On Uniqueness of AdS Black Hole

Wu Xiaoning
Institute of Mathematics, AMSS, CAS

@Dandong University, 2018.8

Joint work with Dr. X. K. He, Prof. J. L. Jing

- Multiple structure of space-time
  - Newton gravity

$$\Delta u = 8\pi\rho, \quad \lim_{r\to\infty} u \to 0$$

$$u(\vec{x}) = \frac{1}{8\pi}\int_{V'} \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} d\vec{x}'$$

$$u(\vec{x}) = \frac{1}{8\pi}\left(\frac{1}{|\vec{x}|}\int_{V'} \rho(\vec{x}')d\vec{x}' + \frac{1}{|x|^2}\int_{V'} \rho(\vec{x}')(\vec{x}'\cdot\hat{x}) + \cdots\right)$$

  - Asymptotic flat case :

$$ds^2 = -du^2 + 2dudr + r^2 d\Omega_0 + O(r^{-\alpha})$$

• Multiple structure for stationary asymptotic flat space-time

I. Near special infinity

1. Geroch-Hanson multiple moments (R. Geroch, 1970, R. Hanson, 1976)

2. Geroch Conjecture : A vacuum stationary asymptotic flat space-time is uniquely determined by its Geroch-Hanson multiples, at least near special infinity.

3. Analyticity near special infinity (R. Beig and W. Simon, 1980)

II. Near null infinity

1. Multiple monent near null infinity (P. Kundu, 1988)

2. Analyticity near null infinity (T. Damour and B. Schmidt, 1990, S. Dain, 2001, P. Tod, 2007)

3. Relation between multiples at null infinity and special infinity (H. Friedrich, 2007)

- Reduced Einstein Equation for stationary case :

1. Static case :

$$ds^2 = e^{2U}\,dt^2 - e^{-2U}\gamma_{ij}\,dx^i\,dx^j \qquad (i,j=1,2,3)$$

$$\Delta_\gamma U = 0$$

$$R_{ij}(\gamma) = -2\partial_i U \partial_j U - \tfrac{1}{2}\gamma_{ij}\gamma^{lm}\partial_l U \partial_m U$$

2. Stationary case :

   Reduced space-like manifold $\Sigma = M/\Gamma$, $\dot{\Gamma} = \xi$ is Killing.

   Ernst Equation

- Uniqueness theorem for asymptotic flat case

1. Birkhoff theorem (Birkhoff, 1923)
2. Israel’s theorem (Israel, 1967)
3. Carter’s theorem (Carter, 1971)
4. Robinsons’ theorem (Robinsons, 1975)
5. Topological Censorship (Hawking, 1973)
6. Hawking rigidity theorem (Hawking, 1973)
7. Recent developments (Review see Chrusciel, 1994; Chrusciel, Costa and Heusler, 2012)

Uniqueness of Kerr metric (Carter, 1971)

The Kerr metric is the only vacuum single black hole solution with $M^2 > a^2$, regular horizon and stationary and axial-symmetric, asymptotic flat domain of outer communications.

More resent result (Chrusciel and Costa, 2008; Chrusciel and Nguyen, 2010)
Let (M, g) be a stationary, asymptotically-flat, $\mathcal{I}^+$-regular, electrovacuum, 4 dimensional analytic spacetime. If the event horizon is connected and either mean non-degenerate or rotating, then M_{ext} is isometric to the domain of outer communications of a Kerr-Newman spacetime.

• Questions :

What is the relation between the uniqueness theorem and gravitational multiple moments ?

- Newman-Penrose formulism (Null tetrad method)

$$\Psi_0 = W_{lmlm}, \quad \Psi_1 = W_{lnlm}, \quad \Psi_2 = \frac{1}{2}(W_{lnln} + W_{lnm\bar{m}}),$$
$$\Psi_3 = W_{nln\bar{m}}, \quad \Psi_4 = W_{n\bar{m}n\bar{m}}$$

- Petrov classification of Weyl curvature

Algebraic special condition :

$$I^3 = 27J^2$$

$$I \equiv \Psi_0\Psi_4 - 4\Psi_1\Psi_3 + 3\Psi_2^2,$$

$$J \equiv \begin{vmatrix} \Psi_4 & \Psi_3 & \Psi_2 \\ \Psi_3 & \Psi_2 & \Psi_1 \\ \Psi_2 & \Psi_1 & \Psi_0 \end{vmatrix}$$

➢ Local Uniqueness and Bondi form of Kerr metric

Add axial symmetric requirement, combining Einstein equations and Killing equations, one can get following control

1. tetrad up to $O(r^{-4})$.
2. connection coefficients up to $O(r^{-5})$.
3. Weyl curvature up to $O(r^{-6})$.

Using Kommar integral, one can find

$$\Psi_2^0 = -M, \quad \Psi_1^0 = \frac{3iMa\sin\theta}{\sqrt{2}}$$

$$
\begin{aligned}
l^a &= \frac{\partial}{\partial r},\\
n^a &= \frac{\partial}{\partial u} + \left[-\frac{1}{2} + \frac{M}{r} - \frac{Ma^2}{2r^3}(3\cos^2\theta - 1) + O(r^{-4})\right]\frac{\partial}{\partial r} + \left[\frac{iMa}{2r^3}\cot\frac{\theta}{2} + O(r^{-4})\right]\frac{\partial}{\partial \zeta} + \left[-\frac{iMa}{2r^3}\cot\frac{\theta}{2} + O(r^{-4})\right]\frac{\partial}{\partial \bar{\zeta}},\\
m^a &= \left[-\frac{3iMa}{2\sqrt{2}r^2}\sin\theta + \frac{Ma^2}{\sqrt{2}r^3}\sin^2\theta + O(r^{-4})\right]\frac{\partial}{\partial r} + O(r^{-4})\frac{\partial}{\partial \zeta} + \left[\frac{(1+\zeta\bar{\zeta})}{\sqrt{2}r} + O(r^{-4})\right]\frac{\partial}{\partial \bar{\zeta}}.
\end{aligned}
\tag{A1}
$$

$$
\begin{aligned}
&\rho = -\frac{1}{r} + O(r^{-5}), \qquad \sigma = -\frac{3Ma^2\sin\theta}{2r^4} + O(r^{-5}), \qquad \alpha = -\frac{\cot\theta}{2\sqrt{2}r} + \frac{3Ma^2\sin\theta\cos\theta}{2\sqrt{2}r^4} + O(r^{-5}),\\
&\beta = \frac{\cot\theta}{2\sqrt{2}r} - \frac{3iMa\sin\theta}{2\sqrt{2}r^3} + \frac{33Ma^2\sin\theta\cos\theta}{2\sqrt{2}r^4} + O(r^{-5}), \qquad \tau = -\frac{3iMa\sin\theta}{2\sqrt{2}r^3} + \frac{18Ma^2\sin\theta\cos\theta}{\sqrt{2}r^4} + O(r^{-5}),\\
&\lambda = -\frac{Ma^2\sin^2\theta}{4r^4} + O(r^{-5}), \qquad \mu = -\frac{1}{2r} + \frac{M}{r^2} + \frac{3iMa\cos\theta}{2r^3} - \frac{Ma^2(3\cos^2\theta - 1)}{r^4} + O(r^{-5}),\\
&\gamma = \frac{M}{2r^2} + \frac{(2\sqrt{2}-1)3iMa\cos\theta}{\sqrt{2}r^3} - \frac{3Ma^2(3\cos^2\theta - 1)}{4r^4} + O(r^{-5}), \qquad \nu = \frac{3iMa\sin\theta}{4\sqrt{2}r^3} - \frac{Ma^2\sin\theta\cos\theta}{\sqrt{2}r^4} + O(r^{-5}).
\end{aligned}
$$

(3) Weyl curvature,

$$\Psi_0 = \frac{3Ma^2\sin^2\theta}{r^5} + O(r^{-6}), \qquad \Psi_1 = \frac{3iMa\sin\theta}{\sqrt{2}r^4} - \frac{12Ma^2\sin\theta\cos\theta}{\sqrt{2}r^5} + O(r^{-6}),$$

$$\Psi_2 = -\frac{M}{r^3} - \frac{3iMa\cos\theta}{4r^4} + \frac{3Ma^2(3\cos^2\theta - 1)}{r^5} + O(r^{-6}),$$

$$\Psi_3 = -\frac{3iMa\sin\theta}{2\sqrt{2}r^4} + \left[-\frac{3i}{2\sqrt{2}}M^2a\sin\theta + \frac{6i}{\sqrt{2}}Ma^2\sin\theta\cos\theta\right]r^{-5} + O(r^{-6}), \qquad \Psi_4 = \frac{3Ma^2\sin^2\theta}{4r^5} + O(r^{-6}).$$

$$\eth\bar{\eth}\Psi_0^1 + 5\Psi_0^1 = 10(\Psi_1^0)^2 - 15\Psi_0^0\Psi_2^0.$$

For higher order coefficients, from Cartan structure equations and Bianchi identities, one can get the k-th order coefficients in terms of Ψ_0^k, its derivative and other coefficients of lower order. From Killing equations, one can get following control equation

$$\eth\bar{\eth}\Psi_0^k + \frac{(k+4)(k+1)}{2}\Psi_0^k = \cdots$$

The general solution of the homogeneous part of above equation is

$$\Psi = D^k\ {}_2Y_{2,0},$$

so the general solution for Ψ_0^k is

$$\Psi_0^k = \tilde{\Psi}_0^k + \Psi.$$

Now the problem is how to fix D^k ?

It is well-known that Kerr metric is algebraic special. Let $\tilde{\Psi}_0^k$ to be the special solution corresponds to Kerr solution, if Ψ_0^k corresponds to another algebraic special solution, the algebraic special condition implies

$$[81(\Psi_2^0)^4\Psi_4^4 - 54(\Psi_2^0)^3(\Psi_3^2)^2]\Psi_0^k + [81(\Psi_2^0)^4\Psi_0^0 - 54(\Psi_2^0)^3(\Psi_1^0)^2]\Psi_4^k + \cdots = 0$$

$$[81(\Psi_2^0)^4\Psi_4^4 - 54(\Psi_2^0)^3(\Psi_3^2)^2]\tilde{\Psi}_0^k + [81(\Psi_2^0)^4\Psi_0^0 - 54(\Psi_2^0)^3(\Psi_1^0)^2]\tilde{\Psi}_4^k + \cdots = 0.$$

which implies $D^k = 0$, i.e. following theorem holds.

Further more, consider the algebraic special condition order by order, one can get the concrete form of $\tilde{\Psi}_0^k$, i.e. one can get the concrete form of Kerr metric in Bondi coordinates, for example

$$D^5 = Ma^2\sqrt{\frac{16\pi}{5}} + i\sqrt{6\pi}M^2a.$$

Theorem (Wu and Bai, Phys. Rev. D 2010)

All stationary, vacuum, asymptotic flat, algebraic special space-time will be isomorphic to Kerr metric in a neighborhood of null infinity.

➢Generalize to Asymptotic AdS Case

- Asymptotic AdS space-time

A d-dimensional space-time $(\hat{M}, \hat{g}_{ab})$ will be said to be *asymptotically anti-de Sitter* if there exists a manifold M with boundary $\mathcal{I}$, equipped with a metric g_{ab} and a diffeomorphism from $\hat{M}$ onto $M - \mathcal{I}$ of M (with which we identify $\hat{M}$ and $M - \mathcal{I}$) and the interior of M such that:

1. there exists a function Ω on M for which $g_{ab} = \Omega^2 \hat{g}_{ab}$ on $\hat{M}$;
2. $\mathcal{I}$ is topologically $S^{d-2} \times R$, Ω vanishes on $\mathcal{I}$ but its gradient $\nabla_a \Omega$ is nowhere vanishing on $\mathcal{I}$;
3. On $\hat{M}$, $\hat{g}_{ab}$ satisfies $\hat{R}_{ab} - \frac{1}{2}\hat{R}\,\hat{g}_{ab} + \Lambda \hat{g}_{ab} = 8\pi G_{(d)} \hat{T}_{ab}$, where Λ is a negative constant, $G_{(d)}$ is Newton's constant in d-dimensions, and the matter stress-energy $\hat{T}_{ab}$ is such that $\Omega^{2-d}\hat{T}_{ab}$ admits a smooth limit to $\mathcal{I}$.
4. The Weyl tensor of g_{ab} is such that $\Omega^{4-d} C_{abcd}$ is smooth on M and vanishes at $\mathcal{I}$.

- Strong asymptotic AdS spacetime : Induced metric on conformal boundary is conformal flat. (Ashtekar, 2015)

- Gravitational wave in Asymptotic AdS space-time (X. Zhang et. al., 2011, 2017)
- Uniqueness for stationary AdS black hole ? (Black ring ? Topological Censorship and Higher Genus Black Holes, Galloway et.al. 1999, the sum of the genera of the cross-sections in which such a hypersurface meets black hole horizons is bounded above by the genus of the cut of infinity defined by the hypersurface.)
- Multiple structure of asymptotic AdS space-time ?
- Local uniqueness for AdS black hole ?

- Bondi coordinates for asymptotic AdS metric (X. Zhang, et. al. G. R. G., 2011)

$$ds^2 = \left(e^{2\beta}\frac{V}{r} - r^2 h_{AB}U^A U^B\right)du^2 + 2e^{2\beta}dudr + 2r^2 h_{AB}U^B dudx^A - r^2 h_{AB}dx^A dx^B \tag{2.1}$$

where $A, B = 2, 3$, $U^2 = U$, $U^3 = W\csc\theta$,

$$h \equiv \begin{pmatrix} h_{22} & h_{23} \\ h_{32} & h_{33} \end{pmatrix} \equiv \begin{pmatrix} e^{2\gamma}\cosh 2\delta & \sinh 2\delta \sin\theta \\ \sinh 2\delta \sin\theta & e^{-2\gamma}\cosh 2\delta \sin^2\theta \end{pmatrix},$$

Asymptotic behavior of associated functions

$$\gamma = \frac{c}{r} + \left(-\frac{1}{6}c^3 - \frac{3}{2}d^2 c + C\right)\left(\frac{1}{r}\right)^3 + O\left(\frac{1}{r}\right)^4,$$

$$\delta = \frac{d}{r} + \left(-\frac{1}{6}d^3 + \frac{1}{2}c^2 d + D\right)\left(\frac{1}{r}\right)^3 + O\left(\frac{1}{r}\right)^4.$$

$$\beta = B - \frac{1}{4}\left(c^2 + d^2\right)\left(\frac{1}{r}\right)^2 + \frac{1}{8}\left(c^4 + 2c^2d^2 + d^4 - 6cC - 6dD\right)\left(\frac{1}{r}\right)^4 + O\left(\frac{1}{r}\right)^5.$$

$$
\begin{aligned}
W =& X + 2e^{2B} B_{,\phi} \csc\theta \frac{1}{r} + e^{2B}\Big(2cB_{,\phi}\csc\theta + c_{,\phi}\csc\theta - 2d\cot\theta + 2dB_{,\theta} - d_{,\theta}\Big)\left(\frac{1}{r}\right)^2 \\
&+ e^{2B}\Big(B_{,\phi}c^2\csc\theta + 3cc_{,\phi}\csc\theta - 2cd_{,\theta} + 2P + d^2 B_{,\phi}\csc\theta + 3dd_{,\phi}\csc\theta \\
&+ 2c_{,\theta}d\Big)\left(\frac{1}{r}\right)^3 + O\left(\frac{1}{r}\right)^4, \\
U =& Y + 2e^{2B} B_{,\theta}\frac{1}{r} - e^{2B}\Big(2dB_{,\phi}\csc\theta + d_{,\phi}\csc\theta + 2c\cot\theta + 2cB_{,\theta} + c_{,\theta}\Big)\left(\frac{1}{r}\right)^2 \\
&+ e^{2B}\Big(c^2 B_{,\theta} + 4c^2\cot\theta + 2cd_{,\phi}\csc\theta + 3cc_{,\theta} + 2N - 2dc_{,\phi}\csc\theta + 4d^2\cot\theta \\
&+ d^2 B_{,\theta} + 3dd_{,\theta}\Big)\left(\frac{1}{r}\right)^3 + O\left(\frac{1}{r}\right)^4.
\end{aligned}
$$

$$
\begin{aligned}
V =& -\frac{e^{2B}\Lambda}{3}r^3 + \Big(\cot\theta Y + \csc\theta X_{,\phi} + Y_{,\theta}\Big)r^2 + e^{2B}\Big(4B_{,\phi}^2\csc^2\theta + 2B_{,\phi\phi}\csc^2\theta \\
&+ \frac{1}{2}\Lambda c^2 + \frac{1}{2}\Lambda d^2 + 2B_{,\theta}\cot\theta + 4B_{,\theta}^2 + 2B_{,\theta\theta} + 1\Big)r - 2M + O\Big(\frac{1}{r}\Big).
\end{aligned}
$$

Some facts :

If one want the metric has following form, i.e. the strong asymptotic AdS condition holds (Schouten tensor of conformal boundary vanishes),

$$ds^2 = ds^2{}_{AdS} + higher\ order\ term$$

which means

$$B = X = Y = 0$$
$$\Lambda(c^2 + d^2) = 0$$

This means no gravitational wave (or Bondi news function) !

Stating from the asymptotic AdS metric, combining Einstein equations and static condition, one can get following control on Weyl curvature :

1. For lower order coefficients ($O(r^{-k})$, $k \leq 5$)

$$\Psi_1^0 = 0, \quad A^2 = 0, \quad \Psi_3^0 = 0. \qquad \Psi_0^1 = 0. \qquad \eth\Psi_2^0 = \frac{\Lambda}{6}\eth'\Psi_0^0$$

$$\Psi_1^0 = \Psi_3^0 = \Psi_0^1 = 0. \qquad \Psi_1^1 = 0,\ \Psi_1^2 = 0,$$

$$\bar{\Psi}_2^0 = \Psi_2^0,\ \Psi_2^1 = 0,\ \Psi_2^2 = -\frac{1}{2}\eth'\Psi_1^1,$$

$$\Psi_2^3 = \frac{\Lambda}{36}\Psi_0^0\bar{\Psi}_0^0,$$

$$\Psi_4^0 = \frac{\Lambda^2}{36}\bar{\Psi}_0^0,\ \Psi_4^1 = 0,\ \Psi_4^2 = -\frac{1}{2}\eth'\Psi_3^1,$$

$$\Psi_4^3 = \frac{\Lambda}{12}\bar{\Psi}_0^0\Psi_2^0,\ \Psi_4^4 = -\frac{1}{4}\eth'\Psi_3^3.$$

$$\Psi_3^1 = -\eth'\Psi_2^0,\ \Psi_3^2 = 0,\ \Psi_3^3 = -\frac{1}{3}\eth'\Psi_2^2,$$

$$\Psi_3^4 = \frac{1}{4}\Big[-\eth'\Psi_2^3 - \frac{1}{2}\Psi_2^0\bar{\Psi}_1^1 + \frac{\Lambda}{6}\bar{\Psi}_0^0\Psi_1^1 - \frac{1}{6}\bar{\Psi}_0^0\eth\Psi_2^0\Big],$$

2. For higher order coefficients

a. Master equation :

$$\frac{\Lambda(k-3)}{6(k-1)(k-2)}\Psi_0^{k-3} = \cdots$$

b. All order coefficients can be got by algebraic combination of Ψ_0^k and its spherical derivatives.

c. Compare with the master equation for asymptotic flat case,

$$\eth\bar{\eth}\Psi_0^k + \frac{(k+4)(k+1)}{2}\Psi_0^k = \cdots.$$

One can find there is no multiple structure in higher order coefficients for asymptotic ads metric. Detail calculation also shows the function Ψ_2^0 and Ψ_0^0 are not independent. Only one is free.

- Multiple structure of asymptotic AdS space-time

One can express function Ψ_2^0 in terms of spherical harmonics, i.e. $\Psi_2^0 = \sum_{l=0,m}^{+\infty} A_{lm}Y_{lm}$, then $\{A_{lm}\}$ are multiples of asymptotic AdS space-time.

- Uniqueness result for asymptotic AdS space-time

To get a uniqueness result for asymptotic AdS space-time, it is clear that the central issue is to fix the only free function Ψ_2^0 .

1. If function Ψ_2^0 only contains finite multiples, i.e.

$$\Psi_2^0 = \sum_{l=0,m}^{k} A_{lm}Y_{lm} \text{ , } k < +\infty \text{ ,}$$

with algevraic special conition, one can prove that such space-time must be Schwarzschild-AdS space-time, at least in a neighborhood of infinity. (He et. al., 2018, in prepation)

2. For general case, the uniqueness problem of asymptotic AdS space-time is still open. More work and new idea are needed.

THANK YOU !