Exact Solution of the Instantaneous Bethe-Salpeter Equation for 1^+ States

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Outline

- Motivation
- Relativistic potential model
- Introduction to intantaneous Behte–Salpeter equation
- Mixing angles, mass spectra and inversion, decay constants
- Summary

Motivation

- Many new states are found recently, including the P−wave 1^+ states
- Excited state has larger relativistic correction than ground state
- Mixing angle of 1^+ states has large uncertainty

${}^1P_1$ and ${}^3P_1$ and $\frac{1}{2}^+$ and $\frac{3}{2}^+$ states

- In a heavy-light axial vector, the heavy quark spin decouple, the angular momentum of light quark is a good quantum number $\vec{j}_\ell = \vec{L} + \vec{S}_q$ (j_q=1/2 or 3/2, when L=1).
- So, in j-j coupling, L=1, there are two doublets (wide and narrow) in Heavy Quark Effective Theory (HQET) :

$$j_q^P = \frac{1}{2}^+ \; S \text{ doublet } (0^+, 1^+)$$
$$j_q^P = \frac{3}{2}^+ \; T \text{ doublet } (1^+, 2^+)$$

- There are two 1^+ states.

Mixing angle

- S–L couping states (relativistic model), $^1P_1$ and $^3P_1$ no longer the physical states, but their mixing.
- We define the rotation matrix：

$$R(\alpha) \equiv \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

- The relation between HQET states and $^1P_1$,$^3P_1$ states

$$\begin{bmatrix} |1,3/2\rangle \\ |1,1/2\rangle \end{bmatrix} = R(\theta_H) \begin{bmatrix} |^1P_1\rangle \\ |^3P_1\rangle \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{2} & 1 \\ -1 & \sqrt{2} \end{bmatrix} \begin{bmatrix} |^1P_1\rangle \\ |^3P_1\rangle \end{bmatrix}$$

where the mixing angle $\theta_H = \arctan\sqrt{1/2} = 35.3^\circ$

* Note: Somebody said there are two equivalent mixing angle, the other is –54.7, is this exactly correct?

Exist results from relativistic potential model

- Mass spectra

Th	D1	Ex	Th	Ds1	Ex
2469	$D_1(2430)$	2427(40)	2574	$D_{s1}(2460)$	2459.6(6)
2426	$D_1(2420)$	2423.4(3.1)	2536	$D_{s1}(2536)$	2535.35(60)

	B1			Bs1	
5774			5865	$B^{*}_{sJ}(5850)$	5853(15)?
5723	$B_1(5721)$	5723.4(2.0)	5831	$B_{s1}(5830)$	5829.4(7)

D. Ebert et al, Eur. Phys. J. C66 (2010) 197

Exist results from relativistic model

- Mixing angle D. Ebert et al, Eur. Phys. J. C66 (2010) 197

State	D	D_s	B	B_s
$1P$	35.5	34.5	35.0	36.0
$2P$	37.5	37.6	37.3	34.0

- Close to 35.3, ideal state in HQET ? but the mass difference between c and s (Ds1) is not very large ?

- Mixing angle can be calculated from the spin-orbit interaction etc between quarks, see for example:
  J. L. Rosner, Comm. Nucl. Part. Phys. 15 (1986) 109

Exist results from relativistic potential model

S. Godfrey, R. Kokoshi, PRD 43 (1991) 1679

State	$s\bar{u}$ (K)	$c\bar{u}$ (D)	$c\bar{s}$ (D_s)	$b\bar{u}$ (B)	$b\bar{s}$ (B_s)	$b\bar{c}$ (B_c)
$^3P_2$	1.43	2.50	2.59	5.80	5.88	6.77
Q_{high}	1.37	2.47	2.56	5.78	5.86	6.75
Q_{low}	1.35	2.46	2.55	5.78	5.86	6.74
$^3P_0$	1.24	2.40	2.48	5.76	5.83	6.71
$^3S_1$	0.90	2.04	2.13	5.37	5.45	6.34
$^1S_0$	0.47	1.88	1.98	5.31	5.39	6.27
$^3P_1$	1.37	2.47	2.55	5.78	5.86	6.74
$^1P_1$	1.35	2.46	2.55	5.78	5.86	6.75
θ	$-5°$	$-26°$	$-38°$	$-31°$	$-40°$	$68°$

- Much difference
- Bc1 has a negative angle？

Exist results from relativistic model

S. Godfrey, K. Moats, PRD 93 (2015) 034035

State	$c\bar{q}$		$c\bar{s}$	
	Mass	β_{eff}	Mass	β_{eff}
$1P_1$	2456	0.475, 0.482	2549	0.498, 0.505
$1P_1'$	2467	0.475, 0.482	2556	0.498, 0.505
1^3P_0	2399	0.516	2484	0.542
θ_{1P}	-25.68°		-37.48°	

- The angle of Ds1 close to the HQET result, not D1.

Exist results from relativistic model

S. Godfrey, et al, PRD 94 (2016) 054025

	GI $b\bar{q}$		GI $b\bar{s}$		ARM $b\bar{q}$		ARM $b\bar{s}$	
State	Mass	β_{eff}	Mass	β_{eff}	Mass	β_{eff}	Mass	β_{eff}
1^3S_1	5371	0.542	5450	0.595	5316	0.586	5400	0.616
1^1S_0	5312	0.580	5394	0.636	5275	0.628	5366	0.651
1^3P_2	5797	0.472	5876	0.504	5754	0.465	5836	0.487
$1P_1$	5777	0.499, 0.511	5857	0.528, 0.538	5738	0.481, 0.492	5822	0.500, 0.507
$1P_1'$	5784		5861		5753		5830	
1^3P_0	5756	0.536	5831	0.563	5720	0.525	5805	0.531
θ_{1P}	30.28°		39.12°		43.6°		37.9°	

Problems of mixing angle

- If the angles 35.3 and −54.7 are equivalent?
- The existing angles are confused, some are positive, others negative, with large range.
- Thus mixing angle is always treated as a free parameter in production or decay process.

Mass problem

broad state	narrow state
K1(1400) heavy	K1(1270) light
D1(2430) heavy	D1(2420) light
Ds1(2460) light	Ds1(2536) heavy

- Only narrow B1、Bs1 have been found.
- Low masses of Ds0(2317) and Ds1(2460)： coupled-channel-effect theory can answer this queation.
- Is there other possible effect to make the mass inverse between Ds1(2460) and Ds1(2536)?

Possible mass inversion

- 1978年，Howard J. Schnitzer, PLB 20 （1978）461

INVERTED CHARMED MESON MULTIPLETS AS A

TEST FOR SCALAR CONFINEMENT

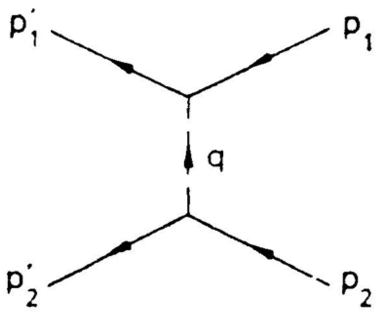
by

Howard J. Schnitzer†

which implies that the spin-orbit force receives most of its contribution from the scalar interaction, which is therefore opposite in sign to the spin-orbit force observed in the self-conjugate mesons. The signal of such an effect would be inverted multiplets for the charmed D and F mesons. As we shall show,

Relativistic potential model

- Relativistic Potential between a electron and positron
- Electron and muon exchange a photon



$$M_{fi} = e^2(\bar{u}'_1\gamma^\mu u_1)D_{\mu\nu}(q)(\bar{u}'_2\gamma^\nu u_2),$$

$$q = p'_1 - p_1 = p_2 - p'_2;$$

V.B.Berestetskii,E.M. Lifshitz and L.P. Pitaevskii “Quantum Electrodynamics”

Coulomb gauge

- The coulomb gauge is chosen

$$D_{00} = -\frac{4\pi}{\mathbf{q}^2}, \qquad D_{0i} = 0, \qquad D_{ik} = \frac{4\pi}{\mathbf{q}^2 - \omega^2/c^2 - i0}\left(\delta_{ik} - \frac{q_i q_k}{\mathbf{q}^2}\right).$$

- Scattering amplitude：

$$M_{fi} = e^2\{(\bar{u}_1'\gamma^0 u_1)(\bar{u}_2'\gamma^0 u_2)D_{00} + (\bar{u}_1'\gamma^i u_1)(\bar{u}_2'\gamma^k u_2)D_{ik}\}$$

- Free spinor： $u = \sqrt{(2m)}\begin{pmatrix}(1 - \mathbf{p}^2/8m^2c^2)w \\ (\boldsymbol{\sigma}\cdot\mathbf{p}/2mc)w\end{pmatrix}$
- First term, expanded

$$\bar{u}_1'\gamma^0 u_1 = u_1'^* u_1$$

with $\mathbf{q} = \mathbf{p}_1' - \mathbf{p}_1 = \mathbf{p}_2 - \mathbf{p}_2'$

$$= 2m_1\left(1 - \frac{\mathbf{p}_1'^2 + \mathbf{p}_1^2}{8m_1^2c^2}\right) w_1'^* w_1 + \frac{1}{2m_1c^2} w_1'^*(\boldsymbol{\sigma}\cdot\mathbf{p}_1')(\boldsymbol{\sigma}\cdot\mathbf{p}_1) w_1$$

$$= 2m_1 w_1'^*\left\{1 - \frac{\mathbf{q}^2}{8m_1^2c^2} + \frac{i\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{p}_1}{4m_1^2c^2}\right\} w_1,$$

- Second term

$$\bar{u}_1'\gamma u_1 = u_1'^*\boldsymbol{\alpha} u_1$$
$$= (1/c)w_1'^*\{\boldsymbol{\sigma}(\boldsymbol{\sigma}\cdot\mathbf{p}_1) + (\boldsymbol{\sigma}\cdot\mathbf{p}_1')\boldsymbol{\sigma}\}w_1$$
$$= (1/c)w_1'^*\{i\boldsymbol{\sigma}\times\mathbf{q} + 2\mathbf{p}_1 + \mathbf{q}\}w_1,$$

- Define potential U：

$$M_{fi} = -2m_1\cdot 2m_2(w_1'^* w_2'^* U(\mathbf{p}_1, \mathbf{p}_2, \mathbf{q})w_1\, w_2)$$

- potential between electron-muon in momentum space

$$U(\mathbf{p}_1, \mathbf{p}_2, \mathbf{q}) = 4\pi e^2\left\{\frac{1}{\mathbf{q}^2} - \frac{1}{8m_1^2c^2} - \frac{1}{8m_2^2c^2} + \frac{(\mathbf{q}\cdot\mathbf{p}_1)(\mathbf{q}\cdot\mathbf{p}_2)}{m_1m_2\mathbf{q}^4} - \right.$$
$$-\frac{\mathbf{p}_1\cdot\mathbf{p}_2}{m_1m_2\mathbf{q}^2} + \frac{i\boldsymbol{\sigma}_1\cdot\mathbf{q}\times\mathbf{p}_1}{4m_1^2c^2\mathbf{q}^2} - \frac{i\boldsymbol{\sigma}_1\cdot\mathbf{q}\times\mathbf{p}_2}{2m_1m_2c^2\mathbf{q}^2} - \frac{i\boldsymbol{\sigma}_2\cdot\mathbf{q}\times\mathbf{p}_2}{4m_2^2c^2\mathbf{q}^2} +$$
$$\left. + \frac{i\boldsymbol{\sigma}_2\cdot\mathbf{q}\times\mathbf{p}_1}{2m_1m_2c^2\mathbf{q}^2} + \frac{(\boldsymbol{\sigma}_1\cdot\mathbf{q})(\boldsymbol{\sigma}_2\cdot\mathbf{q})}{4m_1m_2c^2\mathbf{q}^2} - \frac{\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2}{4m_1m_2c^2}\right\};$$

Potential between electron and positron

$$\hat{H} = \frac{\hat{\mathbf{p}}^2}{m} - \frac{e^2}{r} + \hat{V}_1 + \hat{V}_2 + \hat{V}_3,$$

$$\hat{V}_1 = -\frac{\hat{\mathbf{p}}^4}{4m^3c^2} + 4\pi\mu_0^2\delta(\mathbf{r}) - \frac{e^2}{2m^2c^2r}\left\{\hat{\mathbf{p}}^2 + \frac{\mathbf{r}\cdot(\mathbf{r}\cdot\hat{\mathbf{p}})\hat{\mathbf{p}}}{r^2}\right\},$$

$$\hat{V}_2 = 6\mu_0^2\frac{1}{r^3}\hat{\mathbf{l}}\cdot\hat{\mathbf{S}},$$

$$\hat{V}_3 = 6\mu_0^2\frac{1}{r^3}\left\{\frac{(\hat{\mathbf{S}}\cdot\mathbf{r})(\hat{\mathbf{S}}\cdot\mathbf{r})}{r^2} - \tfrac{1}{3}\hat{\mathbf{S}}^2\right\} + 4\pi\mu_0^2(\tfrac{7}{3}\hat{\mathbf{S}}^2 - 2)\,\delta(\mathbf{r}).$$

- where $\hbar\hat{\mathbf{l}} = \mathbf{r}\times\hat{\mathbf{p}}$ $\quad\hat{\mathbf{S}} = \tfrac{1}{2}(\boldsymbol{\sigma}_+ + \boldsymbol{\sigma}_-)$ $\quad\hat{\mathbf{S}}^2 = \tfrac{1}{2}(3 + \boldsymbol{\sigma}_+\cdot\boldsymbol{\sigma}_-)$
- V1: orbital term， V2： spin-orbit， V3： spin-spin

V.B.Berestetskii,E.M. Lifshitz and L.P. Pitaevskii “Quantum Electrodynamics”

Potential between quarks

- D. Ebert, et al, PRD 79 (2009) 114029
- Schordinger equation：

$$\left(\frac{b^2(M)}{2\mu_R}-\frac{\mathbf{p}^2}{2\mu_R}\right)\Psi_M(\mathbf{p})=\int\frac{d^3q}{(2\pi)^3}V(\mathbf{p},\mathbf{q};M)\Psi_M(\mathbf{q})$$

- potential

$$V(\mathbf{p},\mathbf{q};M)=\bar{u}_1(p)\bar{u}_2(-p)\mathcal{V}(\mathbf{p},\mathbf{q};M)u_1(q)u_2(-q),$$

$$\mathcal{V}(\mathbf{p},\mathbf{q};M)=\frac{4}{3}\alpha_s D_{\mu\nu}(\mathbf{k})\gamma_1^{\mu}\gamma_2^{\nu}+V_{\rm conf}^{V}(\mathbf{k})\Gamma_1^{\mu}\Gamma_{2;\mu}+V_{\rm conf}^{S}(\mathbf{k})$$

- Coulomb gauge

$$D^{00}(\mathbf{k})=-\frac{4\pi}{\mathbf{k}^2},\quad D^{ij}(\mathbf{k})=-\frac{4\pi}{k^2}\left(\delta^{ij}-\frac{k^ik^j}{\mathbf{k}^2}\right),\quad D^{0i}=D^{i0}=0$$

Potential between quarks

- Dirac spinor

$$u^{\lambda}(p) = \sqrt{\frac{\epsilon(p)+m}{2\epsilon(p)}} \begin{pmatrix} 1 \\ \frac{\boldsymbol{\sigma}\mathbf{p}}{\epsilon(p)+m} \end{pmatrix} \chi^{\lambda}$$

$$V(r) = V_{\rm SI}(r) + V_{\rm SD}(r)$$

$$V_{\rm SD}(r) = a_1\,\mathbf{L}\mathbf{S}_1 + a_2\,\mathbf{L}\mathbf{S}_2 + b\left[-\mathbf{S}_1\mathbf{S}_2 + \frac{3}{r^2}(\mathbf{S}_1\mathbf{r})(\mathbf{S}_2\mathbf{r})\right] + c\,\mathbf{S}_1\mathbf{S}_2 + d\,(\mathbf{L}\mathbf{S}_1)(\mathbf{L}\mathbf{S}_2)$$

- Note： relativistic potential， non-relativistic wave function, good to study mass spectra, but not good for production and decay。

- Our method with no expansion, the relativistic effect is kept to all orders, we provide more precise results.

- Further, relativistic wave function from a relativistic dynamic equation is suitable to calculate decays and productions, etc.

Introduction to instantaneous Bethe-Salpeter Eequation

- Bethe-Salpeter (BS) equation:

$$(\not{p}_1 - m_2)\chi(q)(\not{p}_2 + m_2) = i\int \frac{d^4k}{(2\pi)^4} V(P,k,q)\chi(k)$$

This is relativistic dynamic equation.

- Meson momentum P and relative momentum q:

$$p_1 = \alpha_1 P + q, \quad \alpha_1 = \frac{m_1}{m_1 + m_2}$$

$$p_2 = \alpha_2 P - q, \quad \alpha_2 = \frac{m_2}{m_1 + m_2}$$

E.E.Salpeter and H.A.Bethe, Phys. Rev. 84 (1951)1232.

- There is the difficulty about the kernel of BS Eq, it is a time-inspired interaction?
- Reduced version is needed, after instantaneous approximation first made by Salpeter, the BS equation become to the Salpeter equation
- We define :

$$q^{\mu} = q_{||}^{\mu} + q_{\perp}^{\mu} \; ,$$

$$q_P = \frac{(P\cdot q)}{M} \; , \qquad q_T = \sqrt{q_P^2 - q^2} = \sqrt{-q_{\perp}^2} \; .$$

where

$$q_{||}^{\mu} \equiv (P\cdot q/M^2)P^{\mu} \; , \quad q_{\perp}^{\mu} \equiv q^{\mu} - q_{||}^{\mu} \; .$$

In the center of mass system, they turn to q_0 and $|\vec{q}|$

Salpeter Eequation

- The reduced Bethe-Salpeter wave function:

$$\varphi_P(q_\perp^\mu) \equiv i \int \frac{dq_P}{2\pi} \chi(q_\parallel^\mu, q_\perp^\mu) \ ,$$

$$\eta(q_\perp^\mu) \equiv \int \frac{k_T^2 dk_T ds}{(2\pi)^2} V(k_\perp, q_\perp, s) \varphi_p(k_\perp^\mu) \ .$$

and the useful notations

$$\omega_i = \sqrt{m_i^2 + q_T^2} \ , \quad \Lambda_i^\pm(q_\perp) = \frac{1}{2\omega_i}\left[\frac{\not{P}}{M}\omega_i \pm J(i)(m_i + \not{q}_\perp)\right]$$

$$\varphi_P^{\pm\pm}(q_\perp) \equiv \Lambda_1^\pm(q_\perp)\frac{\not{P}}{M}\varphi_P(q_\perp)\frac{\not{P}}{M}\Lambda_2^\pm(q_\perp)$$

E.E.Salpeter, PRD 87 (1952) 328.

Salpeter wave function include four parts

$$\varphi_P(q_\perp) = \varphi_P^{++}(q_\perp) + \varphi_P^{+-}(q_\perp) + \varphi_P^{-+}(q_\perp) + \varphi_P^{--}(q_\perp)$$

And the Salpter equation can be written as:

$$\varphi_P(q_\perp) = \frac{\Lambda_1^+(q_\perp)\eta_P(q_\perp)\Lambda_2^+(q_\perp)}{(M-\omega_1-\omega_2)} - \frac{\Lambda_1^-(q_\perp)\eta_P(q_\perp)\Lambda_2^-(q_\perp)}{(M+\omega_1+\omega_2)} ,$$

Salpeter equation (positive and negative energy wave function)

$$(M-\omega_1-\omega_2)\varphi_P^{++}(q_\perp) = \Lambda_1^+(q_\perp)\eta_P(q_\perp)\Lambda_2^+(q_\perp) ,$$

$$(M+\omega_1+\omega_2)\varphi_P^{--}(q_\perp) = -\Lambda_1^-(q_\perp)\eta_P(q_\perp)\Lambda_2^-(q_\perp) ,$$

$$\varphi_P^{+-}(q_\perp) = \varphi_P^{-+}(q_\perp) = 0 .$$

E.E.Salpeter, PRD 87 (1952) 328.

Wave function and mixing angle

- The general wave function for 1^+ state is：

$$\varphi(1^+) = \frac{q_\perp \cdot \xi}{|\vec{q}\,|}\left(f_1 + f_2\frac{\not{P}}{M} + f_3\frac{\not{q}_\perp}{|\vec{q}\,|} + f_4\frac{\not{P}\not{q}_\perp}{M|\vec{q}\,|}\right)\gamma^5 + i\frac{\epsilon_{\mu P q_\perp \xi}}{M|\vec{q}\,|}\gamma^\mu\left(h_1 + h_2\frac{\not{P}}{M} + h_3\frac{\not{q}_\perp}{|\vec{q}\,|} + h_4\frac{\not{P}\not{q}_\perp}{M|\vec{q}\,|}\right)$$

where，f1、f2、f3、f4 have number of $J^{PC} = 1^{+-}({}^1P_1)$
while h1、h2、h3、h4 are $J^{PC} = 1^{++}({}^3P_1)$

- Then，mixing angle can be defined as：

$$\varphi(1^+) = \cos\theta_{nP}\varphi({}^1P_1) + \sin\theta_{nP}\varphi({}^3P_1)$$

Mixing angle

- 8 wave functions are not all independent

$$f_3 = -c_- f_1, \quad f_4 = -c_+ f_2, \quad h_3 = c_- h_1, \quad h_4 = c_+ h_2$$

with

$$\omega_i = \sqrt{m_i^2 + \vec{q}^{\,2}}, i = 1, 2; \; c_\pm = \frac{|\vec{q}\,|(\omega_1 \pm \omega_2)}{m_1\omega_2 + m_2\omega_1}$$

- Normalization condition

$$\int \frac{\mathrm{d}\vec{q}}{(2\pi)^3} \frac{A(\omega)}{3M}(f_1 f_2 - 2h_1 h_2) = 1 \qquad A(\omega) = \frac{8\omega_1\omega_2}{(m_1\omega_2 + m_2\omega_1)}$$

- So mixing angle

$$\cos^2\theta_{nP} = \int \frac{\mathrm{d}\vec{q}}{(2\pi)^3} \frac{A(\omega)}{3M} f_1 f_2,$$

$$\sin^2\theta_{nP} = \int \frac{\mathrm{d}\vec{q}}{(2\pi)^3} \frac{A(\omega)}{3M} (-2h_1 h_2)$$

Decay constant

- Definition of decay constant

$$f_{1^+} M \xi^\mu \equiv \langle 0|\bar{q}_1 \Gamma^\mu q_2|M,\xi\rangle$$

- In this method

$$\langle 0|\bar{q}_1 \Gamma^\mu q_2|M,\xi\rangle = -\sqrt{N_c}\int \frac{\mathrm{d}^3 q_\perp}{(2\pi)^3}\mathrm{Tr}[\varphi(q_\perp)\Gamma^\mu] = \frac{4\sqrt{N_c}}{3}\xi^\mu \int \frac{\mathrm{d}^3 q_\perp}{(2\pi)^3}(f_3 + 2h_4)$$

So

$$f_{1^+} = \frac{4\sqrt{N_c}}{3M}\int \frac{\mathrm{d}^3\vec{q}}{(2\pi)^3}(f_3 + 2h_4)$$

Potential

- Cornell potential (Coulomb term plus linear one)

$$\begin{aligned} V(r) &= V_s(r) + V_0 + \gamma_0 \otimes \gamma^0 V_v(r) \\ &= \frac{\lambda}{\mu}(1 - e^{-\mu r}) + V_0 - \gamma_0 \otimes \gamma^0 \frac{4}{3} \frac{\alpha_s}{r} e^{-\sigma r} \end{aligned}$$

- We don’t need a “relativistic” potential, because the “relativitic corrections” are already included in the wave function, otherwise “double counting” happens.

Numerical results

- Choice of parameters

$$\sigma = 0.060\ \text{GeV},\ \lambda = 0.125\ \text{GeV}^2,\ \Lambda_{\text{QCD}} = 0.252\ \text{GeV},\ \mu = 0.040\ \text{GeV}$$

$$m_u = 0.305\ \text{GeV},\ m_d = 0.311\ \text{GeV},\ m_s = 0.5\ \text{GeV},\ m_c = 1.72\ \text{GeV},\ m_b = 4.96\ \text{GeV}$$

$$V_0 = -0.1\ \text{GeV for}\ c\bar{q}\ (q = u, d, s),\ -0.41\ \text{GeV for}\ b\bar{u}(\bar{d}),$$
$$-0.18\ \text{GeV for}\ b\bar{s},\ \text{and}\ -0.05\ \text{GeV for}\ b\bar{c}.$$

- To see the difference clearly, we choose same parameter V0 for light quarks, if we
- First I show you if we choose different V0, we can give good consistence mass spectra.

TABLE I: Parameter of V_0 in unit of MeV

	$c\bar{c}$	$b\bar{b}$
$\mathbf{n}J^{PC} = \mathbf{n}\ 0^{-+}(^1S_0)$	-0.314	-0.240
$\mathbf{n}J^{PC} = \mathbf{n}\ 1^{--}(^3S_1)$	-0.176	-0.166
$\mathbf{n}J^{PC} = \mathbf{n}\ 0^{++}(^3P_0)$	-0.282	-0.174
$\mathbf{n}J^{PC} = \mathbf{n}\ 1^{++}(^3P_1)$	-0.162	-0.141
$\mathbf{n}J^{PC} = \mathbf{n}\ 2^{++}(^3P_2)$	-0.110	-0.121
$\mathbf{n}J^{PC} = \mathbf{n}\ 1^{+-}(^1P_1)$	-0.144	-0.135

Wave Fuctions

- The relativistic wave function for a pseudoscalar meson $0^{-}(0^{-+})$

$$\varphi_{^1S_0}(\vec{q}) = M\left[\gamma_0\varphi_1(\vec{q}) + \varphi_2(\vec{q}) + \frac{\not{q}_\perp}{M}\varphi_3(\vec{q}) + \frac{\gamma_0\not{q}_\perp}{M}\varphi_4(\vec{q})\right]\gamma_5$$

- Wave function for $0^{+}(0^{++})$ meson

$$\varphi_{0^+}(q_\perp) = f_1(q_\perp)\not{q} + f_2(q_\perp)\frac{\not{P}\not{q}_\perp}{M} + f_3(q_\perp)M + f_4(q_\perp)\not{P}$$

- Wave function for $^3S_1$ $1^-(1^{--})$ vector meson

$$\varphi^{\lambda}_{1^-}(q_\perp) = q_\perp \cdot \epsilon^{\lambda}_\perp \left[f_1(q_\perp) + \frac{\not{P}}{M} f_2(q_\perp) + \frac{\not{q}_\perp}{M} f_3(q_\perp) + \frac{\not{P}\not{q}_\perp}{M^2} f_4(q_\perp) \right] + M \not{\epsilon}^{\lambda}_\perp f_5(q_\perp)$$

$$+ \not{\epsilon}^{\lambda}_\perp \not{P} f_6(q_\perp) + (\not{q}_\perp \not{\epsilon}^{\lambda}_\perp - q_\perp \cdot \epsilon^{\lambda}_\perp) f_7(q_\perp) + \frac{1}{M}(\not{P}\not{\epsilon}^{\lambda}_\perp \not{q}_\perp - \not{P} q_\perp \cdot \epsilon^{\lambda}_\perp) f_8(q_\perp),$$

- Wave function for $2^+(2^{++})$ meson

$$\Psi_{2^+}(\vec{q}) = \varepsilon_{\mu\nu} q^{\nu}_\perp \left\{ q^{\mu}_\perp \left[f_1(\vec{q}) + \frac{\not{P}}{M} f_2(\vec{q}) + \frac{\not{q}_\perp}{M} f_3(\vec{q}) + \frac{\not{P}\not{q}_\perp}{M^2} f_4(\vec{q}) \right] \right.$$

$$\left. + \gamma^{\mu} \left[M f_5(\vec{q}) + \not{P} f_6(\vec{q}) + \not{q}_\perp f_7(\vec{q}) \right] + \frac{i}{M} f_8(\vec{q}) \epsilon^{\mu\alpha\beta\gamma} P_\alpha q_{\perp\beta} \gamma_\gamma \gamma_5 \right\},$$

n $J^{PC}(^{(2S+1)}L_J)$	Th($c\bar{c}$)	Ex($c\bar{c}$)	Th($b\bar{b}$)	Ex($b\bar{b}$)
1 $0^{-+}(^1S_0)$	2980.3(input)	2980.3	9390.2(input)	9388.9
2 $0^{-+}(^1S_0)$	3576.4	3637	9950.0	
3 $0^{-+}(^1S_0)$	3948.8		10311.4	
1 $1^{--}(^3S_1)$	3096.9(input)	3096.916	9460.5(input)	9460.30
2 $1^{--}(^3S_1)$	3688.1	3686.09	10023.1	10023.26
3 $1^{--}(^3D_1)$	3778.9	3772.92	10129.5	
4 $1^{--}(^3S_1)$	4056.8	4039	10368.9	10355.2
5 $1^{--}(^3D_1)$	4110.7	4153	10434.7	
6 $1^{--}(^3S_1)$	4329.4	4421	10635.8	10579.4
7 $1^{--}(^3S_1)$	4545.9		10852.1	10865

n $J^{PC}(^{2S+1})L_J$	Th($c\bar{c}$)	Ex($c\bar{c}$)	Th($b\bar{b}$)	Ex($b\bar{b}$)
1 $0^{++}(^3P_0)$	3414.7(input)	3414.75	9859.0	9859.44
2 $0^{++}(^3P_0)$	3836.8		10240.6	10232.5
3 $0^{++}(^3P_0)$	4140.1		10524.7	
1 $1^{++}(^3P_1)$	3510.3(input)	3510.66	9892.2	9892.78
2 $1^{++}(^3P_1)$	3928.7		10272.7	10255.46
3 $1^{++}(^3P_1)$	4228.8		10556.2	

n $J^{PC}(^{2S+1})L_J$	Th($c\bar{c}$)	Ex($c\bar{c}$)	Th($b\bar{b}$)	Ex($b\bar{b}$)
1 $2^{++}(^3P_2)$	3556.1(input)	3556.20	9914.4	9912.21
2 $2^{++}(^3P_2)$	3972.4		10293.6	10268.65
3 $2^{++}(^3F_2)$	4037.9		10374.4	
4 $2^{++}(^3P_2)$	4271.0		10561.5	
1 $1^{+-}(^1P_1)$	3526.0(input)	3525.93	9900.2	
2 $1^{+-}(^1P_1)$	3943.0		10280.4	
3 $1^{+-}(^1P_1)$	4242.4		10562.0	

	Th	Ex		Th	Ex
$D^-(1S)$	1869.4	1869.6±0.16	$\bar{D}^0(1S)$	1865.0	1864.83±0.14
$D^-(2S)$	2560 ± 110		$\bar{D}^0(2S)$	2550 ± 109	$2539.4 \pm 4.5 \pm 6.8$
$M(2S) - M(1S)$	691 ± 110		$M(2S) - M(1S)$	685 ± 109	$674.6 \pm 4.5 \pm 6.8$
$B^0(1S)$	5279.5	5279.5±0.3	$D_s^-(1S)$	1968.2	1968.47±0.33
$B^0(2S)$	5930 ± 279		$D_s^-(2S)$	2641 ± 123	
$M(2S) - M(1S)$	651 ± 279		$M(2S) - M(1S)$	673 ± 123	
$B^+(1S)$	5279.0	5279.17±0.29	$B_s^0(1S)$	5367.9	5366.3±0.6
$B^+(2S)$	5930 ± 280		$B_s^0(2S)$	6020 ± 281	
$M(2S) - M(1S)$	651 ± 280		$M(2S) - M(1S)$	652 ± 281	

Notation

- Rotation matrix $$R(\alpha) \equiv \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$
- Relation between HQET and $^1P_1$, $^3P_1$ states

$$\begin{bmatrix} |1,3/2\rangle \\ |1,1/2\rangle \end{bmatrix} = R(\theta_H) \begin{bmatrix} |^1P_1\rangle \\ |^3P_1\rangle \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{2} & 1 \\ -1 & \sqrt{2} \end{bmatrix} \begin{bmatrix} |^1P_1\rangle \\ |^3P_1\rangle \end{bmatrix}$$

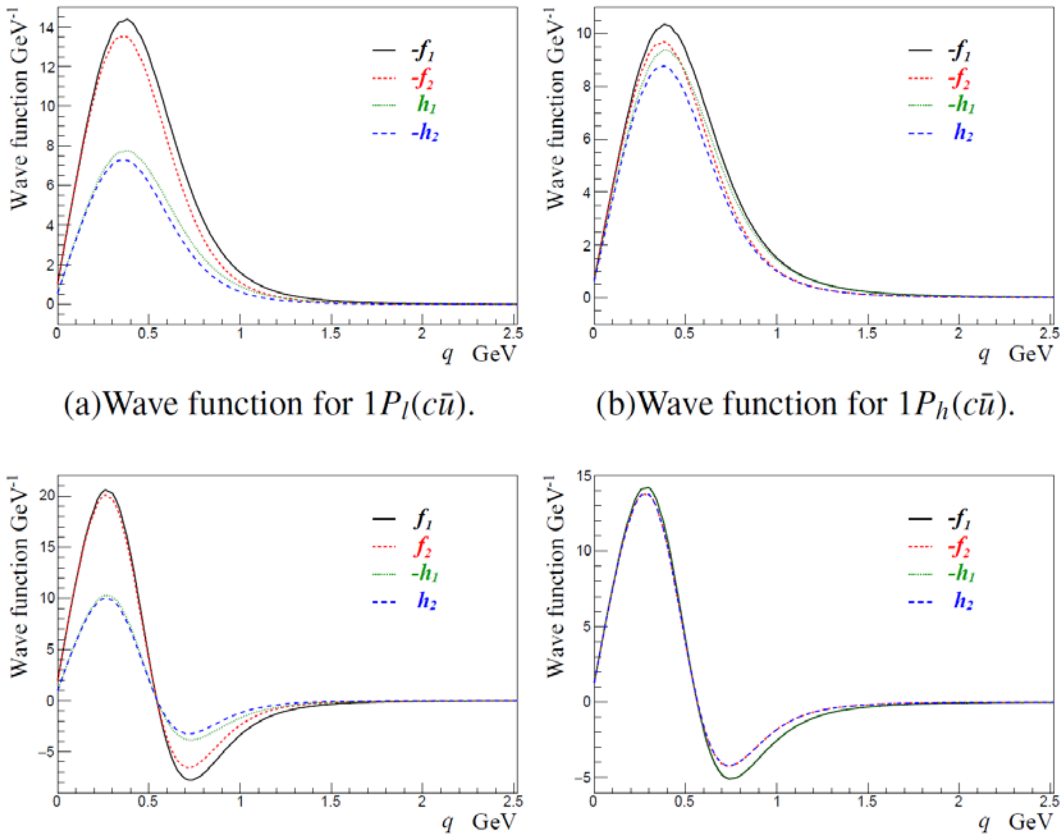
- Physics states $$\begin{bmatrix} |nP_l\rangle \\ |nP_h\rangle \end{bmatrix} = R(\theta_{nP}) \begin{bmatrix} |n^1P_1\rangle \\ |n^3P_1\rangle \end{bmatrix} = R(\theta_{nH}) \begin{bmatrix} |3/2\rangle \\ |1/2\rangle \end{bmatrix}$$
- In the limit of HQET，

$$\theta_{nP} = 35.3^\circ \text{ if } |\tfrac{3}{2}\rangle \text{ is the lower mass state}$$

$$\theta_{nP} = \theta_H - 90 = -54.7^\circ \text{ if } |\tfrac{1}{2}\rangle \text{ is the lower mass one}$$

Numerical results of 1^+ wave functions

Two sets of eigenvalues and wave functions



(a)Wave function for $1P_l(c\bar{u})$.　(b)Wave function for $1P_h(c\bar{u})$.

(c)Wave function for $2P_l(c\bar{u})$.　(d)Wave function for $2P_h(c\bar{u})$.

FIG. 1: Radial wave functions for 1^+ state $D_1^{(\prime)}$ mesons.

Mixing angles, mass spectra and inversion, decay constants

Table 1: $\theta_{nH} = \theta_{nP} - 35.3$, where θ_{nH} is under basis $|\frac{3}{2}\rangle$ and $|\frac{1}{2}\rangle$, while θ_{nP} is under basis $|^1P_1\rangle$ and $|^3P_1\rangle$.

$Q\bar{q}$	$c\bar{u}$	$c\bar{d}$	$c\bar{s}$	$b\bar{u}$	$b\bar{d}$	$b\bar{s}$	$b\bar{c}$
M_{1l}	2398^{-72}_{+72}	2405^{-72}_{+72}	2600^{-79}_{+79}	5710^{-176}_{+177}	5717^{-177}_{+177}	5810^{-179}_{+179}	6819^{-211}_{+211}
M_{1h}	2403^{-73}_{+73}	2409^{-73}_{+74}	2604^{-78}_{+78}	5724^{-174}_{+175}	5732^{-175}_{+175}	5826^{-177}_{+177}	6830^{-210}_{+210}
M_{2l}	2687^{-75}_{+75}	2695^{-75}_{+75}	2915^{-81}_{+81}	5998^{-177}_{+177}	6007^{-177}_{+177}	6122^{-180}_{+180}	7143^{-212}_{+212}
M_{2h}	2701^{-75}_{+75}	2709^{-75}_{+75}	2920^{-81}_{+81}	6006^{-177}_{+177}	6015^{-178}_{+178}	6123^{-179}_{+179}	7149^{-212}_{+211}
f_{1l}	$70.2^{-8.4}_{+53.6}$	$72.1^{-9.4}_{+87.2}$	$250.5^{-8.6}_{+8.9}$	$228.8^{-8.2}_{+8.3}$	$229.2^{-8.2}_{+8.3}$	$214.5^{-7.2}_{+7.2}$	$176.0^{-6.2}_{+6.3}$
f_{1h}	$247.1^{-22.0}_{+7.5}$	$246.7^{-44.9}_{+7.7}$	$48.3^{-46.1}_{+7.4}$	$21.8^{-1.9}_{+2.0}$	$22.1^{-1.9}_{+2.1}$	$25.7^{-2.0}_{+2.2}$	$43.9^{-2.5}_{+2.5}$
f_{2l}	$65.4^{-4.1}_{+4.2}$	$66.1^{-4.2}_{+4.2}$	$90.0^{-7.5}_{+11.8}$	$28.8^{-2.6}_{+3.1}$	$29.3^{-2.6}_{+3.3}$	$190.0^{-136.1}_{+6.3}$	$176.4^{-6.0}_{+6.0}$
f_{2h}	$198.1^{-5.6}_{+5.5}$	$198.5^{-5.5}_{+5.5}$	$202.2^{-6.1}_{+5.7}$	$181.1^{-6.4}_{+6.4}$	$181.7^{-6.4}_{+6.4}$	$5.9^{-4.2}_{+177.0}$	$45.1^{-2.4}_{+2.4}$
θ_{1P}	$36.7^{-1.5}_{+13.1}$	$37.0^{-1.7}_{+22.1}$	$-62.1^{-10.6}_{+1.8}$	$-55.3^{-0.2}_{+0.1}$	$-55.3^{-0.2}_{+0.1}$	$-55.5^{-0.1}_{+0.1}$	$-57.9^{-0.4}_{+0.4}$
θ_{2P}	$35.2^{-0.3}_{+0.4}$	$35.2^{-0.3}_{+0.4}$	$38.0^{-1.7}_{+3.4}$	$35.9^{-0.3}_{+0.5}$	$36.0^{-0.3}_{+0.5}$	$-62.5^{-20.8}_{+140.4}$	$-58.7^{-0.5}_{+0.4}$
θ_{1H}	$1.4^{-1.5}_{+13.1}$	$1.7^{-1.7}_{+22.1}$	$82.4^{-10.6}_{+1.8}$	$89.4^{-0.2}_{+0.1}$	$89.4^{-0.2}_{+0.1}$	$89.2^{-0.1}_{+0.1}$	$86.8^{-0.4}_{+0.4}$
θ_{2H}	$-0.1^{-0.3}_{+0.4}$	$-0.1^{-0.3}_{+0.4}$	$2.7^{-1.7}_{+3.4}$	$0.6^{-0.3}_{+0.5}$	$0.7^{-0.3}_{+0.5}$	$82.2^{-20.8}_{+140.4}$	$86.0^{-0.5}_{+0.4}$

Numerical results

- The exact solution include two states

$Q\bar{q}$	$c\bar{u}$
M_{1l}	2398^{-72}_{+72}
M_{1h}	2403^{-73}_{+73}
M_{2l}	2687^{-75}_{+75}
M_{2h}	2701^{-75}_{+75}

- Broad state with large decay constant, narrow state with small one

f_{1l}	$70.2^{-8.4}_{+53.6}$
f_{1h}	$247.1^{-22.0}_{+7.5}$

- There are negative value of mixing angle

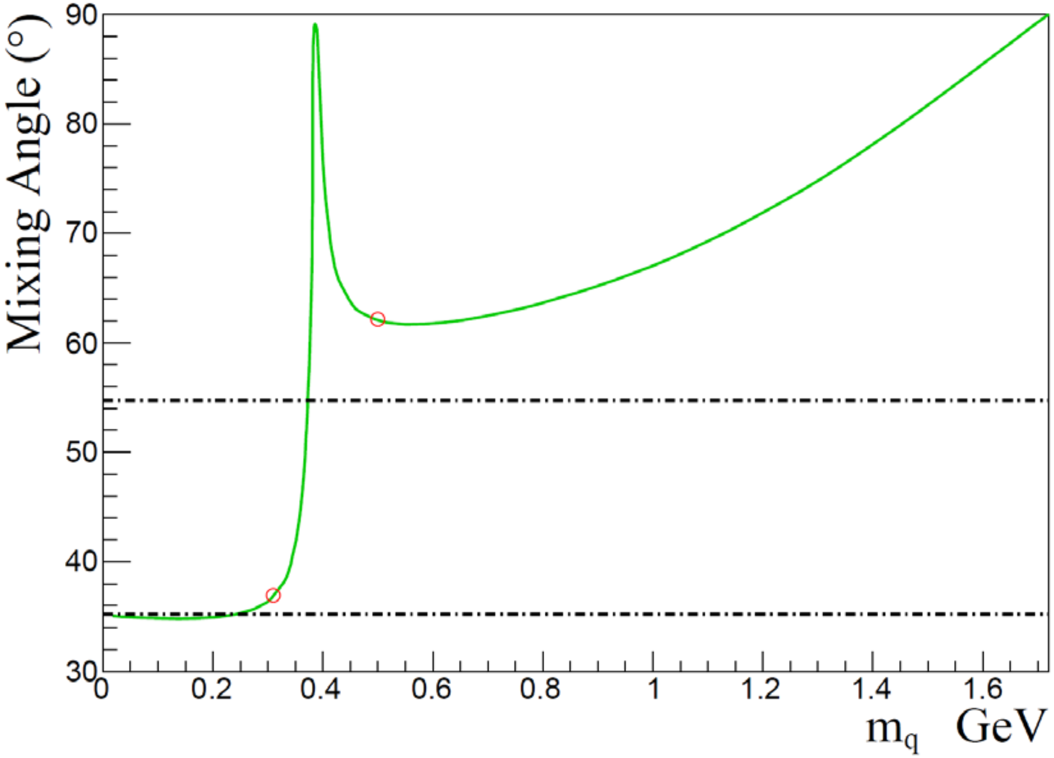
θ_{1P}	$36.7^{-1.5}_{+13.1}$	$37.0^{-1.7}_{+22.1}$	$-62.1^{-10.6}_{+1.8}$

Negative mixing angle means the mass inversion

- Keep the charm mass unchanged， vary the light quark mass

- After peak point ~0.4 GeV, add a “-” sign

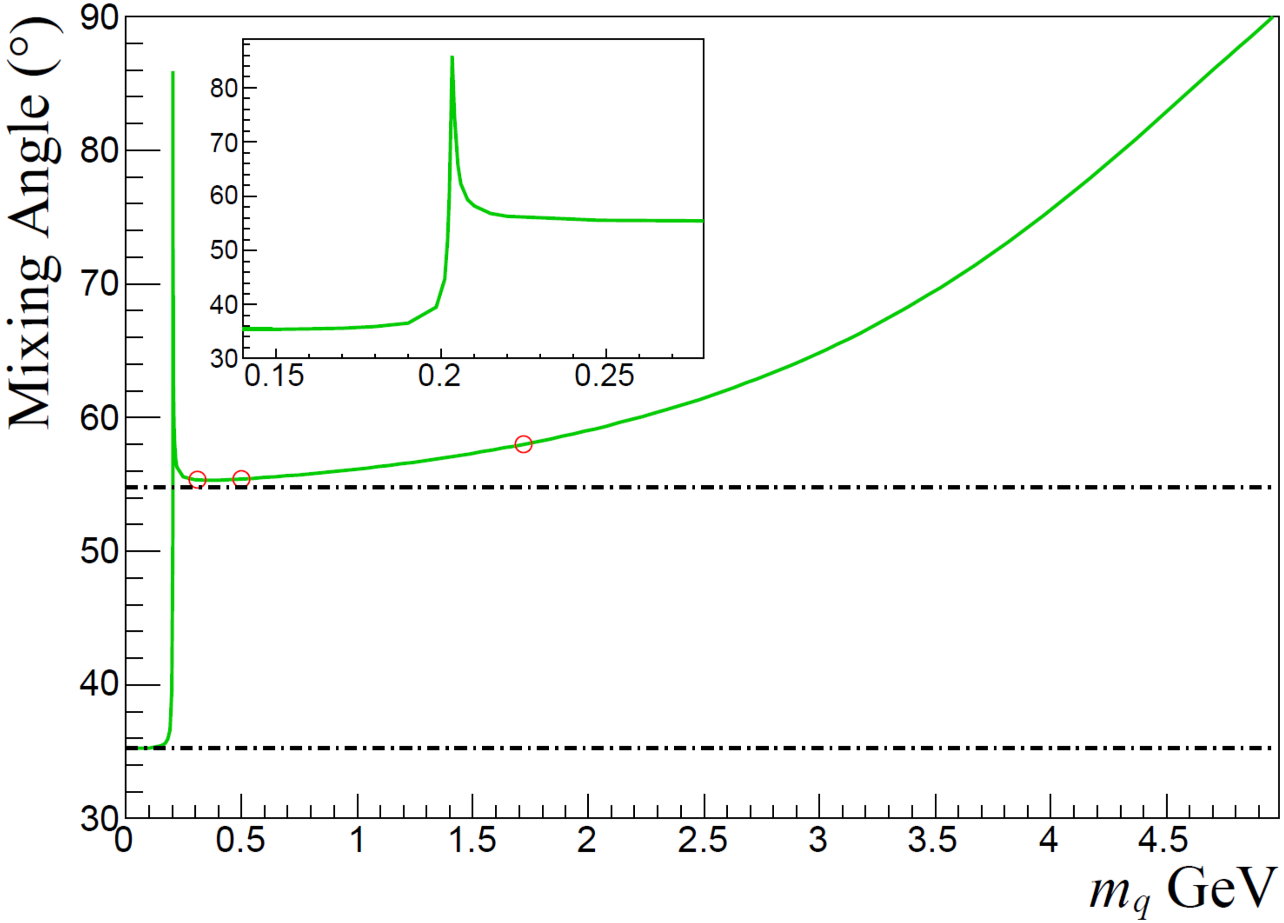
- When mq=mc, mixing angle is -90, definite C parity



(a)Mixing angle versus m_q for $1P(c\bar{q})$.

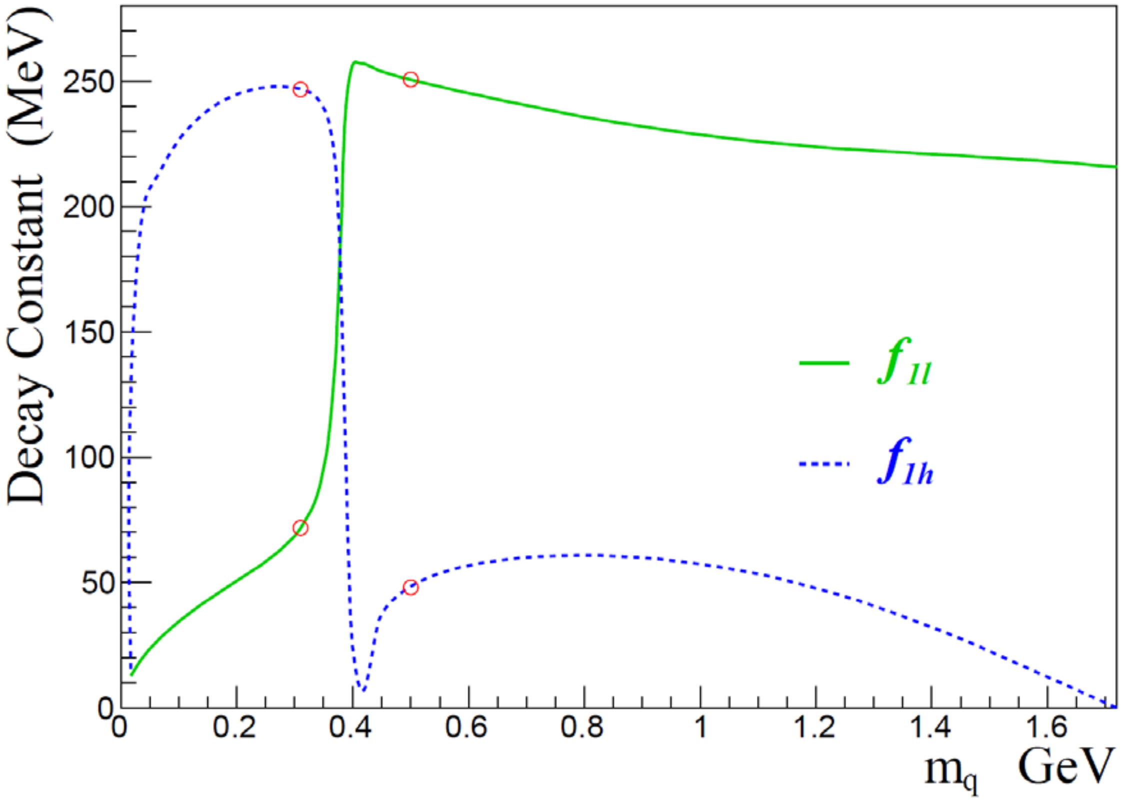
Mass inversion

- Bottom mass unchanged， varying the light quark mass



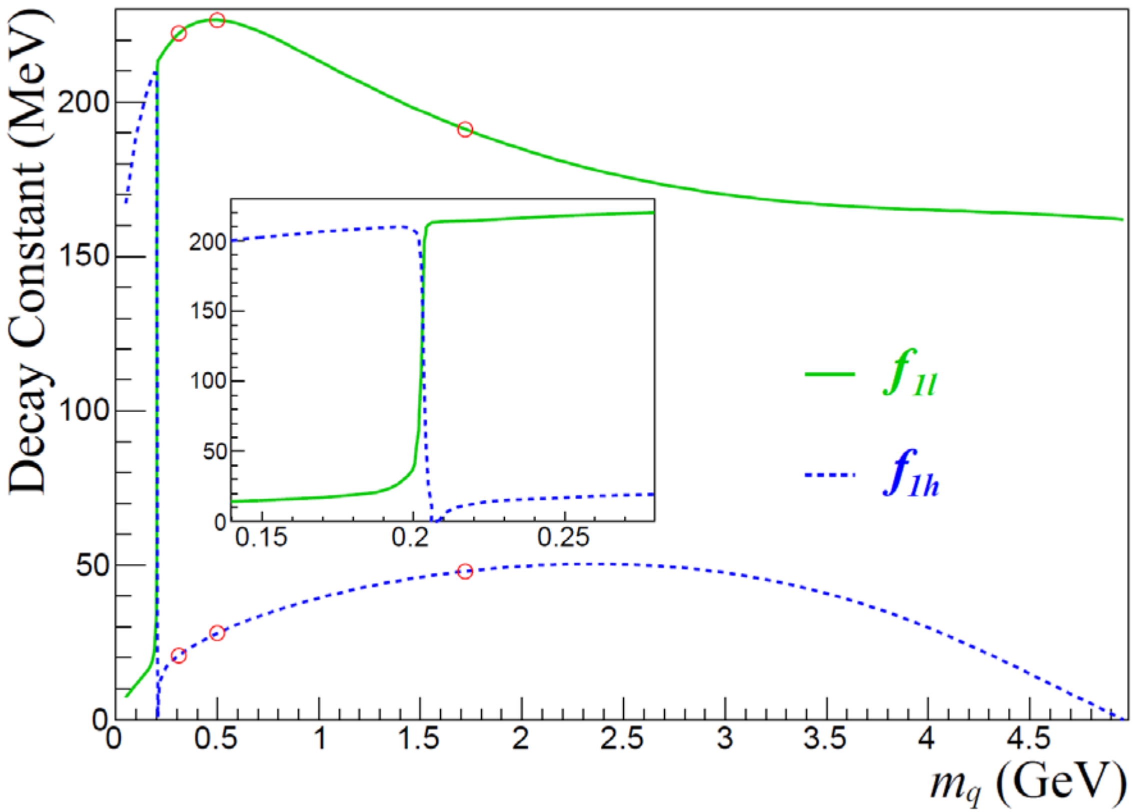
(a)Mixing angle versus m_q for $1P(b\bar{q})$.

Decay costant inversion



(b)Decay constant versus m_q for $1P(c\bar{q})$.

Decay costant inversion



(b)Decay constant versus m_q for $1P(b\bar{q})$.

Large range because of the peak structure

$Q\bar{q}$	$c\bar{u}$	$c\bar{d}$	$c\bar{s}$	$b\bar{u}$	$b\bar{d}$	$b\bar{s}$	$b\bar{c}$
θ_{1P}	$36.7^{-1.5}_{+13.1}$	$37.0^{-1.7}_{+22.1}$	$-62.1^{-10.6}_{+1.8}$	$-55.3^{-0.2}_{+0.1}$	$-55.3^{-0.2}_{+0.1}$	$-55.5^{-0.1}_{+0.1}$	$-57.9^{-0.4}_{+0.4}$
θ_{2P}	$35.2^{-0.3}_{+0.4}$	$35.2^{-0.3}_{+0.4}$	$38.0^{-1.7}_{+3.4}$	$35.9^{-0.3}_{+0.5}$	$36.0^{-0.3}_{+0.5}$	$-62.5^{-20.8}_{+140.4}$	$-58.7^{-0.5}_{+0.4}$
θ_{1H}	$1.4^{-1.5}_{+13.1}$	$1.7^{-1.7}_{+22.1}$	$82.4^{-10.6}_{+1.8}$	$89.4^{-0.2}_{+0.1}$	$89.4^{-0.2}_{+0.1}$	$89.2^{-0.1}_{+0.1}$	$86.8^{-0.4}_{+0.4}$
θ_{2H}	$-0.1^{-0.3}_{+0.4}$	$-0.1^{-0.3}_{+0.4}$	$2.7^{-1.7}_{+3.4}$	$0.6^{-0.3}_{+0.5}$	$0.7^{-0.3}_{+0.5}$	$\boxed{82.2^{-20.8}_{+140.4}}$	$86.0^{-0.5}_{+0.4}$

- Varying the parameters (6%), large ranges are obtained.
- But except Bs1(2P) case, the relations are un-changed.

Mention

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State	$s\bar{u}$ (K)	$c\bar{u}$ (D)	$c\bar{s}$ (D_s)	$b\bar{u}$ (B)	$b\bar{s}$ (B_s)	$b\bar{c}$ (B_c)
$^3P_2$	1.43	2.50	2.59	5.80	5.88	6.77
Q_{high}	1.37	2.47	2.56	5.78	5.86	6.75
Q_{low}	1.35	2.46	2.55	5.78	5.86	6.74
$^3P_0$	1.24	2.40	2.48	5.76	5.83	6.71
$^3S_1$	0.90	2.04	2.13	5.37	5.45	6.34
$^1S_0$	0.47	1.88	1.98	5.31	5.39	6.27
$^3P_1$	1.37	2.47	2.55	5.78	5.86	6.74
$^1P_1$	1.35	2.46	2.55	5.78	5.86	6.75
θ	$-5°$	$-26°$	$-38°$	$-31°$	$-40°$	$68°$

- Mixing angle with large range come from the inverted effect.
- Bc1 has a negative mixing angle, means the mass inversion.

Summary

- Exact solution to the instantaneous BS equation for 1^+ state, two physical states are obtained.
- Precise study show that there are the inverted mass phenomenon or mixing angle inversion, which result in large range of mixing angle.
- Mixing angles 35.3 and −54.7 are not equivalent, means different mass order of two 1^+ states.
- The result show that the mass of Ds1(2460) is lower than Ds1(2536).
- The not found Bottom axial vectors are the lower mass broad states.

Thank you