Gauge invariant regularization for perturbative chiral gauge theory

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Based on
YH, H. Kawai, arXiv:1705.01317,
YH, H. Kawai, K. Sakai, arXiv:1806.00349
YH, H. Kawai, K. Sakai, to appear

Motivation

Standard Model is a chiral gauge theory (CGT)

$$SU(3)_C \times SU(2)_L \times U(1)_Y \quad \text{(EW sector)}$$

However, regularization of CGT is difficult!

- No lattice regulator [cf. Nielsen-Ninomiya's thm]
- No manifestly gauge-invariant perturbative regulator because fermion mass term is forbidden by chiral gauge symmetry...

Regularization problem for CGT

Eg. Dimensional Regularization

$$\mathcal{L}_{reg.} = \bar{\psi}\left(\not{D}_{(4)} + \not{D}_{(\epsilon)}\right) P_L \psi, \qquad \left[\not{D}_{(\epsilon)}, \gamma_5\right] = 0$$

- ϵ-dimensional kinetic term behaves as “mass term”
- Gauge sym. is broken even when anomaly-free theory
- Need extra local counter terms to restore gauge sym:

$$\Gamma[A] + \Delta\Gamma[A] \qquad \text{s.t.} \qquad \delta_\omega\left(\Gamma[A] + \Delta\Gamma[A]\right) = 0$$

- However, the procedure is rather complicated...

Is there a gauge-invariant regularization for CGT?

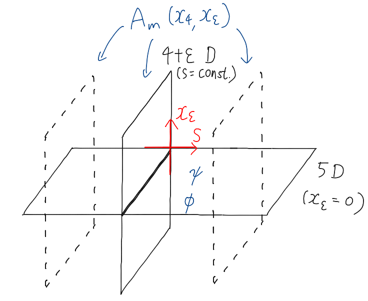
(except for gauge anomaly)

Note: The naive prescription $\left\{\not{D}_{(\epsilon)}, \gamma_5\right\} = 0$ can be used only for one-loop calculations.

5D Domain-Wall fermion with PV regulators

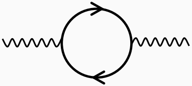
+

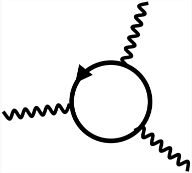
$(4+\epsilon)$-D gauge field

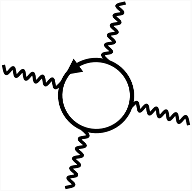


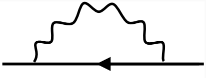
This regularization is quite useful!

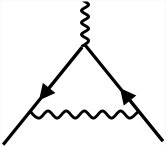
One-loop diagrams

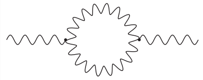


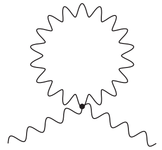












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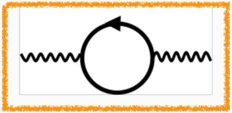
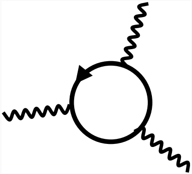
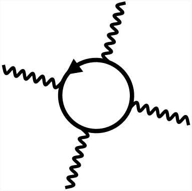
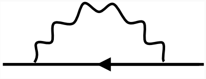
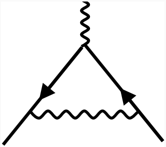
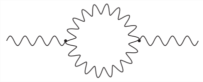


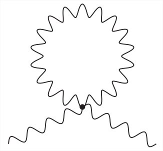








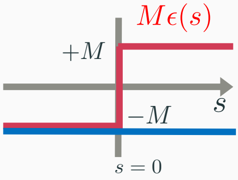




Domain-wall fermion [Kaplan, 1992]

$$S_{DW} = \int d^4x \int_{-\infty}^{\infty} ds\ \bar{\psi}(x,s)\left[\partial\!\!\!/_{(4)} + \gamma_5 \partial_s - M\epsilon(s)\right]\psi(x,s)$$

- $\epsilon(s)$ is the sign function $(M > 0)$
  $\rightarrow$ induce LH massless mode $\propto e^{-M|s|}$
- s-direction's size is infinitely large
  $\rightarrow$ RH massless mode is decoupled
- Massive modes form a continuous spectrum and give rise to IR divergence
  $\rightarrow$ cancel by bosonic field $\phi(x,s)$ with constant mass $(-M)$

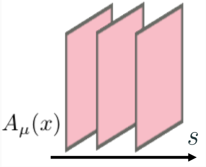


Action

$$S = \int d^4x \int ds\ \bar{\psi}(x,s)\left[\not{D}_{(4)} + \gamma_5\partial_s - M\epsilon(s)\right]\psi(x,s)$$
$$+ \int d^4x \int ds\ \bar{\phi}(x,s)\left[\not{D}_{(4)} + \gamma_5\partial_s + M\right]\phi(x,s)$$
$$+ \frac{1}{4g^2}\int d^4x\ \mathrm{tr}(F_{\mu\nu}(x)F_{\mu\nu}(x))$$

- boson ϕ will cancel IR div. from ψ
- Gauge field is 4-dimensional one $A_\mu(x)$:

  s-indep. & $A_5 = 0$
- Dirac fermion (boson) are expected to be regularized in a gauge-invariant way



Action in 4D form [Narayanan-Neuberger, 1993]

$$S = \int d^4x \int ds\ \bar{\psi}(x,s)\left[\not{D}_{(4)} + \gamma_5 \partial_s - M\epsilon(s)\right]\psi(x,s)$$
$$+ \int d^4x \int ds\ \bar{\phi}(x,s)\left[\not{D}_{(4)} + \gamma_5 \partial_s + M\right]\phi(x,s)$$
$$+ \frac{1}{4g^2}\int d^4x\ \mathrm{tr}(F_{\mu\nu}(x)F_{\mu\nu}(x))$$

$\Downarrow$ "mass operators" $\begin{cases} \hat{\mathcal{M}}_\psi \equiv -\partial_s - M\epsilon(s) \\ \hat{\mathcal{M}}_\phi \equiv -\partial_s + M \end{cases}$

$$S = \int d^4x \int ds\ \bar{\psi}(x,s)\left[\not{D}_{(4)} + \hat{\mathcal{M}}_\psi P_L + \hat{\mathcal{M}}_\psi^\dagger P_R\right]\psi(x,s)$$
$$+ \int d^4x \int ds\ \bar{\phi}(x,s)\left[\not{D}_{(4)} + \hat{\mathcal{M}}_\phi P_L + \hat{\mathcal{M}}_\phi^\dagger P_R\right]\phi(x,s)$$
$$+ \frac{1}{4g^2}\int d^4x\ \mathrm{tr}(F_{\mu\nu}(x)F_{\mu\nu}(x)),$$

s-space looks like an internal space of 4D spinors ψ, ϕ

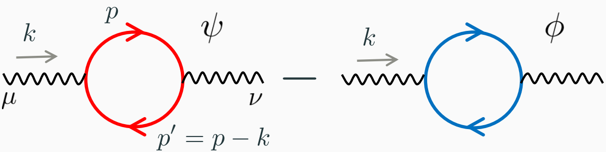
Vacuum polarization diagram

- Propagator

$$\begin{cases} G_\psi(p) = \left(-i\not{p} + \hat{\mathcal{M}}_\psi\right) \frac{1}{p^2+\hat{\mathcal{M}}_\psi^\dagger \hat{\mathcal{M}}_\psi} P_R + \left(-i\not{p} + \hat{\mathcal{M}}_\psi^\dagger\right) \frac{1}{p^2+\hat{\mathcal{M}}_\psi \hat{\mathcal{M}}_\psi^\dagger} P_L \\ G_\phi(p) = \frac{-i\not{p}+\hat{\mathcal{M}}_\phi P_R+\hat{\mathcal{M}}_\phi^\dagger P_L}{p^2+\hat{\mathcal{M}}_\phi^\dagger \hat{\mathcal{M}}_\phi} \end{cases}$$

- Vaccum polarization diagram

$\Pi_{\mu\nu}(k) =$



$$= \int \frac{d^4p}{(2\pi)^4} \left(\mathrm{Tr}\left[G_\psi(p)\gamma_\mu G_\psi(p')\gamma_\nu\right] - \mathrm{Tr}\left[G_\phi(p)\gamma_\mu G_\phi(p')\gamma_\nu\right]\right)$$

- Tr includes the trace over s-space where $\hat{\mathcal{M}}_\psi$ and $\hat{\mathcal{M}}_\phi$ act.
- We will regulate the UV divergence later.

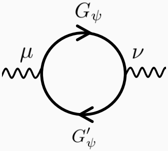
How to define Tr ?

$$\mathrm{Tr}\left[G_\psi(p)\gamma_\mu G_\psi(p')\gamma_\nu\right] - \mathrm{Tr}\left[G_\phi(p)\gamma_\mu G_\phi(p')\gamma_\nu\right]$$

- Each trace is IR divergent $\rightarrow$ First subtract these contents, and then take the trace over s-space. $\rightarrow$ IR finite result
- However, there is an ambiguity in the choice of the starting point of the loop.

(i) $$\int ds\ \langle s| \left[G_\psi\gamma_\mu G'_\psi\gamma_\nu - G_\phi\gamma_\mu G'_\phi\gamma_\nu\right]|s\rangle$$

(ii) $$\int ds\ \langle s| \left[G'_\psi\gamma_\nu G_\psi\gamma_\mu - G'_\phi\gamma_\nu G_\phi\gamma_\mu\right]|s\rangle$$



$$\mathrm{Tr}\left[G_\psi(p)\gamma_\mu G_\psi(p')\gamma_\nu\right] - \mathrm{Tr}\left[G_\phi(p)\gamma_\mu G_\phi(p')\gamma_\nu\right]$$

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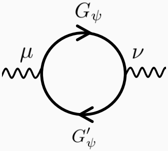
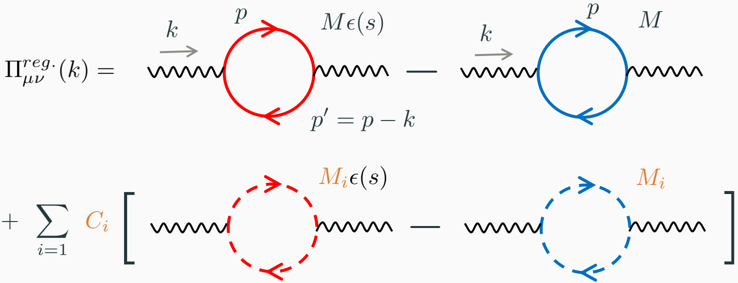
(i) $\int ds\ \langle s| \left[G_\psi \gamma_\mu G'_\psi \gamma_\nu - G_\phi \gamma_\mu G'_\phi \gamma_\nu\right] |s\rangle$

(ii) $\int ds\ \langle s| \left[G'_\psi \gamma_\nu G_\psi \gamma_\mu - G'_\phi \gamma_\nu G_\phi \gamma_\mu\right] |s\rangle$



- As seen later, the ambiguity vanishes when anomaly-free.
- We adopt the symmetric choice (iii) $\equiv \frac{1}{2}$(i) $+ \frac{1}{2}$(ii) for the moment.

To make the loop UV-finite, we introduce Pauli-Villars fields



- C_i, M_i have to be chosen such that the sum is UV-finite.
- Need another condition to eliminate the extra chiral fermions :

$$\sum_{i=1} C_i = 0$$

Result

$$\Pi^{reg.}_{\mu\nu}(k)$$

$$= \int \frac{d^4p}{(2\pi)^4} \left\{ \frac{1}{2}\mathrm{tr}\left[\frac{i\not{p}}{p^2}\gamma_\mu \frac{i\not{p}'}{p'^2}\gamma_\nu\right] - \frac{1}{2}\mathrm{tr}\left[\frac{i\not{p}+M}{p^2+M^2}\gamma_\mu \frac{i\not{p}'+M}{p'^2+M^2}\gamma_\nu\right] \right.$$

$$\left. - \sum_i C_i \frac{1}{2}\mathrm{tr}\left[\frac{i\not{p}+M_i}{p^2+{M_i}^2}\gamma_\mu \frac{i\not{p}'+M_i}{p'^2+{M_i}^2}\gamma_\nu\right] + f(p,p')\mathrm{tr}\left[\frac{i\not{p}}{p^2}\gamma_\mu \frac{i\not{p}'}{p'^2}\gamma_\nu\gamma_5\right] \right\}$$

$$f(p,p') \equiv \sum_{i=0} C_i \left(1 - \frac{\sqrt{p^2+{M_i}^2}\left(\sqrt{p^2+{M_i}^2}\sqrt{p'^2+{M_i}^2}+{M_i}^2\right)}{\sqrt{p'^2+{M_i}^2}\left(\sqrt{p^2+{M_i}^2}+\sqrt{p'^2+{M_i}^2}\right)^2}\right) \frac{M_i}{2\sqrt{p^2+{M_i}^2}} + (p \leftrightarrow p')$$

$$(C_0 \equiv 1,\ M_0 \equiv M)$$

Parity-even Regularized via 4D Pauli-Villars fields in a gauge-invariant way

Parity-odd Multiplied by a non-local regularization factor

The entire is UV-finite!

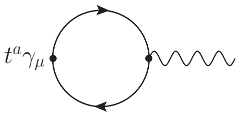
Gauge anomaly

- The parity-odd part reproduces the correct gauge anomaly.

Of course, 4D anomaly is obtained from the triangle and rectangle diagrams.
For simplicity, let us consider 2D non-Abelian CGT.

$$\begin{aligned}\delta_\omega S_{eff}[A] &= \omega^a(x)\left(D_\mu \langle J_\mu(x)\rangle\right)^a \\ &= \omega^a(x)\,\partial_\mu\,\mathrm{tr}\left[t^a A_\nu(x)\right]\ \ \Pi^{reg.}_{\mu\nu}\Big|_{parity-odd} \\ &\to -\frac{1}{4\pi}\epsilon_{\mu\nu}\,\omega^a(x)\mathrm{tr}\left[t^a\partial_\mu A_\nu(x)\right] \qquad (M^2\to\infty)\end{aligned}$$

2D consistent anomaly!



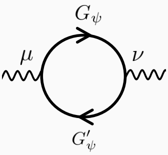
How to define Tr? -revisited-

$$\mathrm{Tr}\left[G_\psi(p)\gamma_\mu G_\psi(p')\gamma_\nu\right] - \mathrm{Tr}\left[G_\phi(p)\gamma_\mu G_\phi(p')\gamma_\nu\right]$$

(i) $\int ds\ \langle s|\left[G_\psi\gamma_\mu G'_\psi\gamma_\nu - G_\phi\gamma_\mu G'_\phi\gamma_\nu\right]|s\rangle$

(ii) $\int ds\ \langle s|\left[G'_\psi\gamma_\nu G_\psi\gamma_\mu - G'_\phi\gamma_\nu G_\phi\gamma_\mu\right]|s\rangle$

(iii) $\frac{1}{2}$(i) $+ \frac{1}{2}$(ii)

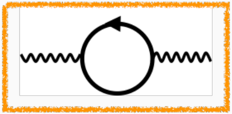


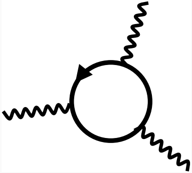
(i) $\rightarrow$ covariant anomaly: $(D_\mu J_\mu(x))^a = -\frac{1}{2\pi}\epsilon_{\mu\nu}\ \mathrm{tr}\left[t^a F_{\mu\nu}\right]$

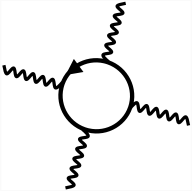
$\therefore$ The choices correspond to the consistent and covariant anomaly!

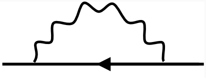
For anomaly-free cases, no ambiguity because both are 0.

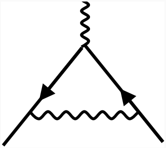
One-loop diagrams

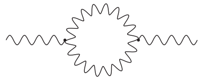


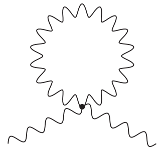












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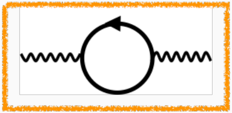
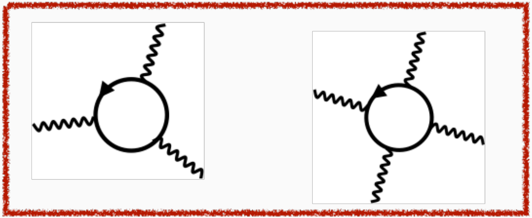
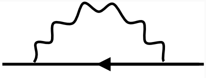
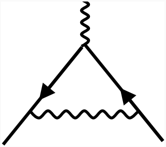
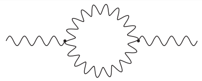


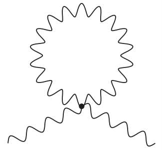


Can be regularized similarly

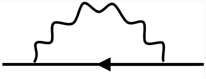


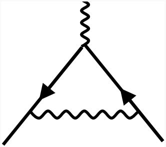


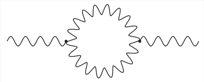


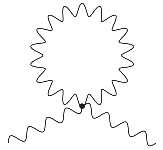


Other one-loop diagrams









- Apply the dim. reg. only to the gauge field.

$$\frac{1}{4g^2}\int d^4x\ \mathrm{tr}\left(F_{\mu\nu}(x)\right)^2 \to \frac{1}{4g^2}\int d^{4+\epsilon}x\ \mathrm{tr}\left(F_{mn}(\tilde{x})\right)^2$$

$$\tilde{x} = (x_{(4)}, x_\epsilon), \qquad m, n = 1, \cdots, (4+\epsilon)$$

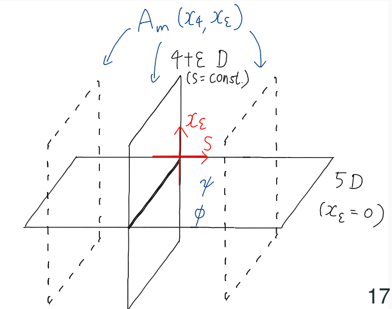
Full Lagrangian

$$
\begin{aligned}
S = &\int d^4x \int ds \;\bar{\psi}(x,s)\left[\not{D}_{(4)} + \hat{\mathcal{M}}_\psi P_L + \hat{\mathcal{M}}_\psi^\dagger P_R\right]\psi(x,s) \\
&+ \int d^4x \int ds \;\bar{\phi}(x,s)\left[\not{D}_{(4)} + \hat{\mathcal{M}}_\phi P_L + \hat{\mathcal{M}}_\phi^\dagger P_R\right]\phi(x,s) \\
&+ (\text{Pauli-Villars fields}) \\
&+ \frac{1}{4g^2}\int d^{4+\epsilon}\tilde{x}\; \mathrm{tr}(F_{mn}(\tilde{x}))^2
\end{aligned}
$$

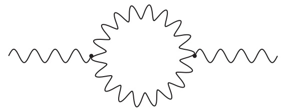
- ψ and ϕ couple with the $(4+\epsilon)$-D gauge field on the brane $x_\epsilon = 0$.

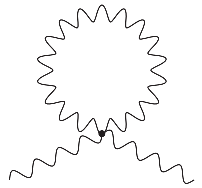
$$\not{D}_{(4)} = \gamma_\mu\left(\partial_\mu - iA_\mu(x_{(4)}, x_\epsilon = 0)\right)$$

$$\mu = 1, \cdots, 4$$



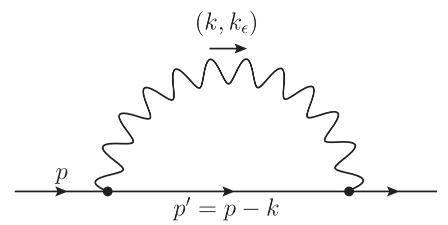
Gauge boson loop





- The gauge boson's loops can be regularized as the ordinary dimensional regularization.

Fermion self-energy

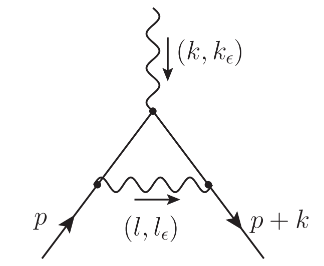


- The ϵ-dimensional momentum k_ϵ is not conserved at the vertices because fermion's momentum is 4-dimensional.

$$
\begin{aligned}
& -t^a t^a \int \frac{d^{4+\epsilon}k}{(2\pi)^{4+\epsilon}} \frac{1}{k^2 + (k_\epsilon)^2} \gamma_\mu G_\psi(p-k)\gamma_\mu \\
&= -t^a t^a \int \frac{d^4 k}{(2\pi)^4} \frac{\Gamma(1-\epsilon/2)}{(4\pi)^{\epsilon/2}} \frac{1}{(k^2)^{1-\epsilon/2}} \gamma_\mu G_\psi(p-k)\gamma_\mu \\
&= -t^a t^a \frac{\Gamma(-\epsilon/2)}{16\pi^2} \left[i\not{p} + 4\left(\hat{\mathcal{M}}_\psi P_L + \hat{\mathcal{M}}_\psi^\dagger P_R \right) \right] + \text{(finite terms)}
\end{aligned}
$$

- Regularized via the parameter ϵ

Vertex correction



- Can be regularized similarly

$$= -i\frac{\Gamma(1-\epsilon/2)}{(4\pi)^{\epsilon/2}} \int \frac{d^4 l}{(2\pi)^4} \frac{t^a t^b t^a}{(l^2)^{1-\epsilon/2}} \gamma_\mu G_\psi(p+k-l)\gamma_\nu G_\psi(p-l)\gamma_\mu$$

Summary

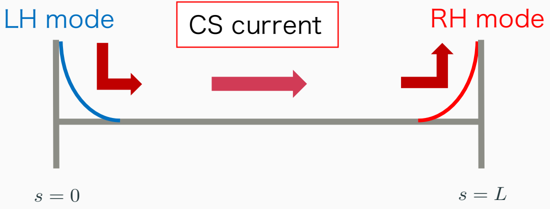
- We propose a gauge-invariant regularization for perturbative chiral gauge theory.
- 5D domain-wall fermion with PV regulators can describe the 4D chiral fermion's loop in a gauge-invariant way.
- The expression for the gauge anomaly has an ambiguity, which vanishes for anomaly-free cases.
- The other loop diagrams can be regularized by the $(4+\epsilon)$-dimensional gauge field.
- (Fermion number anomaly can be obtained in the gauge-invariant form.)

Please use this regularization !

Backup Slides

Anomaly inflow [Callan-Harvey, 1985]

- 5D parity anomaly
  $\rightarrow$ Chern-Simons action in bulk (parity-odd)
- Gauge current flows from $s = 0$ to $s = L$ through bulk
- We decoupled RH mode by $L \rightarrow \infty$
  $\rightarrow$ Flowing out of gauge current looks anomaly!
- This current is sensitive to boundary condition at $s = \infty$ (=IR regulator)

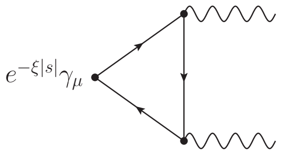


Fermion number anomaly

Fermion number anomaly is physical, so it should be defined without any ambiguity.

$\rightarrow$ Define the fermion number current with a damping factor:

$$\langle j_\mu^{U(1)}(x)\rangle \equiv \lim_{\xi\to 0}\int ds\, e^{-\xi|s|}\left[\langle\bar{\psi}\gamma_\mu\psi\rangle + \langle\bar{\phi}\gamma_\mu\phi\rangle\right] + \text{(PV-fields)}$$



- The IR divergence is regularized by ξ and then canceled.
- The regularized current is automatically gauge invariant without any ambiguity:

$$\langle\partial_\mu j_\mu^{U(1)}(x)\rangle = \frac{-1}{32\pi^2}\mathrm{tr}\left[F\tilde{F}\right]$$

Computation of Tr :(1)

$$\int ds\ \langle s| \left[G_\psi \gamma_\mu G'_\psi \gamma_\nu - G_\phi \gamma_\mu G'_\phi \gamma_\nu\right] |s\rangle$$

$$= \int ds\ \langle s| \left[\frac{\mathrm{tr}\left(i\not{p}\gamma_\mu i\not{p}'\gamma_\nu P_R\right) + \mathrm{tr}\left(\gamma_\mu\gamma_\nu P_R\right)\hat{\mathcal{M}}_\psi^\dagger \hat{\mathcal{M}}_\psi}{(p^2+\hat{\mathcal{M}}_\psi^\dagger\hat{\mathcal{M}}_\psi)(p'^2+\hat{\mathcal{M}}_\psi^\dagger\hat{\mathcal{M}}_\psi)} \right.$$

$$\left. + \frac{\mathrm{tr}\left(i\not{p}\gamma_\mu i\not{p}'\gamma_\nu P_L\right) + \mathrm{tr}\left(\gamma_\mu\gamma_\nu P_L\right)\hat{\mathcal{M}}_\psi \hat{\mathcal{M}}_\psi^\dagger}{(p^2+\hat{\mathcal{M}}_\psi\hat{\mathcal{M}}_\psi^\dagger)(p'^2+\hat{\mathcal{M}}_\psi\hat{\mathcal{M}}_\psi^\dagger)} - \frac{\mathrm{tr}\left(i\not{p}\gamma_\mu i\not{p}'\gamma_\nu\right) + \mathrm{tr}\left(\gamma_\mu\gamma_\nu\right)\hat{\mathcal{M}}_\phi^\dagger \hat{\mathcal{M}}_\phi}{(p^2+\hat{\mathcal{M}}_\phi^\dagger\hat{\mathcal{M}}_\phi)(p'^2+\hat{\mathcal{M}}_\phi^\dagger\hat{\mathcal{M}}_\phi)} \right] |s\rangle$$

In the 1st and 2nd terms, insert the complete sets of $\hat{\mathcal{M}}_\psi^\dagger \hat{\mathcal{M}}_\psi$ and $\hat{\mathcal{M}}_\psi \hat{\mathcal{M}}_\psi^\dagger$:

$$1 = |0\rangle\langle 0| + \int_0^\infty d\omega\, |l_\omega\rangle\langle l_\omega|, \quad 1 = \int_0^\infty d\omega\, |r_\omega\rangle\langle r_\omega|$$

$$\begin{cases} \hat{\mathcal{M}}_\psi^\dagger \hat{\mathcal{M}}_\psi |0\rangle = 0, \quad \hat{\mathcal{M}}_\psi^\dagger \hat{\mathcal{M}}_\psi |l_\omega\rangle = (M^2+\omega^2)|l_\omega\rangle \\ \hat{\mathcal{M}}_\psi \hat{\mathcal{M}}_\psi^\dagger |r_\omega\rangle = (M^2+\omega^2)|r_\omega\rangle \end{cases}$$

In the 3rd term, insert the complete set of $\hat{\mathcal{M}}_\phi^\dagger \hat{\mathcal{M}}_\phi$:

$$1 = \int_0^\infty d\omega\, |\phi_\omega\rangle\langle\phi_\omega|, \qquad \hat{\mathcal{M}}_\phi^\dagger \hat{\mathcal{M}}_\phi |\phi_\omega\rangle = (M^2+\omega^2)|\phi_\omega\rangle$$

Note: $\left[\hat{\mathcal{M}}_\phi, \hat{\mathcal{M}}_\phi^\dagger\right] = 0$

Computation of Tr : (2)

$$= \int ds \, |\langle s|0\rangle|^2 \frac{\mathrm{tr}\left(i\not{p}\gamma_\mu i\not{p}'\gamma_\nu P_L\right)}{p^2 p'^2}$$

$$+ \int ds \int_0^\infty d\omega \Bigg[|\langle s|r_\omega\rangle|^2 \frac{\mathrm{tr}\left(i\not{p}\gamma_\mu i\not{p}'\gamma_\nu P_R\right) + \mathrm{tr}\left(\gamma_\mu\gamma_\nu P_R\right)(M^2+\omega^2)}{(p^2+M^2+\omega^2)(p'^2+M^2+\omega^2)}$$

$$+ |\langle s|l_\omega\rangle|^2 \frac{\mathrm{tr}\left(i\not{p}\gamma_\mu i\not{p}'\gamma_\nu P_L\right) + \mathrm{tr}\left(\gamma_\mu\gamma_\nu P_L\right)(M^2+\omega^2)}{(p^2+M^2+\omega^2)(p'^2+M^2+\omega^2)}$$

$$-|\langle s|\phi_\omega\rangle \frac{\mathrm{tr}\left(i\not{p}\gamma_\mu i\not{p}'\gamma_\nu\right) + \mathrm{tr}\left(\gamma_\mu\gamma_\nu\right)(M^2+\omega^2)}{(p^2+M^2+\omega^2)(p'^2+M^2+\omega^2)} \Bigg]$$

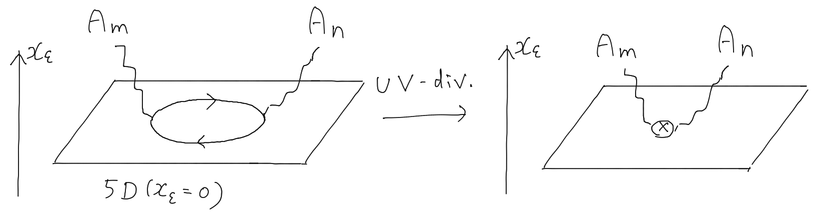
$$= \frac{\mathrm{tr}\left(i\not{p}\gamma_\mu i\not{p}'\gamma_\nu P_L\right)}{p^2 p'^2}$$

$$+ \underline{\int ds} \int_0^\infty d\omega \underbrace{\left[|\langle s|r_\omega\rangle|^2 + |\langle s|l_\omega\rangle|^2 - 2|\langle s|\phi_\omega\rangle|^2\right]}_{=2\delta(\omega)} \frac{\mathrm{tr}\left(i\not{p}\gamma_\mu i\not{p}'\gamma_\nu\right) + \mathrm{tr}\left(\gamma_\mu\gamma_\nu\right)(M^2+\omega^2)}{2(p^2+M^2+\omega^2)(p'^2+M^2+\omega^2)}$$

$$+ \underline{\int ds} \int_0^\infty d\omega \underbrace{\left[|\langle s|r_\omega\rangle|^2 - |\langle s|l_\omega\rangle|^2\right]}_{=\frac{2M}{M^2+\omega^2}} \frac{\mathrm{tr}\left(i\not{p}\gamma_\mu i\not{p}'\gamma_\nu\gamma_5\right) + \mathrm{tr}\left(\gamma_\mu\gamma_\nu\gamma_5\right)(M^2+\omega^2)}{2(p^2+M^2+\omega^2)(p'^2+M^2+\omega^2)}$$

Renormalizability (One-loop) (1)

- Vacuum polarization (with external line)



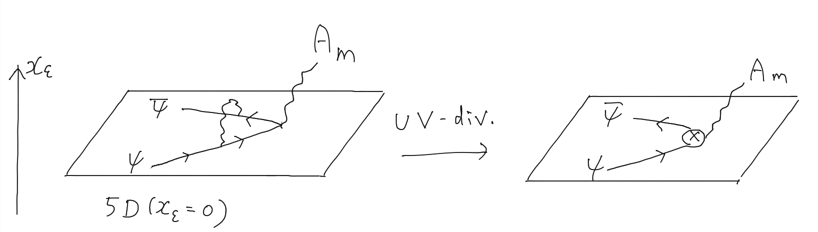
$$\int \frac{d^4k}{(2\pi)^4}\frac{d^\epsilon k_\epsilon}{(2\pi)^\epsilon}\frac{d^\epsilon k'_\epsilon}{(2\pi)^\epsilon} A_\mu(k,k_\epsilon)A_\nu(-k,k'_\epsilon)\Pi_{\mu\nu}(k)$$

$$\underset{\text{div. term}}{\longrightarrow} \int d^4x\, C \log M^2\, \delta_{\mu\nu}\, A_\mu(x, x_\epsilon = 0)A_\nu(x, x_\epsilon = 0)$$

- The (regularized) UV divergence is renormalized by a counter term on the brane $x_\epsilon = 0$.

Renormalizability (One-loop) (2)

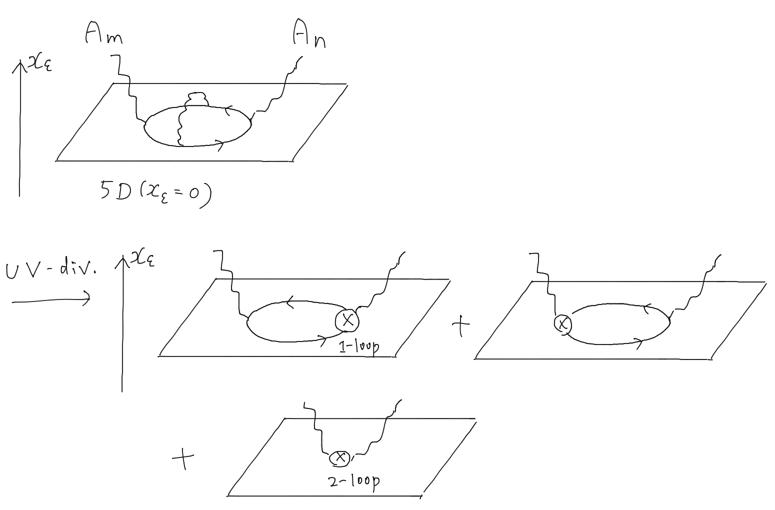
- Vertex correction (with external line)



- This UV divergence is also renormalized by a counter term on the brane $x_\epsilon = 0$.

Renormalizability (Two-loop)

- BPHZ scheme can be applied except that loop sub-diagrams with fermion internal lines should be replaced by counter terms on the brane.



Gauge invariance

- Fermion loop induces 4D gauge-invariant term:

$$\int d^4x \, C \log(\partial^2/\mu_R^2) \, \mathrm{tr} \left(F_{\mu\nu}(x,0)\right)^2 , \qquad (\mu, \nu = 1, \cdots, 4)$$

This term is also invariant under the following $(4+\epsilon)$-D gauge transformation:

$$A_m(x, x_\epsilon) \to e^{-i\omega(x,x_\epsilon)} \left[A_m(x, x_\epsilon) + \partial_m\right] e^{i\omega(x,x_\epsilon)}$$

$$(m, n = 1, \cdots, 4+\epsilon)$$

- Therefore, $(4+\epsilon)$-D gauge invariance is preserved.

Unitarity

The net effect of the mixed dimension is the following:

For external line
Non-conservation of the ϵ-dimensional momentum of A_m
$\rightarrow$ affect unitarity!

For internal line
Change the power of 4D propagators :

$$\frac{1}{k^2} \rightarrow \frac{1}{(k^2)^{1-\epsilon/2}}$$

$\rightarrow$ does not!

Therefore, unitarity would be restored in the limit of $\epsilon \rightarrow 0$.