_s(x)$$

with

$$\psi_c(x) \equiv \xi(x) + \eta(x) = \frac{\not{n}\not{\bar{n}}}{4}\psi_c + \frac{\not{\bar{n}}\not{n}}{4}\psi_c \equiv P_+\psi_c + P_-\psi_c$$

details:[27]

power counting

We are interested at how the different field components scale. We can investigate the two point correlator:

$$\langle 0|T\{\xi(x)\bar{\xi}(0)\}|0\rangle = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2+i0}e^{-ip\cdot x}\frac{\not{n}\not{\bar{n}}}{4}\not{p}\frac{\not{\bar{n}}\not{n}}{4} \sim \lambda^2$$

$$\langle 0|T\{\eta(x)\bar{\eta}(0)\}|0\rangle = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2+i0}e^{-ip\cdot x}\frac{\not{\bar{n}}\not{n}}{4}\not{p}\frac{\not{n}\not{\bar{n}}}{4} \sim \lambda^4$$

$$\langle 0|T\{\psi_s(x)\bar{\psi}_s(0)\}|0\rangle = \int \frac{d^4p}{(2\pi)^4}\frac{i\not{p}}{p^2+i0}e^{-ip\cdot x} \sim \lambda^6$$

$$\langle 0|T\{A^\mu(x)A^\nu(0)\}|0\rangle = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2+i0}e^{-ip\cdot x}\left[-g^{\mu\nu}+\xi\frac{p^\mu p^\nu}{p^2}\right]$$

We see that the gluon fields scales like its momentum. Therefore, we have:

collinear quark: $\xi \sim \lambda^1$

collinear quark: $\eta \sim \lambda^2$

soft quark: $\psi_s \sim \lambda^3$

collinear gluon: $A_c \sim (\lambda^2, \lambda^0, \lambda^1)$

soft gluon: $A_s \sim (\lambda^2, \lambda^2, \lambda^2)$

Effective Lagrangian

We first look at the following Lagrangian which includes collinear and soft fields:

$$\mathcal{L}_c = \bar{\psi}_c i \not{D} \psi_c = \bar{\xi} \frac{\not{\bar{n}}}{2} i n \cdot D \xi + \bar{\xi} i \not{D}_\perp \frac{1}{i \bar{n} \cdot D} i \not{D}_\perp \frac{\bar{n}}{2} \xi$$

$$\mathcal{L}_s = \bar{\psi}_s i \not{D}_s \psi_s - \frac{1}{4} (F_s^a)_{\mu\nu} (F_s^a)^{\mu\nu}$$

Note that the interaction term are difficult to derive. One can also look at label formalism (field transformation in momentum space). Also note that since η field is power suppressed relative to ξ field, we can integrate it out. The general SCET Lagrangian is in a form:

$$\mathcal{L}^{(0)}_{SCET} = \bar{\psi}_s i \not{D}_s \psi_s + \bar{\xi} \frac{\not{\bar{n}}}{2} \left[i n \cdot D + i \not{D}_{c\perp} \frac{1}{i \bar{n} \cdot D_c} i \not{D}_{c\perp} \right] \xi - \frac{1}{4} (F^{s,a}_{\mu\nu})^2 - \frac{1}{4} (F^{c,a}_{\mu\nu})^2$$

where the covariant derivative is:

$$i D_\mu \equiv i \partial_\mu + g A_\mu = i \partial_\mu + g ({A_c}^a_\mu + {A_s}^a_\mu) t^a$$

and the field strength is $i g F_{\mu\nu} = [i D_\mu, i D_\nu]$. But this is not all...

details:[28]

Symmetries

So far we have decompose the fields with some basic vector notations and field redefinitions, forgetting about the symmetries in the original theory. We now move on to restoring those symmetries in the effective theory in order to move beyond tree level.

- *Reparameterization Invariance*
  To guarantee Lorentz invariance when choosing the vectors n, $\bar{n}$. One must investigate the different types of vector transformations on these vectors, and imply conditions on how to use them.
- *Gauge Symmetry*
  To preserve gauge symmetry, one introduce Wilson line along with current tensor, to provide gauge linking between collinear and soft interactions.
- *Spin symmetry*
  One also needs to be careful when defining spinors for the fermion field (to spin along direction of collinear motion).

Symmetries

Other important notes:

- consider the covariant gauge fixing term for collinear gluon, for it to preserve soft gauge invariance, the $i\partial$ is replace with $i\mathcal{D}_s$ which has $\frac{\bar{n}^\mu}{2}(in \cdot \partial + gnA_s)$
- gauge transformation is simpler in position space, easier to verify symmetry. But label formalism which constructs Lagrangian in momentum space is intuitive in constructing Feynman rules.
- keep track of power counting when writing interaction terms between collinear and soft particles since some components do not contribute to the transformation.
- $\cdots$ [see Prof. Christopher lecture]

Applications and Resources

applications

- B-physics
- jet physics
- Electroweak decay (Higgs, top ...)

resources

- Peskin and Schroeder
  - Radiative corrections, Renormalization, QCD and OPE
- Introduction to Soft-Collinear Effective Theory
  - (Becher, Broggio, Ferrolia) arXiv:1410.1892 [hep-ph]
- Effective Field Theory
  - (Iain Stewart - MIT ocw) iTunes U

Summary

- What we did in QFT course is extremely simplified:
  - negligible fermion mass, zero boson mass
- Simple, but ineffective QCD Lagrangian.
- In high energy colliders, one need to take into account multi-scale problem.
- Need to understand divergences:
  - types of divergences
  - origin of divergences
  - how to treat (resum) divergences
- Need to have fun in QCD calculations.

Thank You!

self-energy

We have

$$\Sigma_2(p) = -ie^2\mu^{2\epsilon}\int \frac{d^n k}{(2\pi)^n}\frac{\gamma^\mu(\not{k}+m_0)\gamma_\mu}{[k^2-m_0^2+i\epsilon][(p-k)^2+i\epsilon]}$$

then use Feynman parameter:

$$\frac{1}{AB} = \int_0^1 dxdy\,\delta(x+y-1)\frac{1!}{(Ax+By)^2}$$

Let $l \equiv k - py$, and $\Delta = p^2(y^2-y) + m_0^2(1-y)$, then

$$\Sigma_2(p) = -ie^2\mu^{2\epsilon}\int_0^1 dy\int \frac{d^n l}{(2\pi)^n}\frac{[(2-n)\not{l}+(2-n)\not{p}y+nm_0]}{(l^2-\Delta+i\epsilon)^2}$$

Wick rotate:

$$\int \frac{d^n l}{(2\pi)^n}\frac{\not{l}}{(l^2)^\alpha} = 0 \quad , \quad \int \frac{d^n l}{(2\pi)^n}\frac{1}{(l^2-\Delta+i\epsilon)^2} = \frac{i}{(4\pi)^{n/2}}\frac{\Gamma(2-\frac{n}{2})}{\Gamma(2)}\left(\frac{1}{\Delta}\right)^{2-\frac{n}{2}}$$

use identity:

$$\int_0^1 dx\; x^{\alpha-1}(1-x)^{\beta-1} = B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

return:[7]

Feynman Parametrization

Note that SCET can have propagators which are linear in the loop momentum from expansions in small momentum components. To combine linear and quadratic propagators, we can use:

$$\frac{1}{ab} = \int_0^\infty dy \frac{1}{(a+by)^2}$$

where a is quadratic and b is linear. Generalizing:

$$\frac{1}{a_1 a_2 \cdots a_n} = \prod_1^{n-1} \int_{i=0}^\infty (dy_i) \frac{(n-1)!}{(a_1 + a_2 y_1 + \cdots + a_n y_{n-1})^n} \delta(\sum y_i - 1)$$

For higher power, one can use:

$$\frac{1}{a^n b^m} = \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)} \int_0^\infty dy \frac{y^{m-1}}{(a+yb)^{n+m}}$$

return:[13]

Details derivation for I_h

applying Feynman parametrization

$$\frac{1}{abc} = 2\int_0^1 dx \int_0^x dy \frac{1}{[ay + b(x-y) + c(1-x)]^3}$$

Then

$$I_h = i\pi^{-d/2}\mu^{4-d} \int_0^1 dx \int_0^x dy \int d^d k \frac{2}{\chi^3(x,y,k)}$$

where

$$\chi(x,y,z) = k^2 + 2k \cdot [py + l(1-x)] + \mathcal{O}(\lambda)$$

Using identity (wick rotate):

$$\int d^d k \frac{1}{(k^2 + 2k \cdot Q - M^2)^\alpha} = (-1)^\alpha \frac{i\pi^{d/2}}{(M^2 + Q^2)^{\alpha - d/2}} \frac{\Gamma(\alpha - d/2)}{\Gamma(\alpha)}$$

we then have $d = 4 - 2\epsilon$:

$$\begin{aligned} I_h &= \frac{\Gamma(1+\epsilon)}{2l_+ \cdot p_-} \left(\frac{\mu^2}{2l_+ \cdot p_-}\right)^\epsilon \int_0^1 dx \int_0^x dy \frac{1}{[y(1-x)]^{1+\epsilon}} \\ &= \frac{\Gamma(1+\epsilon)}{2l_+ \cdot p_-} \frac{\Gamma^2(-\epsilon)}{\Gamma(1-2\epsilon)} \left(\frac{\mu^2}{2l_+ \cdot p_-}\right)^\epsilon \end{aligned}$$

return:[13]

vector notations

Using the most minus metric, and define the following light vectors:

$$n_\mu = (1, 0, 0, 1), \quad \bar{n}_\mu = (1, 0, 0, -1)$$

with properties: $n^2 = \bar{n}^2 = 0$, $n \cdot \bar{n} = 2$. Then we can decompose

$$p^\mu = p_+^\mu + p_-^\mu + p_\perp^\mu = \left[(n \cdot p) \frac{\bar{n}^\mu}{2} + (\bar{n} \cdot p) \frac{n^\mu}{2} + p_\perp^\mu \right]$$

and its square $p^2 = (n \cdot p)(\bar{n} \cdot p) - p_T^2$. With these vectors, the integration over momentum space is given by:

$$\int d^d k = \frac{1}{2} \int_{-\infty}^{+\infty} dk_+ \int_{-\infty}^{+\infty} dk_- \int d^{d-2} k_\perp$$

$$\int d^d k \delta(k^2) \theta(k^0) = \frac{1}{2} \int_0^\infty dk_+ \int_0^\infty dk_- \int d^{d-2} k_\perp \delta(k_+ k_- - k_T^2)$$

return:[17]

effective Lagrangian

The Lagrangian is constructed directly in position space using multipole expansion. But we can also use a hybrid position-momentum-space formalism called label formalism, which is similar to heavy quark effective theory. and decompose the momentum as $p^\mu = p_l^\mu + p_r^\mu$, with p_l discrete and p_r continuous, and the summation is done by $\int d^4 p = \sum_{p_l \neq 0} \int d^4 p_r$ and the field transform as:

$$\hat{\xi}_n(x) = \sum_{p_l \neq 0} \int \frac{d^4 p_r}{(2\pi)^4} e^{-ip_l \cdot x} e^{-ip_r \cdot x} \xi_{n,p_l}(p_r) = e^{-i\mathcal{P}\cdot x} \xi_n(x)$$

with $\mathcal{P}$ the momentum operator. The collinear Lagrangian:

$$\begin{aligned} \mathcal{L}_c = \bar{\psi}_c i \not{D} \psi_c &= (\bar{\xi} + \bar{\eta}) \left[\frac{\not{n}}{2} i\bar{n} \cdot D + \frac{\not{\bar{n}}}{2} in \cdot D + i\not{D}_\perp \right] (\xi + \eta) \\ &= \bar{\xi} \frac{\not{\bar{n}}}{2} in \cdot D\xi + \bar{\xi} i \not{D}_\perp \eta + \bar{\eta} i \not{D}_\perp \xi + \bar{\eta} \frac{\not{n}}{2} i\bar{n} \cdot D\eta \end{aligned}$$

where $\not{n}\xi$, $\bar{\xi}\not{n}$, $\not{\bar{n}}\eta$, $\bar{\eta}\not{\bar{n}}$, $\bar{\xi}\not{D}_\perp\xi$, $\bar{\eta}\not{D}_\perp\eta$ terms vanishes. Since η is power suppressed, we can integrate it out using the equation of motion $(-\frac{\partial \mathcal{L}}{\partial \bar{\xi}} = 0)$ and solve for η in terms of ξ, then using the identities $\{\not{\bar{n}}, \not{D}_\perp\} = 0$ and $P_+ \not{D}_\perp \xi = \not{D}_\perp P_+ \xi = \not{D}_\perp \xi$ to obtain the collinear Lagrangian.
We can see that the path integral:

$$\int \mathcal{D}[\eta]\mathcal{D}[\bar{\eta}] exp \left[\int d^4x \; \bar{\eta} \frac{\not{n}}{2} i\bar{n} \cdot D\eta \right] = \det \left(\frac{\not{n}}{2} i\bar{n} \cdot D \right)$$

can be proven to be gauge independent, thus η can be integrated out.
return:[19]