A novel method to compute Multi-loop Feynman integrals

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Based on:

X. Liu, Y.Q. Ma, C. Y. Wang, Phys.Lett. B779 (2018) 353

X. Liu, Y.Q. Ma, arXiv:1801.10523

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Outline

I. Introduction

II. A series representation

III. New reduction

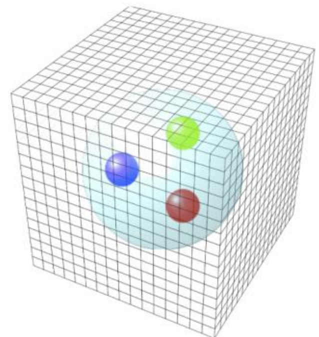
III. Analytical continuation

IV. Summary

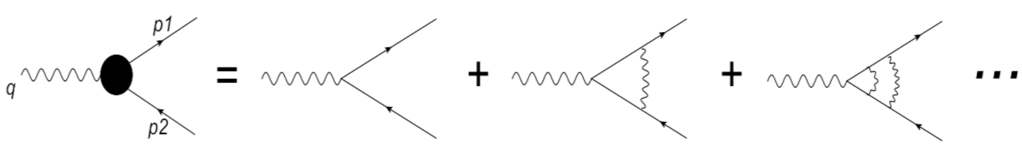
Perturbative QFT

- **QFT: the underlying theory of modern physics**
- **Solving QFT:**

  - **Nonperturbatively (lattice field theory):**
    discretize spacetime, numerical simulation
    very hard, application limited



  - **Perturbatively (Feynman diagram expansion):**
    generate and calculate Feynman integrals,
    relatively easier, the primary method



Super computer

Calculation of Feynman loop integrals

➢ Main task in applying perturbative QFT

$$I(D;\{\nu_\alpha\};\eta) \equiv \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \prod_{\alpha=1}^{N} \frac{1}{(q_\alpha^2 - m_\alpha^2 + \mathrm{i}\eta)^{\nu_\alpha}}$$

$\eta = 0^+$

Causality

- **Test of the standard model**
- **Discovery of new physics**

Multi-loop: a challenge to intelligence

- **One-loop calculation: (up to 4 legs) satisfactory approaches existed as early as 1970s**

’t Hooft, Veltman, NPB (1979)
Passarino, Veltman, NPB (1979)
Oldenborgh, Vermaseren (1990)

Developments of unitarity-based method in the past decade made the calculation efficient for multi-leg problems

Britto, Cachazo, Feng, 0412103
Ossola, Papadopoulos, Pittau, 0609007
Giele, Kunszt, Melnikov, 0801.2237

- **About 40 years later, a satisfactory approach for multi-loop calculation is still missing**

Main strategy

1) Reduce any loop integral to master integrals

- Integration-by-parts (IBP) reduction:

  Chetyrkin, Tkachov, NPB (1981)
  Laporta, 0102033

  currently, the only way, main bottleneck

  brute force algorithm, extremely inefficient for complicated problems

  unitarity-based reduction is efficient but cannot give complete reduction

2) Calculate MIs/original integrals

- differential equation (depends on reduction and BCs) Kotikov, PLB (1991)
- difference equation (depends on reduction and BCs) Laporta, 0102033
- sector decomposition (extremely time-consuming) Binoth, Heinrich, 0004013
- Mellin-Barnes representation (hard for nonplanar integrals) Usyukina (1975) Smirnov, 9905323

State-of-the-art computation

- $2 \to 2$ process with massive particles at two-loop order is already the frontier
  - $g + g \to t + \bar{t},\ g + g \to H + H,\ g + g \to H + g, \ldots$
- Very time-consuming
  - Two-loop $g + g \to H + H\ (g)$: complete IBP reduction cannot be achieved within tolerable time
    Borowka et. al., 1604.06447
    Jones, Kerner, Luisoni, 1802.00349
  - Two-loop decay $Q + \bar{Q} \to g + g$, MIs cost $O(10^5)$ CPU core-hour
    Feng, Jia, Sang, 1707.05758
  - Four-loop nonplanar cusp anomalous dimension, within tolerable computational expense, calculated MIs have 10% uncertainty
    Boels, Huber, Yang, 1705.03444

New ideas are badly needed to give a better solution!!!

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Feynman integrals with an auxiliary variable

- **Dimensional regularized scalar Feynman loop integral with an auxiliary variable η**

$$I(D;\{\nu_\alpha\};\eta) \equiv \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \prod_{\alpha=1}^{N} \frac{1}{(q_\alpha^2 - m_\alpha^2 + \mathrm{i}\eta)^{\nu_\alpha}}$$

- Think it as an analytical function of η
- Physical result is defined by

$$I(D;\{\nu_\alpha\};0) \equiv \lim_{\eta \to 0^+} I(D;\{\nu_\alpha\};\eta)$$

Expansion of propagators

➢ Expansion of propagators around $\eta \to \infty$

$$\frac{1}{(\ell+p)^2-m^2+\mathrm{i}\eta}=\frac{1}{\ell^2+\mathrm{i}\eta}\sum_{j=0}^{\infty}\left(-\frac{2\ell\cdot p+p^2-m^2}{\ell^2+\mathrm{i}\eta}\right)^j$$

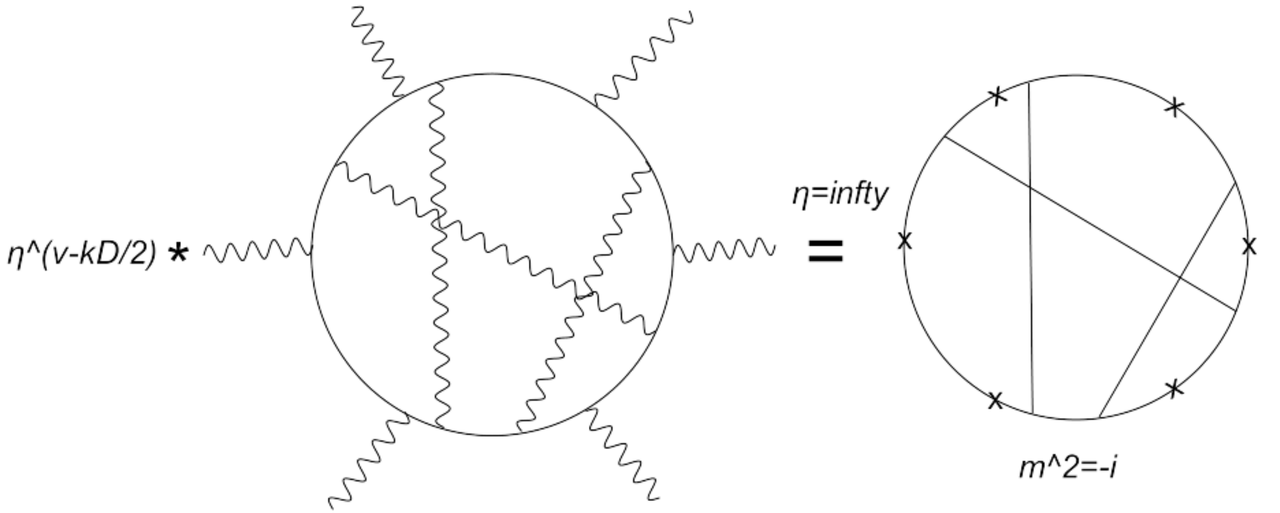
- We proved the validity of the expansion rigorously

➢ After expansion: no external momenta, equal squared masses $-i\,\eta$

Asymptotic expansion

➢ Essentially Taylor expansion

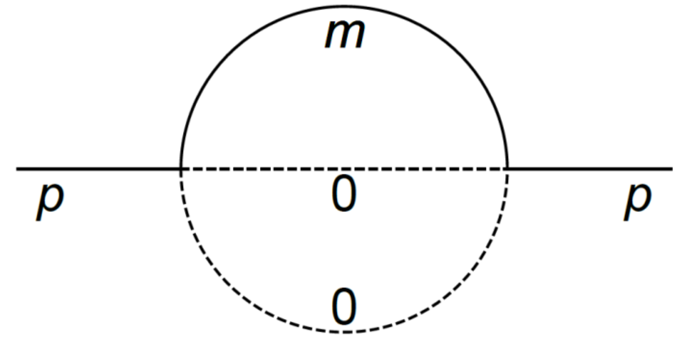
$$I_0^{\rm bub}(D;\eta)+\frac{1}{\eta}\sum_k c_{1k}I_{1k}^{\rm bub}(D+2;\eta)+\frac{1}{\eta^2}\sum_k c_{2k}I_{2k}^{\rm bub}(D+4;\eta)+\cdots\,,\quad \eta\to\infty$$



- **All terms are combinations of vacuum bubble integrals**

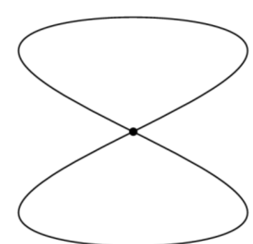
➢ Sunrise integral

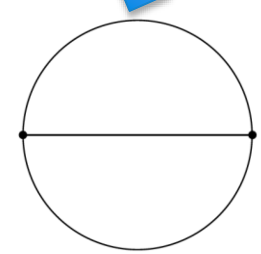
$$\hat{I}_{\nu_1\nu_2\nu_3} \equiv \int \prod_{i=1}^{2} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \frac{1}{\mathcal{D}_1^{\nu_1}\mathcal{D}_2^{\nu_2}\mathcal{D}_3^{\nu_3}}$$



$$\mathcal{D}_1 = (\ell_1 + p)^2 - m^2,\ \mathcal{D}_2 = \ell_2^2,\ \mathcal{D}_3 = (\ell_1 + \ell_2)^2$$

$$I_{111} = \eta^{D-3}\left\{\left[1 - \frac{D-3}{3}\frac{m^2}{\mathrm{i}\eta} + \frac{(D+4)(D-3)}{9D}\frac{p^2}{\mathrm{i}\eta}\right] I_{2,2}^{\mathrm{bub}} - \mathrm{i}\left[\frac{(D-2)^2}{3D}\frac{p^2}{\mathrm{i}\eta}\right] I_{2,1}^{\mathrm{bub}} + \mathcal{O}(\eta^{-2})\right\}$$

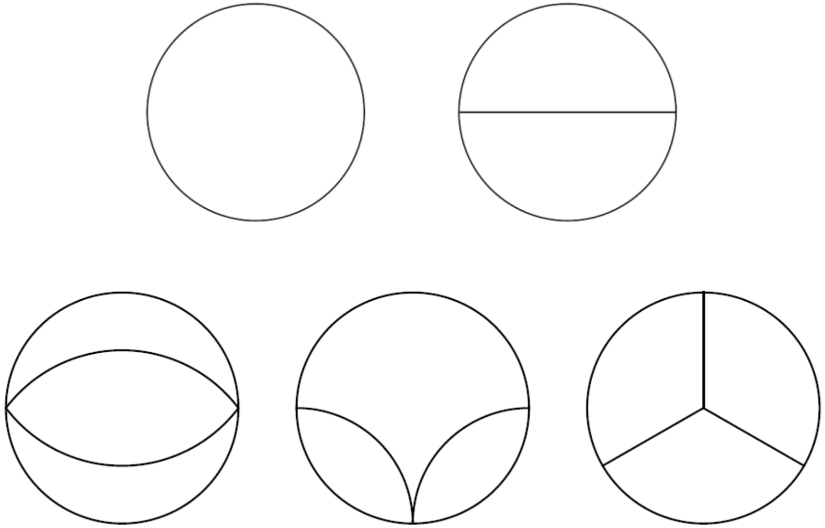




Vacuum bubble integrals

- Vacuum bubble integrals up to 3-loop, analytic results are known

Davydychev,Tausk, NPB(1993)
Broadhurst, 9803091
Kniehl, Pikelner, Veretin, 1705.05136



Numerical results known to 5-loop order!!!

Schroder, Vuorinen, 0503209
Luthe, PhD thesis (2015)
Luthe, Maier, Marquard, Ychroder, 1701.07068

A series representation

- **The infinite series at $\eta \to \infty$ uniquely defines the analytical function $I(\eta)$**

$$I(\eta) = \eta^{LD/2-\nu}\left[I^{(0)}_{\text{bub}} + \frac{1}{\eta} I^{(1)}_{\text{bub}} + \cdots\right]$$

- **Analytical continuation defines $I(0)$**

$$I(D;\{\nu_\alpha\};0) \equiv \lim_{\eta\to 0^+} I(D;\{\nu_\alpha\};\eta)$$

- **Physical Feynman integral can be defined as analytical continuation of a calculable series**

  - The series contains only vacuum integrals, easy to obtain

New strategy to calculate Feynman integrals

1) Construct the series representation (Easy)

2) Perform analytical continuation (How?)

Analytical continuation: usually very hard

Yet another unsolved problem?

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The existence of MIs

➢ The Number of Master Integrals is Finite

Smirnov, Petukhov, 1004.4199
Georgoudis, Larsen, Zhang, 1612.04252

- Feynman integrals form a finite dimensional linear space

➢ Decomposition to bases

- Suppose the dimension of the linear space is n

$$\sum_{i=1}^{n+1} Q_i(D,\vec{s},\eta)\,\mathcal{M}_i(D,\vec{s},\eta) = 0$$

- Q_i polynomials in $D, \vec{s}, \eta$, with mass dimension $2d_i$

$$2d_1 + \mathrm{Dim}(\mathcal{M}_1) = \cdots = 2d_{n+1} + \mathrm{Dim}(\mathcal{M}_{n+1})$$

Perturbative matching

- Parametrization, finite unknown numbers

$$Q_i(D,\vec{s},\eta)=\sum_{(\lambda_0,\vec{\lambda})\in\Omega_{d_i}^{r+1}} Q_i^{\lambda_0\ldots\lambda_r}(D)\,\eta^{\lambda_0}s_1^{\lambda_1}\cdots s_r^{\lambda_r}$$

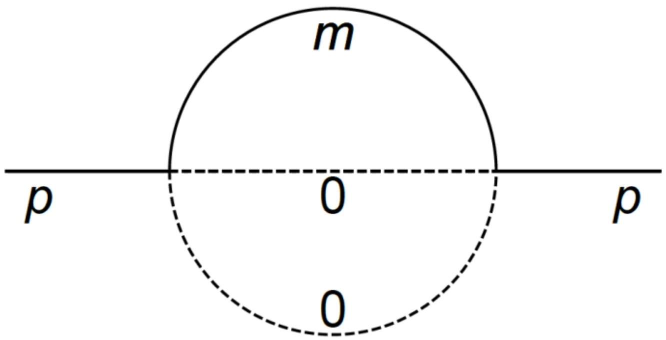
- Determine unknown parameters by matching both sides of the relation at large η, using series representation

$$\sum_{i=1}^{n+1} Q_i(D,\vec{s},\eta)\,\mathcal{M}_i(D,\vec{s},\eta)=0$$

- Note: although obtained at large η, the relation is valid for any value of η

- **Reduction of $I_{\nu 11}$ v.s. IBP**

$$\hat{I}_{\nu_1\nu_2\nu_3} \equiv \int \prod_{i=1}^{2} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \frac{1}{\mathcal{D}_1^{\nu_1}\mathcal{D}_2^{\nu_2}\mathcal{D}_3^{\nu_3}}$$



ν	Our reduction		FIRE5	
	Time/s	# of relations	Time/s	# of relations
5	0.14	14	12	203
10	0.33	34	42	1313
15	0.48	54	346	4073
20	0.75	74	2169	9233
100	5.43	394	-	-

New method can generate more efficient relations comparing with IBP (FIRE 5), thus much faster

37 years later, the second (and more efficient) reduction method is finally available

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Differential equations

- **Perturbative matching reduces any loop integrals to MIs**
    - Only need analytical continuation of MIs
- **Perturbative matching can also set up differential equations for MIs**

$$\frac{\partial}{\partial\eta}\vec{I}(D;\eta) = A(D;\eta)\vec{I}(D;\eta)$$

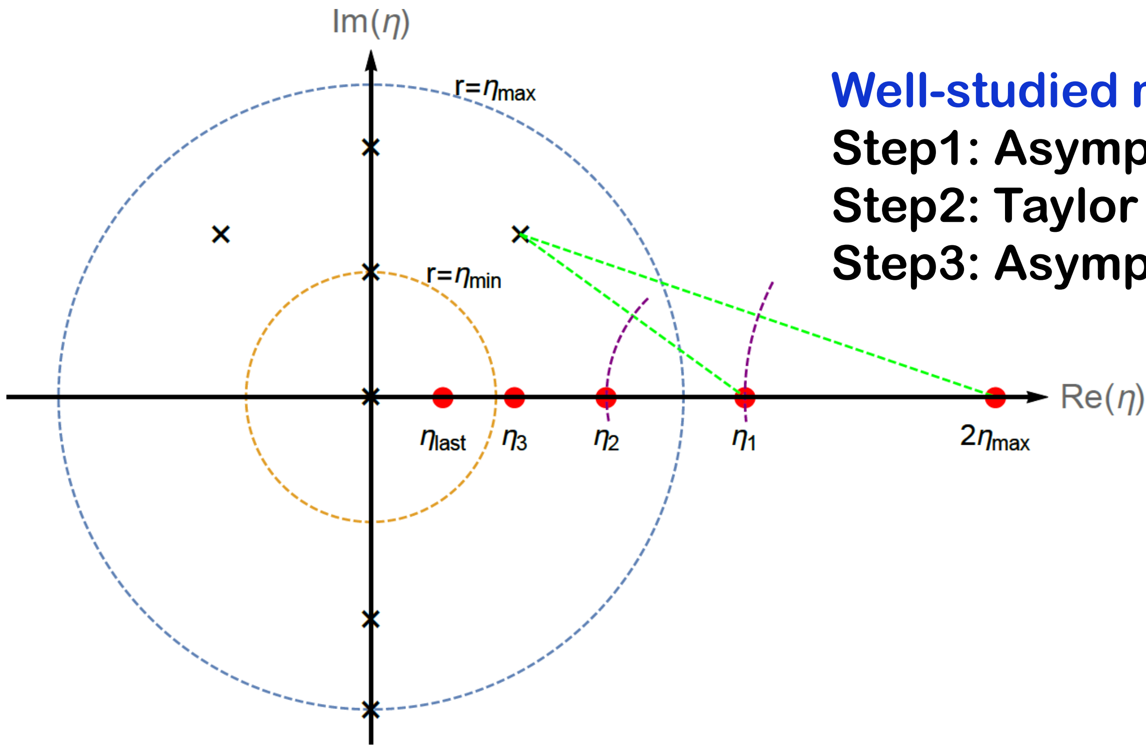
- **Boundary conditions at $\eta = \infty$: leading term of the series representation, known**

Solving the DEs

- **DEs**

$$\frac{\partial}{\partial\eta}\vec{I}(D;\eta) = A(D;\eta)\vec{I}(D;\eta)$$ **With known** $\vec{I}(D;\infty)$

- **Singularity structure**



Well-studied mathematic problem:

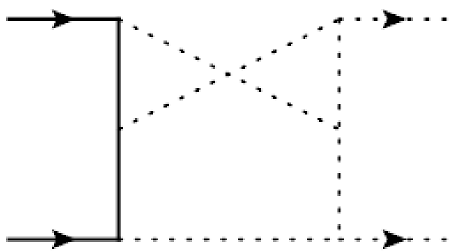
Step1: Asymptotic expansion at $\eta = \infty$

Step2: Taylor expansion at analytical points

Step3: Asymptotic expansion at $\eta = 0$

A 2-loop example

- **Test for nonplanar 2-loop box integral**



with $m^2 = 1, s = 4, t = -1$

- **168 master integrals, 26 steps to go from $\eta = \infty$ to $\eta = 0^+$, 30 orders in expansion**
- **The 168 integrals can be evaluated within a few minutes on Mathematica**

A 2-loop example

- Result agree with sector decomposition

$$
\begin{aligned}
I_{\mathrm{np}}(4-2\epsilon) = {}& 0.0520833\epsilon^{-4} - (0.131616 - 0.147262\mathrm{i})\epsilon^{-3} \\
& - (0.741857 + 0.185602\mathrm{i})\epsilon^{-2} + (3.73984 - 4.15756\mathrm{i})\epsilon^{-1} \\
& - (4.75677 - 12.0749\mathrm{i}) + (23.9674 - 55.4214\mathrm{i})\epsilon + \cdots ,
\end{aligned}
$$

- It takse a few minutes
- To compare with, FIESTA4: $O(10^4)$ CPU core-hour

Faster by at least 10^5 times!!

Summary

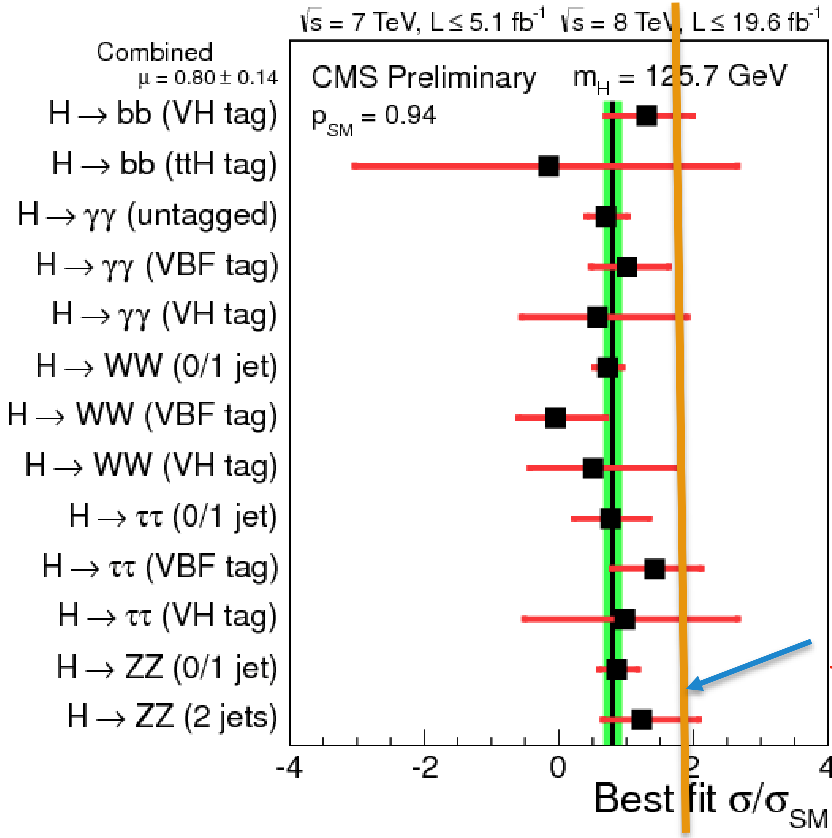
- A series representation of Feynman integrals: analytical continuation of a calculable series, which contains only vacuum integrals

1) Construct the series representation
2) Perturbative matching to setup reduction relations and DEs (the second reduction method)
3) Analytical continuation by solving DEs

Thank you!

Why precision is needed?

- **Interpretation of SM physics**



LO @ 8 TeV: 9.6 +/- 25% pb
N3LO @ 8 TeV: 19.47 + 0.32% - 2.99% pb

Anastasiou, Duhr, Dulat, Herzog, Mistlberger, 1503.06056

If w/o higher order results,
we can NOT interpret the Higgs properly

- **No significant new physics signal at LHC, precision is needed for NP discovery**

Analytic structure at infinity

➢ Feynman parametric rep.

$$I(\eta) = (-1)^{\nu} \frac{\Gamma(\nu - LD/2)}{\prod_i \Gamma(\nu_i)} \int \prod_{\alpha} (x_{\alpha}^{\nu_\alpha - 1} \mathrm{d}x_\alpha)\, \delta\left(1 - \sum_j x_j\right) \frac{\mathcal{U}^{-D/2}}{(\mathcal{F}/\mathcal{U} - \mathrm{i}\eta)^{\nu - LD/2}}$$

- $\mathcal{U}$: graph polynomial of 1-tree
- $\mathcal{F}$: graph polynomial of 2-tree

➢ Observation: $|\mathcal{F}/\mathcal{U}|$ is bounded in the Feynman parameter space!

$$|\mathcal{F}_i| < |t_i||\mathcal{U}_i| < |t_i||\mathcal{U}| \text{ and } |\mathcal{F}| < \textstyle\sum_i |t_i||\mathcal{U}|$$

➢ Thus: $J(D;\eta) \equiv \eta^{\nu - LD/2} I(D;\eta)$ is analytic at $\eta = \infty$

A stupid mathematician

Review of QCD factorization

- Factorize observables to nonperturbative functions; RGEs for nonperturbative functions

$$\sigma = \hat{\sigma} \otimes f, \qquad\qquad df = C \otimes f$$

- Calculate quantities in perturbative region

$$\sigma = \sum_n \sigma^{(n)} \alpha^n, \quad df = \sum_n (df)^{(n)} \alpha^n, \quad f = \sum_n f^{(n)} \alpha^n$$

- Perturbative matching to determine coefficients of nonperturbative relations
  - For $n = 0$: $\sigma^{(0)} = \hat{\sigma}^{(0)} \otimes f^{(0)} \rightarrow \hat{\sigma}^{(0)} = \sigma^{(0)}/f^{(0)}$
  - For $n = 1$: $\sigma^{(1)} = \hat{\sigma}^{(1)} \otimes f^{(0)} + \hat{\sigma}^{(0)} \otimes f^{(1)} \rightarrow \hat{\sigma}^{(1)} = (\sigma^{(1)} - \hat{\sigma}^{(0)} \otimes f^{(1)})/f^{(0)}$
  - And so on. Similar for $C^{(n)}$
- Everything is determined by BC of f

Step1: Expansion at the infinity

- **Transformation**

$$J(D;\eta) = \eta^{\nu - LD/2} I(D;\eta)$$

- $\eta \to x^{-1}$

$$x \frac{\partial}{\partial x} \vec{J}(x) = B_1(x) \vec{J}(x)$$

- **"Outside of the large circle"**

$$\vec{J}(x) = \sum_{n=0}^{\infty} \vec{J}_n x^n, \quad B_1(x) = \sum_{n=0}^{\infty} B_{1n} x^n$$

Step1: Expansion at the infinity

- **Recurrence relations**

$$(n - B_{10})\vec{J}_n = \sum_{k=0}^{n-1} B_{1n-k}\vec{J}_k$$

- **Can be used to determine any order of** $\vec{J}_n$

- **Estimation of** $\vec{J}(x)$

$$\vec{J}(x) \sim \sum_{n=0}^{n_0} \vec{J}_n x^n$$

e.g. at $x = \frac{1}{2}r, \vec{J}\left(\frac{\mathrm{r}}{2}\right) \Rightarrow \vec{I}\left(\frac{2}{r}\right)$

Step2: Expansion at analytical points

- **At $\eta = \eta_k$:**
  - Expand the differential equation and obtain the recurrence relations
  - Solve for high-order expansion coefficients
  - Estimate the value of $\vec{I}(\eta)$ at $\eta = \eta_{k+1}$

- **End if we have entered the small circle**

Step3: Expansion at $\eta = 0$

- $\vec{I}(\eta_0)$ is known. How to determine $\vec{I}(0)$?
- DE

$$\eta \frac{\partial}{\partial \eta} \vec{I} = \tilde{A} \vec{I}$$

- Asymptotic behavior

$$\vec{I}(\eta) \sim \eta^{\tilde{A}(0)} \vec{v}_0$$

with $\vec{v}_0$ being constant

- In general

$$\vec{I}(\eta) \equiv P(\eta) \eta^{\tilde{A}(0)} \vec{v}_0$$

Step3: Expansion at $\eta = 0$

- Expand and obtain recurrence relations

$$nP_n + [P_n, \tilde{A}_0] = \sum_{k=0}^{n-1} \tilde{A}_{n-k} P_k$$

- Can be used to determine any order of P_n
- $\vec{v}_0$ contains all information of boundary
- Determine $\vec{v}_0$ via matching

$$\vec{I}(\eta_0) = P(\eta_0)\eta_0^{\tilde{A}(0)}\vec{v}_0$$

then

$$\vec{I}(0) = \lim_{\eta \to 0} \eta^{A(0)}\vec{v}_0$$

1-Loop Test

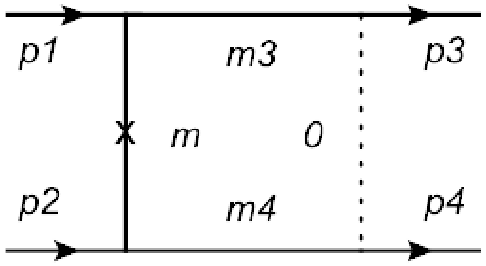
- **Set up DE** Duplancic et al. hep-ph/0303814

$$C I_1^N(D-2;\{\nu_\beta\};\eta) = \sum_{\alpha=1}^{N} z_\alpha I_1^N(D-2;\{\nu_\beta-\delta_{\alpha\beta}\};\eta) + (D-1-\nu)\, z_0 I_1^N(D;\{\nu_\beta\};\eta)$$

$$\frac{\partial}{\partial\eta} I_1^N(D;\{\nu_\beta\};\eta) = -\mathrm{i}\sum_{\alpha} \nu_\alpha I_1^N(D;\{\nu_\beta+\delta_{\alpha\beta}\};\eta) = \mathrm{i} I_1^N(D-2;\{\nu_\beta\};\eta)$$

$$\frac{\partial}{\partial\eta} I_1^N(D;\{\nu_\beta\};\eta) = \frac{\mathrm{i}}{C}\Big((D-1-\nu) z_0 I_1^N(D;\{\nu_\beta\};\eta) + \sum_{\alpha} z_\alpha I_1^N(D-2;\{\nu_\beta-\delta_{\alpha\beta}\};\eta)\Big)$$

- $D_0(D+4)$, $C_0(D+2)$, $B_0(D)$, $A_0(D-2)$
- **Test for D_0 function**



with $p_1^2 = 1.2, p_2^2 = 3.1,\ p_3^2 = m_3^2 = 0.75,\ p_4^2 = m_4^2 = 7.5, m^2 = 5.4, t = (p_1 - p_3)^2 = -1$

$s = (p_1 + p_2)^2 = 4,$

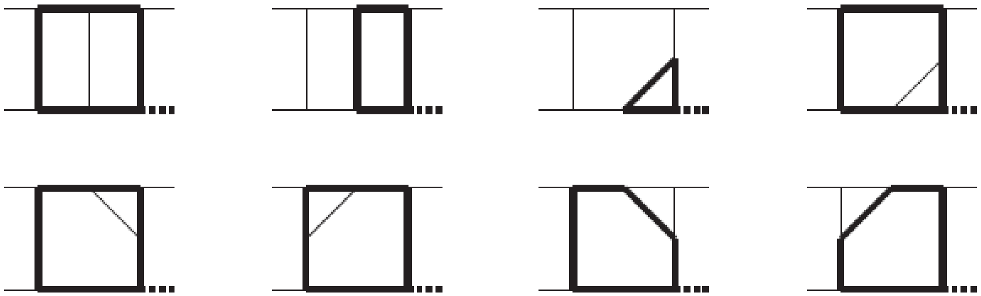
1-Loop Test

With $s = (p_1 + p_2)^2 = (m_3 + m_4)^2(1+\delta) = 19.2(1+\delta)$
$\eta_{min} \approx 4.5|\delta|$ **and** $\eta_{max} \approx 10.2$**, we can run around** $4.2 + 1.4\ln(1/|\delta|)$ **steps to go from** $\eta = \infty$ **to** $\eta = 0^+$

- **By taking Taylor expansion or asymptotic expansion to 30 orders at each step, we get results with at least 10 correct significant digits for any choice in** $10^{-7} \leq |\delta| \leq 1$
- **To compare with, sector decomposition FIESTA4 can get result with tolerable uncertainty only for** $|\delta| \geq 10^{-3}$

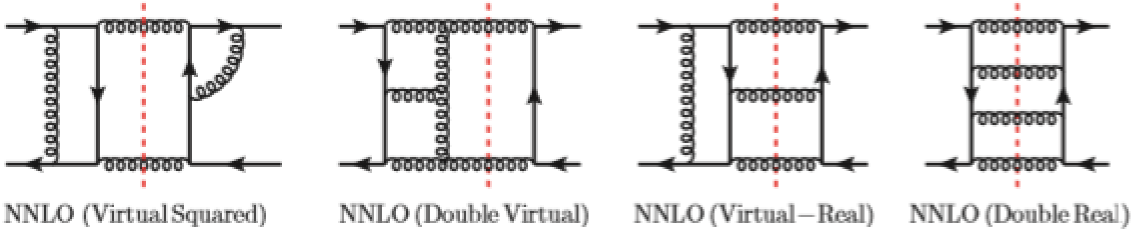
High Order Correction

- Higgs → 3 partons (Euclidean Rigion) [R. Bonciani, et.al 2016]



200MB, 10 min

- NNLO QCD corrections to the hadronic decay rates of the pseudo-scalar quarkonia [F. Feng, Y. Jia, W.L. Sang 2017]



10^5 CPU core-hour

Sector Decomposition

➢ Feynman parametric representation

$$I(D;\{\nu_\alpha\}) \equiv \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \prod_{\alpha=1}^{N} \frac{1}{(q_\alpha^2 - m_\alpha^2)^{\nu_\alpha}}$$ **where** $q_\alpha = c_\alpha^i \ell_i + d_\alpha^i p_i$

$$I(D;\{\nu_\alpha\}) = (-1)^\nu \frac{\Gamma(\nu - LD/2)}{\prod_k \Gamma(\nu_k)} \int \prod_\alpha (x_\alpha^{\nu_\alpha - 1} \mathrm{d}x_\alpha) \times \delta(x-1) \frac{\mathcal{U}^{\nu-(L+1)D/2}}{\mathcal{F}^{\nu-LD/2}}$$

$$\mathcal{U}(\vec{x}) = \sum_{T\in T_1} \prod_{i\notin T_1} x_i$$

$\mathcal{U} \sim x^L$ **Spanning 1-tree, sub UV divergences**

$$\mathcal{F}_0(\vec{x}) = -\sum_{T\in T_2} s_T \prod_{i\notin T_2} x_i$$

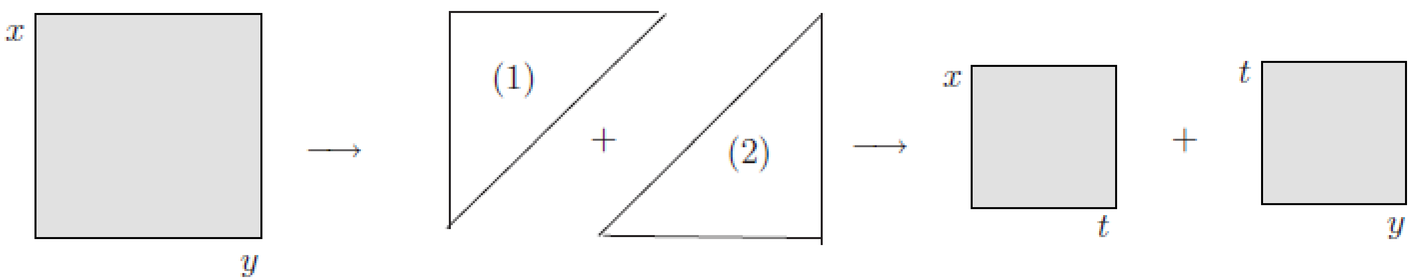
$\mathcal{F} \sim x^{L+1}$ **Spanning 2-tree, IR divergences**

$$\mathcal{F}(\vec{x}) = \mathcal{F}_0(\vec{x}) + \mathcal{U}(\vec{x}) \sum_{\alpha=1}^{N} x_\alpha m_\alpha^2$$

See e.g. [Heinrich2008]

➢ Sector decomposition: basic example

$$I = \int_0^1 dx \int_0^1 dy\, x^{-1-a\epsilon}\, y^{-b\epsilon} \Big(x + (1-x)\, y\Big)^{-1}$$



$$I = \int_0^1 dx\, x^{-1-(a+b)\epsilon} \int_0^1 dt\, t^{-b\epsilon} \Big(1 + (1-x)\, t\Big)^{-1}$$

$$+ \int_0^1 dy\, y^{-1-(a+b)\epsilon} \int_0^1 dt\, t^{-1-a\epsilon} \Big(1 + (1-y)\, t\Big)^{-1}$$

Sector Decomposition

➢ Apply to Calculation of Feynman Integrals

- Generate primary sectors
- Generate subsectors iteratively
- Take epsilon expansion

[Binoth, Heinrich 2000]
…

$$I = (-1)^{\nu}\Gamma(\nu - LD/2)\sum_{i=1}^{N}\sum_{j=1}^{\Lambda(i)} I_{ij}, \quad I_{ij} = \sum_{k=-2L}^{r} C_{ij,k}\epsilon^{k} + \mathcal{O}(\epsilon^{r+1})$$

- Evaluate the finite integrals numerically

$$C_{ij,k} \quad \xrightarrow{\text{M-C}} \quad \text{number}$$

Sector Decomposition

➢ History

- **1966 K. Hepp (BPHZ)**

  "Proof of the Bogoliubov-Parasiuk Theorem on Renormalization"

- **2000 T. Binoth, G. Heinrich**

  "An automatized algorithm to compute infrared divergent multi-loop integrals"

- **2008 A. Smirnov, M.N. Tentyukov, et.al ->** FIESTA

  "Feynman Integral Evaluation by a Sector decompasiTion Approach (FIESTA)"

- **2010 J. Carter, G. Heinrich, et.al ->** SecDec

  "SecDec: A general program for sector decomposition"

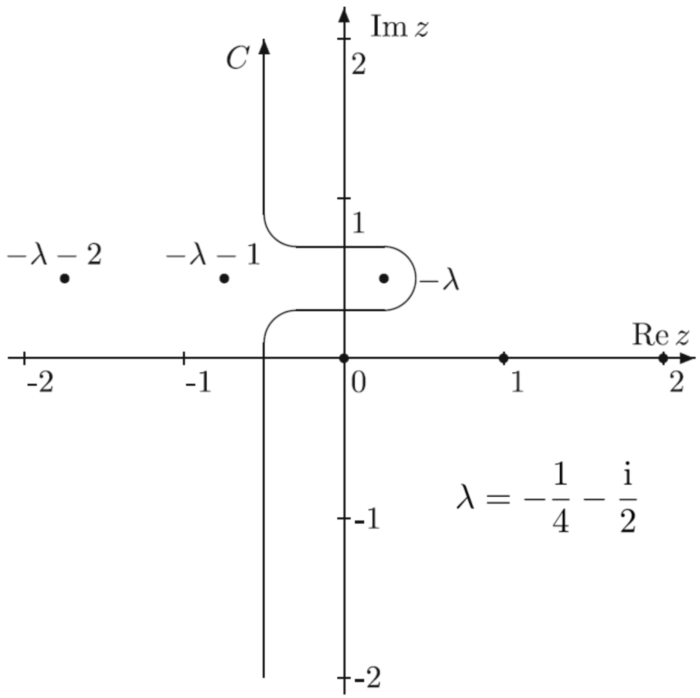
- **2017 S. Borowka, G. Heinrich, et.al ->** pySecDec

  "pySecDec: a toolbox for the numerical evaluation of multi-scale integrals"

Mellin-Barnes Representation

➢ Basic Relation

$$\frac{1}{(X+Y)^\lambda} = \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \Gamma(\lambda+z)\Gamma(-z) \frac{Y^z}{X^{\lambda+z}}$$

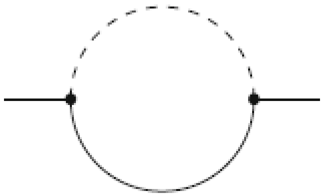


Rules:
Poles of $\Gamma(\cdots + z)$ are to the left of the contour.
Poles of $\Gamma(\cdots - z)$ are to the right of the contour.

Mellin-Barnes Representation

- Apply to massive propagator

$$\frac{1}{(\ell^2-m^2)^\lambda}=\frac{1}{(\ell^2)^\lambda}\frac{1}{\Gamma(\lambda)}\frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\mathrm{d}z\Gamma(\lambda+z)\Gamma(-z)\left(-\frac{m^2}{\ell^2}\right)^z$$



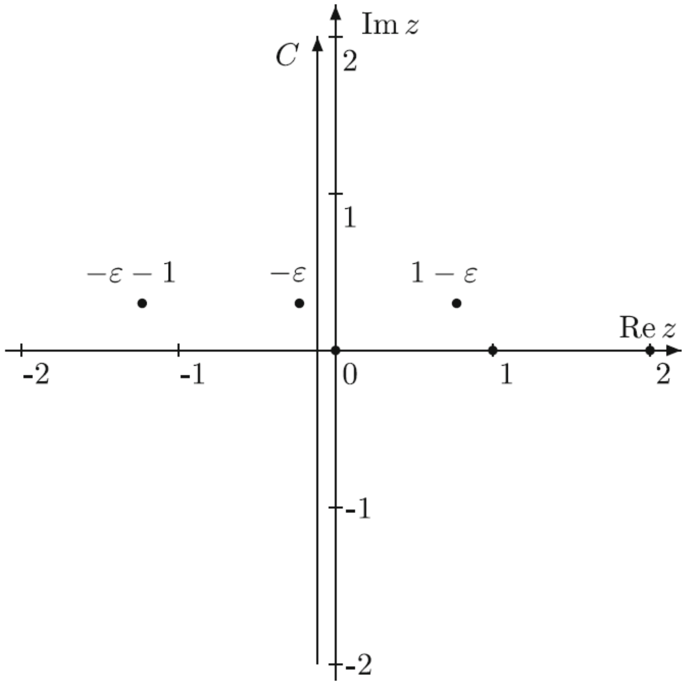
$$=\int\frac{\mathrm{d}^D\ell}{\mathrm{i}\pi^{D/2}}\frac{1}{(\ell^2-m^2)(\ell+p)^2}$$

$$\longrightarrow\quad\int\mathrm{d}z\frac{\Gamma(\epsilon+z)\Gamma(-z)\Gamma(1-\epsilon-z)}{\Gamma(2-2\epsilon-z)}\left(-\frac{m^2}{p^2}\right)^z$$

The contour is pinched.

There is a UV divergence. We need to resolve the singularity.

Mellin-Barnes Representation



Strategy A: MBresolve.m [A. Smirnov, V. Smirnov 2009]

Deform the integration contours.

Strategy B: MB.m [M. Czakon 2005]

Fix the integration contours and tends ϵ to 0.

➢ Practical procedure

- Obtain MB representation
- Resolve epsilon singularities
- Perform epsilon expansion
- Evaluate the finite integrals numerically

[M. Czakon 2005]
[J.Gluza, et.al 2007]
[A. Smirnov, V. Smirnov 2009]
…

Mellin-Barnes Representation

➢ History

- **1975 N. Usyukina**
  "On a representation for the three-point function"
- **1999 V. Smirnov**
  "Analytical result for dimensionally regularized massless on-shell double box"
- **2005 M. Czakon->** MB.m
  "Automatized analytic continuation of Mellin-Barnes integrals"
- **2007 J. Gluza, K. Kajda, T. Riemann ->** AMBRE.m
  "AMBRE – a Mathematica package for the construction of Mellin-Barnes representations for Feynman integals"
- **2009 A. Smirnov, V. Smirnov, et.al ->** MBresolve.m
  "On the resolution of singularities of multiple Mellin-Barnes integrals"
- **2014 J. Blumlein, I. Dubovyk, et.al**
  "Non-planar Feynman integals, Mellin-Barnes representations, multiple sums"
- **2015 M. Ochman, T. Riemann ->** MBsums.m
  "Mbsums – a Mathematica package for the representation of Mellin-Barnes integrals by multiple sums"

Integration-By-Parts (IBP)

➢ **A direct result of DR scheme**

$$\int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \frac{\partial}{\partial \ell_j^\mu} \left(l_k^\mu \prod_{\alpha=1}^{N} \frac{1}{(q_\alpha^2 - m_\alpha^2)^{\nu_\alpha}} \right) = 0, \quad \forall j, k$$

$$\Downarrow$$

$$\sum_i c_i(D; \{m_\alpha^2\}; \{s_\beta\}) \times I(D; \{\nu_\alpha + \lambda_\alpha^i\}) = 0$$

Linear equations $\Rightarrow$ **Basis**

$$V = \mathrm{span}\{I_1, I_2, \ldots, I_n\}$$

Finite-dimension "linear space"!

[A. Smirnov, Petukhov 2011]

Integration-By-Parts (IBP)

➢ History

- **1981 K. G. Chetyrkin, F. V. Tkachov**
  "Integration by parts: the algorithm to calculate beta function in 4 loops"
- **2000 S. Laporta**
  "High precision calculation of multiloop Feynman integrals by difference equations"
- **2004 C. Anastasiou, A. Lazopoulos ->** AIR
  "Automatic Integral Reduction for higher order perturbative calculations"
- **2008 A. Smirnov ->** FIRE
  "Algorithm FIRE – Feynman Integral REduction"
- **2009 C. Studerus ->** Reduze
  "Reduze- Feynman integral reduction in C++"
- **2012 R. Lee ->** LiteRed
  "Presenting LiteRed: a tool for the Loop InTEgrals REDuction"
- **2015 A. Manteuffel, R. Schabinger**
  "A novel approach to integration by parts reduction"
- **2016 K. Larsen, Y. Zhang**
  "Integration-by-parts reductions from unitarity cuts and algebraic geometry"
- **2017 A. Georgoudis, K. Larsen, Y. Zhang ->** AZURITE
  "AZURITE: an algebraic geometry based package for finding bases of loop integrals"

Unitarity Cuts

➢ Integrand-level reduction

$$\text{Integrand} = \sum c_i \times I_i$$

Physical singularities $\Rightarrow$ **Coefficients**

$$\mathcal{M}^{(1)}(2\to 2) = \int \frac{\mathrm{d}^D\ell}{\mathrm{i}\pi^{D/2}} \left(\frac{\Delta_4(\ell)}{\mathcal{D}_1\mathcal{D}_2\mathcal{D}_3\mathcal{D}_4} + \sum_{i_1 i_2 i_3} \frac{\Delta_{3,i_1 i_2 i_3}(\ell)}{\mathcal{D}_{i_1}\mathcal{D}_{i_2}\mathcal{D}_{i_3}} + \sum_{i_1 i_2} \frac{\Delta_{2,i_1 i_2}(\ell)}{\mathcal{D}_{i_1}\mathcal{D}_{i_2}} + \text{tadpoles} \right)$$

$$D_1 = D_2 = D_3 = D_4 = 0 \quad \Rightarrow \quad \Delta_4$$

$$D_{i_1} = D_{i_2} = D_{i_3} = 0 \quad \Rightarrow \quad \Delta_{3,i_1 i_2 i_3}$$

$$\vdots$$

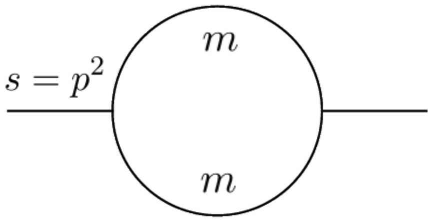
Unitarity Cuts

➢ History

- **1994 Z. Bern, L. Dixon, D. Dunbar, D. Kosower**
  "One-loop n-point gauge theory amplitudes, unitarity and collinear limits"
- **2005 R. Britto, F. Cachazo, B. Feng**
  "Generalized unitarity and one-loop amplitudes in N=4 super-Yang-Mills"
- **2007 G. Ossola, C. Papadopoulos, R. Pittan ->** OPP method
  "Reducing full one-loop amplitudes to scalar integrals at the integrand level"
- **2008 G. Ossola, C. Papadopoulos, R. Pittan ->** CutTools
  "CutTools: a program implementing the OPP reduction method to compute one-loop amplitudes"
- **2011 P. Mastrolia, G. Ossola**
  "On the integrand-reduction method for two-loop scattering amplitudes"
- **2012 Y. Zhang**
  "Integrand-level reduction of loop amplitudes by computational algebraic geometry methods"
- **2017 J. Bosma, M. Sogaard, Y. Zhang**
  "Maximal cuts in arbitrary dimension"

Differential Equation Method

➢ **Differential Equation + Boundary Condition**



$$I(D;\{1,1\}) = \int \frac{\mathrm{d}^D \ell}{\mathrm{i}\pi^{D/2}} \frac{1}{(\ell^2-m^2)[(\ell+p)^2-m^2]}$$

$$\begin{aligned}
\frac{\partial}{\partial m^2} I(D;\{1,1\}) &= I(D;\{2,1\}) + I(D;\{1,2\}) \\
&\overset{\text{IBP}}{=} \frac{2(D-3)}{4m^2-s} I(D;\{1,1\}) - \frac{D-2}{m^2(4m^2-s)} I(D;\{1,0\})
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial m^2} I(D;\{1,0\}) &= I(D;\{2,0\}) \\
&\overset{\text{IBP}}{=} \frac{D-2}{2m^2} I(D;\{1,0\})
\end{aligned}$$

$$I(D;\{1,1\})|_{m^2=0} = \Gamma(2-D/2)(-s)^{D/2-2} \frac{\Gamma(D/2-1)^2}{\Gamma(D-2)}, \quad I(D;\{1,0\})|_{m^2=0} = \cdots$$

Differential Equation Method

➢ Step1: Set up the differential equation

- Differentiate w.r.t. invariants, such as m^2, p^2
- IBP relations $\frac{\partial}{\partial x}\vec{I}(x;\epsilon) = A(x;\epsilon)\vec{I}(x;\epsilon)$

➢ Step2: Calculate boundary condition

- Calculate integrals at special value of m^2, p^2

➢ Step3: Solve the differential equation

- Canonical form $\partial_x\vec{I}(x;\epsilon) = \epsilon A(x)\vec{I}(x;\epsilon)$ [J. Henn 2013]
- …

Differential Equation Method

➢ History

- **1991 A. Kotikov**
  "Differential equations method: the calculation of N point Feynman diagrams"
- **1991 A. Kotikov**
  "Differential equations method: new technique for massive Feynman diagrams calculation"
- **1997 E. Remiddi**
  "Differential equations for Feynman graph ampltides"
- **2000 T. Gehrmann, E. Remiddi**
  "Differential equations for two-loop four-point functions"
- **2013 J. Henn ->** Canonical form
  "Multiloop integrals in dimensional regularization made simple"
- **2014 R. Lee**
  "Reducing differential equations for multiloop master integrals"
- **2017 L. Adams, E. Chaubey, S. Weinzierl**
  "Simplifying differential equations for multiscale Feynman integrals beyond multiple polylogarithms"