QCD and conformal symmetry

V. M. Braun

University of Regensburg

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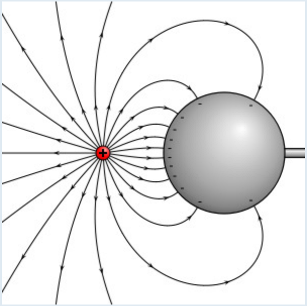
Prologue

L.I. Magnus (1831):
Inversion transformation

$$x^\mu \to \frac{x^\mu}{x^2}$$



Conformal field theories

$$\langle O_1(x_1)\, O_2(x_2)\rangle = \frac{\text{const}}{|x_1 - x_2|^{2\Delta_1}}\,\delta_{\Delta_1\Delta_2}$$

$$\langle O_1(x_1)\, O_2(x_2)\, O_3(x_3)\rangle = \frac{\text{const}}{|x_1 - x_2|^{\Delta_1+\Delta_2-\Delta_3}\,|x_1 - x_3|^{\Delta_1+\Delta_3-\Delta_2}\,|x_2 - x_3|^{\Delta_3+\Delta_2-\Delta_1}}$$

- $\leftarrow$ Δ_k is a scaling dimension (canonical + anomalous)
- $\leftarrow$ Can be taken as a definition

based on a review

- V.M. Braun, G.P. Korchemsky, D. Müller, *Prog. Part. Nucl. Phys.* **51** *(2003) 311*

and new development:

- V. M. Braun, A. N. Manashov and J. Rohrwild, *Nucl. Phys. B* **826** *(2010) 235*
- V. M. Braun and A. N. Manashov, *JHEP* **1201** *(2012) 085*
- ...
- V. M. Braun, A. N. Manashov, S. Moch and M. Strohmaier *JHEP* **1706** *(2017) 037*
- V. M. Braun, Yao Ji and A. N. Manashov, *JHEP* **1806** *(2018) 017*

Outline

Conformal Transformations

1. conserve the interval $ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$
2. only change the scale of the metric $g'_{\mu\nu}(x') = \omega(x)g_{\mu\nu}(x)$

$\Rightarrow$ preserve the angles and leave the light-cone invariant

- Translations
- Rotations and Lorentz boosts
- **Dilatation** (global scale transformation) $x^\mu \to x'^\mu = \lambda x^\mu$
- **Inversion** $x^\mu \to x'^\mu = x^\mu / x^2$

Special conformal transformation

$$x^\mu \to x'^\mu = \frac{x^\mu + a^\mu x^2}{1 + 2a\cdot x + a^2 x^2}$$

$=$ inversion, translation $x^\mu \to x^\mu + a^\mu$, inversion

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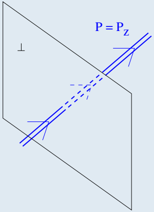
Conformal Algebra

The full conformal algebra includes 15 generators SO(4,2)

- $\mathbf{P}_\mu$ (4 translations)
- $\mathbf{M}_{\mu\nu}$ (6 Lorentz rotations)
- $\mathbf{D}$ (dilatation)
- $\mathbf{K}_\mu$ (4 special conformal transformations)

Collinear Subgroup



- Special conformal transformations
  $$z \to z' = \frac{z}{1+2az}$$
- translations $z \to z' = z + c$
- dilatations $z \to z' = \lambda z$

$$p_+ = \frac{1}{\sqrt{2}}(p_0 + p_z) \to \infty$$
$$p_- = \frac{1}{\sqrt{2}}(p_0 - p_z) \to 0$$
$$px \to p_+ x_-$$

write

$$x_- = zn\,, \qquad n^2 = 0\,, \qquad z \in \mathbb{R}$$

form the so-called collinear subgroup $SL(2,R)$

$$z \to z' = \frac{a\,z+b}{c\,z+d}\,, \quad ad - bc = 1$$

SL(2) algebra

$SL(2,R)$ algebra

$$S_- = -\frac{d}{dz}$$
$$S_+ = z^2\frac{d}{dz} + 2j\,z$$
$$S_0 = z\frac{d}{dz} + j$$

$SL(2)$ commutation relations

$$[S_+,S_-] = 2S_0 \qquad [S_0,S_\pm] = \pm S_+$$

Conformal spin

$$j = \frac{1}{2}(l+s)$$

A (quantum) field $\Phi(z)$ with scaling dimension ℓ and spin projection s on "plus" direction transforms as

$$\Phi(z) \to \Phi'(z) = (c\,z+d)^{-2j}\,\Phi\left(\frac{az+b}{cz+d}\right)$$

is an eigenstate of the quadratic Casimir operator

$$S^2 = S_0^2 + S_1^2 + S_2^2 = S_+S_- + S_0(S_0-1)$$

$$S^2\Phi(z) = j(j-1)\Phi(z)$$

$\Phi(z=0)$ is an eigenstate of S_0 and annihilated by S_+

$$S_+\Phi(0) = 0\,, \qquad S_0,\Phi(0) = j\,\Phi(0)$$

A complete operator basis can be obtained from $\Phi(0)$ by applying the "raising" operator S_-

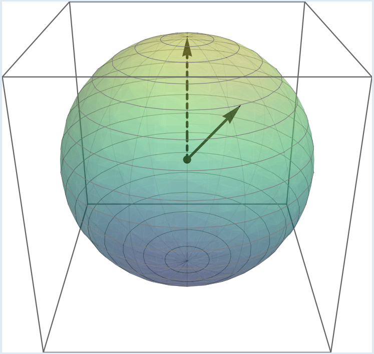
$$\mathcal{O}_k = S_-^k\,\Phi(0) \equiv (-\partial_z)^k\,\Phi(z)\Big|_{z=0}\,, \qquad \mathcal{O}_0 \equiv \Phi(0)$$

From the commutation relations it follows that

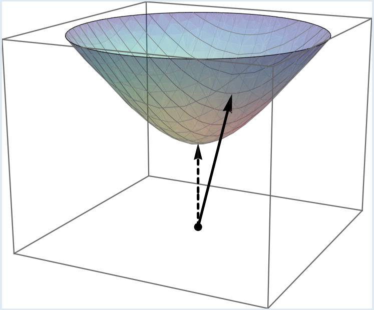
$$S_0\mathcal{O}_k = (k+j)\mathcal{O}_k$$

— conformal tower

Non-compact spin



$SO(3)$



$SL(2,\mathbb{R}) \sim O(2,1)$

Adjoint Representation

- Two different complete sets of states:
  - $\Phi(z)$ for arbitrary z
  - $\mathcal{O}_k$ for arbitrary k

Relation = Taylor series:

$$\Phi(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!} \mathcal{O}_k \,,$$

- In general: Two equivalent descriptions
  - light-ray operators (nonlocal)
  - local operators with derivatives (Wilson)

Characteristic polynomial

Any local composite operator can be specified by a polynomial that details the structure and the number of derivatives acting on the fields. For example

$$\mathcal{O}_k = [\mathcal{P}_k(\partial_z)\Phi(z)]\Big|_{z=0}$$

$$\mathcal{P}_k(u) = (-u)^k$$

where $\partial_z \equiv d/dz$

The algebra of generators acting on light-ray operators can equivalently be rewritten as algebra of differential operators acting on the space of characteristic polynomials. One finds the following 'adjoint' representation of the generators acting on this space:

$$\begin{aligned}
\widetilde{S}_0 \mathcal{P}(u) &= (u\partial_u + j)\,\mathcal{P}(u)\,,\\
\widetilde{S}_- \mathcal{P}(u) &= -u\,\mathcal{P}(u)\,,\\
\widetilde{S}_+ \mathcal{P}(u) &= \left(u\partial_u^2 + 2j\partial_u\right)\mathcal{P}(u)\,,
\end{aligned}$$

Light-ray quark-antiquark operators

$$\begin{aligned}
\mathcal{O}(0,z) &= \bar{q}(0)[0,zn]\not{n}\,q(zn)\,, \qquad n^2 = 0 \\
&= \sum_{N=0}^{\infty} \frac{z^N}{N!} n_{\mu_1}\ldots n_{\mu_N}\left[\bar{q}\gamma_{\mu_1}D_{\mu_2}\ldots D_{\mu_N}q\right] \\
&= \sum_{N=0}^{\infty} \frac{z^N}{N!} n_{\mu_1}\ldots n_{\mu_N}\left[\bar{q}\gamma_{\mu_1}D_{\mu_2}\ldots D_{\mu_N}q - \text{Traces}\right] \\
&= \sum_{N=0}^{\infty} \frac{z^N}{N!} n_{\mu_1}\ldots n_{\mu_N}\,\mathcal{O}^{t=2}_{\mu_1\ldots\mu_N}
\end{aligned}$$

⇐ **Light–ray operator**

$$[0,zn] = \text{Pexp}\left\{-ig_s z\int_0^1 du\, n_\mu A^\mu(uzn)\right\}$$

Two-point correlation function

$$\Phi(z) \to \Phi'(z) = (c\,z + d)^{-2j}\,\Phi\left(\frac{az+b}{cz+d}\right)$$

$$\langle \Phi_1(z_1)\,\Phi_2(z_2)\rangle = \int \mathcal{D}\Phi_k\,\Phi_1(z_1)\,\Phi_2(z_2)\,e^{iS(\Phi)} = f(z_1 - z_2) =?$$

- Dilatation $\quad b = c = 0, a/d = \lambda, ad - bc = 1 \quad \Phi_k(z) \mapsto \lambda^{j_k}\,\Phi_k(\lambda z)$

$$f(z_1 - z_2) := \lambda^{j_1}\lambda^{j_2} f(\lambda(z_1 - z_2)) \qquad \Rightarrow \qquad f(z_1 - z_2) = \frac{\text{const}}{(z_1 - z_2)^{j_1+j_2}}$$

- Inversion $\quad a = d = 0, c = 1, b = -1 \quad \Phi_k(z) \mapsto z^{-2j_k}\,\Phi_k(-1/z)$

$$\frac{\text{const}}{(z_1 - z_2)^{j_1+j_2}} := z_1^{-2j_1} z_2^{-2j_2} \frac{\text{const}}{(1/z_2 - 1/z_1)^{j_1+j_2}} = z_1^{j_2-j_1} z_2^{j_1-j_2} \frac{\text{const}}{(z_1 - z_2)^{j_1+j_2}}$$

$$\Rightarrow \quad j_1 = j_2$$

It follows

$$\langle \Phi_1(z_1)\,\Phi_2(z_2)\rangle = \frac{\text{const}}{(z_1 - z_2)^{j_1+j_2}}\,\delta_{j_1-j_2}$$

[subtlety: $(z_1 - z_2)n \to (z_1 - z_2)n + \epsilon\bar{n}, \quad [(z_1 - z_2)n + \epsilon\bar{n}]^2 = \epsilon(z_1 - z_2)n\bar{n}$]

to summarize:

- A one-particle local operator (quark or gluon) can be characterized by twist t , conformal spin j, and conformal spin projection $j_0 = j + k$
- Conformal spin $j = 1/2(\ell + s)$ involves the spin projection on "plus" direction. This is not helicity! Example:

$$\psi_+ : \qquad j = \frac{1}{2}(3/2 + 1/2) = 1$$

$$\psi_- : \qquad j = \frac{1}{2}(3/2 - 1/2) = 1/2$$

- The difference $k = j_0 - j \geq 0$ is equal to the number of (total) derivatives
- Exact analogy to the usual spin, but conformal spin is noncompact, corresponds to hyperbolic rotations $SL(2) \sim SO(2,1)$
- **We are dealing with the infinite-dimensional representation of the collinear conformal group**

Separation of variables

In Quantum Mechanics: O(3) rotational symmetry $\Rightarrow$ Angular vs. radial dependence

$$\left[-\frac{\hbar^2}{2m}\Delta + V(|r|)\right]\Psi = E\Psi \quad \Rightarrow \quad \Psi(\vec{r}) = R(r)Y_{lm}(\theta,\phi)$$

♡

$Y_{lm}(\theta,\phi)$ are eigenfunctions of $L^2 Y_{lm} = l(l+1)Y_{lm}$, $[\mathcal{H}, L^2] = 0$.

In Quantum Chromodynamics: SL(2,R) conformal symmetry $\Rightarrow$ Longitudinal vs. transverse dependence

Migdal ‘77
Brodsky,Frishman,Lepage,Sachrajda ‘80
Makeenko ‘81
Ohrndorf ‘82

Separation of variables — *continued*

Momentum fraction distribution of quarks in a pion at small transverse separations:

- **Dependence on transverse coordinates is traded for the scale dependence:**

$$\mu^2\frac{d}{d\mu^2}\,\phi_\pi(u,\mu) = \int_0^1 dw\,\mathcal{H}(u,w)\,\phi_\pi(u,\mu) \;\Rightarrow\; \begin{array}{l}\phi_\pi(u,\mu) = 6u(1-u)\left[1+\phi_\pi^2(\mu)C_2^{3/2}(2u-1)+\ldots\right]\\ \phi_\pi^n(\mu) = \phi_\pi^n(\mu_0)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\gamma_n/b_0}\end{array}$$

- **Summation goes over total conformal spin of the $\bar{q}q$ pair:** $j = j_q + j_{\bar{q}} + n = n+2$
- **Gegenbauer Polynomials $C_n^{3/2}$ are 'spherical harmonics' of the $SL(2,R)$ group**
- **Exact analogy: Partial wave expansion in Quantum Mechanics**

For Baryons: Summation of three conformal spins

$$\begin{aligned}\phi_N(u_i) &= 120u_1u_2u_3\sum_{J=3}^{\infty}\sum_{j=2}^{J-1}\phi_N^{J,j}(\mu)\;Y_{J,j}^{(12)3}(u)\\ J &= 3+N, N=0,1,\ldots \quad \text{total spin of three quarks}\\ j &= 2+n, n=0,1,\ldots \quad \text{spin of the (12)-quark pair}\end{aligned}$$

Spin Summation

Two-particle:

$$[j_1] \otimes [j_2] = \bigoplus_{n\geq 0} [j_1 + j_2 + n]$$

For two fields "living" on the light-cone

$$O(z_1, z_2) = \Phi_{j_1}(z_1)\, \Phi_{j_2}(z_2)$$

or, equivalently, local composite operators

$$\mathcal{O}_j(0) = \mathcal{P}_j(\partial_1, \partial_2)\Phi_{j_1}(z_1)\, \Phi_{j_2}(z_2)\Big|_{z_1=z_2=0}$$

the two-particle SL(2) generators are

$$\begin{aligned} \mathbf{S}_a &= \mathbf{S}_{1,a} + \mathbf{S}_{2,a}\,, \qquad a = 0,1,2 \\ \mathbf{S}^2 &= \sum_{a=0,1,2} (\mathbf{S}_{1,a} + \mathbf{S}_{2,a})^2 \end{aligned}$$

We define a conformal operator

$$\begin{aligned} \widetilde{S}^2\, \mathcal{P}_j &= j(j-1)\mathcal{P}_j \\ \widetilde{S}_0\, \mathcal{P}_j &= j_0\, \mathcal{P}_j \\ \widetilde{S}_+ \mathcal{P}_j &= 0 \end{aligned}$$

General solution is $j_0 = j = j_1 + j_2 + n$ and

$$\mathbb{P}_j^{j_1,j_2}(u_1, u_2) = \\ = (u_1+u_2)^n P_n^{(2j_1-1,2j_2-1)}\left(\frac{u_2-u_1}{u_1+u_2}\right)$$

where $P_n^{(a,b)}(x)$ are Jacobi polynomials

Call the corresponding operator $\mathbb{O}_j^{j_1,j_2}$

Spin Summation *-continued*

Three-particle:

$$[j_1]\otimes[j_2]\otimes[j_2] = \bigoplus_{n\geq 0}[j_1+j_2+j_3+N]$$

local composite operators

$$\mathcal{O}(0) = \mathcal{P}(\partial_1,\partial_2,\partial_2)\Phi_{j_1}(z_1)\,\Phi_{j_2}(z_2)\,\Phi_{j_3}(z_3)\Big|_{[z_i]=0}$$

- Impose following conditions:
  - fixed total spin $J = j_1+j_2+j_3+N$
  - annihilated by $\widetilde{S}_+$
  - fixed spin $j = j_1+j_2+n$ in the (12)-channel

$\Rightarrow$

$$\mathbb{P}^{(12)3}_{Jj}(u_i) \quad = \quad (1-u_3)^{j-j_1-j_2}\,P^{(2j_3-1,2j-1)}_{J-j-j_3}(1-2u_3)\,P^{(2j_1-1,2j_2-1)}_{j-j_1-j_2}\left(\frac{u_2-u_1}{1-u_3}\right)$$

if a different order of spin summation is chosen

$$\mathbb{P}^{(31)2}_{Jj}(u_i) = \sum_{j_1+j_2\leq j'\leq J-j_3} \Omega_{jj'}(J)\,\mathbb{P}^{(12)3}_{Jj'}(u_i)$$

the matrix Ω defines Racah $6j$-symbols of the SL$(2,\mathbb{R})$ group

Braun, Derkachov, Korchemsky, Manashov '99

Conformal Scalar Product

Correlation function of two conformal operators with different spin must vanish:

$$\langle 0|\mathrm{T}\,\{\mathbb{O}^{j_1,j_2}_{j}(x)\mathbb{O}^{j_1,j_2}_{j'}(0)\}|0\rangle \sim \delta_{jj'}$$

Evaluating this in a free theory:

$$\langle \mathbb{P}_n|\mathbb{P}_m\rangle \equiv \int_0^1 [du]\, u_1^{2j_1-1}u_2^{2j_2-1}\mathbb{P}_n(u_1,u_2)\mathbb{P}_m(u_1,u_2) \sim \delta_{mn}\,,$$

where $[du]=du_1du_2\,\delta(u_1+u_2-1)$

similar for three particles

$$\langle \mathbb{P}_{J,j}|\mathbb{P}_{J',j'}\rangle \equiv \int_0^1 [du]\, u_1^{2j_1-1}u_2^{2j_2-1}u_3^{2j_3-1}\,\mathbb{P}_{N'q'}(u_1,u_2,u_3)\mathbb{P}_{N,q}(u_1,u_2,u_3) \sim \delta_{JJ'}\delta_{jj'}$$

Complete set of orthogonal polynomials w.r.t. the conformal scalar product

Pion Distribution Amplitude

Definition:

$$\langle 0|\bar{d}(0)\gamma_+\gamma_5 u(z)|\pi^+(p)\rangle = if_\pi p_+ \int_0^1 du\, \mathrm{e}^{-iuzp_+}\phi_\pi(u,\mu)$$

$\Rightarrow$

$$\langle 0|\bar{d}(0)\gamma_+\gamma_5 \left(i\overleftrightarrow{D}_+\right)^n u(0)|\pi^+(p)\rangle = if_\pi (p_+)^{n+1}\int_0^1 du\,(2u-1)^n\phi_\pi(u,\mu)$$

It follows

$$\int_0^1 du\, P_n^{1,1}(2u-1)\phi_\pi(u,\mu) = \langle\!\langle \mathbb{Q}_n^{1,1}\rangle\!\rangle, \qquad \langle 0|\mathbb{Q}_n^{1,1}(0)|\pi^+(p)\rangle = if_\pi p_+^{n+1}\langle\!\langle \mathbb{Q}_n^{1,1}\rangle\!\rangle$$

Since $P_n^{1,1}(x) \sim C_n^{3/2}(x)$ form a complete set on the interval $-1 < x < 1$

$$\phi_\pi(u,\mu) = 6u(1-u)\sum_{n=0}^{\infty}\phi_n(\mu)\, C_n^{3/2}(2u-1)\,,$$

$$\phi_n(\mu) = \frac{2(2n+3)}{3(n+1)(n+2)}\langle\!\langle \mathbb{Q}_n^{1,1}(\mu)\rangle\!\rangle, \qquad \phi_n(\mu) = \phi_n(\mu_0)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\gamma_n^{(0)}/\beta_0}$$

ERBL '79 – '80

to summarize:

- **Conformal symmetry offers a classification of different components in hadron wave functions**
- The contributions with different conformal spin cannot mix under renormalization to one-loop accuracy
  Whether this is enough to achieve multiplicative renormalization depends on the problem
- QCD equations of motion are satisfied
- **Allows for model building: LO, NLO in conformal spin**
- The validity of the conformal expansion is not spolied by quark masses
- Convergence of the conformal expansion is not addressed

Close analogy: Partial wave expansion in nonrelativistic quantum mechanics

Leading-order QCD Evolution Equations

ERBL evolution equation

$$\mu^2 \frac{d}{d\mu^2}\phi_\pi(u,\mu) = \int_0^1 dv\, V(u,v;\alpha_s(\mu))\,\phi_\pi(v,\mu)$$

$$V_0(u,v) = C_F \frac{\alpha_s}{2\pi}\left[\frac{1-u}{1-v}\left(1+\frac{1}{u-v}\right)\theta(u-v) + \frac{u}{v}\left(1+\frac{1}{v-u}\right)\theta(v-u)\right]_+$$

How to make maximum use of conformal symmetry?

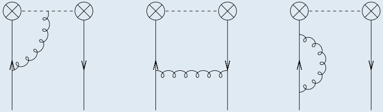
Bukhvostov, Frolov, Kuraev, Lipatov '85

Where is the symmetry?

RG equation for the nonlocal operator $\mathcal{Q}(\alpha_1,\alpha_2) = \bar\psi(\alpha_1)\gamma_+\,\psi(\alpha_2)$

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g}\right)\mathcal{Q}(\alpha_1,\alpha_2) = -\frac{\alpha_s C_F}{4\pi}\left[\mathbb{H}\cdot\mathcal{Q}\right](\alpha_1,\alpha_2)$$

$$\mathbb{H} = 2\mathcal{H}_{\rm v}^{(12)} - 2\mathcal{H}_{\rm e}^{(12)} + 1$$



Explicit calculation

$$[\mathcal{H}_{\rm v}^{(12)}\cdot\mathcal{Q}](\alpha_1,\alpha_2) = -\int_0^1\frac{du}{u}(1-u)\left\{\mathcal{Q}(\alpha_{12}^u,\alpha_2)+\mathcal{Q}(\alpha_1,\alpha_{21}^u)-2\mathcal{Q}(\alpha_1,\alpha_2)\right\},$$

$$[\mathcal{H}_{\rm e}^{(12)}\cdot\mathcal{Q}](\alpha_1,\alpha_2) = \int_0^1[du]\,\mathcal{Q}(\alpha_{12}^{u_1},\alpha_{21}^{u_2})\qquad \alpha_{12}^u\equiv\alpha_1(1-u)+\alpha_2 u$$

Balitsky, Braun '88

Now verify

$$[\mathbb{H}\cdot S_a\,\mathcal{Q}](\alpha_1,\alpha_2) = S_a\,[\mathbb{H}\cdot\mathcal{Q}](\alpha_1,\alpha_2)\,,\qquad S_a\,\mathcal{Q}(\alpha_1,\alpha_2) = (S_{1,a}+S_{2,a})\mathcal{Q}(\alpha_1,\alpha_2)$$

$$\boxed{[\mathbb{H},S_+] = [\mathbb{H},S_-] = [\mathbb{H},S_0] = 0}$$

Balitsky, Braun, 1989

General expression

$$\mathbb{H}[\mathcal{O}](z_1,z_2) = \int_0^1 d\alpha \int_0^1 d\beta\, h(\alpha,\beta)\, [\mathcal{O}](z_{12}^\alpha, z_{21}^\beta)$$

$$z_{12}^\alpha \equiv z_1\bar\alpha + z_2\alpha \qquad \bar\alpha = 1-\alpha$$

$$[S_+,\mathbb{H}] = 0 \qquad \Longrightarrow \qquad h(\alpha,\beta) = h\left(\frac{\alpha\beta}{\bar\alpha\bar\beta}\right) = h^{(1)}(\tau)$$

i.e. function of two variables reduces to a function of one variable
$\longrightarrow$ can be restored from anomalous dimensions

$$\gamma_N = \int_0^1 d\alpha \int_0^1 d\beta\, (1-\alpha-\beta)^{N-1} h(\alpha,\beta)$$

Braun, Korchemsky, Manashov, 1999

$$h(\alpha,\beta) = -4C_F\left[\delta_+(\tau) + \theta(1-\tau) - \frac{1}{2}\delta(\alpha)\delta(\beta)\right]$$

- **Combined LO DGLAP, ERBL and GPD evolution equations in the most compact form**

Covariant Representation

if $[\mathbb{H}, S_k] = 0$, $\mathbb{H}$ must be a function of the two-particle Casimir operator

$$S_{12}^2 = -\partial_{\alpha_1}\partial_{\alpha_2}(\alpha_1 - \alpha_2)^2 \,,$$

To find this function, compare action of $\mathbb{H}$ and S_{12}^2 on the set of functions $(\alpha_1 - \alpha_2)^n$

$$\mathcal{H}_{\rm v}^{(12)} = 2\,[\psi(J_{12}) - \psi(2)]\,, \qquad S_{12}^2 = J_{12}(J_{12}-1)$$
$$\mathcal{H}_{\rm e}^{(12)} = 1/[J_{12}(J_{12}-1)] = 1/S_{12}^2$$

where $\psi(x)$ is the Euler's digamma function

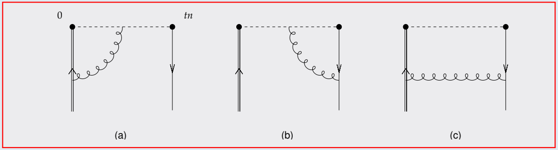
$$\mathbb{H}_{\rm ERBL} = 4\,[\psi(J_{12}) - \psi(2)] - 2/[J_{12}(J_{12}-1)] + 1$$

For local operators take S_{12}^2 in the adjoint representation $\widetilde{S}_{12}^2$

Bukhvostov, Frolov, Kuraev, Lipatov '85

Application: heavy-light mesons

Very heavy quarks fly along their classical trajectory; in static limit propagator → Wilson line



ERBL ⇒ Lange-Neubert (LN) evolution equation '03

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \frac{\alpha_s C_F}{\pi} H_{LN}\right)\Phi_+(\omega,\mu) = 0\,,$$

$$[H_{LN}f](\omega) \quad = \quad -\int_0^\infty d\omega' \left[\frac{\omega}{\omega'}\frac{\theta(\omega'-\omega)}{\omega'-\omega} + \frac{\theta(\omega-\omega')}{\omega-\omega'}\right]_+ + \left[\ln\frac{\mu}{\omega} - \frac{5}{4}\right] f(\omega)$$

- Explicit solution *G. Bell et. al., '13*

$$\Phi_+(\omega,\mu) \quad = \quad \int_0^\infty ds\,\sqrt{\omega s}\,J_1(2\sqrt{\omega s})\,\eta_+(s,\mu)$$

Recall $(j=1)$ *V.Braun, Yao Ji, A. Manashov JHEP* **1806**, *017 (2018)*

$$S_- = -\partial_z\,, \qquad S_+ = z^2\partial_z + 2\,z\,, \qquad S_0 = z\partial_z + 1$$

For heavy quarks $z \sim 1/m \to 0$, suggests to rescale

$$S_-^{(h)} \to \lambda S_-^{(h)}\,, \qquad S_+^{(h)} \to \lambda^{-1} S_+^{(h)}\,, \qquad \lambda \sim m \to \infty$$

The two-particle generators for a system consisting of one heavy quark (h) and one light quark (q)

$$S_+^{(qh)} \equiv S_+^{(q)} + S_+^{(h)} \mapsto S_+^{(q)} + \lambda^{-1} S_+^{(h)} = S_+^{(q)} + \mathcal{O}(\lambda^{-1})\,,$$
$$S_-^{(qh)} \equiv S_-^{(q)} + S_-^{(h)} \mapsto S_-^{(q)} + \lambda S_-^{(h)} = \lambda S_-^{(h)} + \mathcal{O}(1)\,,$$
$$S_0^{(qh)} \equiv S_0^{(q)} + S_0^{(h)} = \mathcal{O}(1)\,.$$

The two-particle quadratic Casimir operator

$$S_{qh}^2 = S_+^{(qh)} S_-^{(qh)} + S_0^{(qh)}(S_0^{(qh)} - 1) \mapsto \lambda S_+^{(q)} S_-^{(h)} + \mathcal{O}(1)$$

Hence $J_{(qh)} \sim \sqrt{\lambda}$ and the ERBL kernel simplifies to

$$\mathcal{H}_{qh} = 2\Big[\psi(J_{qh}) - \psi(1)\Big] \mapsto \ln\Big(\lambda S_+^{(q)} S_-^{(h)}\Big) + \mathcal{O}(\lambda^{-1})$$

Since the “heavy” and “light” generators act on different spaces

$$\mathcal{H}_{qh} \mapsto \ln\Big(i\mu S_+^{(q)}\Big) + \ln\Big(-i\mu^{-1}\lambda S_-^{(h)}\Big)$$

where μ is an arbitrary parameter with dimension of mass. In HQET only light degrees of freedom remain

$$\boxed{\mathcal{H}_{qh}^{\rm HQET} = \ln\Big(i\mu S_+^{(q)}\Big) + \text{ const}}$$

in $\overline{\text{MS}}$ const $= \gamma_E - 5/4$

V.Braun, A. Manashov Phys. Lett. B **731** *(2014) 316*

Solution

- $\mathcal{H}_{LN}$ and S^+ share the same eigenfunctions:

$$iS^+Q_s = sQ_s$$

in position space

$$S_+ = z^2\partial_z + 2z \;\Rightarrow\; Q_s(z) = -\frac{1}{z^2}\exp\left\{\frac{is}{z}\right\}$$

in momentum space

$$\mathcal{S}_+ = i\Big[\omega\partial_\omega^2 + 2\partial_\omega\Big] \;\Rightarrow\; Q_s(\omega) = \frac{1}{\sqrt{\omega s}}J_1(2\sqrt{\omega s})$$

eigenvalues $\equiv$ anomalous dimensions

$$\mathcal{H}_{LN}Q_s(z) = \Big[\ln(i\mu\, S^+) - \psi(1) - \frac{5}{4}\Big]Q_s(z) = \Big[\ln(\mu s) - \psi(1) - \frac{5}{4}\Big]Q_s(z)$$

to summarize:

- **Leading-order QCD evolution equations are $SL(2)$ invariant**

- Evolution kernels in position space are functions of conformal ratio(s)

- Knowledge of anomalous dimensions is sufficient to restore the whole mixing matrix

- Covariant representation in terms of quadratic Casimir operators
  — Convenient to search for more symmetries (integrability)

Beyond leading order?

Consider YM theory with a hard cutoff M, integrate out the fields with frequencies above μ

$$S_{\rm eff} = -\frac{1}{4}\int d^4x \left(\frac{1}{(g^{(0)})^2} - \frac{\beta_0}{16\pi^2}\ln M^2/\mu^2\right)\left[G^a_{\mu\nu}G^{a\mu\nu}\right]_{\rm slow}(x) + \ldots$$

Under the scale transformation $x^\mu \to \lambda x^\mu$, $A_\mu(x) \to \lambda A_\mu(\lambda x)$, $\psi(x) \to \lambda^{3/2}\psi(\lambda x)$ and $\mu \to \mu/\lambda$ with the fixed cutoff

$$\delta S = -\frac{1}{32\pi^2}\beta_0 \ln\lambda \int d^4x\, G^a_{\mu\nu}G^{a\mu\nu}(x) + \ldots$$

Thus dilatation symmetry (scale invariance) is broken. Inversion as well.

Hope is to find a representation, for a generic quantity $\mathcal{Q}$

$$\mathcal{Q} = \mathcal{Q}^{\rm conf} + \frac{\beta(g)}{g}\Delta\mathcal{Q}\,, \quad \text{where} \quad \Delta\mathcal{Q} = \text{power series in } \alpha_s$$

Idea: $\mathcal{Q}^{\rm conf}$ is conformal QCD at the Banks-Zaks critical point in $d\neq 4$ dimensions

V.Braun, A.Manashov, Eur. Phys. J. C **73** *(2013) 2544*

QCD in $d \neq 4$ dimensions

- We consider QCD in the $d = 4 - 2\epsilon$ Euclidean space. The action reads

$$S = \int d^dx \Big\{ \bar{q} \not{D} q + \frac{1}{4} F^a_{\mu\nu} F^{a,\mu\nu} - \bar{c}^a \partial_\mu (D^\mu c)^a + \frac{1}{2\xi} (\partial_\mu A^{a,\mu})^2 \Big\}$$

where $D_\mu = \partial_\mu - igM^\epsilon A^a_\mu T^a$ with T^a in appropriate representation, M is scale parameter

- The renormalized action is obtained by the replacement

$$q \to Z_q q\,, \qquad A \to Z_A A\,, \qquad c \to Z_c c\,, \qquad g \to Z_g g\,, \qquad \xi \to Z_\xi \xi\,,$$

Here $Z_\xi = Z_A^2$ and the renormalization factors are defined using minimal subtraction

$$Z = 1 + \sum_{j=1}^{\infty} \epsilon^{-j} \sum_{k=j}^{\infty} z_{jk} \left(\frac{\alpha_s}{4\pi}\right)^k, \qquad \alpha_s = \frac{g^2}{4\pi}$$

notation: $a = \frac{\alpha_s}{4\pi}$

where z_{jk} are ϵ-independent constants

- Note that we do not send $\epsilon \to 0$ so that renormalized quantities explicitly depend on ϵ

Critical coupling

- Theory has two charges — g and ξ. The corresponding β-functions are

$$\boxed{\beta(a) = 2a\Big(-\epsilon - \gamma_g\Big)}$$

$$\beta_g(g) = M\frac{dg}{dM} = g\Big(-\epsilon - \gamma_g\Big)\,, \qquad \beta_\xi(\xi, g) = M\frac{d\xi}{dM} = -2\xi\gamma_A\,,$$

where

$$\gamma_g = M\partial_M \ln Z_g = \beta_0\left(\frac{\alpha_s}{4\pi}\right) + \beta_1\left(\frac{\alpha_s}{4\pi}\right)^2 + \mathcal{O}(\alpha_s^3)\,, \qquad \beta_0 = \frac{11}{3}N_c - \frac{2}{3}N_f \tag{1}$$

- For a sufficiently large number of flavors, N_f, one obtains $\beta_0 < 0$. Therefore, there exists a special (critical) value of the coupling, $g = g_*(\epsilon)$ such that $\beta_g(g_*) = 0$, alias $\epsilon = -\gamma_g(a_*)$ or, equivalently

$$\boxed{a_*(\epsilon) = \left(\frac{g_*(\epsilon)}{4\pi}\right)^2 = -\frac{\epsilon}{\beta_0} - \left(\frac{\epsilon}{\beta_0}\right)^2\frac{\beta_1}{\beta_0} + O(\epsilon^3)}$$

- The β-function associated with the gauge parameter ξ vanishes identically in the Landau gauge $\xi = 0$. As a consequence QCD in Landau gauge at critical coupling enjoys scale invariance — Banks, '81

- Scale invariance usually implies conformal invariance?

Conformal transformations

- Define

$$\Phi = \{A_\mu, q, \bar{q}, c, \bar{c}\}$$

$$\Phi \mapsto \Phi + \delta_D \Phi\,, \qquad \delta_D \Phi = \Big(x\partial_x + \Delta_\Phi\Big)\Phi(x)\,,$$

$$\Phi \mapsto \Phi + \delta_K^\mu \Phi\,, \qquad \delta_K^\mu \Phi = \Big(2x_\mu(x\partial) - x^2\partial_\mu + 2\Delta_\Phi x_\mu - 2\Sigma_{\mu\nu}x^\nu\Big)\Phi(x)\,,$$

The generator of spin rotations

$$\Sigma_{\mu\nu} c = \Sigma_{\mu\nu}\bar{c} = 0\,, \qquad \Sigma_{\mu\nu} q = \frac{i}{2}\sigma_{\mu\nu} q\,, \qquad \Sigma_{\mu\nu} A_\alpha = g_{\nu\alpha}A_\mu - g_{\mu\alpha}A_\nu$$

"Scaling dimensions"

$$\Delta_q = \frac{3}{2} - \epsilon, \qquad \Delta_A = 1, \qquad \Delta_c = 0\,, \qquad \Delta_{\bar{c}} = 2 - \epsilon\,.$$

— $\Delta_A = 1$ ensures that $F_{\alpha\beta}$ transforms covariantly

$$\delta_K^\mu F_{\alpha\beta} = \Big[2x_\mu(x\partial) - x^2\partial_\mu + 4x_\mu - 2\Sigma_{\mu\nu}x^\nu\Big]F_{\alpha\beta}\,,$$

— $\Delta_c = 0$ ensures is that $D_\rho c(x)$ transforms as as the gluon field A_ρ

- Variation of the QCD action

$$\delta_D S_R = \int d^d x \, \mathcal{N}(x) \,, \qquad \delta_K^\mu S_R = \int d^d x \, 2x^\mu \Big(\mathcal{N}(x) - (d-2)\partial^\rho \mathcal{B}_\rho(x)\Big)$$

where

$$\mathcal{N}(x) = 2\epsilon \, \mathcal{L}_R^{YM+gf} = 2\epsilon \left(\frac{1}{4} Z_A^2 F^2 + \frac{1}{2\xi}(\partial A)^2\right) ,$$

$$\mathcal{B}_\rho(x) = Z_c^2 \bar{c} D^\rho c - \frac{1}{\xi} A^\rho (\partial A) = \delta_{\mathsf{BRST}}(\bar{c}^a A_\mu^a)$$

- Conformal symmetry can be expected only for correlation functions of gauge-invariant operators

What is conformal symmetry ?

- For correlation functions

$$[\mathbf{C}, \langle \mathcal{O}_N(x)\mathcal{O}_M(y)\rangle] = 0\,, \qquad \mathbf{C} = \{\mathbf{P}_\mu, \mathbf{M}_{\mu\nu}, \mathbf{D}, \mathbf{K}_\mu\}$$

- If $\mathcal{O}_N$ is a multiplicatively renormalizable leading-twist operator

$$\left(M\partial_M + \beta(a)\partial_a + \gamma_N(a)\right)[\mathcal{O}_N] = 0 \qquad \mathcal{O}_N = \sum_{m+k=N-1} \Psi_{Nk}\,(n\cdot D)^k \bar{q}\,\not{n}\,(n\cdot D)^m q$$

then at the critical point $\beta(a^*) = 0$

$$i[\mathbf{D}, [\mathcal{O}_N](x)] = \left(x\partial_x + \Delta_N^*\right)[\mathcal{O}_N](x)$$

with

$$\Delta_N^* = \Delta_N + \gamma_N(a^*) = (2\Delta_q + N - 1) + \gamma_N(a^*) = N + 2 - 2\epsilon + \gamma_N(a^*)$$

and from conformal algebra $\mathcal{K}_\mu$ must have the form

$$i[\mathbf{K}^\mu, [\mathcal{O}_N](x)] = \left[2x^\mu(x\partial) - x^2\partial^\mu + 2\Delta_N^* x^\mu + 2x^\nu\left(n^\mu\frac{\partial}{\partial n^\nu} - n_\nu\frac{\partial}{\partial n_\mu}\right)\right][\mathcal{O}_N](x)$$

- If we do not know multiplicatively renormalizable operators, have instead a mixing matrix

$$(M\partial_M + \beta(a)\partial_a + \mathbb{H}(a))\,[\widehat{\mathcal{O}}] = 0 \qquad\qquad \widehat{\mathcal{O}}_{N,k} = (n\cdot D)^k \bar{q}\,\not{n}\,(n\cdot D)^{N-k-1} q$$

At the critical point $\beta(a^*) = 0$

$$i[\mathbf{D}, [\widehat{\mathcal{O}}](x)] = \Big(x\partial_x + (\Delta_N + \mathbb{H}(a^*))\Big)\,[\widehat{\mathcal{O}}](x)$$

and from conformal algebra $\mathbf{K}_\mu$ must have the form

$$i[\mathbf{K}^\mu, [\widehat{\mathcal{O}}](x)] = \left[2x^\mu(x\partial) - x^2\partial^\mu + 2(\Delta_N + \mathbb{H}(a^*))x^\mu + 2x^\nu\left(n^\mu\frac{\partial}{\partial n^\nu} - n_\nu\frac{\partial}{\partial n_\mu}\right)\right]\,[\widehat{\mathcal{O}}](x)$$

$+$ possibly, extra contributions of operators with $N \to N-1$

- To prove conformal symmetry we have to show that this representation exists
  $\to$ conformal Ward identities

Conformal Ward identities

exploit invariance under change of variables $\Phi \to \Phi + \delta_{D,K,\ldots}\Phi$ in the path integral

$$\langle [\mathcal{O}_1](x)[\mathcal{O}_2](y)\rangle = \int \mathcal{D}\Phi\, [\mathcal{O}_1](x)[\mathcal{O}_2](y)\, e^{S(\Phi)}$$

$$\boxed{\langle \delta_C [\mathcal{O}_1](x)[\mathcal{O}_2](y)\rangle + \langle [\mathcal{O}_1](x)\delta_C [\mathcal{O}_2](y)\rangle + \langle \delta_C S\, [\mathcal{O}_1](x)[\mathcal{O}_2](y)\rangle}$$

- For $\mathbf{C} = \mathbf{P}_\mu, \mathbf{M}_{\mu\nu}$ action is invariant and we obtain the usual symmetry statement
- For $\mathbf{C} = \mathbf{D}, \mathbf{K}_\mu$ action is not invariant, we have to calculate the extra term and prove that
  — for $\mathbf{D}$, the effect is modification $\Delta_N \to \Delta_N + \mathbb{H}(a^*)$
  — for $\mathbf{K}_\mu$ also $\Delta_N \to \Delta_N + \mathbb{H}(a^*)$ and the extra terms do not spoil the algebra

one can show that

$$\delta_D S = -\frac{1}{4}\frac{\beta(a)}{a}\int d^dz\, [F^a_{\mu\nu}F^{a,\mu\nu}] + \ldots\,, \qquad \delta^\mu_K S = -\frac{1}{4}\frac{\beta(a)}{a}\int d^dz\, (2z_\mu)[F^a_{\mu\nu}F^{a,\mu\nu}] + \ldots\,,$$

so that for $\beta(a^*) = 0$ only the pair-counterterms for $z \to x$ and $z \to y$ can contribute

Result $\mathcal{O}(\alpha_s)$ in the light-ray operator representation

V. Braun, A. Manashov, Phys. Lett. B **734**, 137 (2014)

$$\begin{aligned} S_- &= S_-^{(0)}\,, \\ S_0 &= S_0^{(0)} - \epsilon + \frac{1}{2}\mathbb{H}(a_s^*)\,, \qquad \mathbb{H}(a_s^*) = a_s^*\,\mathbb{H}^{(1)} + \ldots \\ S_+ &= S_+^{(0)} + (z_1+z_2)\Big(-\epsilon + \frac{1}{2}a_s^*\mathbb{H}^{(1)}\Big) + a_s^*(z_1-z_2)\Delta_+ + \mathcal{O}(\epsilon^2) \end{aligned}$$

where

$$\Delta_+[\mathcal{O}](z_1,z_2) = -2C_F\int_0^1 d\alpha\Big(\frac{\bar\alpha}{\alpha}+\ln\alpha\Big)\Big[[\mathcal{O}](z_{12}^\alpha,z_2) - [\mathcal{O}](z_1,z_{21}^\alpha)\Big]$$

- Two-loop result also available, JHEP **1603**, 142 (2016)

"Hidden" conformal invariance of QCD RG equations in MS-like schemes

- RG kernels in $d = 4$ and $d = 4 - 2\epsilon$ have the same perturbative expansion

$$\Big(\mu\partial_\mu + \mathbb{H}(a_s^*)\Big)[\mathcal{O}] = 0 \qquad \Rightarrow \qquad \Big(\mu\partial_\mu + \beta(g)\partial_g + \mathbb{H}(a_s)\Big)[\mathcal{O}] = 0$$

- Conformal symmetry implies existence of three generators that satisfy usual SL(2) relations and commute with the renormalization kernel

$$[S_k, \mathbb{H}] = 0$$

$$[S_+, S_-] = 2S_0 \qquad [S_0, S_+] = S_+ \qquad [S_0, S_-] = -S_-$$

- True to all orders in perturbation theory (in MS-like schemes)
- Complete RG kernel in $d = 4$, a digression to $d = 4 - \epsilon$ is an intermediate step

RG equations from operator algebra

- **Expanding the commutation relations in powers of a_s^***

$$[S_+^{(0)}, \mathbb{H}^{(1)}] = 0\,,$$
$$[S_+^{(0)}, \mathbb{H}^{(2)}] = [\mathbb{H}^{(1)}, S_+^{(1)}]\,,$$
$$[S_+^{(0)}, \mathbb{H}^{(3)}] = [\mathbb{H}^{(1)}, S_+^{(2)}] + [\mathbb{H}^{(2)}, S_+^{(1)}]\,, \qquad \text{etc.}$$

- A nested set of inhomogenious first order differential equations for $\mathbb{H}^{(k)}$
  Their solution determines $\mathbb{H}^{(k)}$ up to an $SL(2)$-invariant term
- The r.h.s. involves $\mathbb{H}^{(k)}$ and $S_+^{(m)}$ at one oder less compared to the l.h.s. D.Müller

$\longrightarrow$ $\Delta_+^{(2)}$: 1601.05937

$\longrightarrow$ $\mathbb{H}^{(3)}$: 1703.09532

Vladimirov correspondence

Different geometries of Wilson lines

jet physics **multiparton interactions**

are related by a conformal transformation

$$\{x_+, x_-, x_t\} \mapsto \left\{ -\frac{1}{2x_+}, x_- - \frac{x_t^2}{2x_+}, \frac{x_t}{\sqrt{2}x_+} \right\}$$

Implies exact relation between soft and rapidity anomalous dimensions

$$\gamma_{\rm soft}(v_1, \ldots, v_n) = \gamma_{\rm rapidity}(b_1, \ldots, b_n; \epsilon^*)\,, \quad b_k = \frac{1}{\sqrt{2}} \frac{v_t}{v_+}$$

- checked to NNLO

A. Vladimirov, 1610.05791

Vladimirov correspondence

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- checked to NNLO

A. Vladimirov, 1610.05791

to summarize:

- **QCD in $d=4$ and conformal QCD at $d=4-2\epsilon$ at fine-tuned coupling have the same RG equations in $\overline{\text{MS}}$-scheme**
- **The difference, terms $\mathcal{O}(\epsilon)$, can be reexpanded in terms of QCD β-function**

$$\mathcal{Q}^{d=4} = \mathcal{Q}^{d=4-2\epsilon^*} + \frac{\beta(g)}{g}\Delta\mathcal{Q}$$

- — Save one loop in the calculation of evolution kernels
- — Off-forward from forward coefficient functions up to termns in $\beta(a)$
- — Crewther relation
- — Relation between soft and rapidity anomalous dimensions
- — Regge limit, BFKL and beyond, nonglobal logarithms ...

Venturing into the transverse plane

All discussion so far was about operators "living" on the light-cone:

$$\Phi(0)\partial_+^n\Phi(0) \qquad \Longleftrightarrow \qquad \Phi(0)\Phi(zn)$$

What happens if we include transverse or "minus" derivatives?

$$\Phi(0)(\partial_\perp^2)\partial_+^n\Phi(0) \qquad \Phi(0)(\partial_-\partial_+)\partial_+^n\Phi(0)$$

study isnpired by recent developments in $N = 4$ SUSY:

Beisert, 2004; Beisert, Ferretti, Heise, Zarembo, 2005

> Embedding $SL(2,\mathbb{R})$ in $SO(4,2)$?

Applications:

- Complete results for operator renormalization in QCD up to twist four (2010)
- Kinematic corrections in hard exclusive reactions (2011-2014)

Spinor representation

Coordinates:

$$x_{\alpha\dot\alpha} = x_\mu \sigma^\mu_{\alpha\dot\alpha} = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} = \begin{pmatrix} x_+ & w \\ \bar w & x_- \end{pmatrix}, \qquad \sigma^\mu = (\mathbb{1}, \vec\sigma)$$

To maintain Lorentz–covariance, introduce two light-like vectors $n^2 = \tilde n^2 = 0$

$$n_{\alpha\dot\alpha} = \lambda_\alpha \bar\lambda_{\dot\alpha}, \qquad \tilde n_{\alpha\dot\alpha} = \mu_\alpha \bar\mu_{\dot\alpha}$$

with auxiliary spinors λ and μ

$$x_{\alpha\dot\alpha} = z\,\lambda_\alpha\bar\lambda_{\dot\alpha} + \tilde z\,\mu_\alpha\bar\mu_{\dot\alpha} + w\,\lambda_\alpha\bar\mu_{\dot\alpha} + \bar w\,\mu_\alpha\bar\lambda_{\dot\alpha}$$

Fields:

$$q = \begin{pmatrix} \psi_\alpha \\ \bar\chi^{\dot\beta} \end{pmatrix}, \qquad \bar q = (\chi^\beta, \bar\psi_{\dot\alpha}),$$

$$F_{\alpha\beta,\dot\alpha\dot\beta} = \sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta}F_{\mu\nu} = 2\left(\epsilon_{\dot\alpha\dot\beta} f_{\alpha\beta} - \epsilon_{\alpha\beta}\bar f_{\dot\alpha\dot\beta}\right)$$

$f_{\alpha\beta}$ and $\bar f_{\dot\alpha\dot\beta}$ transform according to $(1,0)$ and $(0,1)$ representations of Lorentz group

“Plus” and “Minus” components

$$\begin{aligned} \psi_+ &= \lambda^\alpha \psi_\alpha\,, & \chi_+ &= \lambda^\alpha \chi_\alpha\,, & f_{++} &= \lambda^\alpha \lambda^\beta f_{\alpha\beta}\,, \\ \bar\psi_+ &= \bar\lambda^{\dot\alpha} \bar\psi_{\dot\alpha}\,, & \bar\chi_+ &= \bar\lambda^{\dot\alpha} \chi_{\dot\alpha}\,, & \bar f_{++} &= \bar\lambda^{\dot\alpha} \bar\lambda^{\dot\beta} \bar f_{\dot\alpha\dot\beta}\,, \\ \psi_- &= \mu^\alpha \psi_\alpha\,, & \bar\psi_- &= \bar\mu^{\dot\alpha} \bar\psi_{\dot\alpha} & f_{+-} &= \lambda^\alpha \mu^\beta f_{\alpha\beta} \end{aligned}$$

similar for derivatives $\partial_\mu \to \partial_{\alpha\dot\alpha}$

$$\partial_{++} = 2\partial_z\,, \qquad \partial_{--} = 2\partial_{\bar z}\,, \qquad \partial_{+-} = 2\partial_w\,, \qquad \partial_{-+} = 2\partial_{\bar w}$$

- ψ_+, χ_+, f_{++} and $\bar\psi_+, \bar\chi_+, \bar f_{++}$ are defined as “good” (quasipartonic) components

Conformal operator basis

- Operator basis containing fields and all possible derivatives is overcompleted
- In general fields with derivatives have "bad" $SL(2,\mathbb{R})$ transformation properties

- under infinitesimal special conformal trafo in the light-cone direction: $x = \{z, \tilde{z}, w, \bar{w}\}$

$$\psi_-(x) \quad \to \quad \frac{1}{(1+z\epsilon)}\psi_-\left(\frac{z}{1+\epsilon z}, \tilde{z}, \frac{w}{1+\epsilon z}, \frac{\bar{w}}{1+\epsilon z}\right)$$

where from e.g.

$$[D_w D_{\bar{w}} D_{\tilde{z}} \psi_-](z) \quad = \quad \frac{1}{(1+z\epsilon)^3}[D_w D_{\bar{w}} D_{\tilde{z}} \psi_-]\left(\frac{z}{1+\epsilon z}\right)$$

$\Rightarrow$ $[D_w D_{\bar{w}} D_{\tilde{z}} \psi_-](z)$ is a "primary" field with $j = 3/2$

- but:

$$\psi_+(x) \quad \to \quad \frac{1}{(1+z\epsilon)^2}\left\{\psi_+\left(\frac{z}{1+\epsilon z}, \tilde{z}, \frac{w}{1+\epsilon z}, \frac{\bar{w}}{1+\epsilon z}\right) + \epsilon z \bar{w} \psi_-\Big(\dots\Big)\right\}$$

$\Rightarrow$ e.g. $[D_{\bar{w}} \psi_+](z)$ does not transform homogeneously under $SL(2,\mathbb{R})$

Solution: allow only

V. Braun, A. Manashov, J. Rohrwild, 2008

$$\psi_+(z,\tilde{z},w,0) = \sum_{n,k} \frac{\tilde{z}^k}{k!}\frac{w^n}{n!}[D_w^n D_{\tilde{z}}^k \psi_+](z)$$
$$\psi_-(z,\tilde{z},0,\bar{w}) = \sum_{n,k} \frac{\tilde{z}^k}{k!}\frac{\bar{w}^n}{n!}[D_{\bar{w}}^n D_{\tilde{z}}^k \psi_-](z)$$

and eliminate remaining “half” of transverse derivatives using EOM, e.g.

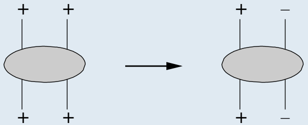
$$[D_{\bar{w}}\psi_+](z) \equiv [D_{-+}\psi_+](z) = [D_{++}\psi_-](z) + EOM = 2\partial_z\psi_-(z) + EOM$$

- similar for gluon fields

Premium

Manifest $SL(2)$ symmetry of higher-twist evolution equations in this basis

$SL(2,\mathbb{R}) \to SO(4,2)$



$$SL(2,\mathbb{R}): \qquad \mathbb{C}_2^{SL(2,R)} = J(J-1)$$

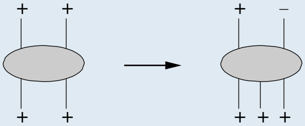
$$SO(4,2): \qquad \mathbb{C}_2^{SO(4,2)} = \mathbf{J}(\mathbf{J}-1)$$

$$\mathbb{H}(J) \quad \to \quad \mathbb{H}(\mathbf{J})$$

the same function !

- This works because two–particle representations are not degenerate w.r.t. both groups

$2 \to 3$ transitions



E.g. $\psi_-\psi_+,\ \psi_+\psi_- \to \psi_+\psi_+\bar{f}_{++}$

Idea:

- Infinitesimal translation in transverse plane $P_{\mu\bar\lambda}$

$$i[\mathbf{P}_{\mu\bar\lambda}, \psi_+] \quad = \quad 2\partial_z\psi_- + igA_{\mu\bar\lambda}\psi_+ + \text{ EOM},$$

- Lorentz Rotation $M_{\mu\mu}$

$$i[\mathbf{M}_{\mu\mu}, \psi_+] \quad \sim \quad (z\partial_z + 1)\psi_- + \frac{1}{2}igzA_{\mu\bar\lambda}\psi_+ + \text{ EOM},$$

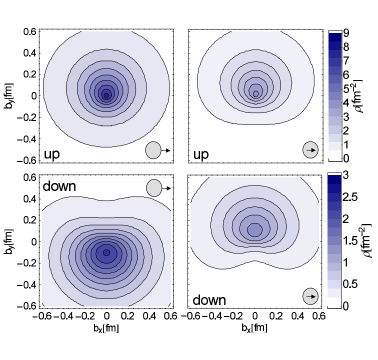
↪ Exact relations between renormalized operators containing “plus” and “minus” fields
↪ The counterterms on the LHS and RHS must coincide

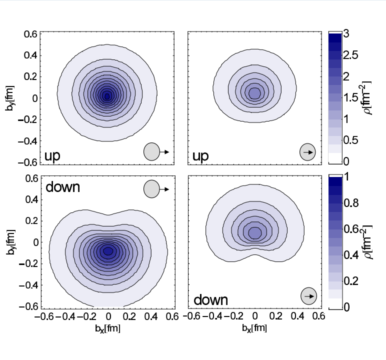
Evolution equations for operators of arbitrary twist are reduced to leading-twist kernels

— no new Feynman diagrams

Electron Ion Collider: Nucleon Tomography

access to three-dimensional picture of the nucleon (M. Burkardt)

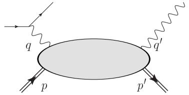




$\hookrightarrow$ **first two moments of transverse spin parton density**

computer simulations: M. Göckeler *et al.*, Phys.Rev.Lett. 98 (2007) 222001

Power corrections $\sim t/Q^2$ and m^2/Q^2 in non-planar kinematics



- Identification of longitudinal and transverse directions not unique
- Leading twist approximation for helicity amplitudes ambiguos
- On the top of it, violation of electromagnetic Ward identities

What is missing?

$$T\{j_\mu(x)j_\nu(0)\} = \sum_N \Big[a_N \mathcal{O}_N + b_N (\partial\mathcal{O})_N + c_N \partial^2 \mathcal{O}_N + \ldots + \text{qqG-operators}\Big]$$

- Matrix elements $\langle(\partial\mathcal{O})_N\rangle$ vanish on free quarks
- b_N cannot be calculated directly(?) even at tree level, but...
- b_N and c_N are related to a_N by conformal algebra

PRL107(2011)202001; PRL109(2012)242001;
JHEP1201(2012)085; PRD89(2014)074022

M. Defurne *et al.* [Hall A Collaboration] arXiv:1504.05453

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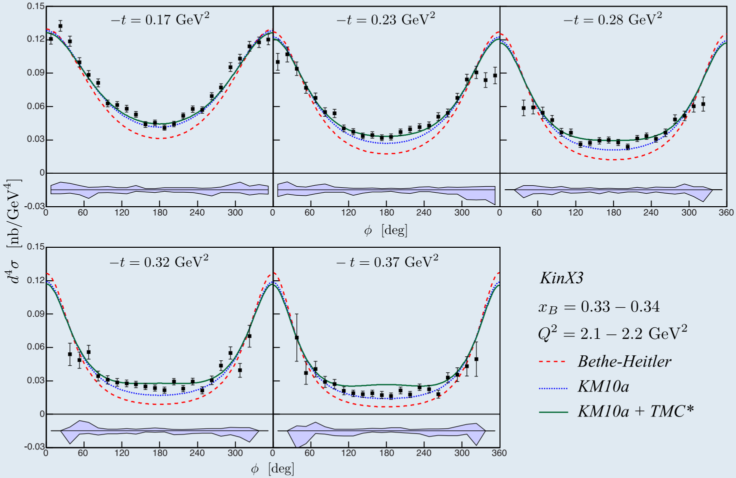

- **TMC* refers to the calculation that includes full kinematic twist-4 corrections**

GPD model: KM10a (Kumericki, Mueller, Nucl. Phys. B **841** (2010) 1)

The problem is to disentangle contributions of “kinematic” and “dynamical” operators

- Using EOM $\partial^\mu \mathcal{O}_{\mu\mu_1\ldots\mu_N}$ can be expressed in terms of quark-gluon operators, e.g.

$$\partial^\mu O_{\mu\nu} = 2ig\bar{q}F_{\nu\mu}\gamma^\mu q\,, \qquad O_{\mu\nu} = (1/2)[\bar{q}\gamma_\mu \overleftrightarrow{D}_\nu q + (\mu \leftrightarrow \nu)]$$

Are matrix elements of $\bar{q}Fq$ operators $\mathcal{O}(\Lambda^2_{QCD})$ or $\mathcal{O}(t \gg \Lambda^2_{QCD})$?

Braun, Manashov 2011; Braun, Manashov, Pirnay, 2012

- Contributions of $(\partial O)_n$ are orthogonal to all other existing multiplicatively renormalizable operators w.r.t. to a conformal scalar product
- The scalar product looks very simple in the conformal operator basis (complicated otherwise)

$$Q_1(z_1,z_2,z_3) = \bar\psi_+(z_1)f_{+-}(z_2)\psi_+(z_3)\,, \qquad T^{j=1}\otimes T^{j=1}\otimes T^{j=1}$$

$$Q_2(z_1,z_2,z_3) = \bar\psi_+(z_1)f_{++}(z_2)\psi_-(z_3)\,, \qquad T^{j=1}\otimes T^{j=3/2}\otimes T^{j=1/2}$$

$$Q_3(z_1,z_2,z_3) = \frac{1}{2}[D_{-+}\bar\psi_+](z_1)f_{++}(z_2)\psi_+(z_3)\,, \qquad T^{j=3/2}\otimes T^{j=3/2}\otimes T^{j=1}$$

$$\langle\langle \vec{\Phi}, \vec{\Psi}\rangle\rangle = 2\langle\Phi_1,\Psi_1\rangle_{111} + \langle\Phi_2,\Psi_2\rangle_{1\frac{3}{2}\frac{1}{2}} + \frac{1}{2}\langle\Phi_3,\Psi_3\rangle_{\frac{3}{2}\frac{3}{2}1}$$

Epilogue

Using hidden symmetries ...

$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu,a} + \bar{\psi}i\not{D}\psi$$

- conformal symmetry — broken on quantum level
- integrability — revealed on quantum level

... to reveal the structure and as a calculational tool