Calculability of Vacuum Energy demands the Scale Invariance of the World

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1 Cosmological Constant Problem

Dark Clouds hanging over the two well-established theories

$$\text{Quantum Field Theory} \Longleftrightarrow \text{Einstein Gravity Theory}$$

I first explain my view point on what is actually the problem.
We know the recently observed Dark Energy Λ_0, which looks like a small Cosmological Constant (CC):

$$\text{Present observed CC} \quad 10^{-29}\text{gr/cm}^3 \sim 10^{-47}\text{GeV}^4 \equiv \Lambda_0 \tag{1}$$

We do not mind this tiny CC now, which will be explained after our CC problem is solved. However, we use it as the scale unit Λ_0 of our discussion in the Introduction.

What is the true problem?

Essential point: multiple mass scales are involved!

There are several dynamical symmetry breakings and they are necessarily accompanied by vacuum condensation energy (potential energy):

In particular, we are confident from the success of the Standard Model of the existence of at least two symmetry breakings:

$$\text{Higgs Condensation} \sim \quad (\,200\,\text{GeV}\,)^4 \sim 10^9 \text{GeV}^4 \sim 10^{56}\Lambda_0$$

$$\text{QCD Chiral Condensation } \langle \bar{q}q \rangle^{4/3} \sim \quad (\,200\,\text{MeV}\,)^4 \sim 10^{-3}\text{GeV}^4 \sim 10^{44}\Lambda_0$$

Nevertheless, these seem not contributing to the Cosmological Constant!

It is a Super fine tuning problem:

$$c \;:\; \text{initially prepared CC } (> 0)$$

$$c - 10^{56}\Lambda_0 \;:\; \text{should cancell, but leaving 1 part per } 10^{12}\text{; i.e.,} \;\sim 10^{44}\Lambda_0$$

$$c - 10^{56}\Lambda_0 - 10^{44}\Lambda_0 \;:\; \text{should cancell, but leaving 1 part per } 10^{44}\text{; i.e.,} \;\sim \Lambda_0$$

$$c - 10^{56}\Lambda_0 - 10^{44}\Lambda_0 \sim \Lambda_0 \;:\; \text{present Dark Energy}$$

$$c = \text{initially prepared CC}$$

$$\underbrace{654321,098765}_{12\,\text{digits}}4321,0987654321,0987654321,0987654321,0987654321 \times \Lambda_0 \sim 10^{56}\Lambda_0$$

$$c + V_{\text{Higgs}} =$$

$$\underbrace{4321,0987654321,0987654321,0987654321,0987654321}_{44\,\text{digits}} \times \Lambda_0 \sim 10^{44}\Lambda_0$$

$$c + V_{\text{Higgs}} + V_{\text{QCD}} = \text{present Dark Energy}$$

$$1 \times \Lambda_0 \sim \Lambda_0$$

Note that the vacuum energy is almost totally cancelled at each stage of spontaneous breaking as far as the the relevant energy scale order.

Contents

Part I: Scale Invariance is a Necessary Condition

Part II: Scale Invariance is Sufficient Condition

First part is the main part which is based on a simple observation on the vacuum energy, and I explain that calculability of the vacuum energy demands the scale invariance of the world.

Second part has many overlaps with my last year's SI talk at Fuji-Yoshida. My idea present there turned essentially be the same as proposed by the following people prior to me,

M. Shaposhnikov and D. Zenhausern, Phys. Lett. B **671** (2009) 162

This is actually a very good paper, which I believe solves the CC problem. I introduce you their scenario and point out the issues to be clarified more.

Part I: SI is Necessary

2 Vacuum Energy $\simeq$ vacuum condensation energy

People may suspect that there are "Two" origins of Cosmological Constant

(Quantum) Vacuum Energy

$$\sum_{\boldsymbol{k},s} \frac{1}{2}\hbar\omega_{\boldsymbol{k}} - \sum_{\boldsymbol{k},s} \hbar E_{\boldsymbol{k}} \tag{2}$$

Infinite, No controle, simply discarded

$\updownarrow$

(Classical) Potential Energy

$$V(\phi_c) : \text{potential} \tag{3}$$

Finite, vacuum condensation energy

They are separately stored in our (or my, at least) memory, but actually, almost the same object, as we see now.

We now show for the vacuum energies in the SM that

$$\text{quantum Vacuum Energy} = \text{Higgs Potential Energy} \tag{4}$$

Let us see this more explicitly. For that purpose, Consider

Simplified SM

$$\mathcal{L}_{\rm r} = \bar{\psi}\big(i\gamma^\mu\partial_\mu - y\phi(x)\big)\psi(x) + \frac{1}{2}\big(\partial^\mu\phi(x)\partial_\mu\phi(x) - m^2\phi^2(x)\big) - \frac{\lambda}{4!}\phi^4(x) - hm^4. \tag{5}$$

Effective Action (Effective Potential) is calculated prior to the vacuum choice.
(i.e., calculable independently of the choice of the vacuum)

Review:

$W[J]$: generating functional of connected Green's functions

$$Z = e^{iW[J]} = \int \mathcal{D}\Phi\, e^{i(S[\Phi]+J\Phi)}$$

$\Gamma[\phi]$: generating functional of 1PI vertex functions

$$\Gamma[\phi] = W[J] - J\cdot\phi$$

$$\phi(x) \equiv \frac{\delta W[J]}{\delta J(x)} \quad \rightarrow \quad \frac{\delta\Gamma[\phi]}{\delta\phi(x)} = -J(x)$$

$V(\phi)$: effective potential

$$V(\phi) = -\frac{\Gamma[\phi(x)=\phi]}{\int d^4x} \tag{6}$$

Calculating Formula:

$$\mathcal{L}(\Phi+\phi)=\mathcal{L}(\phi)+\frac{\partial\mathcal{L}(\phi)}{\partial\phi}\Phi+\frac{1}{2}\Phi\left(iD_{\mathrm{F}}^{-1}(\phi)\right)\Phi+\mathcal{L}_{\mathrm{int.}}(\Phi;\phi) \tag{7}$$

$$\Gamma[\phi]=\int d^4x\mathcal{L}(\phi)+\frac{i}{2}\hbar\ln\mathrm{Det}\left[iD_{\mathrm{F}}^{-1}(\phi)\right]-i\hbar\left\langle\exp\left(\frac{i}{\hbar}\int d^4x\mathcal{L}_{\mathrm{int}}(\Phi;\phi)\right)\right\rangle_{\mathrm{1PI}} \tag{8}$$

$$V[\phi]=V_0(\phi)+\frac{1}{2}\hbar\int\frac{d^4k}{i(2\pi)^4}\ln\det\left[iD_{\mathrm{F}}^{-1}(k;\phi)\right]+i\hbar\left\langle\exp\left(\frac{i}{\hbar}\int d^4x\mathcal{L}_{\mathrm{int}}(\Phi;\phi)\right)\right\rangle_{\mathrm{1PI}} \tag{9}$$

1-loop effective potential in the Simplified SM

Use dimensional regularization and Mass-Independent (MI) renormalization

$$\begin{aligned}\mathcal{L}_0&=\mathcal{L}_{\mathrm{r}}+\delta\mathcal{L}\\ \delta\mathcal{L}&=A\bar{\psi}i\gamma^\mu\partial_\mu\psi+\frac{B}{2}(\partial_\mu\phi)^2-C\,y\bar{\psi}\phi\psi-\frac{D}{4!}\lambda\phi^4\\ &\qquad-\frac{1}{2}(E\,m^2+\delta m^2)\phi^2-(F\,m^4+G\,m^2+H)\end{aligned} \tag{10}$$

$$\psi_0 = Z_\psi \psi,\ \phi_0 = Z_\phi \phi,\ y_0 = Z_y y,\ \lambda_0 = Z_\lambda \lambda,\ m_0^2 = Z_m m^2,\ \delta m_0^2 = Z_\phi^{-1} \delta m^2,\ h_0 = Z_h h,$$
$$1 + A = Z_\psi^{1/2} \psi,\ 1 + B = Z_\phi,\ 1 + C = Z_y Z_\psi Z_\phi^{1/2},\ 1 + D = Z_\lambda Z_\phi^2,$$
$$1 + E = Z_m Z_\phi,\ 1 + F = Z_h Z_m^2,\ G = h_2 Z_m,\ H = h_4 \tag{11}$$

In MI renormalization, renormalization conditions, e.g,

$$\Gamma_{\phi\phi}^{(2,0)}(k^2, m^2)\Big|_{k^2=0, m^2=0} = 0, \quad \leftarrow \text{automatic in DR} \tag{12}$$
$$\frac{\partial}{\partial k^2}\Gamma_{\phi\phi}^{(2,0)}(k^2, m^2)\Big|_{k^2=-\mu^2,\, m^2=0} = 1, \quad \leftarrow \text{drop } 1/\bar{\varepsilon} \text{ in } \overline{MS} \tag{13}$$
$$\frac{\partial}{\partial m^2}\Gamma_{\phi\phi}^{(2,0)}(k^2, m^2)\Big|_{k^2=0,\, m^2=\mu^2} = -1, \quad \leftarrow \text{drop } 1/\bar{\varepsilon} \text{ in } \overline{MS} \tag{14}$$

$(\frac{1}{\bar{\varepsilon}} = \frac{1}{\varepsilon} - \gamma + \ln 4\pi,\ \varepsilon = 2 - \frac{n}{2})$

$$Z = Z\Big(\lambda, \frac{\Lambda}{\mu}\Big) : \quad \text{(independent of } m\text{)} \tag{15}$$

In $\overline{MS}$

$$Z(\lambda, n) = 1 + \frac{a^{(1)}(\lambda)}{\varepsilon} + \frac{a^{(2)}(\lambda)}{\varepsilon^2} + \cdots \tag{16}$$

The 1-loop effective potential in the Simplified SM:

$$\begin{aligned} V(\phi,m^2) &= \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 + hm^4 + V_{\text{1-loop}} + \delta V^{(1)}_{\text{counterterms}} \\ V_{\text{1-loop}} &= \frac{1}{2}\int \frac{d^4k}{i(2\pi)^4}\ln(-k^2+m^2+\frac{1}{2}\lambda\phi^2) - 2\int \frac{d^4p}{i(2\pi)^4}\ln(-p^2+y^2\phi^2) \\ \delta V^{(1)}_{\text{counterterms}} &= \frac{D^{(1)}}{4!}\lambda\phi^4 + \frac{1}{2}(E^{(1)}m^2 + (\delta m^2)^{(1)})\phi^2 + (F^{(1)}m^4 + G^{(1)}m^2 + H^{(1)}) \end{aligned} \tag{17}$$

Using the dimensional integration formula

$$\mu^{4-n}\int \frac{d^nk}{i(2\pi)^n}\ln(-k^2+M^2) = -\frac{\Gamma(-\eta)}{(4\pi)^\eta}(M^2)^\eta(\mu^2)^{2-\eta} = \frac{M^4}{32\pi^2}\left(-\frac{1}{\bar{\varepsilon}} + \ln\frac{M^2}{\mu^2} - \frac{3}{2}\right).$$

$$\eta \equiv \frac{n}{2}, \quad \varepsilon \equiv 2-\eta, \quad \frac{1}{\bar{\varepsilon}} = \frac{1}{\varepsilon} - \gamma + \ln 4\pi \tag{18}$$

and, dropping the $1/\bar{\varepsilon}$ parts in $\overline{MS}$ renormalization scheme,

$$\Gamma^{(4,0)}_{\phi^4}: \qquad D^{(1)}\lambda = \frac{3}{16\pi^2}\frac{\lambda^2}{2}\frac{1}{\bar{\varepsilon}} - \frac{4!}{16\pi^2}g^4\frac{1}{\bar{\varepsilon}}$$

$$\Gamma^{(2,0)}_{\phi^2}: \qquad E^{(1)}m^2 = \frac{\lambda}{32\pi^2}\frac{1}{\bar{\varepsilon}}m^2, \quad (\delta m^2)^{(1)} = 0 \tag{19}$$

$$\Gamma^{(0,0)}: \qquad F^{(1)}m^4 = \frac{1}{32\pi^2}\frac{1}{\bar{\varepsilon}}m^4, \quad G^{(1)}m^2 = 0, \quad H^{(1)} = 0 \tag{20}$$

[Note that, in dimensional regularization, all the dimensionful counterterms automatically vanish:

$$\delta m^2 = G = H = 0. \qquad] \tag{21}$$

Finally, we get finite renormalized 1-loop effective potential:

$$\begin{aligned} V(\phi, m^2) = & \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!}\phi^4 + h m^4 \\ & + \frac{(m^2 + \frac{1}{2}\lambda\phi^2)^2}{64\pi^2}\left(\ln\frac{m^2 + \frac{1}{2}\lambda\phi^2}{\mu^2} - \frac{3}{2}\right) - 4\frac{(y\phi)^4}{64\pi^2}\left(\ln\frac{y^2\phi^2}{\mu^2} - \frac{3}{2}\right) \end{aligned} \tag{22}$$

Note that the general 1-loop contributions are given by

$$V_{\text{1-loop}}(\phi) = \sum_i \pm n_i\, F_{\ln}(M_i^2(\phi)), \qquad F_{\ln}(M^2) \equiv \frac{1}{2}\int \frac{d^4k}{i(2\pi)^4} \ln(-k^2 + M^2) \tag{23}$$

But, this shows it's nothing but Vacuum Energies: Zero-point osc. for boson and Dirac's sea negative energies.

$$\frac{1}{2}\int \frac{d^4k}{i(2\pi)^4} \ln(-k^2 + M^2) = \frac{M^4}{64\pi^2}\Bigg(-\frac{1}{\bar{\varepsilon}} \underbrace{+\ln\frac{M^2}{\mu^2} - \frac{3}{2}}_{\text{Coleman-Weinberg potential}} \Bigg). \tag{24}$$

To show this, evaluate the LHS as follows:

$$
\begin{aligned}
F_{\rm ln}(M^2) - F_{\rm ln}(0) &= \frac{1}{2}\int_0^{M^2} dm^2 \frac{\partial}{\partial m^2}\int \frac{d^4k}{i(2\pi)^4}\ln(-k^2+m^2-i\varepsilon) \\
&= \frac{1}{2}\int_0^{M^2} dm^2 \int \frac{d^4k}{i(2\pi)^4}\frac{1}{-k^2+m^2-i\varepsilon} \\
&= \frac{1}{2}\int_0^{M^2} dm^2 \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\sqrt{\boldsymbol{k}^2+m^2}} \\
&= \int \frac{d^3k}{(2\pi\hbar)^3}\left(\frac{\hbar}{2}\sqrt{\boldsymbol{k}^2+M^2} - \frac{\hbar}{2}\sqrt{\boldsymbol{k}^2}\right)
\end{aligned}
\tag{25}
$$

Note also that $F_{\rm ln}(0)$ in the massless case vanishes in the dimensional regularization. If you apply the dimensional formula to the last expression, you can also recover the original RHS result.

3 Conclusions from these simple observation

We have shown that the equivalence between the (quantum) vacuum energies and ('classical') Higgs potential energy. From this simple observation, we can draw very interesting and important conclusions:

As far as the matter fields and gauge fields are concerned, whose mass comes solely from the Higgs condensation $\langle\phi\rangle$,

Their vacuum energies are calculable and finite quantities
in terms of the renormalized λ and m^2 parameters!

Note that this is because that their divergences are proportional to ϕ^4 and $m^2\phi^2$. (At 1-loop, only ϕ^4 divergences appear.)

However, the Higgs itself is an exception! The divergences of the Higgs vacuum energy are not only $m^2\phi^2$ and ϕ^4 but also the zero-point function proprtional to m^4. In order to cancel that part, we have to prepare the counterterm:

$$\begin{aligned} h_0 m_0^4 &= Z_h Z_m^2 h m^4 = (1+F) h m^4 \\ F^{(1)} h &= \frac{1}{64\pi^2}\frac{1}{\bar{\varepsilon}}. \end{aligned} \tag{26}$$

And the renormalized CC term hm^4 is a Free Parameter. Then, there is no chance to explain CC.

Thus, in order for the calculability of CC, we need $m^2 = 0$, or

No dimensionful parameters in the theory $\Rightarrow$ (Classical) Scale-Invariance

Part II: Scale Invariance is Sufficient Condition

4 Scale Invariance solves the problem!

Our world is almost scale invariant: that is, the standard model Lagrangian is scale invariant except for the Higgs mass term!

If the Higgs mass term comes from the spontaneous breaking of scale invariance at higher energy scale physics, the total system can be really be scale invariant:

$$\lambda(h^\dagger h - m^2)^2 \quad \to \quad (h^\dagger h - \varepsilon\Phi^2)^2. \tag{27}$$

I introduce you here the scenario following the very good work,

M. Shaposhnikov and D. Zenhausern, Phys. Lett. B **671** (2009) 162

since it is essentially the same as mine, done prior to me and may be simpler than mine, unfortunately. I will also point out the issues to be clarified more.

4.1 Classical Scale Invariance

Suppose that our world has no dimensionful parameters.

Suppose that the effective potential V of the total system looks like

$$\begin{array}{ccccccc} V(\phi) = & V_0(\Phi) & + & V_1(\Phi, h) & + & V_2(\Phi, h, \varphi) \\ & \downarrow & & \downarrow & & \downarrow \\ & M & \gg & \mu & \gg & m \end{array}$$

and it is scale invariant. Then, classically, it satisfies the scale invariance relation :

$$\sum_i \phi^i \frac{\partial}{\partial \phi^i} V(\phi) = 4V(\phi), \qquad (28)$$

so that the vacuum energy vanishes at any stationary point $\langle \phi^i \rangle = \phi_0^i$:

$$V(\phi_0) = 0.$$

Important point is that this holds at every stages of spontaneous symmetry breaking.

In the above potential V, we can retain only $V_0(\Phi)$ when discussing the physics at scale M, since h and φ are expected to get VEVs of order μ or lower. Then the scale invariance guarantees $V_0(\Phi_0) = 0$.

If we discuss the next stage spontaneous breaking at energy scale μ, we should take $V_0(\Phi) + V_1(\Phi, h)$, and can conclude $V_0(\Phi_0) + V_1(\Phi_0, h_0) = 0$.

Similarly, at scale m, we have the potential $V_0(\Phi) + V_1(\Phi, h) + V_2(\Phi, h, \varphi)$, and can conclude $V_0(\Phi_0) + V_1(\Phi_0, h_0) + V_2(\Phi_0, h_0, \varphi_0) = 0$.

This miracle is realized since the scale invariance holds at each energy scale of spontaneous symmetry breaking.

For the help of understanding, we now write a toy model of potentials.

$$V_0(\Phi) = \frac{1}{2}\lambda_0(\Phi_1^2 - \varepsilon_0\Phi_0^2)^2,$$

in terms of two real scalars Φ_0, Φ_1, to realize a VEV

$$\langle\Phi_0\rangle = M \quad \text{and} \quad \langle\Phi_1\rangle = \sqrt{\varepsilon_0}M \equiv M_1. \tag{29}$$

This M is totally spontaneous and we suppose it be Planck mass giving the Newton coupling constant via the scale invariant Einstein-Hilbert term

$$S_{\text{eff}} = \int d^4x\,\sqrt{-g}\Big\{c_1\Phi_0^2\,R + c_2R^2 + c_3R_{\mu\nu}R^{\mu\nu} + \cdots\Big\}$$

If GUT stage exists, ε_0 may be a constant as small as 10^{-4} and then Φ_1 gives the scalar field breaking GUT symmetry; e.g., Φ_1 : **24** causing $SU(5) \to SU(3) \times SU(2) \times U(1)$.

$V_1(\Phi, h)$ part causes the electroweak symmetry breaking:

$$V_1(\Phi, h) = \frac{1}{2}\lambda_1\left(h^\dagger h - \varepsilon_1\Phi_1^2\right)^2,$$

with very small parameter $\varepsilon_1 \simeq 10^{-28}$. This reproduces the Higgs potential when h is the Higgs doublet field and $\varepsilon_1 \Phi_1^2$ term is replaced by the VEV $\varepsilon_1 M_1^2 = \mu^2/\lambda_1$.

$V_2(\Phi, h, \varphi)$ part causes the chiral symmetry breaking, e.g., $\mathrm{SU(2)_L{\times}SU(2)_R} \rightarrow \mathrm{SU(2)_V}$. Using the 2×2 matrix scalar field $\varphi = \sigma + i\boldsymbol{\tau} \cdot \boldsymbol{\pi}$ (chiral sigma-model field), we may similarly write the potential

$$V_2(\Phi, h, \varphi) = \tfrac{1}{4}\lambda_2 \left(\mathrm{tr}(\varphi^\dagger \varphi) - \varepsilon_2 \Phi_1^2\right)^2 + V_{\mathrm{break}}(\Phi, h, \varphi)$$

with another small parameter $\varepsilon_2 \simeq 10^{-34}$. The first term reproduces the linear σ-model potential invariant under the chiral $\mathrm{SU(2)_L{\times}SU(2)_R}$ transformation $\varphi \rightarrow g_{\mathrm{L}} \varphi g_{\mathrm{R}}$ when $\varepsilon_2 \Phi_1^2$ is replaced by the VEV $\varepsilon_2 M_1^2 = m^2/\lambda_2$. The last term V_{break} stands for the chiral symmetry breaking term which is caused by the explicit quark mass terms appearing as the result of tiny Yukawa couplings of u, d quarks, y_u, y_d, to the Higgs doublet h; e.g.,

$$V_{\mathrm{break}}(\Phi, h, \varphi) = \frac{1}{2} \varepsilon_2 \Phi_1^2 \, \mathrm{tr} \left(\varphi^\dagger \left(y_u \epsilon h^* \;\; y_d h\right) + \mathrm{h.c.}\right)$$

4.2 Quantum Mechanically

Is there Anomaly for the Scale Invariance?

Usual answer is YES in quantum field theory. If we take account of the renormalization point μ, so that we have dimension counting identity

$$\left(\mu\frac{\partial}{\partial\mu}+\sum_i\phi_i\frac{\partial}{\partial\phi_i}\right)V(\phi)=4V(\phi).$$

and, also have renormalization group equation (RGE):

$$\left(\mu\frac{\partial}{\partial\mu}+\sum_a\beta_a(\lambda)\frac{\partial}{\partial\lambda_a}+\sum_i\gamma_i(\lambda)\phi_i\frac{\partial}{\partial\phi_i}\right)V(\phi)=0$$

From these we obtain

$$\left(\sum_i(1-\gamma_i(\lambda))\phi_i\frac{\partial}{\partial\phi_i}-\sum_a\beta_a(\lambda)\frac{\partial}{\partial\lambda_a}\right)V(\phi)=4V(\phi)$$

which replaces the above naive one:

$$\sum_i\phi_i\frac{\partial}{\partial\phi_i}V(\phi)=4V(\phi)$$

This shows the anomalous dimension $\gamma_i(\lambda)$ is not the problem, but $\beta_a(\lambda)$ terms may be problematic.

Still, if I assume the existence of Infrared Fixed Points: $\beta_a(\lambda_{\rm IR}) = 0$, then, I can prove that the potential value $V(\phi_0)$ at the stationary point $\phi = \phi_0$ is zero at any μ. *The vanishing property of the stationary potential value* $V(\phi)$ *is not injured by the scale-inv anomaly.*

Probably, however, it will not be sufficient to gurantee the vanishing CC.

Stationary point ϕ_0 may be the trivial point $\phi_0 = 0$.

Non-trivial is the existence of the flat direction even after the quantum corrections are included.

Shaposhnikov-Zenhausern's New Idea is: **SI exists even quantum mechanically**.

Quantum Scale Invariance

- Englert-Truffin-Gastmans, Nuc. Phys. B177(1976)407.
- Shaposhnikov-Zenhausern, *ibid*

Extension to n-dimension keeping S.I. is possible by introducing dilaton field Φ $\rightarrow$ **NO ANOMALY**.

1. **Usual dimensional regularization**

$$\begin{aligned} \lambda\,(h^\dagger(x)h(x))^2 &\to \lambda\,\mu^{4-n}(h^\dagger(x)h(x))^2 \qquad [h]=\frac{n-2}{2} \\ y\,\bar\psi(x)\psi(x)h(x) &\to y\,\mu^{\frac{4-n}{2}}\bar\psi(x)\psi(x)h(x) \qquad [\psi]=\frac{n-1}{2} \end{aligned} \tag{30}$$

2. **SI prescription** Using 'dilaton' field $\Phi(x)$,

$$\begin{aligned} \lambda\,(h^\dagger(x)h(x))^2 &\to \lambda\,[\Phi(x)^2]^{\frac{4-n}{n-2}}\,(h^\dagger(x)h(x))^2 \\ y\,\bar\psi(x)\psi(x)h(x) &\to y\,[\Phi(x)]^{\frac{4-n}{n-2}}\,\bar\psi(x)\psi(x)h(x) \end{aligned} \tag{31}$$

This introduces FAINT but Non-Polynomial interactions $\propto \varepsilon = 2-\frac{n}{2}$

$$\Phi = Me^{\phi/M},\ \langle\Phi\rangle \equiv M \quad \to \quad [\Phi(x)]^{\frac{4-n}{n-2}} = M^{\frac{\varepsilon}{1-\varepsilon}}\left(1+\frac{\varepsilon}{1-\varepsilon}\frac{\phi}{M}++\frac{1}{2}(\frac{\varepsilon}{1-\varepsilon})^2\frac{\phi^2}{M^2}+\cdots\right) \tag{32}$$

This scenario would give quantum scale invariant theory, which may realize the vanishing CC.

However, there still remain many points that should be clarified:

1. **Non-renormalizable effective theory** The non-polynomial interactions seem to require higher and higher new counterterms.

   $\Rightarrow$ Still needs to clarify the structure of the higher-order terms.

   $\Rightarrow$ Unitarity?

2. **Flat direction** survives the quantum corrections?

   $V(\Phi, h) = \Phi^4 f(r)$ with $r \equiv h/\Phi$.

$$\text{Since} \quad \begin{cases} \dfrac{\partial}{\partial\Phi} = \dfrac{\partial}{\partial\Phi} - \dfrac{r}{\Phi}\dfrac{\partial}{\partial r} \\ \dfrac{\partial}{\partial h} = 0 + \dfrac{1}{\Phi}\dfrac{\partial}{\partial r} \end{cases} \quad \text{for} \quad (\Phi, h) \rightarrow (\Phi, r = h/\Phi), \tag{33}$$

   the initial flat direction $r_0 \equiv h_0/\Phi_0 \neq 0,\ \Phi_0 \neq 0$ remains iff

$$f(r_0) = f'(r_0) = 0 \quad \text{are satisfied.} \tag{34}$$

   This can be satisfied by two coupling constant freedom. Note that they can no longer be satisfied if $f(r)$ additionally depends on μ.

   But C. Tamarit, JHEP12(2013)098 worries about fine tuning, which should be cleared.

3. **Running coupling constants?** after VEV $\langle\Phi\rangle \neq 0$ appears.

e.g., Chiral symmetry breaking scale in QCD:

Usually the coupling $\alpha_3 \equiv g_3^2/4\pi$ runs according to

$$\mu\frac{d}{d\mu}\alpha_3(\mu) = 2b_3\,\alpha_3^2(\mu) \quad \rightarrow \quad \frac{1}{\alpha_3(\mu)} = \frac{1}{\alpha_3(M)} - b_3 \ln\frac{\mu^2}{M^2}$$

$$\rightarrow \quad \varepsilon_2 = \frac{\Lambda_{\rm QCD}^2}{M^2} = \exp\frac{1}{b_3}\Big(\frac{1}{\alpha_3(M)} - \frac{1}{\alpha_3^{\rm cr}}\Big). \tag{35}$$

where $\alpha_3^{\rm cr} = O(1)$ quantity like $\pi/3$.

Does this running $\alpha_3(\mu)$ really exist in quantum SI theory? $\rightarrow$ probably exists. This should be proven more soundly. cf. C.Tamarit, *ibid*

$\alpha_3(M)$ here is the initial gauge coupling, while M is the VEV $M = \langle\Phi\rangle$.

Then, probably, $\alpha_3(M)$ should be replaced by M-independent initial gauge coupling $\alpha_3^{\rm init}$, while the other M^2 should be replaced by the field Φ^2. Then

$$\frac{\Lambda_{\rm QCD}^2}{\Phi^2} = \exp\frac{1}{b_3}\Big(\frac{1}{\alpha_3^{\rm init}} - \frac{1}{\alpha_3^{\rm cr}}\Big). \tag{36}$$

and the QCD scale $\Lambda_{\rm QCD}$ is always scaled with the dilaton VEV $\langle\Phi\rangle$. Anybody has ever shown this? $\rightarrow$ explains Hierarchy.

4. **Hierarchy and Effective Potential**

This hierarchy should show up in the effective potential. And the effective potential should be calculable prior to the spontaneous breaking.

So we suspect that we should be able to derive the effective potential of the Coleman-Weinberg type like

$$\frac{(\varphi^\dagger\varphi)^2}{64\pi^2}\left(-b_3 \ln\frac{\varphi^\dagger\varphi}{\Phi^2}+\frac{1}{\alpha_3^{\rm init}}-\frac{1}{\alpha_3^{\rm cr}}\right) \tag{37}$$

for the chiral sigma model scalar field φ whose VEV giving the QCD scale.

But no one has ever derived such an effective action.

5 Other Problems

1. More sound proof, for the claim that
   Quantum scale invariance persists by the SI prescription.

2. Gauge hierarchies; how do those potentials appear possessing tiny ε_i's?

3. Global or Local scale invariance?

4. If global, What is ∃Dilaton? $\rightarrow$ Higgs ?

5. The fate of dilaton? $\rightarrow$ does it remain massless?

6. How is the present CC value Λ_0 explained?

7. How does the inflation occur in this scale invariant scenario?

8. Thermal effects.

9. Construct scale invariant Beyond Standard Model.

10. (Super)Gravity theory with (local or global) scale invariance.