Lecture Plan

Lecture 1: Hadrons as laboratory for QCD:

- Introduction to QCD
- Bare vs effective effective quarks and gluons
- Phenomenology of Hadrons

Lecture 2: Complex analysis

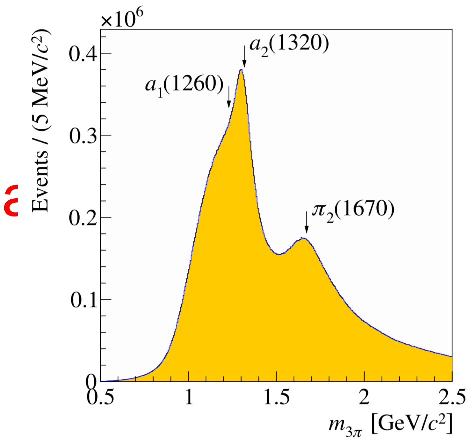
Lecture 3: Phenomenology of hadron reactions

- Kinematics and observables
- Space time picture of Parton interactions and Regge phenomena
- Properties of reaction amplitudes

Lecture 4: How to extract resonance information from the data

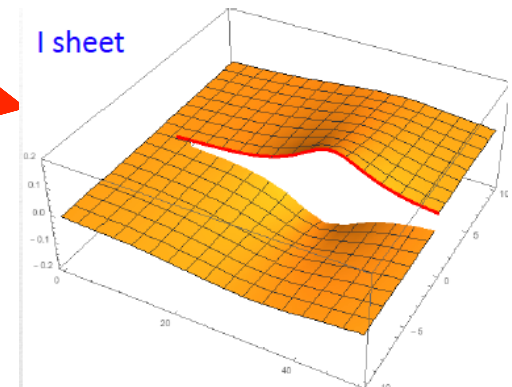
- Partial waves and resonance properties
- Amplitude analysis methods (spin complications)

Identifying resonances



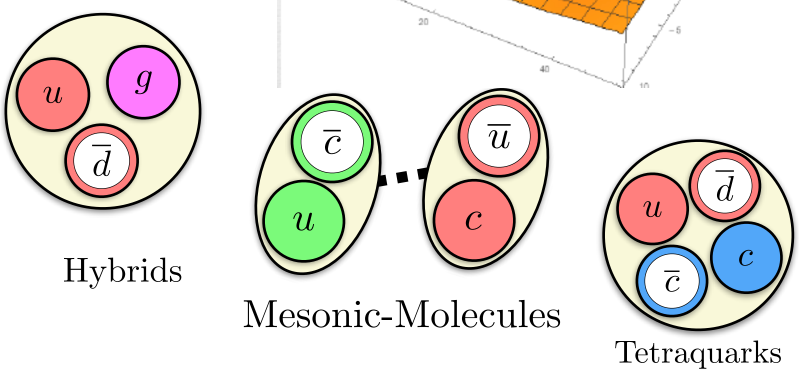
- Experimental or lattice signatures (rea axis data: cross section bumps and dips, energy levels)

Reaction amplitudes



- Theoretical signatures (complex plane singularities: poles, cusps)

Microscopic Models



- What is the interpretation (constituent quarks, molecules, …) ?

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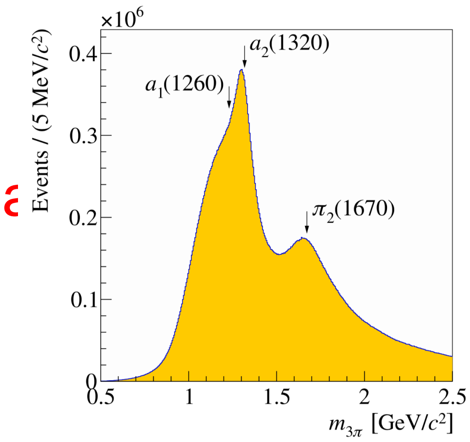

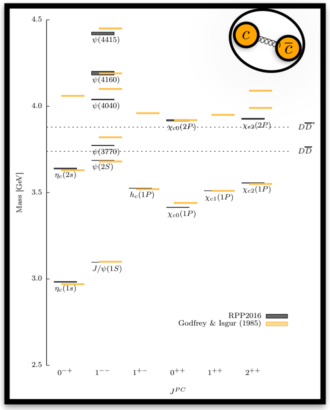

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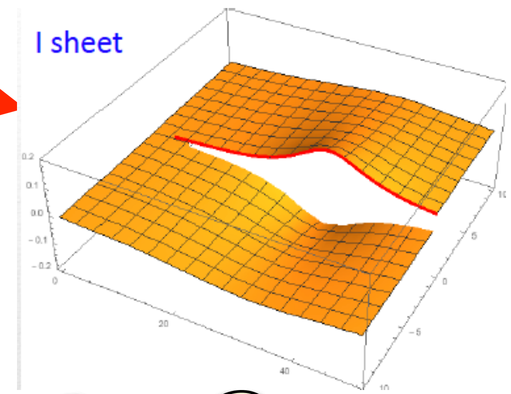

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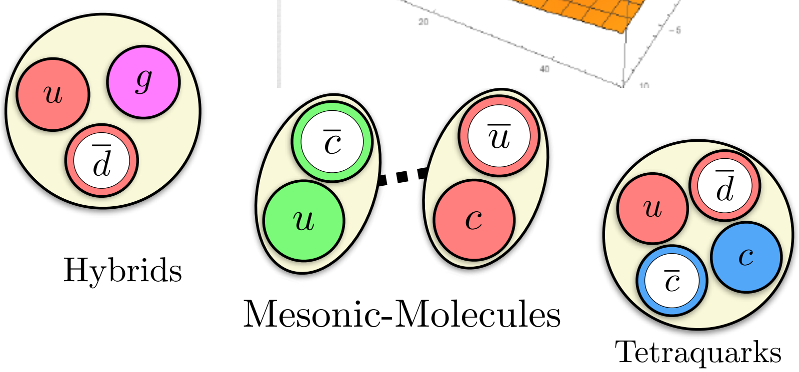


- What is the interpretation (constituent quarks, molecules, …) ?

Probing QCD resonances (using physical states)

- When (color neutral) mesons and baryons a smashed, their quarks overlap, “stick together” and form resonances (quasi QCD eigenstates). They are short lived and decay to lowest energy, asymptotic states (pions, K’s, proton,…)

- Resonances are fundamental to our understanding of QCD dynamics because they are formed by all-order (aka beyond perturbation theory) interactions. Resonances challenge QFT practitioners to develop all orders calculations (still ways to go).

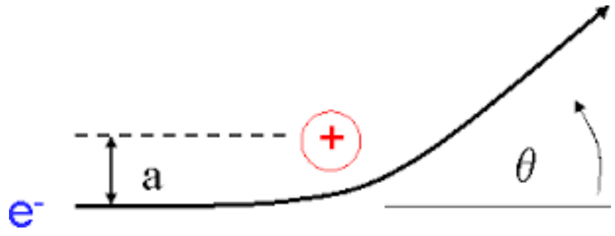
- (QCD) Resonance lead to extremely rich phenomenology, e.g. XYZ states, gluonic excitations, etc.

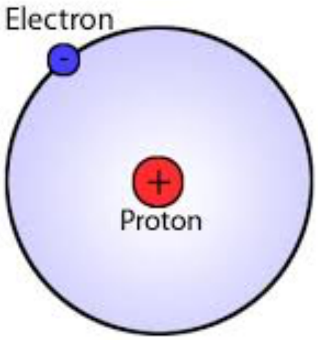
- In practice, one requires tools that relate asymptotic states before collision to asymptotic states after collision that include flexible parametrization of the microscopic dynamics. This is often referred to as amplitude analysis. The rest of these lectures will focus on this topic.

Bound states/Resonances/Asymptotic states

$$\left[\frac{p^2}{2m_e} - \frac{\alpha}{r}\right]\psi(r) = E\psi(r)$$

$$\alpha = \alpha_{QED} = \frac{1}{137}$$





$$\psi(r) = \frac{e^{-ikr}}{r} - S\frac{e^{+ikr}}{r} \qquad (1)$$

$$S = 1 + O(\alpha)$$

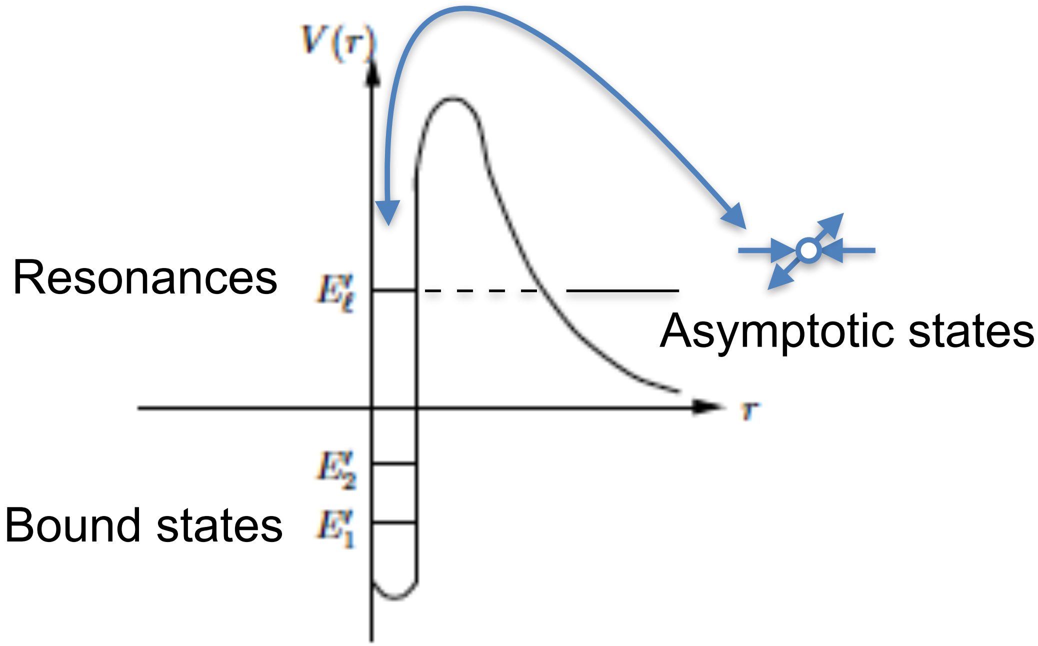
$$\psi(r) = e^{-\alpha m_e r} \qquad (2)$$

Born approximation : “weak” perturbation (lowest order) to free motion

Bound states: compact wave function contains interaction to all orders.

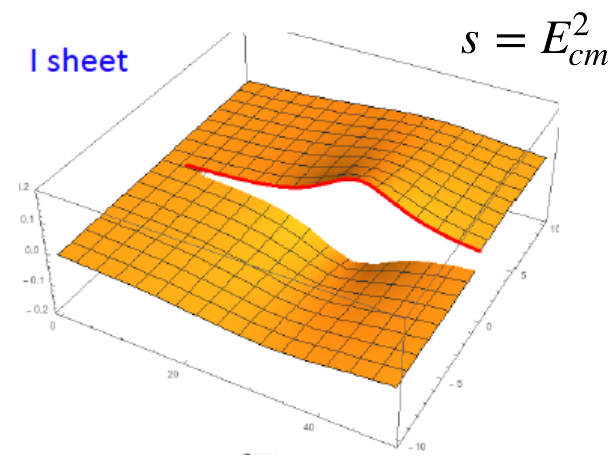
Resonances: particles interact to all orders (like bound states) but eventually decay (connect with asymptotically free states). Their effect appears in the S-matrix : Compare (1) and (2) ! $(k = i\alpha m_e)$

Amplitude analyticity: it is much about complex functions



$$A_{physical} = A(s + i\epsilon) \rightarrow A(s = complex)$$

analytical continuation



- Scattering amplitude describes evolution between asymptotic states. The information related to formation of resonances is “hidden” in unphysical domains (sheets) of the kinematical variables.

- The “bump” in the right figure is an indication of a “hidden” phenomenon. To uncover it one needs to analytically continue outside the physical sheet.

Shrodinger eq.

In non-relativistic potential theory V(x) contains all physics: It determines scattering amplitudes, bound state energies, etc. So one should focus on V(x).

S-amplitude and T (or f) (scattering amplitude) is determined by V but the meaning is more general and definitions can be generalized to relativistic (QFT) theory a

$$\left[-\frac{d^2}{dr^2} - E + \frac{l(l+1)}{r^2} + V\right] u_l(r) = 0$$

$$u_l(r) \to_{r\to\infty} e^{-ikr} - (-1)^l S_l e^{ikr}$$

$$u_l(r) \to_{r\to 0} r^{l+1}$$

$$S_l = 1 + 2ikf_l$$

$$kf(k,\theta) = A(s,t) = \sum_l (2l+1) f_l(s) P_l(\cos\theta)$$

Shrodinger eq.

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$$kf(k,\theta) = A(s,t) = \sum_l (2l+1) f_l(s) P_l(\cos\theta)$$

- The Schrodinger eq. implies analyticity (particular realization of causality). Cauchy theorem enables to reconstruct an analytical function from its singularities. Thus one could imagine recovering the underlying dynamics from the measured S (or f) (Heisenberg program, Mandelstam realization, Bootstrap.)
- Singularity of f has a physical interpretation (bound states, resonances etc.)
- In QFT, use relativistic phase space and kinematics.

Bound states, resonances and poles

$$\left[-\frac{d^2}{dr^2} - E + \frac{l(l+1)}{r^2} + V\right] u_l(r) = 0$$

for l=fixed (i.e. integer) suppose -V is big -> ∞ : then there will be ∞ number of bound states n=1,2,... ∞

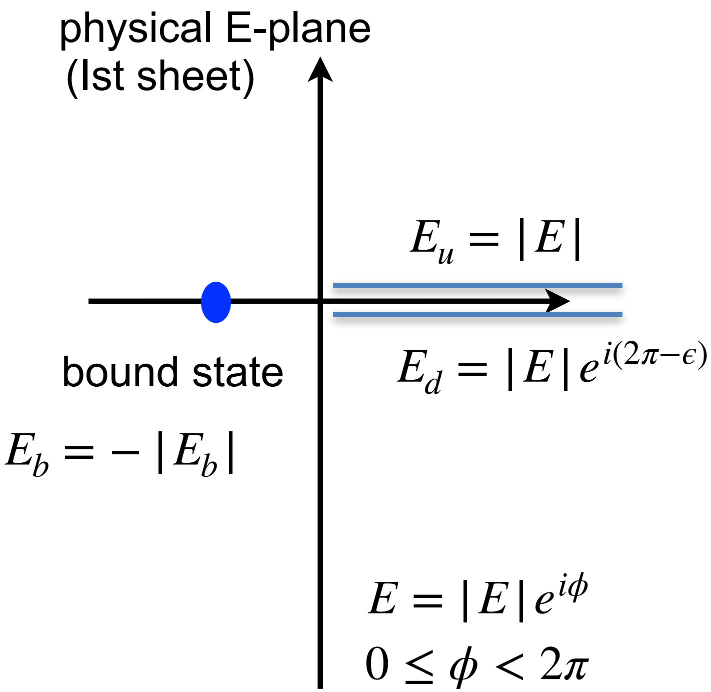
$$u_l(r) \to r\psi_l \to e^{-ikr} - S(l,k)e^{ikr}$$

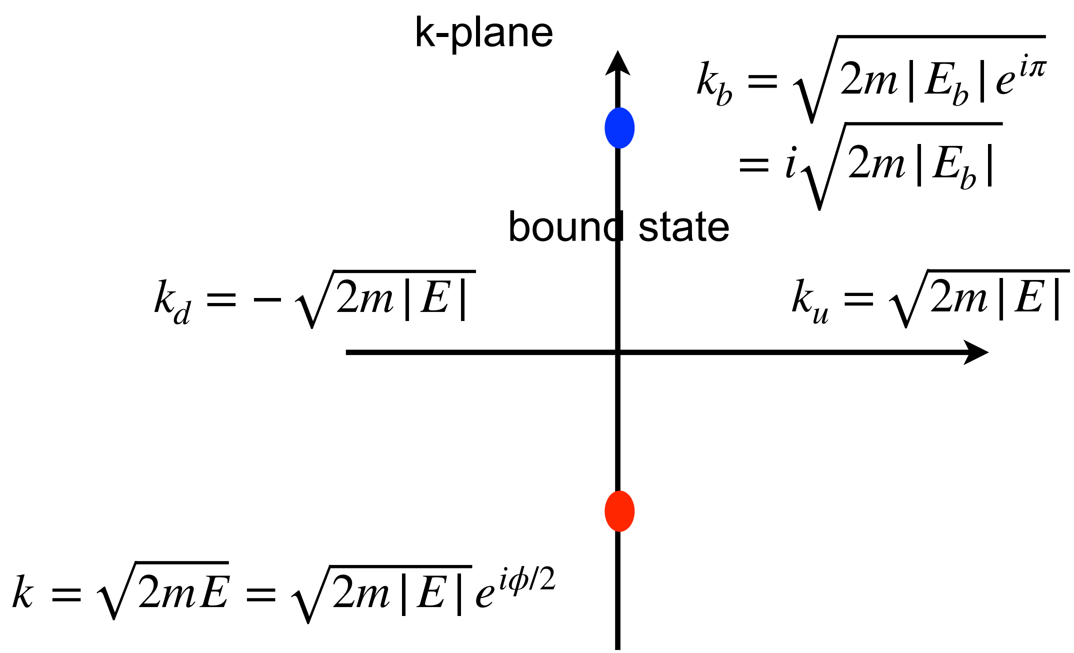
$S_l(E) = \infty$ for a bound state

$$k = \sqrt{2mE} \qquad k_b^2 = \sqrt{-2mE_B} = i\kappa$$

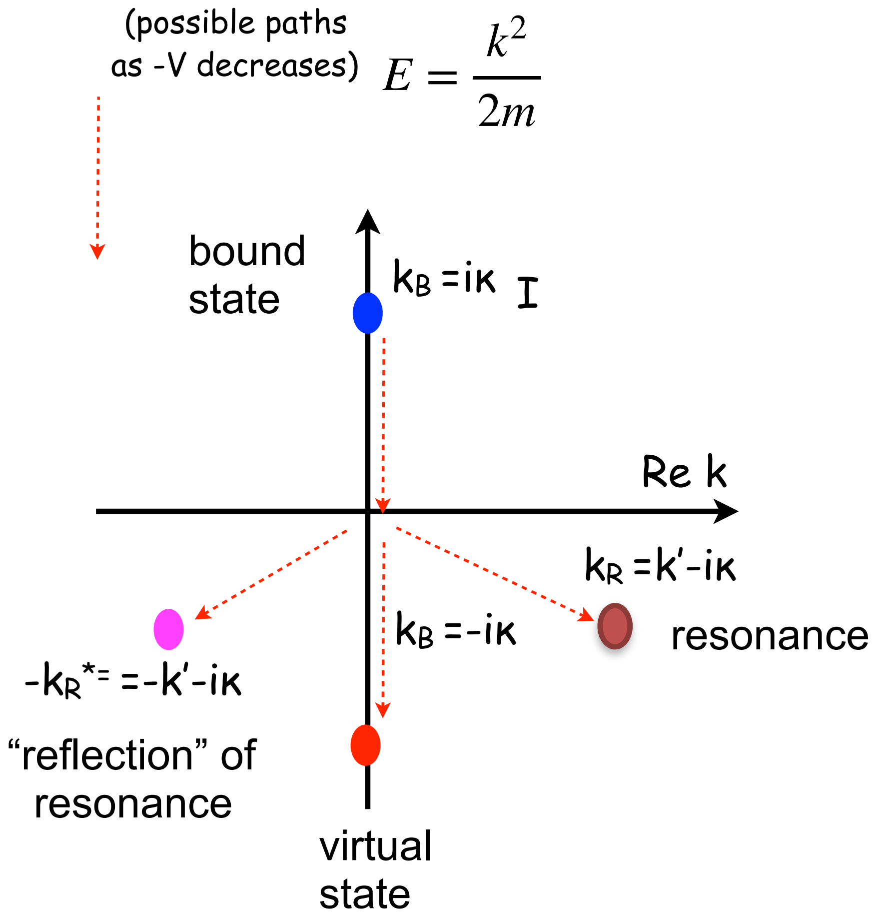
In the E-plane there is a branch point at E=0

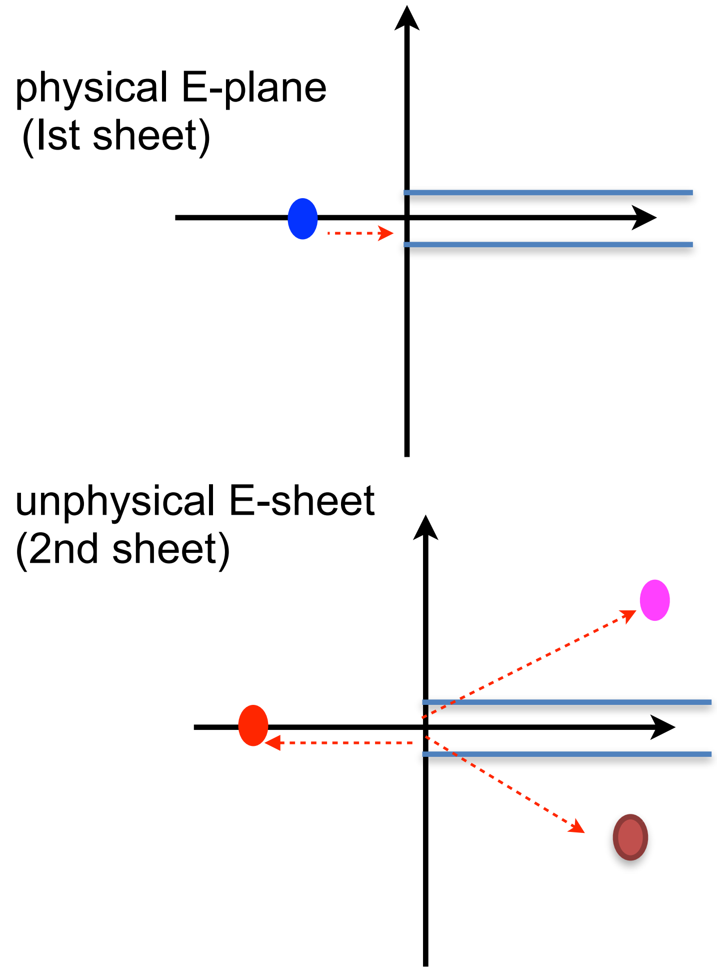
The full plane is cut (to the right) from E=0, it maps onto the Im k>0 half plane





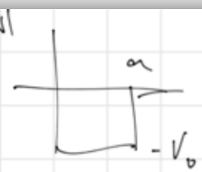
As -V decreases, some bound states disappear. So what happens to the associate poles ?





There is still an infinite number of resonances, even though the potential is finite !

Scattering in a potential well.



$$\psi(\vec r) = R(r)\,Y_{lm}(\Omega) \rightarrow R(r) = \frac{u(r)}{r}$$

$$-\frac{1}{2}\frac{d^2u}{dr^2} + V(r)u = E u \qquad r<a \qquad -\frac{1}{2}\frac{d^2u}{dr^2} - V_0 u = E u$$

$$u(r) = \sin k' r \qquad k' = \sqrt{2(E+V_0)}$$

$$r>a \qquad u(r) = A \sin kr + B\cos kr \qquad k = \sqrt{2E}$$

$$A\sin ka + B\cos ka = \sin k'a$$

$$+kA\cos ka - Bk\sin ka = k'\cos k'a$$

$$\begin{pmatrix} \sin ka & \cos ka \\ k\cos ka & -k\sin ka \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \sin k'a \\ k'\cos k'a \end{pmatrix}$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{k}\begin{pmatrix} k\sin ka & \cos ka \\ k\cos ka & -\sin ka \end{pmatrix}\begin{pmatrix} \sin k'a \\ k'\cos k'a \end{pmatrix}$$

$$A = \frac{k\sin ka \sin k'a + \cos ka \cos k'a\, k'}{k}$$

$$B = \frac{k\cos ka \sin k'a - \sin ka \cos k'a\, k'}{k}$$

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$r>a$

$$\psi(r) = A\,\frac{e^{ikr} - e^{-ikr}}{2ir} + B\,\frac{e^{ikr} + e^{-ikr}}{2r}$$

$$= \frac{(A+Bi)\,e^{ikr}}{2ir} + \frac{(Bi-A)\,e^{-ikr}}{2ir}$$

$$= (-A+Bi)\left[\frac{e^{-ikr}}{2ir} + \frac{A+Bi}{-A+Bi}\,\frac{e^{ikr}}{2ir}\right]$$

$$= \frac{(-A+Bi)}{2i}\left[\frac{e^{-ikr}}{r} - \frac{e^{ikr}}{r}\left[\frac{A+Bi}{A-Bi}\right]\right] \qquad 1+2if = \frac{A+Bi}{A-Bi}$$

$$1 + 2 i f_k = \frac{A + B i}{A - B i} = S \qquad f = \left(\frac{A + B i}{A - B i} - 1 \right) \frac{1}{2 i k}$$

$$f = \frac{B}{A - B i} \frac{1}{k} \quad \left\{ \begin{array}{l} A = \dfrac{k \sin ka \sinh k'a + \cos ka \cosh k'a \, k'}{k} \\ B = \dfrac{k \cos ka \sinh k'a - \sin ka \cosh k'a \, k'}{k} \end{array} \right.$$

$k = \sqrt{2E}$

$k' = \sqrt{2(E+V)}$ $\Rightarrow$ $E = \frac{k^2}{2}$ $\Rightarrow$ whole E plane maps onto upper k-plane.

$$f = \frac{1}{\frac{A}{B} - i} \frac{1}{k} \qquad S = e^{2 i \delta} \qquad f = \frac{e^{2 i \delta} - 1}{2 i k} = \frac{e^{i \delta}}{k} \sin \delta = \frac{\sin \delta}{k (\cos \delta - i \sin \delta)}$$

$$K^{-1} = \cot \delta = \frac{A}{B} = \frac{k \sin ka \sinh k'a + \cos ka \cosh k'a \, k'}{k \cos ka \sinh k'a - \sin ka \cosh k'a \, k'}$$

↳ inverse K-matrix

$$K = \frac{k \cos ka \sinh k'a - \sin ka \cosh k'a \, k'}{k \sin ka \sinh k'a + \cos ka \cosh k'a \, k'} \quad \left\{ f = \frac{1}{k} \frac{1}{K^{-1} - i} \right. , \quad K^{-1} = \cot \delta$$

$=$ K-matrix poles: $k \tan ka = -k' \coth k'a$

if $K^{-1} = \frac{m^2 - s}{\gamma}$ $\Rightarrow$ $f \sim \frac{1}{m^2 - s - i\gamma}$ $\Rightarrow$ Breit-Wigner.

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$$S = \frac{k \sin ka \sinh k'a + \cos ka \cosh k'a \, k'}{+ k \cos ka \sinh k'a \, i - \sin ka \cosh k'a \, k' \, i}$$

$$\frac{k (\sinh k'a) \, e^{-ika} - (\cosh k'a) \, e^{-ika} \, i k'}{-k (\sinh k'a) \, e^{ika} + (\cosh k'a) \, i \, e^{ika} \, k'} \quad \begin{array}{l} \times i \\ \times i \end{array}$$

$$S = \frac{i k \sinh k'a + \cosh k'a \, k'}{- i k \sinh k'a + \cosh k'a \, k'} e^{-2ika} \qquad V_0 = V a^2$$

$$k' = \sqrt{k^2 + V_0^2}$$

$$S = e^{-2ika} \frac{k' \coth k'a + ik}{k' \coth k'a - ik} \qquad \text{poles: } k' \coth k'a = ik$$

Cutoff potential

H.M.Nussenzveig , (1959)

$$V(r) = \begin{cases} -V_0 \text{ for } r < a \\ 0 \text{ for } r > a \end{cases}$$

$$E = k^2/2m$$

$$S_l(\beta) = -\frac{\beta j_l(\alpha) h_l'^{(2)}(\beta) - \alpha j'_l(\alpha) h_l^{(2)}(\beta)}{\beta j_l(\alpha) h_l'^{(1)}(\beta) - \alpha j'_l(\alpha) h_l^{(1)}(\beta)},$$

$$\beta = ka$$

$$\alpha = a\sqrt{k^2 + V_0}$$

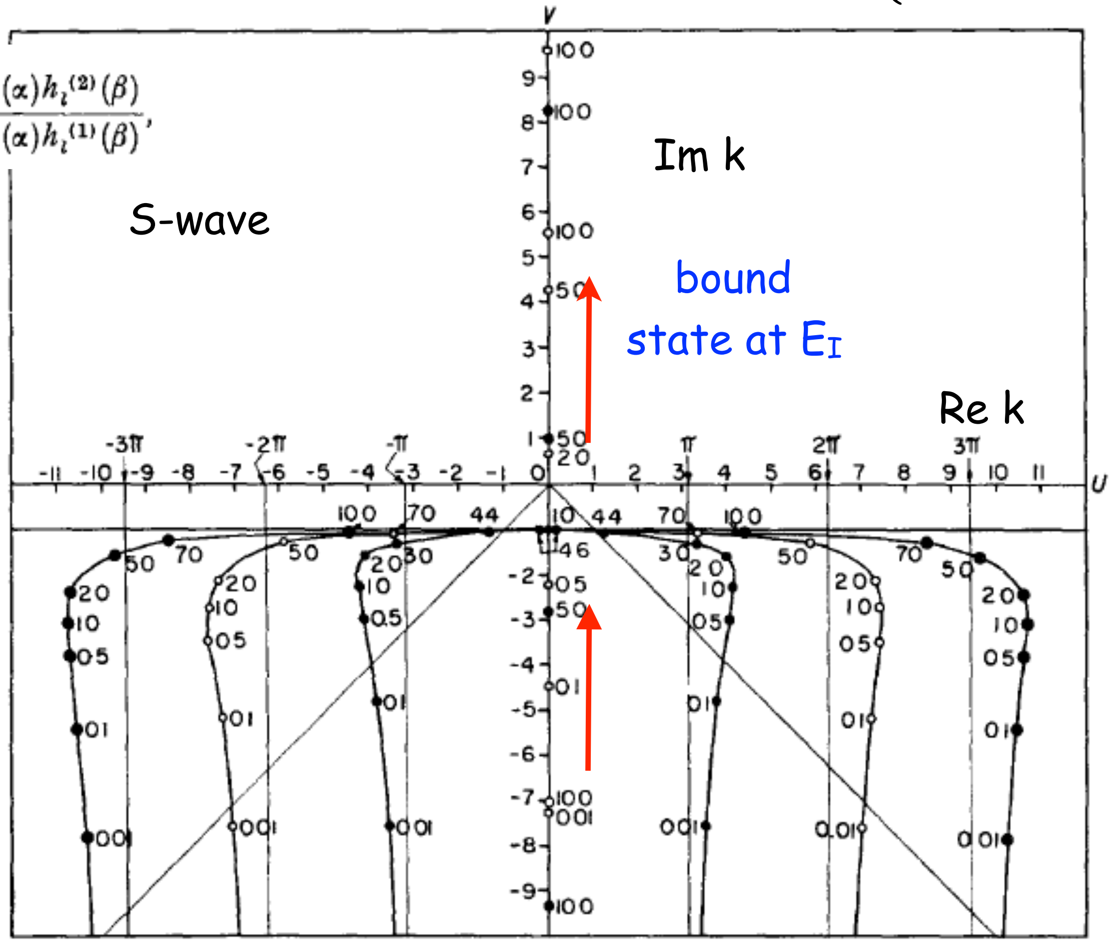

Fig 1 The poles β_n of $S_0(\beta)$ for a potential well
□ $n = 0$, ● $n = \pm 1$ ○ $n = \pm 2$ ◉ $n = \pm 3$
The numbers beside the poles give the corresponding values of A The curves in full line are the paths described by the poles The bisectors of the third and fourth quadrants are also indicated

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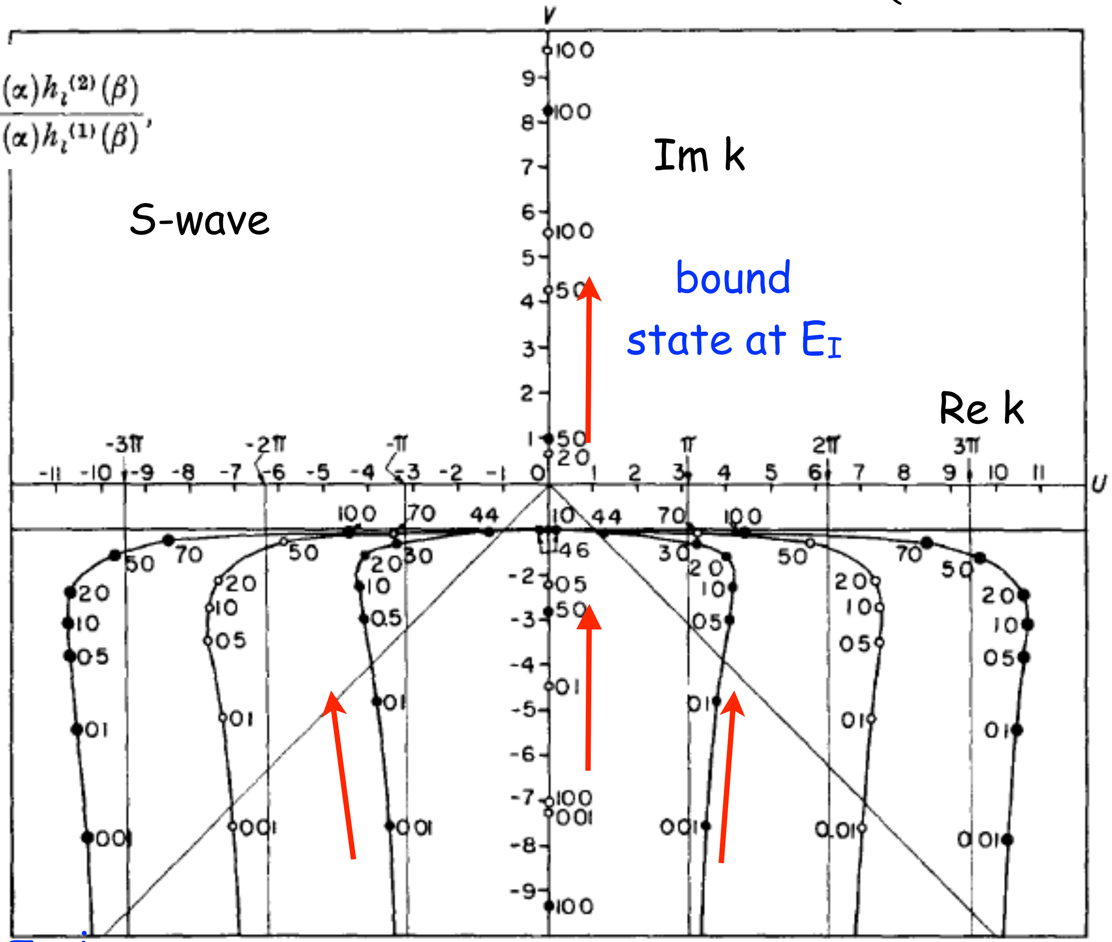
increasing interaction strength



$k_{III} = -k^*_{IV} \sim -n\pi - i\infty$

$E_{II} \sim -\infty^2 + i\infty$

$k_{IV} \sim +n\pi - i\infty$

$E_{II} \sim -\infty^2 - i\infty$

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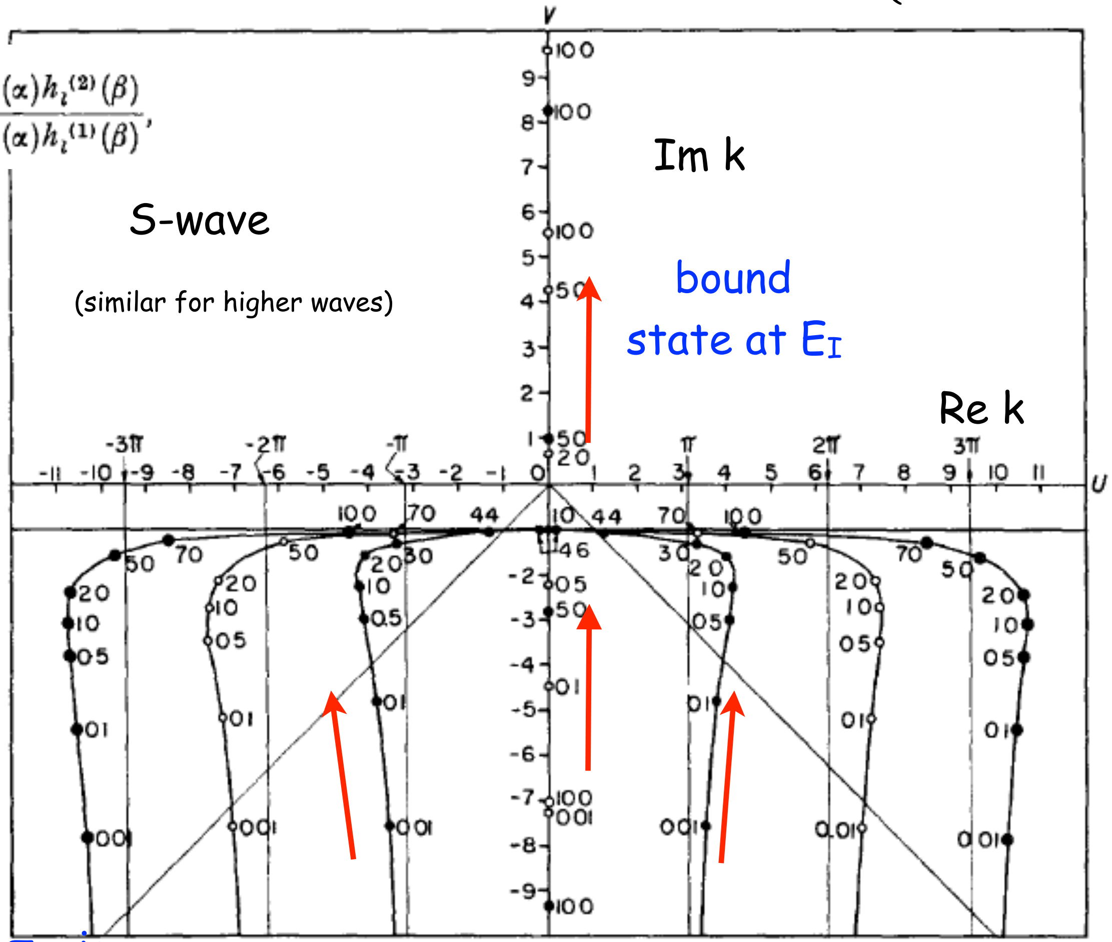
increasing interaction strength



$k_{III} = -k^*_{IV}$ ~ $-n\pi - i\infty$

E_{II} ~ $-\infty^2 + i\infty$

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E_{II} ~ $-\infty^2 - i\infty$

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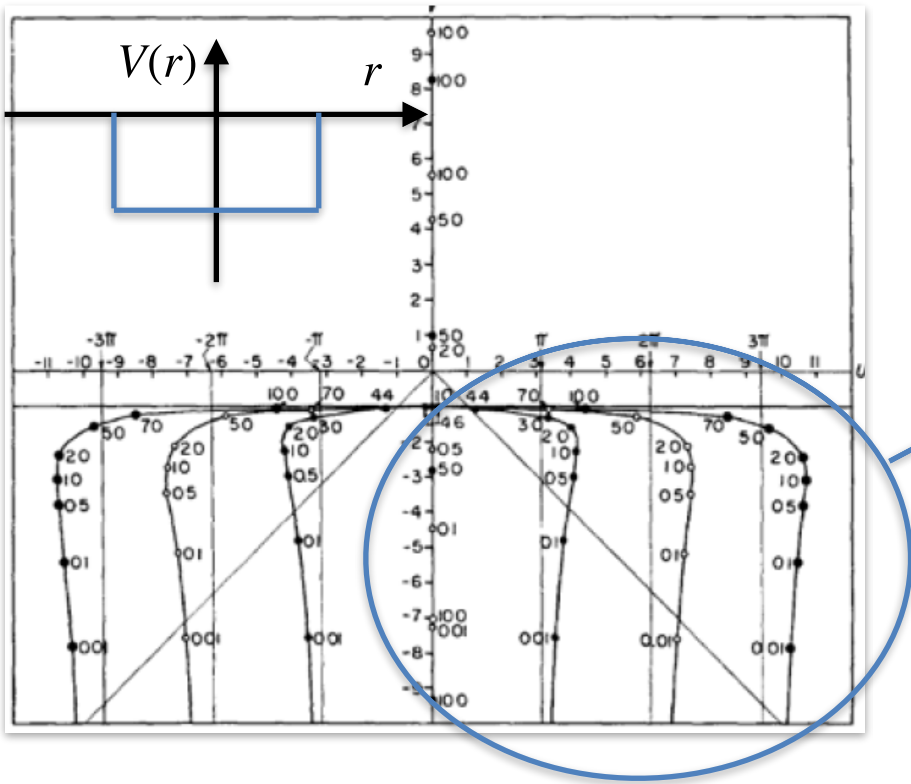
Potential Well vs Barrier



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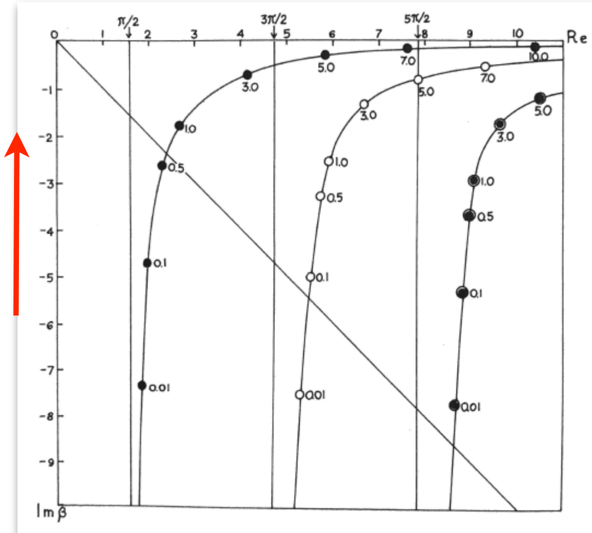
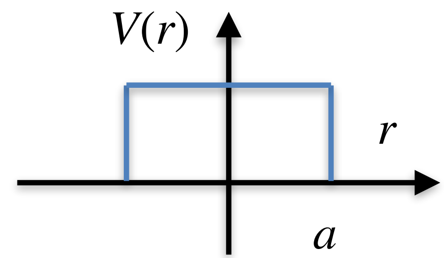




- Resonances have minimum width before they become bound states

$$\Gamma \sim \frac{1}{\tau} \sim \frac{v}{a}$$

- Average velocity inside the Well is always finite

$$\sim \frac{k}{a} \sim \frac{\sqrt{E-V}}{a}$$

- Resonances move to + ∞ with wishing width
- Average velocity of the wave infinitesimal -> long time spend on top of the barrier

Every pole is a resonance (positive energy finite lifetime) but not all resonances (poles) are connected to bound states

Formal theory of scattering

$$H = H_{kin} + V \to H_0 + V(t) \qquad V \to V(t) = V e^{-\epsilon|t|}$$

Interaction is switched on adiabatically at t=0

- Time evolution pictures: Schrodinger, Heisenberg, Interaction

$$O_I(t) = e^{iH_0} O(0) e^{-iH_0 t} \longrightarrow \begin{aligned} H_{0,I}(t) &= H_0 \\ V_I(t) &= e^{iH_0} V e^{-iH_0 t} e^{-\epsilon|t|} \end{aligned}$$

$$|t\rangle_I = e^{iH_0 t}|t\rangle_S \longrightarrow i\frac{d}{dt}|t\rangle_I = V_I(t)|t\rangle_I$$

- As t → ± ∞ interaction picture states evolve to eigenstates of $H_{kin,}$ i.e. to free particles

- At t=0 interactions picture states are solution of the full Hamiltonian

S-matrix and T-matrix : Lippmann-Schwinger

$$i\frac{d}{dt}|t\rangle_I = V_I(t)|t\rangle_I \quad \longrightarrow \quad |t\rangle_I = U(t,-\infty)|initial\rangle \; (t \to -\infty)$$

Evolution operator

- S-matrix

$$S_{fi} = \langle f(t=+\infty)|i(t=-\infty)\rangle = \langle f,(out)|i,(in)\rangle = \langle f|U(+\infty,-\infty)|i\rangle$$

$$U(+\infty,-\infty) = \mathcal{P}\exp\left(-i\int_{-\infty}^{+\infty} dt V_I(t)\right) = I - 2\pi i\delta(E_f - E_i)T$$

- T-matrix

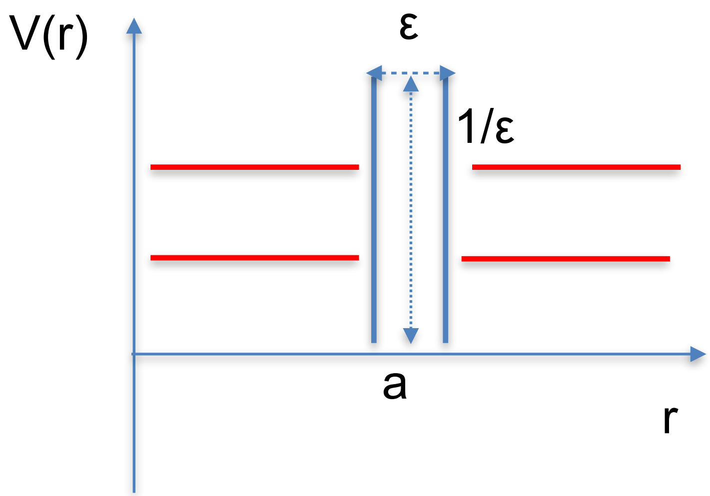
$$T = V + VG_0V + \cdots \qquad G_0 = \frac{1}{E - H_0}$$

$$E = E_i = E_f$$

Infinite, thin shell

$$H = \frac{p^2}{2\mu} + V \qquad V = \frac{\lambda}{2\mu a^2}\delta(r-a) \qquad \dim\lambda = -1$$

“Relation” to QCD



Inside the shell (0<r<a) particles are confined (like quarks in hadrons)
The shell is thin allowing for free asymptotic states (hadron decays)

Method 1: In coordinate space (as before)

Method 2: Lippmann-Schwinger

$$T = V + VG_0V + \cdots$$

Solution

From method 1

Looks like a K-matrix parametrization

$$f(k) = \frac{\left[-\lambda \frac{\sin^2(ka)}{(ka)^2}\right]}{\left[1 + \frac{\lambda}{a}\frac{\sin(ka)\cos(ka)}{ka}\right] - ik\left[-\lambda \frac{\sin^2(ka)}{(ka)^2}\right]}$$

$E = k^2/2\mu$

$$f(k) = \frac{K(E)}{1 - iK(E)k} = \frac{1}{K^{-1}(E) - ik}$$

$$K(E) = \frac{-\lambda \frac{\sin^2(ka)}{(ka)^2}}{1 + \frac{\lambda}{a}\frac{\sin(ka)\cos(ka)}{ka}} = \frac{P(k)}{Q(k)}$$

∞ of zeros

∞ of zeros → Poles of the amplitude

Solution

From method 2

Looks like a Chew-Mandelstam (dispersive) parametrization

$$f(k) = \frac{-\lambda \frac{\sin^2(ka)}{(ka)^2}}{1 - \frac{1}{\pi}\int_0^\infty dE' k' \frac{-\lambda \frac{\sin^2(k'a)}{(k'a)^2}}{E' - E(k)}}$$

(∞) zeros of K !

$$k' = k(E') = \sqrt{2\mu E'}$$

Compare with the K-matrix

$$f(k) = \frac{K(E)}{1 - ikK(E)}$$

$$K(E) = \frac{-\lambda \frac{\sin^2(ka)}{(ka)^2}}{1 - \frac{1}{\pi}\Re\int\cdots}$$

$$K(E) = \frac{-\lambda \frac{\sin^2(ka)}{(ka)^2}}{1 + \frac{\lambda}{a}\frac{\sin(ka)\cos(ka)}{ka}}$$

Analyticity of f

$$f(k) = \frac{\left[-\lambda \frac{\sin^2(ka)}{(ka)^2}\right]}{\left[1 + \frac{\lambda}{a}\frac{\sin(ka)\cos(ka)}{ka}\right] - ik\left[-\lambda \frac{\sin^2(ka)}{(ka)^2}\right]}$$

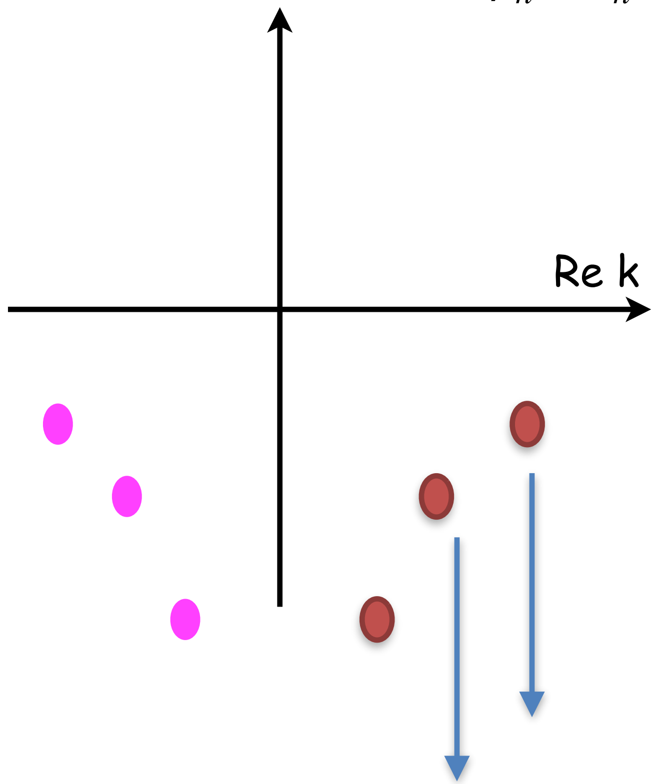
$$K(E) = \frac{-\lambda \frac{\sin^2(ka)}{(ka)^2}}{1 + \frac{\lambda}{a}\frac{\sin(ka)\cos(ka)}{ka}}$$

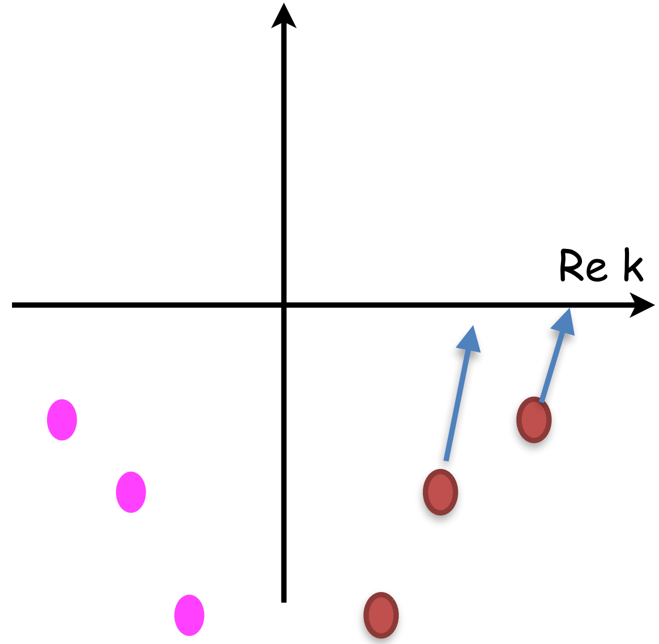
- "Conspiracy" between zeros and poles. f has an infinite number of zeros and poles (so does K). The ∞ number of zeros of K(s) is because of the by geometry of the sphere ("dynamics") and this specific "physics" fixes poles of the amplitude. Zeros of the amplitude and poles are related (CDD ambiguity)
- There is an essential singularity at infinity in the physical sheet ! Difficulty in writing dispersion relations. This is typical for cut-off potential and possibly similar in confining theories (?) (see relation with causality).

$$f(k = k_R + i(k_I \to \infty)) = O(e^{+2k_I a})$$

- For any strength of the potential there is an infinite number of resonances
- There is one pole in each strip $(n-1)\pi < \Re(\beta_n) < n\pi \quad (n = 1,2,\cdots)$

$$\beta_n = k_n a$$





- as potential strength decreases :

$$\beta_n \to (n - \frac{1}{2}) - i\infty$$

- as potential strength increases :

$$\beta_n \to n\pi\left(1 - \frac{1}{1+A}\right) - i\left(\frac{n\pi}{A}\right)^2$$

$$A = \lambda / a$$

Simple model for how external particles (e.g. π's) can access "quark model" states

$2\mu E = k^2$

$V(r) = \frac{\hat{\lambda}}{2\mu}\delta(r-a) \quad -\frac{d^2u}{dr^2} + V(r) = E\,u(r)$

$\lambda = \frac{\hat{\lambda}}{a^2}$

$u = \sin ka$ ← regular at $r=0$

$r < a$

$u = A\sin ka + B\cos ka \quad r > a$ free wave.

at a: $A\sin ka + B\cos ka = \sin ka$

$akA\cos ka - akB\sin ka = ak\cos ka + \lambda^a \sin ka$

$-u_+' + u_-' + V = 0$

$u_+' = u_-' + V$

$\Rightarrow f = \ldots \quad$ Bound states $= \ldots$

$$A \sin ka + B \cos ka = \sin ka$$

$$+ A' k \cos ka - B' k \sin ka = k \cos ka + \lambda s$$

$$\begin{pmatrix} -ks & -c \\ -kc & s \end{pmatrix} \begin{pmatrix} c \\ kc + \lambda s \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} \qquad A = -\frac{k}{a} - \lambda c s, \quad B = \lambda s^2 a$$

$$u(r>a) = A\left[\frac{e^{ikr} - e^{-ikr}}{2i}\right] + B\left[\frac{e^{ikr} + e^{-ikr}}{2}\right]$$

(large distances away from scatterers)

$$= -\frac{1}{2i} e^{-ikr}[A - iB] + \frac{1}{2i} e^{+ikr}[A + iB]$$

outgoing $\times e^{-iEt}$

$$= -\frac{1}{2i}[A - iB]\left[e^{-ikr} - \frac{A+iB}{A-iB} e^{+ikr}\right]$$

incoming

$-S$

compare with free: $e^{-ikr} - e^{+ikr}$

$$S = \frac{A+iB}{A-iB} \qquad f = \frac{S-1}{2i} = \frac{B}{A - iB} = \frac{a\lambda s^2}{-\frac{k}{a} - \frac{\lambda}{a} c s - i\frac{\lambda}{a} s^2}$$

$$f = \frac{a\lambda \sin^2 ka}{-\frac{k}{a} - \frac{\lambda}{a}\sin ka \cos ka - i\frac{\lambda}{a}\sin^2 ka} ; \qquad f = \frac{1}{-\frac{[k + \lambda s c]}{\lambda s^2} - i}$$

$$\tilde{f} = \frac{f}{k} = \frac{1}{K^{-1} - ik} \qquad K^{-1} = -\left[1 + \lambda\left(\frac{s}{k}\right)c\right]\left(\frac{k}{s}\right)^2 \frac{1}{\lambda}$$

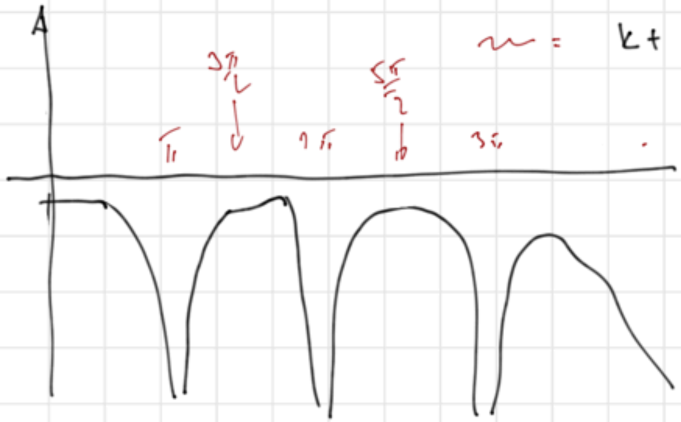
$$k^{-1} = -[1+\lambda(\frac{s}{k})c](\frac{k}{s})^2\frac{1}{\lambda} \qquad \lambda=1 \quad a=1$$

$\sin ak \quad k = \frac{\pi}{a}n$

$\sim = k + T/2 \sin 2ka$

oscillatory maxima

$k \sim \frac{\pi(n+1/2)}{a}$



$k^{-1} - ik$ will have zeros in the strip

$n\frac{\pi}{a} < \mathrm{Re}\, k < \frac{(n+1)\pi}{a} \quad n=0,1,\ldots$

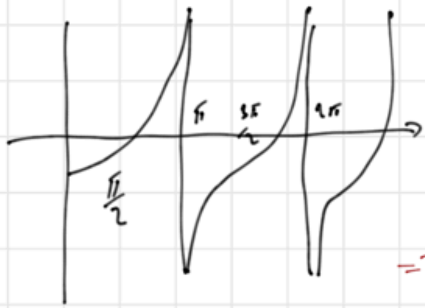
at $k=0 \Rightarrow \tilde{f}^{-1} = -1-\lambda$ is finite.

(for $\lambda<0$, $\lambda\approx-1$ there is an additional state, near zero energy bound state)

(1) Poles of $k^{-1} \Rightarrow$ zeros of amplitude $\Rightarrow$ CDD poles.

near zeros of f^{-1} there are poles $\Rightarrow$ resonances

In the limit of $A\to\infty \Rightarrow$



⇒ poles (resonances) come to the real axis $k = (n+\frac{1}{2})\frac{\pi}{a}$

⇒ this violates unitarity.

It is interesting to study asymptotic behavior of $f(k)$

take $k = iR \quad R \to \infty$ (upper half plane)

$$f = \frac{x\sin^2 ka \to e^{2R}}{-k - x\sin ka\cos ka - i x \sin^2 ka}$$

$$+ \frac{x}{4}\frac{1}{i}e^{2R} + \frac{ix}{4}e^{2R} \to \sim e^{2R}$$

$\sin k \to \frac{1}{2i}(-e^{R})$

$\cos k \to \frac{1}{2}(e^{R})$

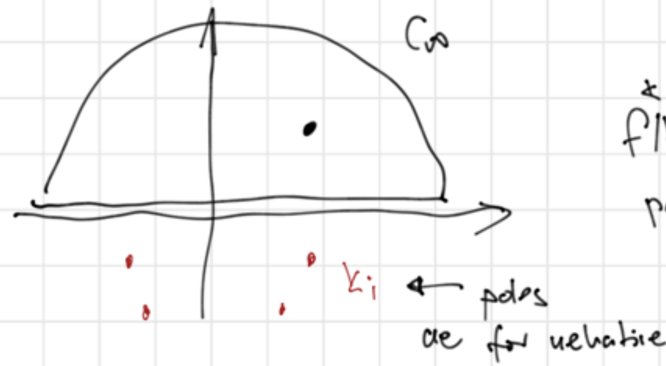
f → blows up exponentially as e^{R} in the upper plane.

$k \to -iR \quad R \to \infty \quad \sin k \to \frac{1}{2i}e^{R} \quad \cos k \to \frac{1}{2}e^{R}$

$$\sim -\frac{x}{4}\frac{1}{i}e^{2R} + i\frac{x}{4}e^{2R} \sim e^{2R} \qquad f \to \text{const}$$

⇒ the blow up is related to existence of ∞ of poles/res

Often one emphasizes dispersion relation.



$f^*(k) = f(-k^*)$

pole at k
has a mirror
at $-k^*$

Consider $\hat{f} = \tilde{f} e^{+2ika}$ ⇒ convergent in the upper plane.

there is ∞ of poles in the upper plane.

$$\hat{f}(k) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} dk' \frac{\hat{f}(k')}{k'-k} + \int_{C_\infty} \to 0$$

to write this one needs to know the assymptotic
behavior ⇒ this is manifestation of the (∞) zeros/poles.

Alternatively one could use the lower half plane
⇒ there one needs the resonances explicitly,

(as we'll see the data is on upper plane)

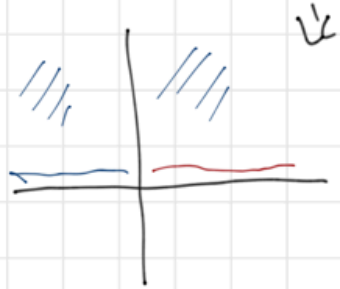
Use energy plane

$k = \sqrt{E 2\mu}$ Use the following determination $\sqrt{z} = \sqrt{|z|e^{i\phi}} = \sqrt{|z|}e^{i\phi/2}$

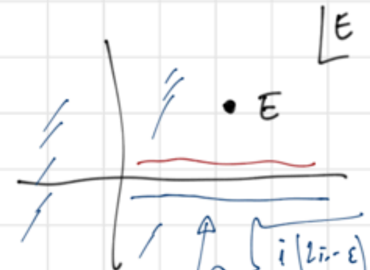
$0 < \phi < 2\pi$

$k>0$

$f^*(k+i\varepsilon) = f(-k+i\varepsilon)$ $\qquad f^*(E+i\varepsilon) = f(E-i\varepsilon)$



$\rightarrow$



$\sqrt{E} = \sqrt{|E|}e^{i(2\pi-\varepsilon)} = \sqrt{|E|}e^{i\phi}e^{-i\varepsilon}$

$= -\sqrt{|E|}(1-i\varepsilon) = -\sqrt{|E|}+i\varepsilon$

one can write dispersion for $f(E)$

$$\frac{1}{2\pi i}\left[\int_{+\infty}^{0} dE' \frac{\hat{f}(E'-i\varepsilon)}{E'-E} + \int_0^{\infty} dE' \frac{\hat{f}(E'+i\varepsilon)}{E'-E}\right] = \frac{1}{2\pi i}\int_0^{\infty} \frac{2i\,\mathrm{Im} f(E+i\varepsilon)}{E'-E}$$

$$\rightarrow f(E) = \frac{1}{\pi}\int_0^{\infty} dE' \frac{\mathrm{Im} f(E')}{E'-E}$$

$$\hat{f} = \frac{\lambda \sin^2 ka\, e^{2ika}}{-k - \lambda \sin 2ka \cos ka - i\lambda \sin^2 ka} \times \frac{1}{k} \quad \Big| \quad \hat{f} = e^{2ika}\tilde{f}$$

$\mathrm{Im}\,\tilde{f} = k|\tilde{f}|^2 \Leftarrow$ unitarity. $\Rightarrow$ dispersion relation would involve f only!

but we need $\hat{f}$ $\Rightarrow$ more information than dispersion relation:

What about $\frac{1}{\hat{f}}$ $\Rightarrow$ $\mathrm{Im}\frac{1}{\hat{f}} = ik$, It converges asymptotically

but needs ∞ of poles (CDD).

Supposed the subtraction were not need.

$\mathrm{Im} f = k|f|^2 \Rightarrow$ reason.

$$f(E) = \frac{1}{\pi}\int dE' \frac{\mathrm{Im}\, f(E')}{E' - E} \Rightarrow \text{continuation}$$

if $\mathrm{Im} f$ is known

$$\hat{f}_B = \frac{-\hat{\lambda} \sin^2 ka}{(ka)^2}$$

$$\hat{f} = \frac{\gamma \sin^2 ka \, e^{2ika}}{-k - \gamma \sin 2ka \cos ka - i\gamma \sin^2 ka} \times \frac{1}{k}$$

vs unitarity Born: $f_B = \gamma \sin^2 ka$

$$f = \frac{-\gamma \sin^2 ka/k}{1 + \int \frac{\gamma \sin^2 ka/k}{E' - E}} \frac{1}{k}$$

$$f = \frac{(-\gamma \sin^2 ka/k)}{\left(1 - \int \frac{(-\gamma \sin^2 ka/k')}{E' - E}\right)} \frac{1}{k} = \frac{N}{1 - \int \frac{\rho N}{E' - E}} \quad \rho = k$$

$$\frac{1}{f} = \frac{1}{N} - \frac{1}{N} \int \rho \frac{N}{E' - N} \quad \mathrm{Im} \frac{1}{f} = -i\rho$$

$\frac{1}{f}$ = dispt poles $\quad \frac{1}{f}$ converged

$$\frac{k}{\sin k} + \frac{\gamma \cos ka}{\sin ka}$$

$$\langle \vec{k}'|\vec{k}\rangle = (2\pi)^3 \delta^3(k'-k) = (2\pi)^3 \frac{1}{k^2}\delta(k'-k)\,\delta^2(\Omega'-\Omega) =$$

$$= (2\pi)^3 \frac{1}{k^2}\left(\frac{dk}{dE}\right)^{-1} \delta(E'-E)\,\delta^2(\Omega'-\Omega) \qquad E = \frac{k^2}{2\mu}$$

$$= (2\pi)^3 \frac{1}{k\mu}\,\delta(E'-E)\,\delta^2(\Omega'-\Omega) = (2\pi)^3 \frac{1}{k\mu}\,\delta(E'-E)\sum_{lm} Y^*_{lm}(\Omega')\,Y_{lm}(\Omega)$$

$V + VG_0V + \dots$

$$\langle \vec{k}'|S|\vec{k}\rangle : \sim - (2\pi)\,\delta(E'-E)\; i\, \langle k'|T|k\rangle$$

$$\langle k'|T|k\rangle = \sum T_l\, Y_{lm} Y_{lm}$$

$$\langle \vec{k}'|S|\vec{k}\rangle = (2\pi)^3 \frac{1}{k\mu}\,\delta(E'-E)\sum_{lm} Y_{lm} Y_{lm} \times$$

$$\Big[\, 1 - \underbrace{\frac{1}{(2\pi)^2}\,\frac{(2\pi)}{4}\,(2\mu)\,k\,T_l}\,\Big]$$
$\quad S_l$

$$f_l = \frac{(S-1)}{2i}$$

$$S_l S_l^* = 1 \qquad S_l = e^{2i\delta} \qquad [\dots] = 1 + 2ikf_l = e^{2i\delta}$$

$$\Rightarrow f_l = \frac{(e^{2i\delta}-1)}{2ik} = \frac{e^{i\delta}\sin\delta}{k}$$

$$f_l = -\frac{1}{(4\pi)^2}\,(2\underbrace{\mu T_l}_{-1})$$

units $T_l = \langle k|V|k\rangle : -3+1 = -2$

$$\langle \vec{k}'|T|\vec{k}\rangle = \sum T_l Y_{lm} Y_{lm} = \sum_l \frac{(2l+1)}{4\pi}\,T_l\,P_l(\cos\theta)$$

$$T_l = \frac{1}{2}\int dz\, P_l(\cos\theta)\, 4\pi\, \langle \vec{k}'|T|\vec{k}\rangle \Rightarrow f_l = -\frac{1}{2}\left(\frac{1}{4\pi}\right)\int dz\, P_l(z)\,\langle k'|2\mu V|k\rangle$$

$$f_\ell = -\frac{1}{2}\left(\frac{1}{4\pi}\right)\int dz\, P_\ell(z)\, \langle k' | 2\mu V | k \rangle$$

$$f_\ell = -\frac{1}{2}\left(\frac{1}{4\pi}\right)\int dz\, P_\ell\underset{\hat{k}\cdot\hat{k}'}{(z)}\, \langle k' | 2\mu T | k \rangle$$

$$2\mu T = V + VG_0V + \dots \qquad G_0 = \frac{1}{2\mu H - 2\mu E}$$

Take $V = \frac{A}{2\mu a^2}\delta(r-a)$ $\qquad [A] = -1$

$$\langle k' | 2\mu V | k \rangle = \int d^3r\, e^{ik'x}\, e^{-ikx}\, \frac{A}{a^2}\delta(r-a) \qquad \left\{ e^{ikx} = \sum_\ell j_\ell(ka)\, i^\ell\, 4\pi YY \right.$$

$$= \sum_\ell [j_\ell(ka)]^2 A\, (2\ell+1)\, P_\ell(x_{kk'})\, 4\pi$$

$$-\frac{1}{2}\frac{1}{4\pi}\int dz\, P_\ell(x) = -j_\ell^2(ka) A = -\frac{\sin^2 ka}{(ka)^2} A$$

$$f_0 = -\frac{1}{4\pi} \frac{8\mu^2 ka}{(ka)^2} A$$

$$f_\ell = -\frac{1}{2}\left(\frac{1}{4\pi}\right)\int dt\, P_\ell(t) \langle k' | 2\mu V | k\rangle$$

compute in 2^{nd} order:

$$\langle k' | 2\mu V G_0 2\mu V | k\rangle = \int d^3x\, e^{ik'x} \frac{A}{a^2}\delta(r-a)\, G_0 \frac{A}{a^2}\delta(r'-a) e^{-ikx'}$$

$$G_0(x,x') = \frac{1}{(2\pi)^3} e^{-ik''(x-x')} \frac{1}{k''^2 - 2\mu E} \quad (-)$$

$$\langle k' | 2\mu V | k''\rangle = \sum_\ell j_\ell^2 A (2\ell+1) P_\ell(x) 4\pi = \sum_\ell j_\ell^2 A (4\pi)^2 YY$$

$$\int \frac{k^2 dk}{(2\pi)^3} j_\ell^4 A^2 (4\pi)^4 \underbrace{Y(k)Y(k')}\frac{1}{k^2 - 2\mu E} \quad (-)$$

$-j^2 \cdot i^\ell j$

$-j^\ell$

$\frac{1}{r} j$

$$\int \frac{k^2 dk}{(2\pi)^3} (4\pi)^3 j_\ell^4 A^2 (2\ell+1) \frac{1}{k^2 - 2\mu E} \times P_\ell \left(-\frac{1}{2}\right)\frac{1}{4\pi} \quad (-)$$

$j^2 j^2$

$$\odot = \frac{\odot (4\pi)^2}{2(2\pi)^3} \int k^2 dk\, A^2 \frac{j_\ell^4}{k^2 - 2\mu E} = (-)\frac{2}{\pi}\int k^2 dk \frac{j_\ell^4 A^2}{k^2 - 2\mu E}$$

$$\frac{dE}{dk} = \frac{k}{\mu} \qquad dk = \frac{dk}{dE} dE = \frac{\mu}{k} dE = \frac{2\mu k\, dE}{2\mu\left(\frac{k^2}{2\mu} - 2\mu E\right)} = \frac{2}{2}\frac{k\, dE}{E}$$

S-matrix properties (in relativistic theory)

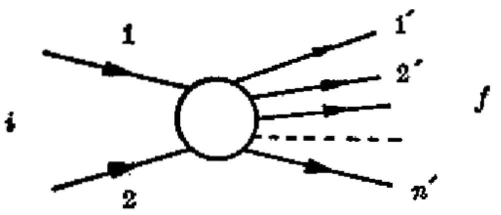
- There are no potentials
- Particles and antiparticles are related by crossing
- There are NO exact, non perturbative methods in QFT (major challenge for mathematicians)
- Physics lows are manifested as singularities of analytical functions (observables)

First order of business: understand properties of reactions enforced by these general principles.

S-matrix properties (in relativistic theory)

- Related to transition probability

$$P_{fi} = |\langle f|S|i\rangle|^2 = \langle i|S^\dagger|f\rangle\langle f|S|i\rangle$$



- Conservation of Probability = Unitarity

$$\sum_f P_{fi} = 1$$

$$S^\dagger S = I$$

$$2ImT_{ft} = \sum_n 2\pi\delta(E_i - E_n)T^*_{fn}T_{ni}$$

- Lorentz symmetry: T is a product of Lorentz scalars and covariant factors representing wave functions of external states, e.g for $\pi(k_1) + N(p_1, \lambda_1) \to \pi(k_2) + N(p_2, \lambda_2)$

$$\bar{u}(p_1, \lambda_1)[A(s,t) + (k_1 + k_2)_\mu \gamma^\mu B(s,t)]u(p_2, \lambda_2)$$

- Crossing symmetry: the same scalar functions describe all process related by permutation of legs between initial and final states (only the wave function change)

$\pi(k_1) + \pi(-k_2) \to \bar{N}(-p_1, \mu_1) + N(p_2, \mu_2)$

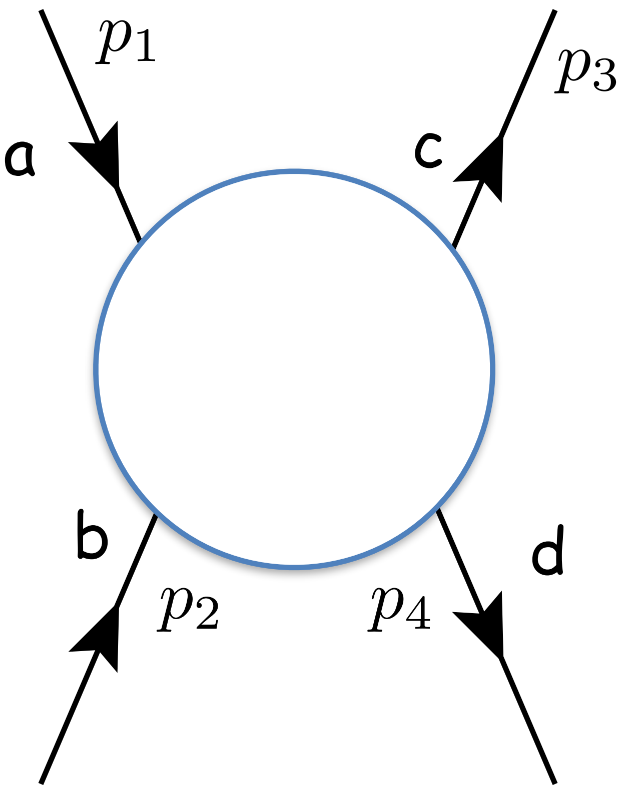
$$\bar{v}(p_1, \mu_1)[A(s,t) + (k_1 + k_2)_\mu \gamma^\mu B(s,t)]u(p_2, \mu_2)$$

- Analyticity: The scalar functions are analytical functions of invariants

Lorentz symmetry

N-to-M scattering depends on 4(N+M)-(N+M)-10 = 3(N+M)-10 invariants

e.g for 2-to-2: 2 invariants related to the c.m. energy and scattering angle



$$s = (p_1 + p_2)^2 = (E_{1,cm} + E_{2,cm})^2$$

$$t = (p_1 - p_3)^2$$

$$t = m_1^2 + m_2^2 - 2E_{1,cm}E_{2,cm} + 2|p_{1,cm}||p_{2,cm}|z_s$$

$$u = (p_1 - p_4)^2 \qquad s + t + u = \sum_i m_i^2$$

$$u = m_1^2 + m_4^2 - 2E_{1,cm}E_{4,cm} - 2|p_{1,cm}||p_{4,cm}|z_s$$

$$2\pi\delta(E_f - E_i)iT = \langle c, d|(S - 1)|a, b\rangle$$

Dimensions $\langle p', \beta|p, \alpha\rangle = 2E(\mathbf{p})\delta(\mathbf{p}_f - \mathbf{p}_i)\delta_{\alpha,\beta}$

$$T = (2\pi)^3\delta(\mathbf{p}_f - \mathbf{p}_i)A(s, t, u)$$

r.h.s has dim = -4

A(s,t,u) is a scalar function of mass dimension =0

Question

How many independent variables describe

- Decay proces A → a + b +c
- Three particle production A +B → a + b + c

Helicity amplitudes

We work in the c.m. frame $$\frac{\vec{S}\cdot\vec{p}}{|\vec{p}|}|p,\lambda\rangle = \lambda|p,\lambda\rangle$$

$$\langle p_3,\lambda_3;p_4,\lambda_4|A|p_1,\lambda_1;p_2,\lambda_2\rangle = A_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}(s,t,u)$$

Helicity states vs canonical spin states: $$S_z|p,m\rangle_z = m|p,m\rangle_z$$

$$|p,m\rangle_z = \Lambda(\vec{p}\leftarrow 0)|0,m\rangle_z$$

$$|p,\lambda\rangle = R(\hat{p})\Lambda(|\vec{p}|\hat{z}\leftarrow 0)|0,m\rangle_z$$

Exercise show this: $$|p,\lambda\rangle_z = \sum_{m=-S}^{S}|p,m\rangle_z D^S_{m,\lambda}(\hat{p})$$

- Even though this looks non relativistic it is relativistic. Notion of LS amplitudes, LS vs. helicity relations are relativistic

η = naturally

Parity $$A_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}(s,t,u) = \eta A_{-\lambda_1,-\lambda_2,-\lambda_3,-\lambda_4}(s,t,u)$$

Question

How many independent scalar functions describe

$J/\psi \rightarrow \pi^+ \pi^- \pi^0$

$\gamma\, p \to \pi^0\, p$

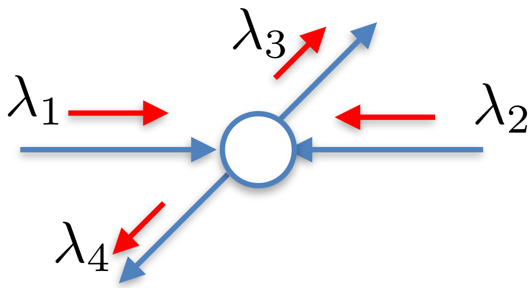
Kinematical vs Dynamical Singularities

For particles with spin

$$A_{\lambda_i}(s,t) = 16\pi \sum_{J=-M}^{M} (2J+1) f^J_{\lambda_i}(s) d^J_{\lambda,\lambda'}(\theta)$$

$$M = \max(|\lambda|, |\lambda'|)$$

$$\lambda' = \lambda_3 - \lambda_4 \quad \lambda = \lambda_1 - \lambda_2$$



$$f^J_{\lambda_i}(s) = \frac{1}{32\pi} \int_{-1}^{1} dz_s A_{\lambda_i}(s, t(s,\theta)) d^J_{\lambda,\lambda'}(\theta)$$

- Wigner d-functions lead to kinematical singularities

- Threshold (barrier factors) originate from kinematical factors in relation between t and cos(θ) (through dependence of A_λ on t)

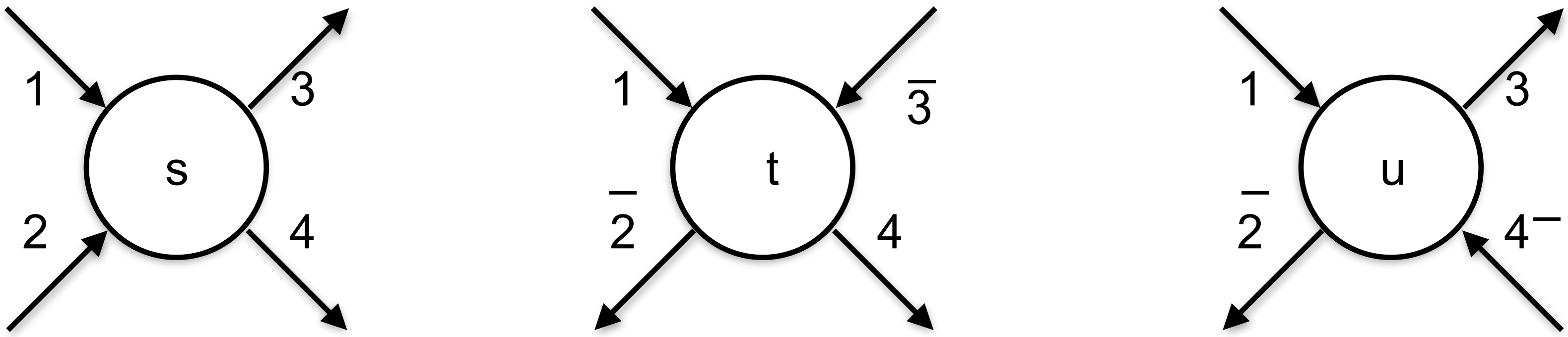
- Unequal masses give lead to "daughter poles"

- Dynamical singularities : from dynamical (unitary cuts) in A(s,t).

Crossing symmetry

$\bar{p}_i = -p_i = (-\vec{p}_i, -E_i)$

$a(p_1) + \bar{c}(\bar{p}_3) \to \bar{b}(\bar{p}_2) + d(p_4)$



$a(p_1) + b(p_2) \to c(p_3) + d(p_4)$

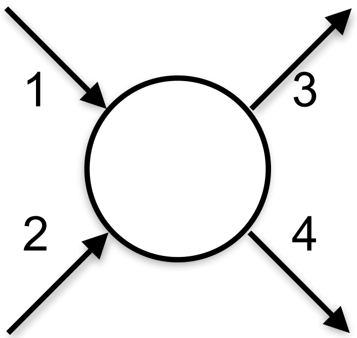
$a(p_1) + \bar{d}(\bar{p}_4) \to c(p_3) + \bar{b}(\bar{p}_2)$

$E_{c.m}$	$s = (p_1 + p_2)^2$	$t = (p_1 + p_{\bar{3}})^2$	$u = (p_1 + p_{\bar{4}})^2$
Cos(θ)	$t = (p_1 - p_3)^2$	$s = (p_1 - p_{\bar{2}})^2$	$t = (p_1 - p_3)^2$
Cos(θ)	$u = (p_1 - p_4)^2$	$u = (p_1 - p_4)^2$	$s = (p_1 - p_{\bar{2}})^2$

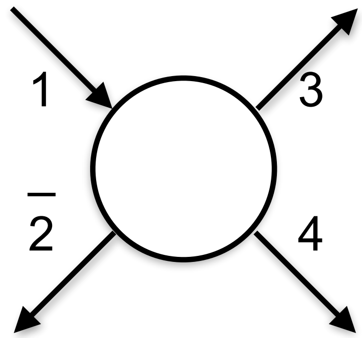
$$A^{(s)}_{\lambda_1,\cdots}(s + i\epsilon, t, u) \to \sum_{\lambda'_1,\cdots} [D^{S_1}_{\lambda_1,\lambda'_1} \cdots] A^{(t)}_{\lambda'_1,\cdots}(s, t + i\epsilon, u) \to \cdots$$

- The iε is important. Function values at, e.g. s + iε vs s - iε are different !

Crossing Symmetry : Decays $M_1 > m_2 + m_3 + m_4$



$a(p_1) + b(p_2) \rightarrow c(p_3) + d(p_4)$



$a(p_1) \rightarrow \bar{b}(\bar{p}_2) + c(p_3) + d(p_4)$

$$A(s,t,u) \rightarrow A(M_1^2 + i\epsilon, s + i\epsilon, t + i\epsilon, u + i\epsilon)$$

- In decay kinematics, the decaying mass becomes a dynamical variable, (iε important)

- Crossing from one kinematical region (e.g. s-channel) to another (e.g. t-channel) requires taking the corresponding variables off the real axis and to the complex plane : analytical continuation.

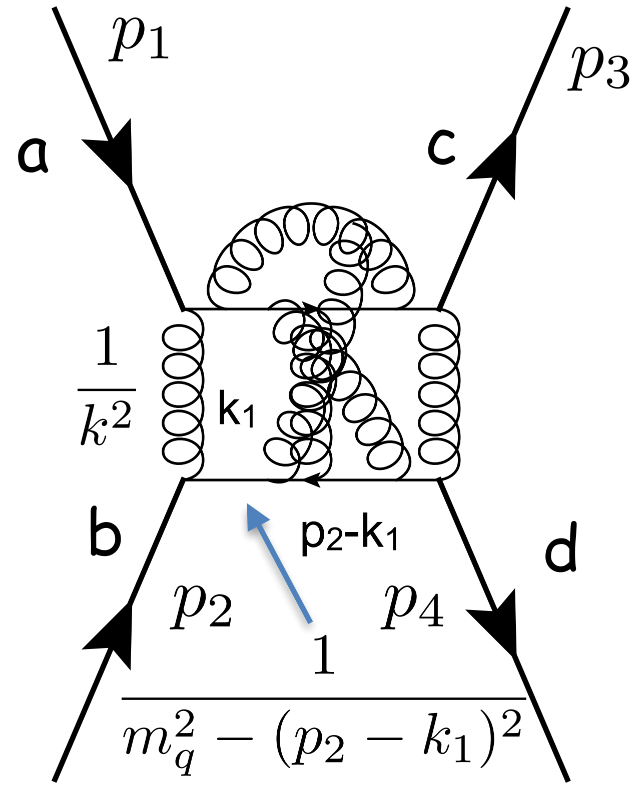
Feynman diagrams

$$A(p_1,\cdots) \propto \int [\Pi_j d^4k_j] \frac{\text{polynomial in } k_j}{(m_q^2 - (p_i - k_j)^2 - i\epsilon)((k_i - k_j)^2 - i\epsilon)\cdots}$$

$$m^2 - p^2 = [m^2 + \mathbf{p}^2] - (p^0)^2$$

$$m^2 - p^2 = 0 \to p^0 = \pm(m^2 + \mathbf{p}^2)^{1/2}$$

- Integrand becomes singular when intermediate states go on shell.
- Thresholds for producing physical intermediate are the only reason why amplitudes are singular.
- Production of intermediate states is related to unitarity. Thus we expect unitarity to determine singularities of the amplitudes.

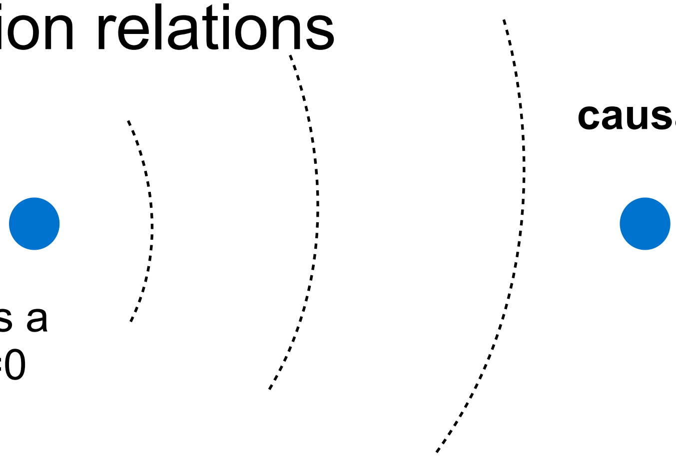


On the role of iε

$$\mathrm{Im}\left[\frac{1}{\sqrt{m^2+\mathbf{p}^2} \mp i\epsilon - p^0}\right] = \pm\pi\delta(p_0 - \sqrt{m^2+\mathbf{p}^2})$$

Analyticity and Causality

Dispersion relations



source emits a signal at t=0

causality: receiver receives at t>0 and not at t<0

amplitude of the signal

$$f(t) \propto \theta(t)$$

consider the Fourier transform (E $\rightarrow$ energy)

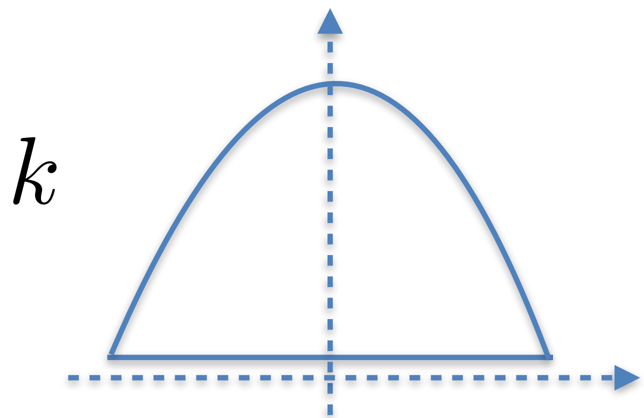
$$f(E) \equiv \int dt e^{iEt} f(t)$$

and extend definition to complex plane E $\rightarrow$ z, then f(z) is holomorphic for Im E > 0

Causality: The outgoing wave cannot appear before the incoming one. Causality determines analytical properties of the scattering amplitude as function on energy/momenta/scattering angle. The specific from of these conditions depend on the type of interactions and kinematics (e.g. relativistic vs non relativistic)

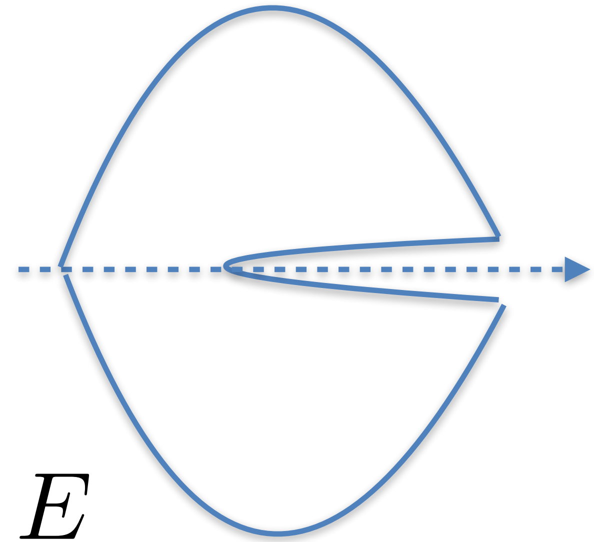
momentum vs energy planes

$$f^*(k) = f(-k^*) \qquad\qquad f^*(E) = f(E^*)$$



$$E = \frac{k^2}{2\mu}$$

$$k = \sqrt{2\mu E}$$



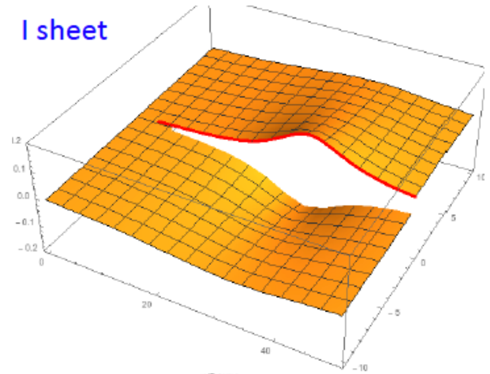
The function is analytical in the whole E-plane not only the upper half

How unitarity constrains singularities

- Unitarity "operates" in the physical domain, i.e. s real and above threshold and |Cos(θ)|<1. This domain is the boundary of the complex plane where analytical amplitude are defined

$$A(s+i\epsilon) = A_{\text{physical}}(s = \text{real and above threshold})$$

sign fixed by "arrow of time V(t) = V exp(-t |ε|)



- The difference (discontinuity) A(s + iε) - A(s - iε) ≠ 0 (cf. Feynman diagrams), comes from particle production this we expect it being determined by unitarity.

$$2ImT_{ft} = \sum_n 2\pi\delta(E_i - E_n)T^*_{fn}T_{ni}$$

- Cauchy theorem : singularities determine the amplitude !!!

Summary of Lecture 3

- In potential scattering partial wave amplitudes, in the energy plane, have branch points on the real axis and cuts. They are analytical everywhere else

- Resonances correspond to poles on the unphysical sheet of partial waves

- Some resonance are bound states. These are poles on the real axis on the physical sheet

- Opening of cuts is due to unitarity. This makes sense. When bound states become resonances they need to decay, a process "controlled" by conservation of probability. They need to move away from the physical sheet and unitarity give the option to exist by "opening" a cut so that they can dive to an unphysical sheet

- The same happens in relativistic theory. The extra complication is existence of "left hand cuts" from crossing symmetry.

- The number of invariant amplitude and variables are constrained by Lorentz symmetry + parity.