ETH zürich

Precision Physics at the LHC: QCD Corrections

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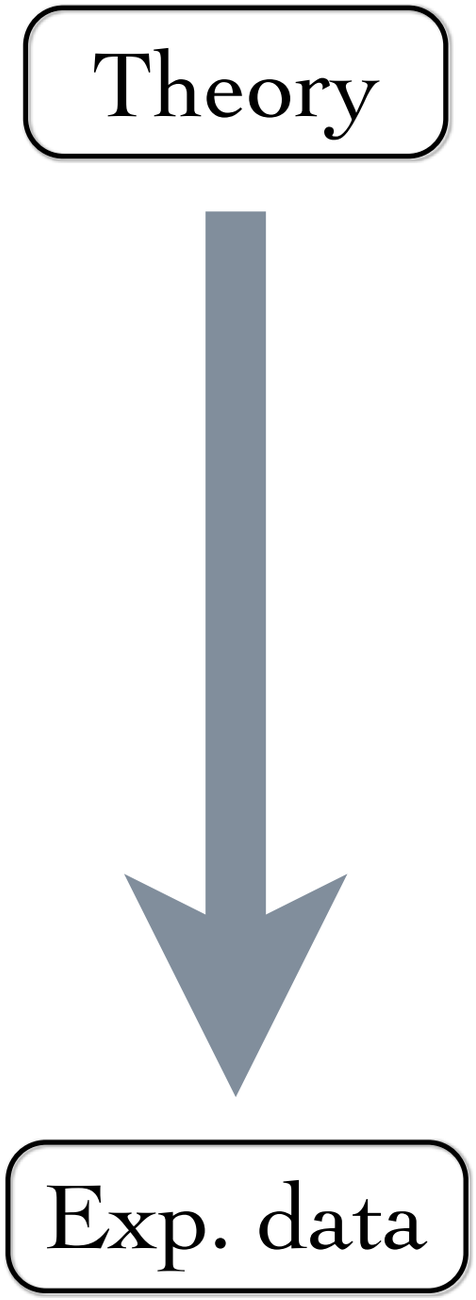
Hefei USTC MadGraph School 2018

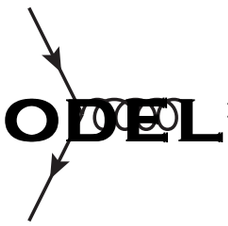
21st November 2018

Prediction chain



Prediction chain

FR + NLOCT — Model — Model



$= i\gamma^\mu t^a_{ij} \quad , \ldots$

MG5 ING. MadLoop — Matrix Element — Matrix Element

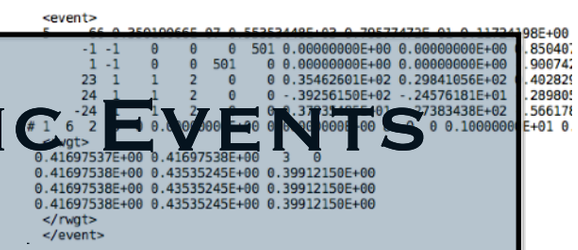
$M^2_{gg\to dd}$, …

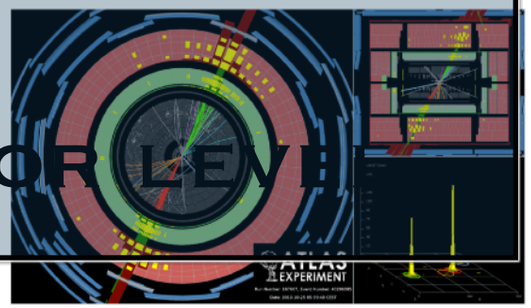
MadFKS INC/MC@NLO — Partonic Events — Partonic Events

Pythia/ Herwig — Hadron level — Hadron level

$\{\pi^0, K^+, \pi^+, p, \cdots\}$

PGS/DELPHES — Detector level — Detector level





MadAnalysis 5

Plan

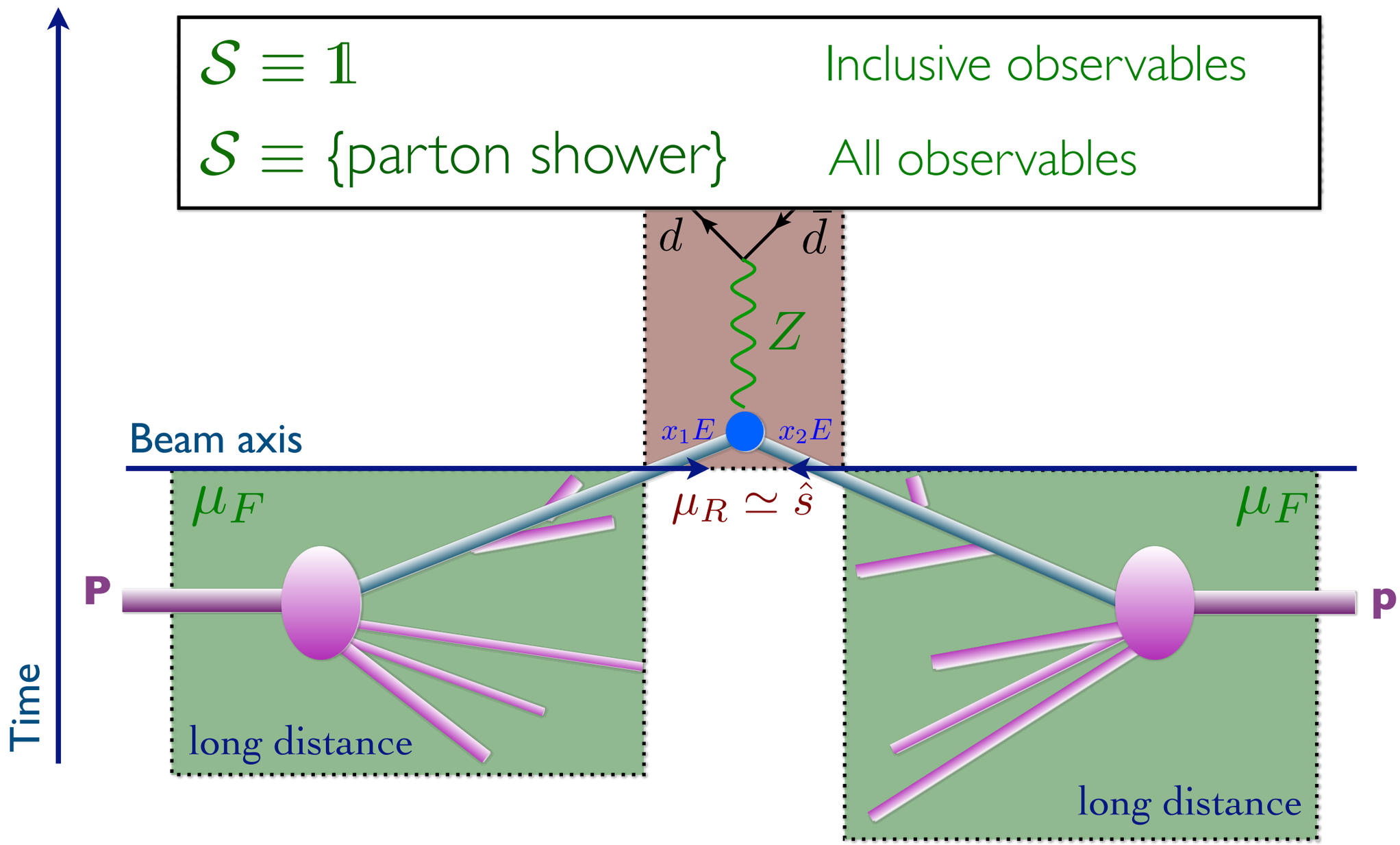
- Why and what are **higher order corrections** ?
- Computing **one-loop** Feynman diagrams
- **Renormalisation** and **rational terms**
- **Subtraction** techniques
- **Matching** to Parton showers beyond LO

Factorisation : $d\sigma_{pp\to\{H\}} =$

$$\sum_{a,b}\int dx_1 dx_2 d\Phi_{\rm FS}\, f_a(x_1,\mu_F) f_b(x_2,\mu_F)\, \hat{\sigma}_{pp\to\{P\}}(\mu_F,\mu_R,\mu_S)\, \mathcal{S}_{\{P\}\to\{H\}}(\mu_S)$$

Phase-space integral — Parton density functions — Parton-level differential cross section — Parton evolution operator

$\mathcal{S} \equiv \mathbb{1}$	Inclusive observables
$\mathcal{S} \equiv$ {parton shower}	All observables



PERTURBATIVE EXPANSIONS

The differential cross section can be written as a perturbation series, using the coupling constant as an expansion parameter :

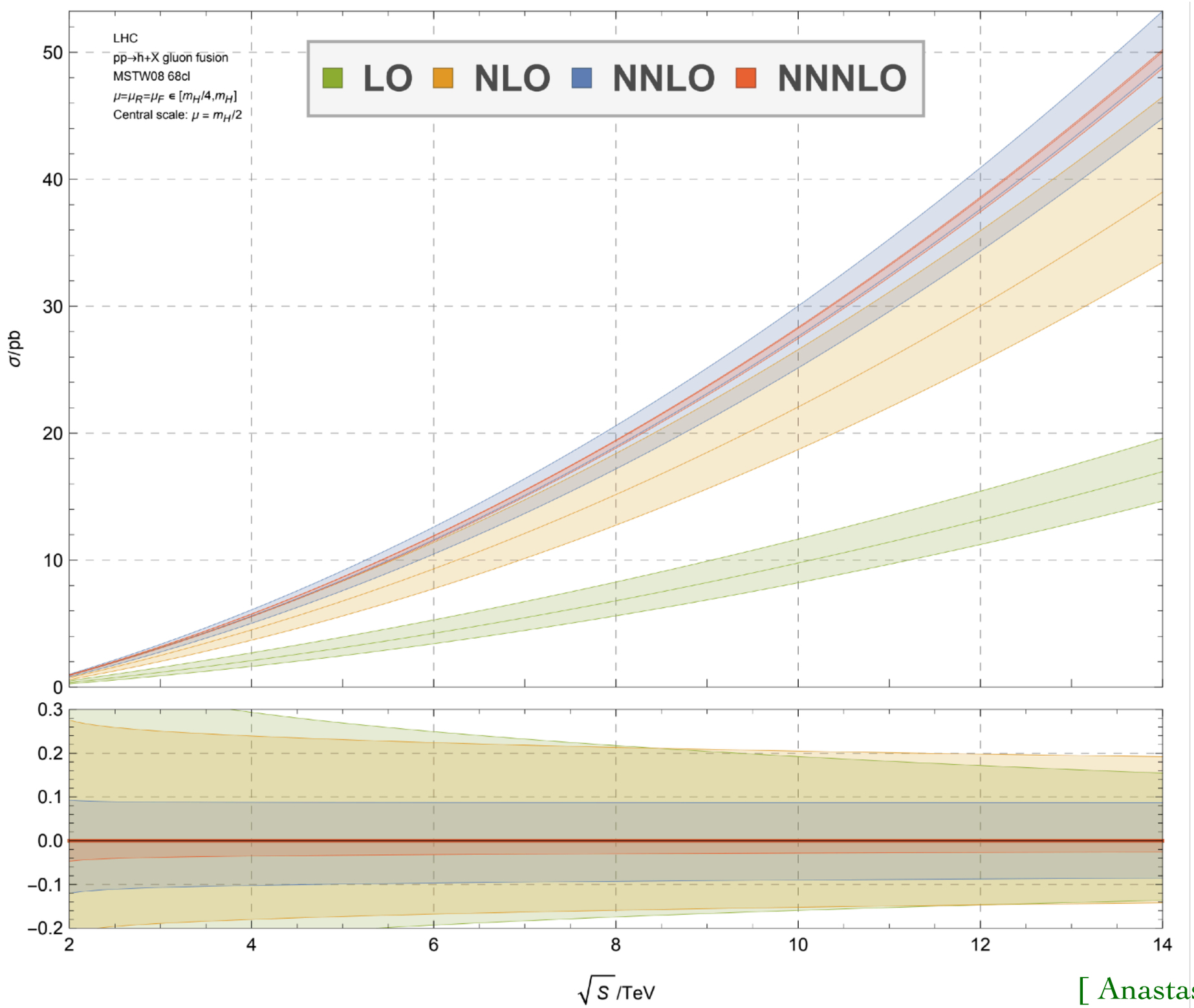
$$\hat{\sigma} = \sigma^{\text{Born}} \left(1 + \frac{\alpha_s}{2\pi} \sigma^{(1)} + \left(\frac{\alpha_s}{2\pi} \right)^2 \sigma^{(2)} + \left(\frac{\alpha_s}{2\pi} \right)^3 \sigma^{(3)} + \dots \right)$$

LO predictions	NLO corrections	NNLO corrections	NNNLO corrections
"Easy"	Difficult but **automated.**	Case-by-case only	Only g g > H

By construction the all-order differential cross-section is scale-independent, but this is not longer true when truncated : assess theoretical uncertainties.

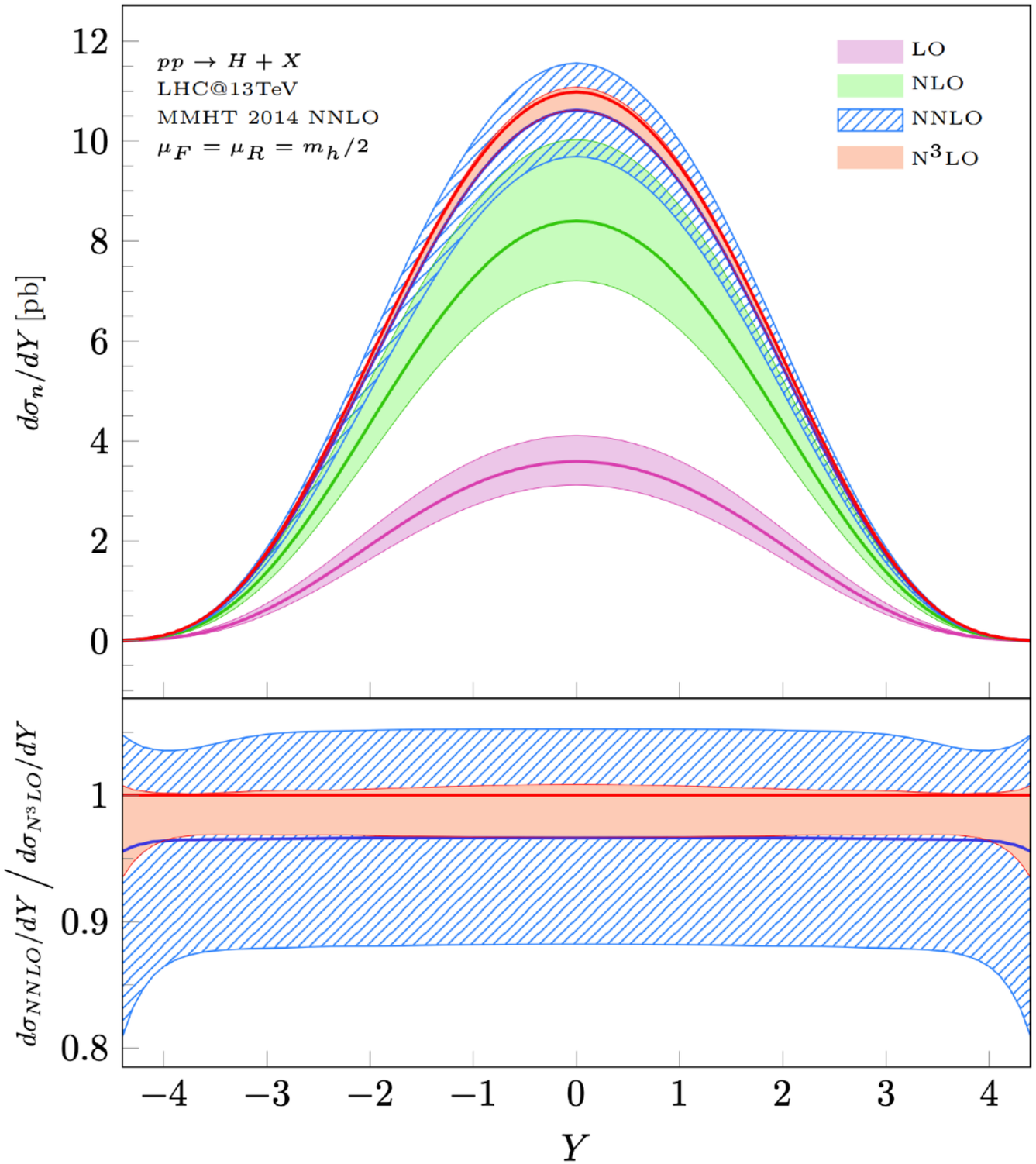
$$\frac{d\sigma_{pp\to X}}{d\log(\mu_R)} = 0 \qquad \textbf{but} \qquad \frac{d\sigma_{pp\to X}|_{N^kLO}}{d\log(\mu_R)} \sim \sigma^{\text{Born}}\, \mathcal{O}(\alpha_s^{k+1})$$

Credible total rates - p p > h



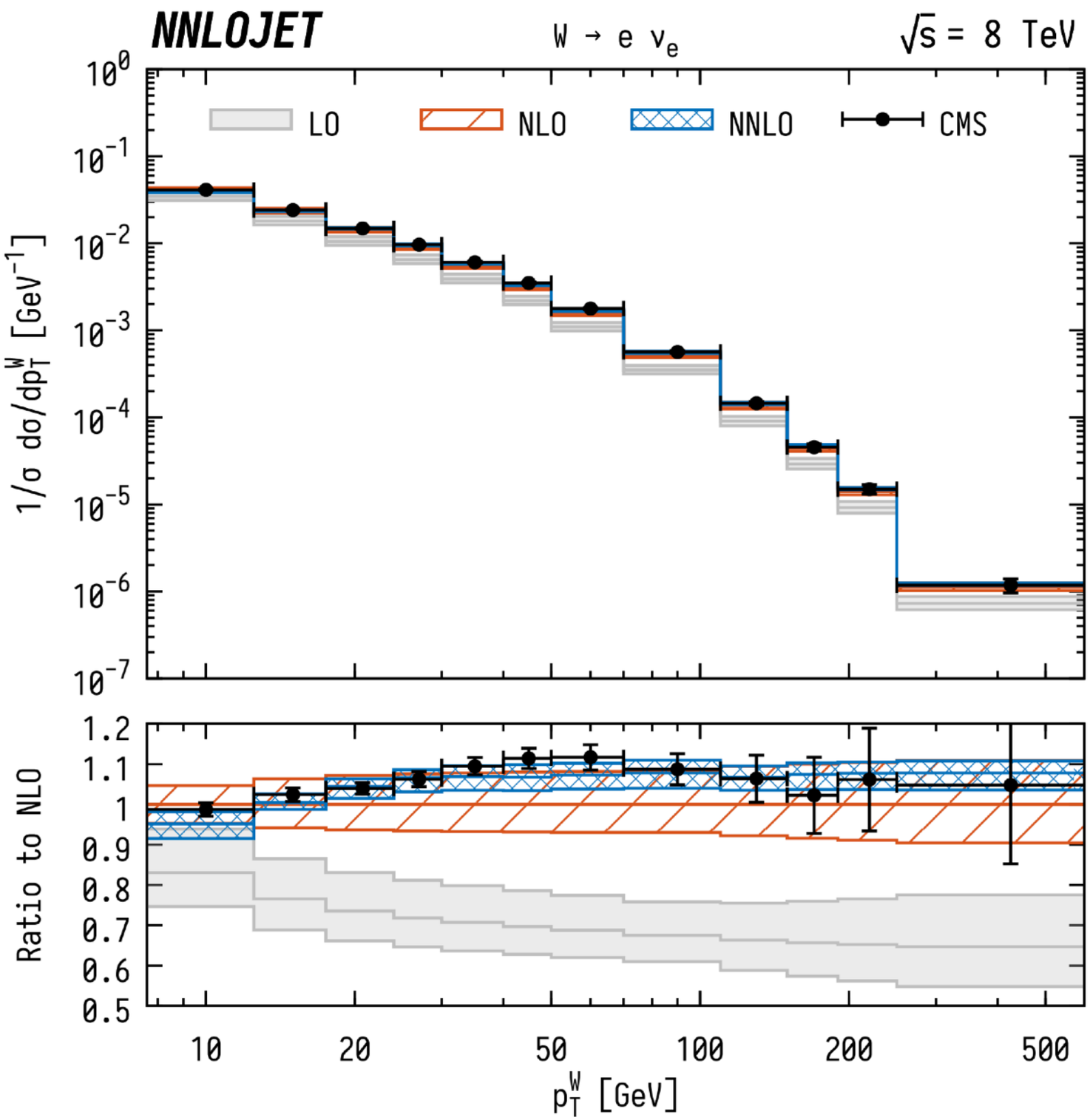
[Anastasiou & al., '15]

Mild impact on rapidity - pp > H

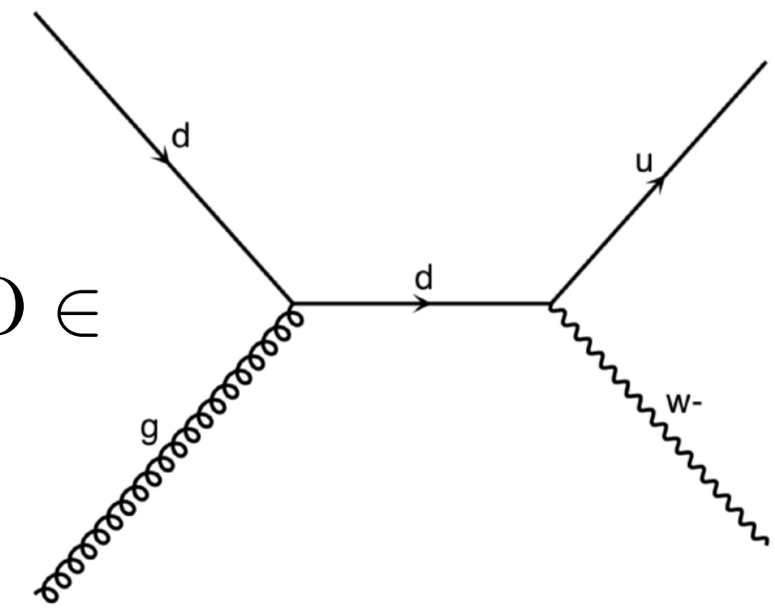


[Dulat & al., '18]

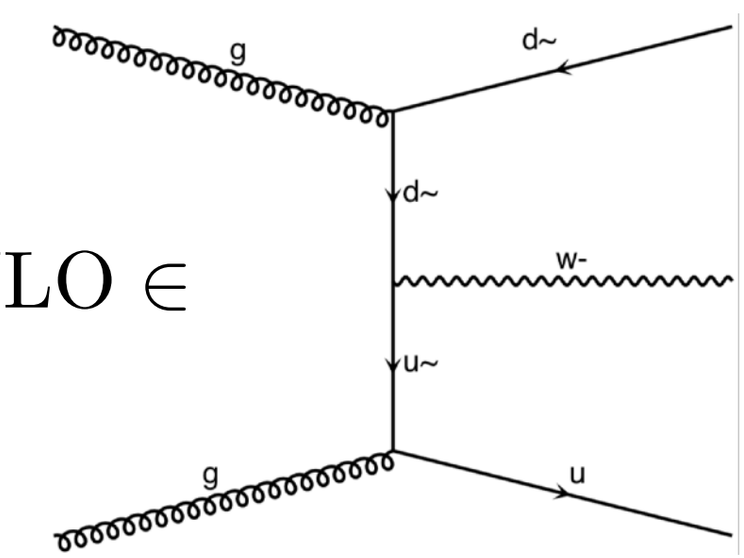
Sometimes significant impact - pp > W J



NLO ∈



NNLO ∈



[Ghermann & al., '17]

PERTURBATIVE EXPANSION

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LO predictions

NLO corrections

NNLO corrections

NNNLO corrections

Summary : why bothering to compute N^{k}LO corrections?

- ➡ Smaller theoretical uncertainty (μ_R var.) when including higher orders.
- ➡ Better descriptions of the shape of highly energetic observables.
- ➡ Credible prediction of total (i.e. inclusive) cross sections of various scattering processes characterised by a set of partonic final-states.

Perturbative expansion

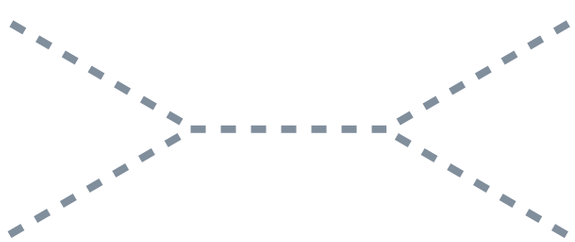
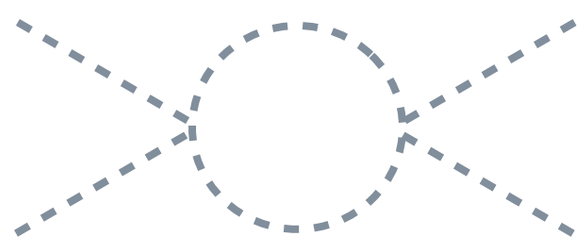
Consider the four-point Green function:

$$\langle \phi_{x_{i1}} \phi_{x_{i2}} | \phi_{x_{f1}} \phi_{x_{f2}} \rangle = Z_0^{-1} \int \mathcal{D}[\phi] \phi_{x_{i1}} \phi_{x_{i2}} \phi_{x_{f1}} \phi_{x_{f2}} e^{-i \int d^4x \mathcal{L}_I[\phi_x]}$$

And expand the exponential of the action:

$$e^{-i \int d^4y \mathcal{L}_I[\phi_y]} = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int \mathcal{L}_I[\phi_{y_1}] d^4y_1 \cdots \int \mathcal{L}_I[\phi_{y_n}] d^4y_n$$

Using Wick theorem and considering $\mathcal{L}_I[\phi_x] \equiv i\lambda\phi_x^3$, we get Feyn. diags:

$n = 2$	$n = 4$	$n > 5$
		Multi-loop
$\mathcal{O}(\lambda^2)$	$\mathcal{O}(\lambda^4)$	$\mathcal{O}(\lambda^n)$

Perturbative expansion

Is this the only contribution however, in a prediction for observable **J**

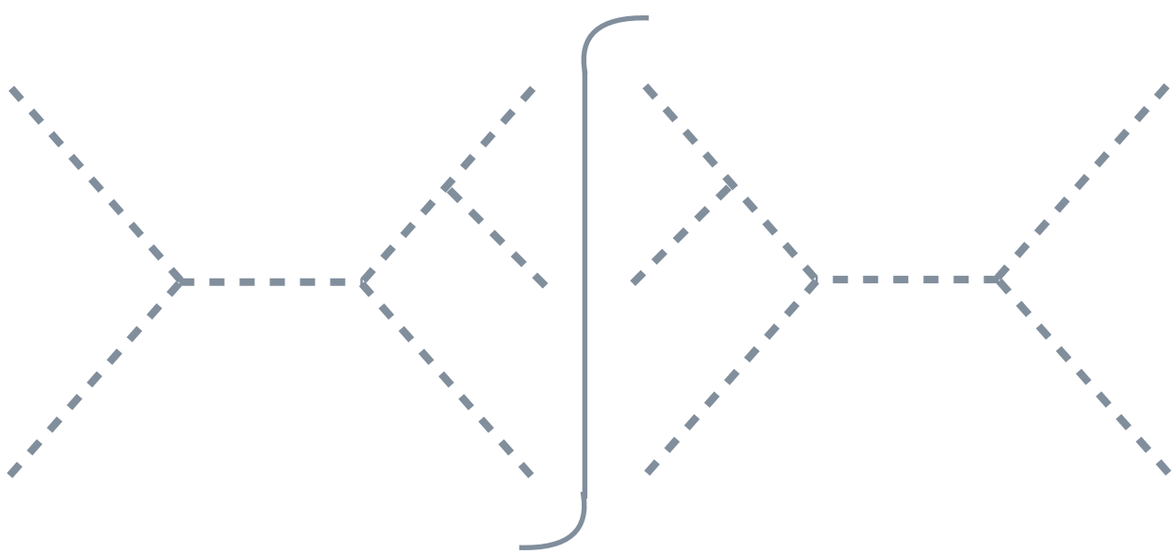
$$\text{Prediction} = \mathrm{J} \otimes \left|\langle \phi_{\mathrm{x_{i1}}} \phi_{\mathrm{x_{i2}}} | \phi_{\mathrm{x_{f1}}} \phi_{\mathrm{x_{f2}}} \rangle\right|^2$$

This assumes that the observable only select that particular final state:

$$J \sim \delta(|\Phi_f\rangle - |\phi_{xf1}\phi_{xf2}\rangle)?$$

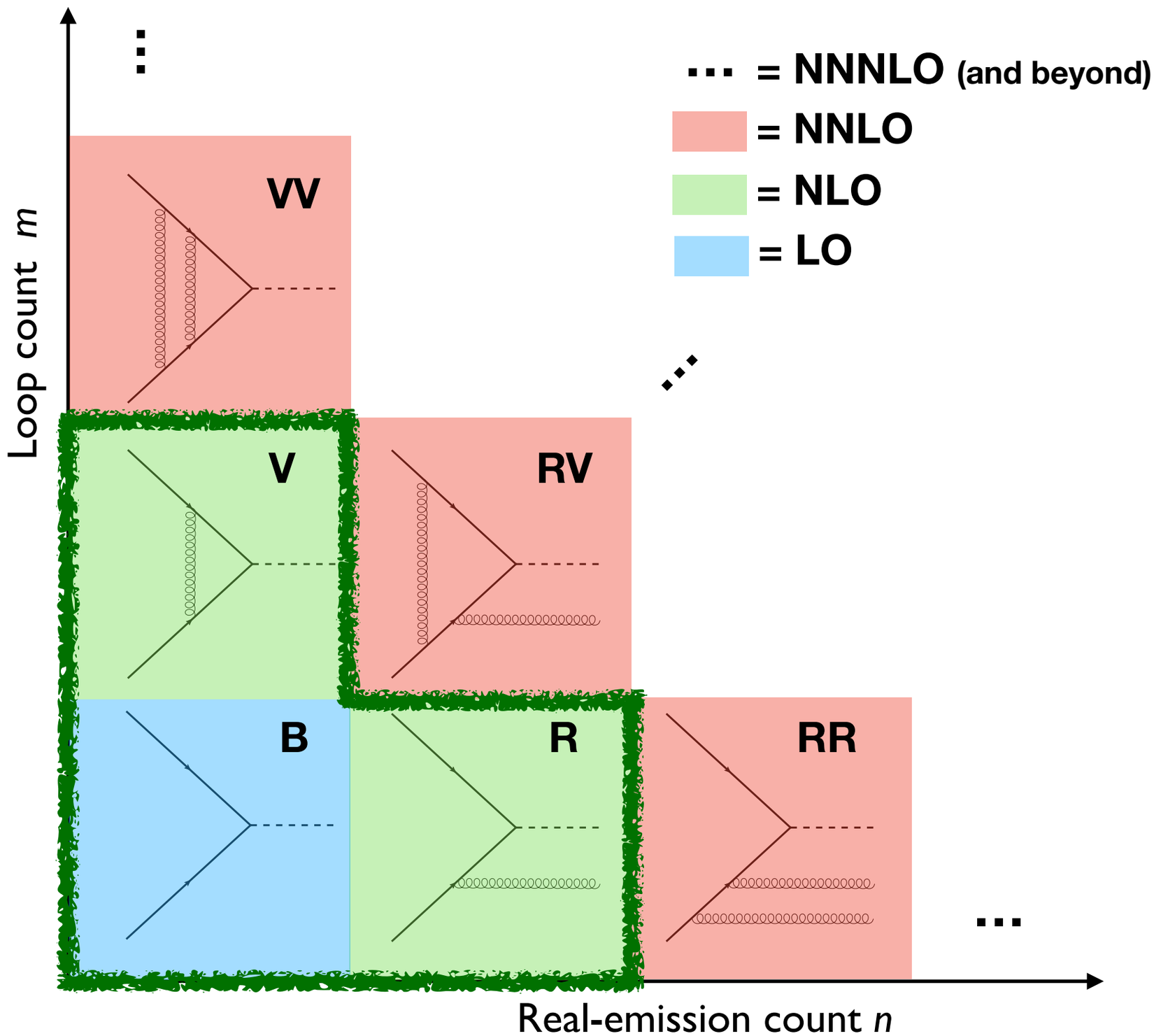
This is **not** reasonable for a theory like QCD (see jets lecture)!
The higher-multiplicity **real-emission** must be considered too :

$$\left|\langle \phi_{x_{i1}} \phi_{x_{i2}} | \phi_{x_{f1}} \phi_{x_{f2}} \phi_{x_{f3}} \rangle\right|^2 \simeq$$

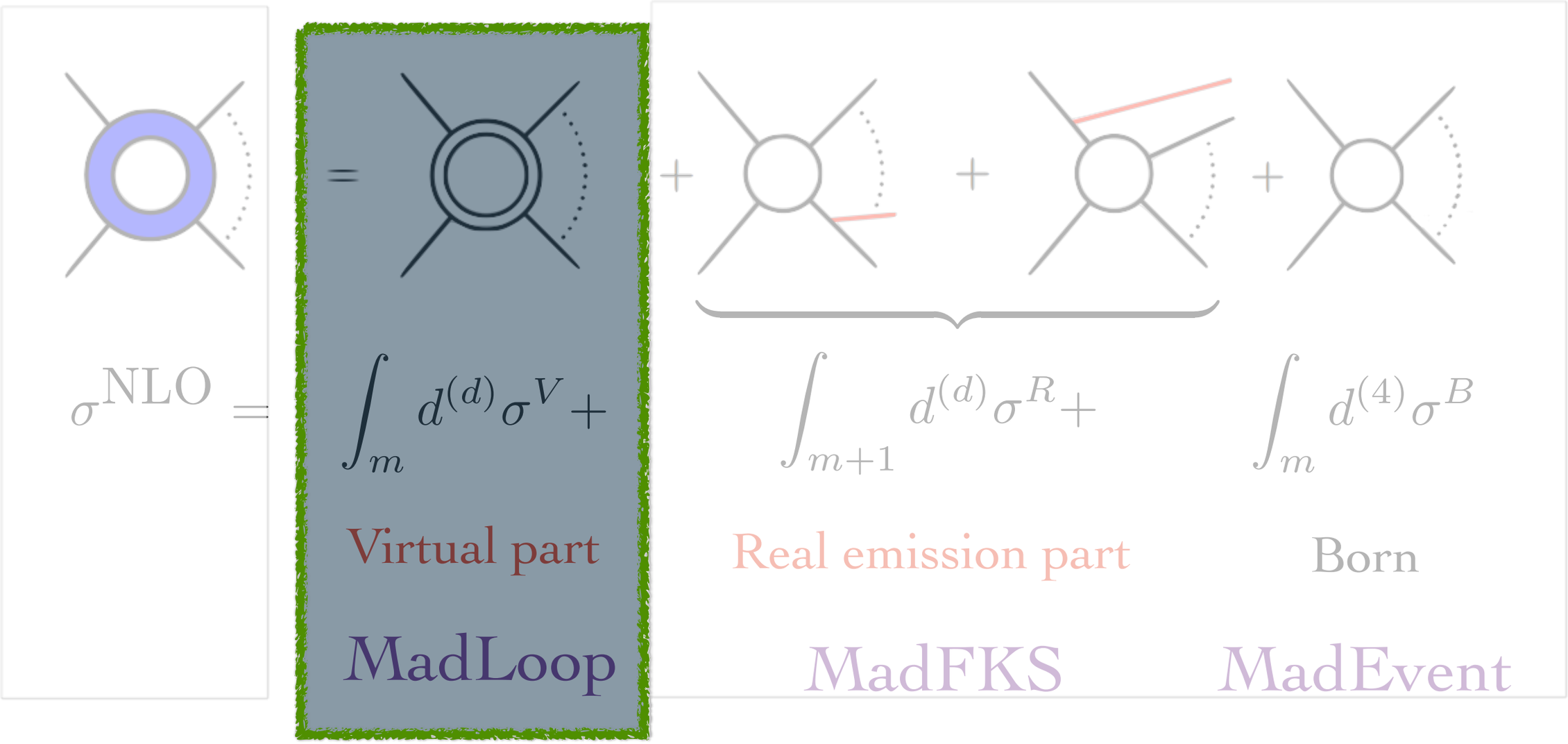


$$\sim (\lambda^3)^2 \quad [\, + \mathcal{O}(\lambda^8) \,]$$

Higher order corrections



Loop computations



Plan

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MadLoop in MG5aMC

- Process generation
  - `import model <model_name>-<restrictions>`
  - `generate <process> <amp_orders_and_option> [<mode>=<pert_orders>] <squared_orders>`
  - `output <format> <folder_name>`
  - `launch <options>`
- Examples, starting from a default MG5aMC interface
  - Very simple one (in this case, generates the full code for NLO computations) :

```
[ 2.5s   ] generate p p > t t~ [QCD]
[ 6.1s   ] output
[ ~ mins*] launch
  * timing for 10k unweighted events on a laptop
```

  - With options specified (in this case, generates the one-loop matrix element code only):

```
[ 0.01s ] import model loop_sm-no_hwidth
[ 0.01s ] set complex_mass_scheme
[ 5min  ] generate g g > e+ ve mu- vm~ b b~ / h QED=2 [virt=QCD]
[ 2min  ] output MyProc
[ ~1 s* ] launch -f
  * time per phase-space point, summed over helicity configurations and colors.
```

Details on how to generate and use a MadLoop standalone library available @

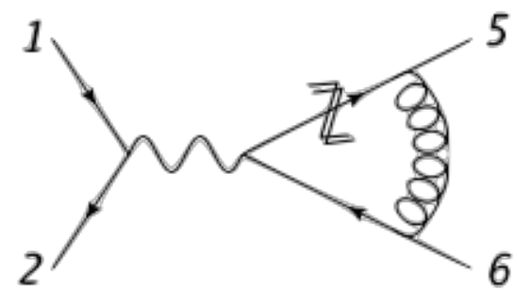
cp3.irmp.ucl.ac.be/projects/madgraph/wiki/MadLoopStandaloneLibrary

Generating loop diagrams

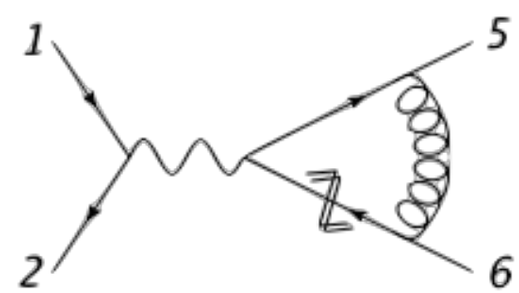
- No external tool for loop diagram generation:
  Reuse MG5_aMC efficient tree level diagram generation!
- Cut loops have two extra external particles

Trees ($e^+e^- \to u\, u\sim\, u\, u\sim$) ≡ Loops ($e^+e^- \to u\, u\sim$)

≡



≡



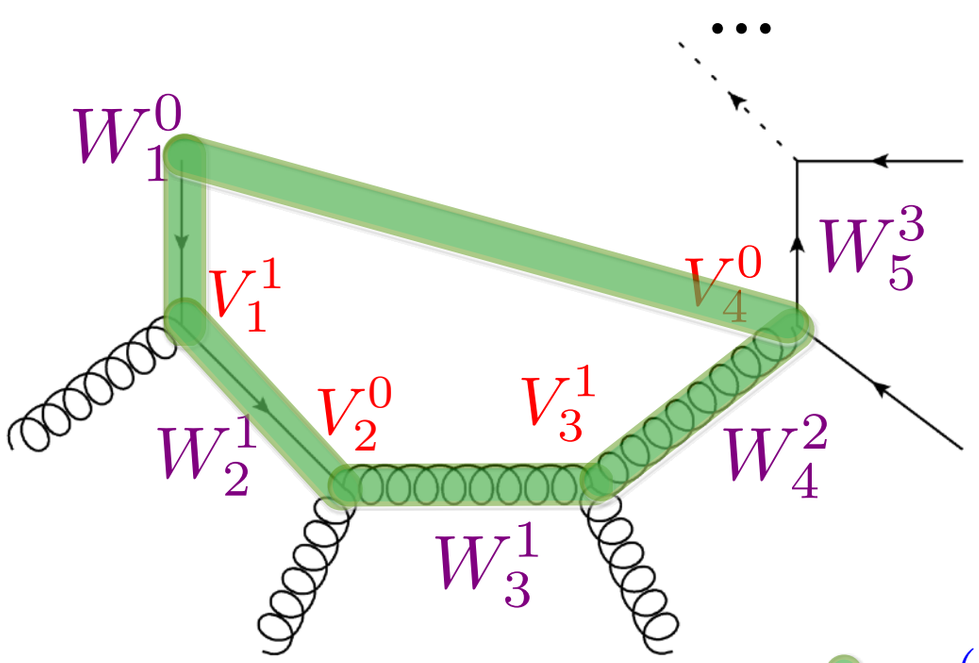
Open-loops

[S. Pozzorini & al. hep-ph/1111.5206]

- Lite-Motive: Be Numerical where you can and analytical where you should.

$$\mathcal{N}(l^\mu) = \sum_{r=0}^{r_{max}} C^{(r)}_{\mu_0\mu_1\cdots\mu_r} l^{\mu_0} l^{\mu_1} \cdots l^{\mu_r}$$

- How to get these coefficients? (Wavefunction and 4-momenta indices now omitted)



$$W_j^{(r)} = \sum_{i=0}^{r} w_j^i l^i \qquad V_j^{(r=0,1)} = \sum_{i=0}^{r} v_j^i l^i$$

$$W_1^{(0)} = w_1^0 = 1$$

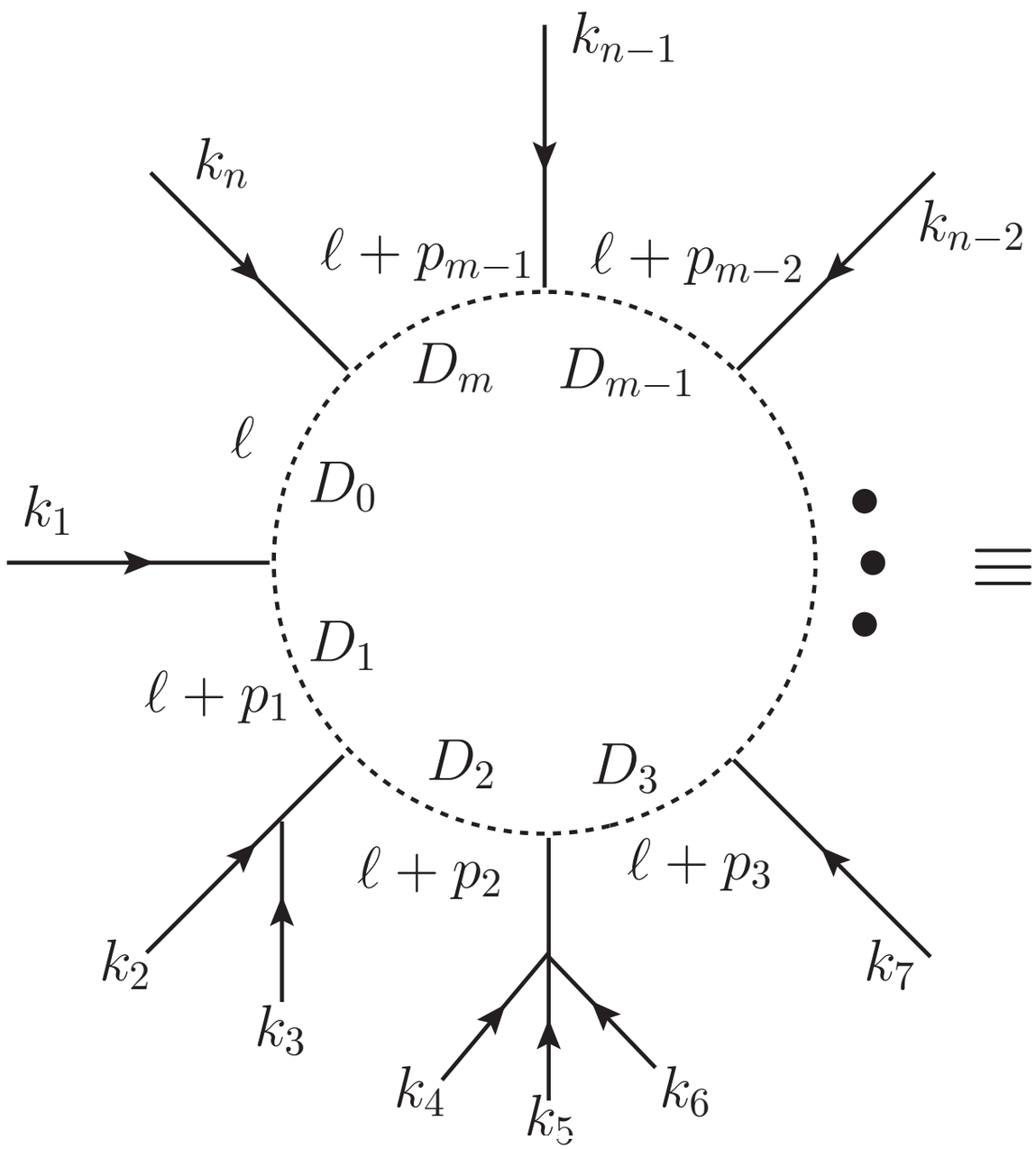
$$W_2^{(1)} = (v_1^1 l + v_1^0) w_1^0$$

$$W_3^{(1)} = v_2^0 W_2^{(1)} = v_2^0 (v_1^1 l + v_1^0) w_1^0$$

$$W_4^{(1)} = V_3^{(1)} W_2^{(1)} = (v_3^1 l + v_3^0) v_2^0 (v_1^1 l + v_1^0) w_1^0$$

... or end of loop and $C^{(2)} = v_3^1 v_2^0 v_1^1 w_1^0, C^{(1)} = v_2^0 w_1^0 (v_3^1 v_0^1 + v_3^0 v_1^1), C^0 = \cdots$

ONE-LOOP INTEGRAL



- Consider this m-point loop diagram with n external momenta

$$\equiv \int \frac{d^d\ell}{(2\pi)^d} \frac{\mathcal{N}(\ell)}{D_0 D_1 D_2 D_3 \cdots D_{m-2} D_{m-1}}$$

with $D_i = (\ell + p_i)^2 - m_i^2$

We will denote by $\mathcal{C}$ this integral.

Scalar integral Basis

$$\mathcal{C}^{\text{1-loop}} = \sum_{i_0<i_1<i_2<i_3} d_{i_0 i_1 i_2 i_3} \text{Box}_{i_0 i_1 i_2 i_3} + \sum_{i_0<i_1<i_2} c_{i_0 i_1 i_2} \text{Triangle}_{i_0 i_1 i_2} + \sum_{i_0<i_1} b_{i_0 i_1} \text{Bubble}_{i_0 i_1} + \sum_{i_0} a_{i_0} \text{Tadpole}_{i_0} + R + \mathcal{O}(\epsilon)$$

$$\text{Box}_{i_0 i_1 i_2 i_3} = \int d^d l \frac{1}{D_{i_0} D_{i_1} D_{i_2} D_{i_3}}$$

$$\text{Triangle}_{i_0 i_1 i_2} = \int d^d l \frac{1}{D_{i_0} D_{i_1} D_{i_2}}$$

$$\text{Bubble}_{i_0 i_1} = \int d^d l \frac{1}{D_{i_0} D_{i_1}}$$

$$\text{Tadpole}_{i_0} = \int d^d l \frac{1}{D_{i_0}}$$

The a, b, c, d and R coefficients depend only on external parameters and momenta.

Reduction of the loop to these scalar coefficients can be achieved using either Tensor Integral Reduction or Reduction at the integrand level

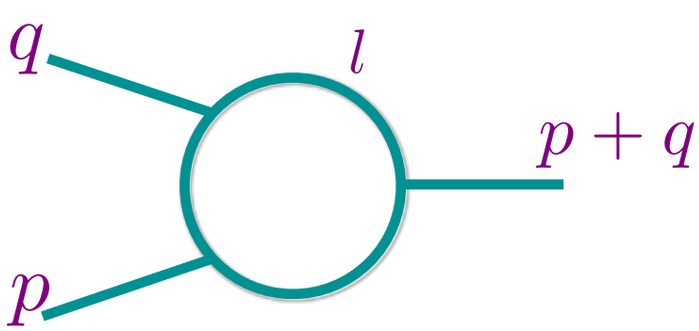
TIR: Passarino-Veltman

- Passarino-Veltman reduction:

$$\int d^d l \, \frac{N(l)}{D_0 D_1 D_2 \cdots D_{m-1}} \to \sum_i \text{coeff}_i \int d^d l \, \frac{1}{D_0 D_1 \cdots}$$

  - Reduce a general integral to "scalar integrals" by "completing the square"

- Example:
  Application of PV to this triangle rank-1 integral



$$\int \frac{d^n l}{(2\pi)^n} \frac{l^\mu}{(l^2 - m_1^2)((l+p)^2 - m_2^2)((l+q)^2 - m_3^2)}$$

- Implemented in codes such as:

COLLIER [A. Denner, S .Dittmaier, L. Hofer, 1604.06792]
GOLEM95 [T. Binoth, J.Guillet, G. Heinrich, E.Pilon, T.Reither, 0810.0992]

$$\int \frac{d^n l}{(2\pi)^n} \frac{l^\mu}{(l^2 - m_1^2)((l+p)^2 - m_2^2)((l+q)^2 - m_3^2)}$$

- The only independent four vectors are p^μ and q^μ . Therefore, the integral must be proportional to those. We can set-up a system of linear equations and try to solve for C_1 and C_2

$$\int \frac{d^n l}{(2\pi)^n} \frac{l^\mu}{(l^2 - m_1^2)((l+p)^2 - m_2^2)((l+q)^2 - m_3^2)} = \begin{pmatrix} p^\mu & q^\mu \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

- We can solve for C_1 and C_2 by contracting with p and q

$$\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = \begin{pmatrix} [2l \cdot p] \\ [2l \cdot q] \end{pmatrix} = G \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \equiv \begin{pmatrix} 2p \cdot p & 2p \cdot q \\ 2p \cdot q & 2q \cdot q \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

where $[2l \cdot p] = \int \frac{d^n l}{(2\pi)^n} \frac{2l \cdot p}{l^2 (l+p)^2 (l+q)^2}$ (For simplicity, the masses are neglected here)

- By expressing $2l.p$ and $2l.q$ as a sum of denominators we can express R_1 and R_2 as a sum of simpler integrals, *e.g.*

$$R_1 = \int \frac{d^n l}{(2\pi)^n} \frac{2l \cdot p}{l^2 (l+p)^2 (l+q)^2} = \int \frac{d^n l}{(2\pi)^n} \frac{(l+p)^2 - l^2 - p^2}{l^2 (l+p)^2 (l+q)^2}$$

$$= \int \frac{d^n l}{(2\pi)^n} \frac{1}{l^2 (l+q)^2} - \int \frac{d^n l}{(2\pi)^n} \frac{1}{(l+p)^2 (l+q)^2} - p^2 \int \frac{d^n l}{(2\pi)^n} \frac{1}{l^2 (l+p)^2 (l+q)^2}$$

- And similarly for R_2

$$R_2 = \int \frac{d^n l}{(2\pi)^n} \frac{2l \cdot q}{l^2 (l+p)^2 (l+q)^2} = \int \frac{d^n l}{(2\pi)^n} \frac{(l+q)^2 - l^2 - q^2}{l^2 (l+p)^2 (l+q)^2}$$
$$= \int \frac{d^n l}{(2\pi)^n} \frac{1}{l^2 (l+p)^2} - \int \frac{d^n l}{(2\pi)^n} \frac{1}{(l+p)^2 (l+q)^2} - q^2 \int \frac{d^n l}{(2\pi)^n} \frac{1}{l^2 (l+p)^2 (l+q)^2}$$

- Now we can solve the equation

$$\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = \begin{pmatrix} [2l \cdot p] \\ [2l \cdot q] \end{pmatrix} = G \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \equiv \begin{pmatrix} 2p \cdot p & 2p \cdot q \\ 2p \cdot q & 2q \cdot q \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

by inverting the "Gram" matrix G

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = G^{-1} \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}$$

- We have re-expressed, reduced, our original integral

$$\int \frac{d^n l}{(2\pi)^n} \frac{l^\mu}{(l^2 - m_1^2)((l+p)^2 - m_2^2)((l+q)^2 - m_3^2)} = \begin{pmatrix} p^\mu & q^\mu \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

in terms of known, simpler *scalar* integrals

PV-Reduction chain

D_{ijkl}	$\to$	$D_{00ij}, D_{ijk}, C_{ijk}, C_{ij}, C_i, C_0$
D_{00ii}	$\to$	$D_{ijk}, D_{ij}, C_{ij}, C_i$
D_{0000}	$\to$	D_{00i}, D_{00}, C_{00}
D_{ijk}	$\to$	$D_{00i}, D_{ij}, C_{ij}, C_i$
D_{00i}	$\to$	D_{ij}, D_i, C_i, C_0
D_{ij}	$\to$	D_{00}, D_i, C_i, C_0
D_{00}	$\to$	D_i, D_0, C_0
D_i	$\to$	D_0, C_0
C_{ijk}	$\to$	$C_{00i}, C_{ij}, B_{ij}, B_i$
C_{00i}	$\to$	C_{ii}, C_i, B_i, B_0
C_{ij}	$\to$	C_{00}, C_i, B_i, B_0
C_{00}	$\to$	C_i, C_0, B_0
C_i	$\to$	C_0, B_0
B_{ii}	$\to$	B_{00}, B_i, A_0
B_{00}	$\to$	B_i, B_0, A_0
B_i	$\to$	B_0, A_0

Table from K.Ellis & al. hep-ph/1105.4319

Integrand reduction

- The integrand (or OPP [Ossola, Papadopoulos, Pittau 2006]) reduction method is a purely numerical algorithm that has been automated in computer codes such as

  CutTools [G.Ossola, C.Papadopoulos, R.Pittau, 0711.3596]
  NINJA [T. Peraro, 1403.1229] (interface to MadLoop in [VH, T. Peraro, 1604.01363]
  SAMURAI [P. Mastrolia, G. Ossola, T. Reiter, F. Tramontano 1006.0710]

  to find the scalar loop coefficients

- Both OPP and Tensor Integral Reduction techniques are interfaced in MadLoop to compute loop diagrams.

How does OPP work?

INTEGRAND LEVEL

- The decomposition to scalar integrals presented before works at the level of the integrals

$$\mathcal{M}^{\text{1-loop}} = \sum_{i_0<i_1<i_2<i_3} d_{i_0 i_1 i_2 i_3} \text{Box}_{i_0 i_1 i_2 i_3} + \sum_{i_0<i_1<i_2} c_{i_0 i_1 i_2} \text{Triangle}_{i_0 i_1 i_2} + \sum_{i_0<i_1} b_{i_0 i_1} \text{Bubble}_{i_0 i_1} + \sum_{i_0} a_{i_0} \text{Tadpole}_{i_0} + R + \mathcal{O}(\epsilon)$$

If we would know a similar relation at the **integrand** level, we would be able to manipulate the integrands and extract the coefficients without doing the integrals

$$N(l) = \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d_{i_0 i_1 i_2 i_3} - \tilde{d}_{i_0 i_1 i_2 i_3}(l)\right] \prod_{i\neq i_0,i_1,i_2,i_3}^{m-1} D_i + \sum_{i_0<i_1<i_2}^{m-1} \left[c_{i_0 i_1 i_2} - \tilde{c}_{i_0 i_1 i_2}(l)\right] \prod_{i\neq i_0,i_1,i_2}^{m-1} D_i + \sum_{i_0<i_1}^{m-1} \left[b_{i_0 i_1} - \tilde{b}_{i_0 i_1}(l)\right] \prod_{i\neq i_0,i_1}^{m-1} D_i + \sum_{i_0}^{m-1} \left[a_{i_0} - \tilde{a}_{i_0}(l)\right] \prod_{i\neq i_0}^{m-1} D_i + \tilde{P}(l) \prod_{i}^{m-1} D_i$$

Spurious term

INTEGRAND LEVEL

- The functional form of the spurious terms is known (it depends on the rank of the integral and the number of propagators in the loop) [del Aguila, Pittau 2004]
  - for example, a box coefficient from a rank 1 numerator is

$$\tilde{d}_{i_0 i_1 i_2 i_3}(l) = \tilde{d}_{i_0 i_1 i_2 i_3}\, \epsilon^{\mu\nu\rho\sigma}\, l^\mu p_1^\nu p_2^\rho p_3^\sigma$$

    (remember that p_i is the sum of the momentum that has entered the loop so far, so we always have $p_0 = 0$)

  - The integral is zero

$$\int d^d l \frac{\tilde{d}_{i_0 i_1 i_2 i_3}(l)}{D_0 D_1 D_2 D_3} = \tilde{d}_{i_0 i_1 i_2 i_3} \int d^d l \frac{\epsilon^{\mu\nu\rho\sigma}\, l^\mu p_1^\nu p_2^\rho p_3^\sigma}{D_0 D_1 D_2 D_3} = 0$$

Example - box coefficients

$$N(l^{\pm}) = d_{0123} + \tilde{d}_{0123}(l^{\pm}) \prod_{i\neq 0,1,2,3}^{m-1} D_i(l^{\pm})$$

- Two values are enough given the functional form for the spurious term. We can immediately determine the Box coefficient

$$d_{0123} = \frac{1}{2}\left[\frac{N(l^{+})}{\prod_{i\neq 0,1,2,3}^{m-1} D_i(l^{+})} + \frac{N(l^{-})}{\prod_{i\neq 0,1,2,3}^{m-1} D_i(l^{-})}\right]$$

- By choosing other values for l, that set other combinations of 4 "denominators" to zero, we can get all the Box coefficients

Example - box coefficients

- Compute this integral: $$\int d^d l \frac{1}{D_0 D_1 D_2 D_3 D_4 D_5 D_6}$$

- So we that the numerator is $N(l) = 1 \quad D_i = (l + p_i)^2 - m_i^2$
- We know that we need only Box, Triangle, Bubble (and Tadpole) contributions. Let's find the first Box integral coefficient.
- Take the two solutions of

$$D_0(l^\pm) = D_1(l^\pm) = D_2(l^\pm) = D_3(l^\pm) = 0$$

- And use the relation we found before and we directly have

$$d_{0123} = \frac{1}{2}\left[\frac{1}{D_4(l^+)D_5(l^+)D_6(l^+)} + \frac{1}{D_4(l^-)D_5(l^-)D_6(l^-)}\right]$$

OPP Reduction

$$N(l) = \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d_{i_0 i_1 i_2 i_3} + \tilde{d}_{i_0 i_1 i_2 i_3}(l)\right] \prod_{i\neq i_0,i_1,i_2,i_3}^{m-1} D_i$$

$$+ \sum_{i_0<i_1<i_2}^{m-1} \left[c_{i_0 i_1 i_2} + \tilde{c}_{i_0 i_1 i_2}(l)\right] \prod_{i\neq i_0,i_1,i_2}^{m-1} D_i$$

$$+ \sum_{i_0<i_1}^{m-1} \left[b_{i_0 i_1} + \tilde{b}_{i_0 i_1}(l)\right] \prod_{i\neq i_0,i_1}^{m-1} D_i$$

$$+ \sum_{i_0}^{m-1} \left[a_{i_0} + \tilde{a}_{i_0}(l)\right] \prod_{i\neq i_0}^{m-1} D_i$$

$$+ \tilde{P}(l) \prod_{i}^{m-1} D_i$$

$$= 0$$

To solve the OPP reduction, choosing special values for the loop momentum helps a lot

For example, choosing *l* such that

$$D_0(l^{\pm}) = D_1(l^{\pm}) = D_2(l^{\pm}) = D_3(l^{\pm}) = 0$$

sets all the terms in this equation to zero except the first line

There are two (complex) solutions to this equation due to the quadratic nature of the propagators

OPP Reduction

$$N(l) = \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d_{i_0 i_1 i_2 i_3} + \tilde{d}_{i_0 i_1 i_2 i_3}(l)\right] \prod_{i\neq i_0,i_1,i_2,i_3}^{m-1} D_i$$

$$+ \sum_{i_0<i_1<i_2}^{m-1} \left[c_{i_0 i_1 i_2} + \tilde{c}_{i_0 i_1 i_2}(l)\right] \prod_{i\neq i_0,i_1,i_2}^{m-1} D_i$$

$$+ \sum_{i_0<i_1}^{m-1} \left[b_{i_0 i_1} + \tilde{b}_{i_0 i_1}(l)\right] \prod_{i\neq i_0,i_1}^{m-1} D_i$$

$$+ \sum_{i_0}^{m-1} \left[a_{i_0} + \tilde{a}_{i_0}(l)\right] \prod_{i\neq i_0}^{m-1} D_i$$

$$+ \tilde{P}(l) \prod_{i}^{m-1} D_i$$

$$= 0$$

Now we choose l such that

$$D_0(l^i) = D_1(l^i) = D_2(l^i) = 0$$

sets all the terms in this equation to zero except the first and second line

Coefficient computed in a previous step

OPP Reduction

$$
\begin{aligned}
N(l) = & \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d_{i_0 i_1 i_2 i_3} + \tilde{d}_{i_0 i_1 i_2 i_3}(l)\right] \prod_{i\neq i_0,i_1,i_2,i_3}^{m-1} D_i \\
& + \sum_{i_0<i_1<i_2}^{m-1} \left[c_{i_0 i_1 i_2} + \tilde{c}_{i_0 i_1 i_2}(l)\right] \prod_{i\neq i_0,i_1,i_2}^{m-1} D_i \\
& + \sum_{i_0<i_1}^{m-1} \left[b_{i_0 i_1} + \tilde{b}_{i_0 i_1}(l)\right] \prod_{i\neq i_0,i_1}^{m-1} D_i \\
& + \sum_{i_0}^{m-1} \left[a_{i_0} + \tilde{a}_{i_0}(l)\right] \prod_{i\neq i_0}^{m-1} D_i \\
& + \tilde{P}(l) \prod_{i}^{m-1} D_i \\
= & \; 0
\end{aligned}
$$

Now, choosing *l* such that

$$D_0(l^i) = D_1(l^i) = 0$$

sets all the terms in this equation to zero except the first, second and third line

Coefficient computed in a previous step

OPP Reduction

$$
\begin{aligned}
N(l) = & \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d_{i_0 i_1 i_2 i_3} + \tilde{d}_{i_0 i_1 i_2 i_3}(l)\right] \prod_{i\neq i_0,i_1,i_2,i_3}^{m-1} D_i \\
& + \sum_{i_0<i_1<i_2}^{m-1} \left[c_{i_0 i_1 i_2} + \tilde{c}_{i_0 i_1 i_2}(l)\right] \prod_{i\neq i_0,i_1,i_2}^{m-1} D_i \\
& + \sum_{i_0<i_1}^{m-1} \left[b_{i_0 i_1} + \tilde{b}_{i_0 i_1}(l)\right] \prod_{i\neq i_0,i_1}^{m-1} D_i \\
& + \sum_{i_0}^{m-1} \left[a_{i_0} + \tilde{a}_{i_0}(l)\right] \prod_{i\neq i_0}^{m-1} D_i \\
& + \tilde{P}(l) \prod_{i}^{m-1} D_i \\
& = 0
\end{aligned}
$$

Now, choosing *l* such that

$$D_1(l^i) = 0$$

sets the last line to zero

Plan

- Why and what are **higher order corrections** ?
- Computing **one-loop** Feynman diagrams
- **Renormalisation** and **rational terms**
- **Subtraction** techniques
- **Matching** to Parton showers beyond LO

Complications in d-dimensions

- The previous expression should in fact be written in d-dimensions
- In the t'HV scheme, external momenta and polarisation vectors are in 4 dimensions; only the loop momentum is in d dimensions

- The integral to be computed should therefore read

$$\int d^d l \, \frac{N(l,\tilde{l})}{\bar{D}_0 \bar{D}_1 \bar{D}_2 \cdots \bar{D}_{m-1}}$$

$$\bar{l} = l + \tilde{l}$$

d dim — 4 dim — epsilon dim

$$\bar{D}_i = (\bar{l} + p_i)^2 - m_i^2 = (l + p_i)^2 - m_i^2 + \tilde{l}^2 = D_i + \tilde{l}^2$$

$$l \cdot \tilde{l} = 0 \qquad \bar{l} \cdot p_i = l \cdot p_i \qquad \bar{l} \cdot \bar{l} = l \cdot l + \tilde{l} \cdot \tilde{l}$$

Complications in d-dimensions

- The d-dimensional contribution gives rise to the rational term which splits into two contributions

$$R = R_1 + R_2$$

- R_1 can be directly computed by the reduction algorithm, while R_2 can be computed from a finite set of process-independent additional Feynman rules.
  - R1: originates from the propagator (calculated in the reduction)
  - R2: originates from the numerator (additional Feynman rules)

R_1

- The origin of R_1 is coming is the denominators of the propagators in the loop

$$\frac{1}{D_i} \to \frac{1}{\bar{D}_i} = \frac{1}{D}\left(1 - \frac{\tilde{l}^2}{D_i}\right)$$

- Of course, the propagator structure is known, so these contributions can be included in the OPP reduction
- They give contributions proportional to

$$\int d^d\bar{l} \frac{\tilde{l}^2}{\bar{D}_i \bar{D}_j} = -\frac{i\pi^2}{2}\left[m_i^2 + m_j^2 - \frac{(p_i - p_j)^2}{3}\right] + \mathcal{O}(\epsilon)$$

$$\int d^d\bar{l} \frac{\tilde{l}^2}{\bar{D}_i \bar{D}_j \bar{D}_k} = -\frac{i\pi^2}{2} + \mathcal{O}(\epsilon)$$

$$\int d^d\bar{l} \frac{\tilde{l}^4}{\bar{D}_i \bar{D}_j \bar{D}_k \bar{D}_l} = -\frac{i\pi^2}{6} + \mathcal{O}(\epsilon)$$

R2

Loop amplitude: $$\frac{1}{(2\pi)^4}\int d^d\bar{q}\frac{\bar{N}(\bar{q})}{\bar{D}_0\bar{D}_1\cdots\bar{D}_{m-1}} \quad , \quad \bar{D}_i=(\bar{q}+p_i)^2-m_i^2$$

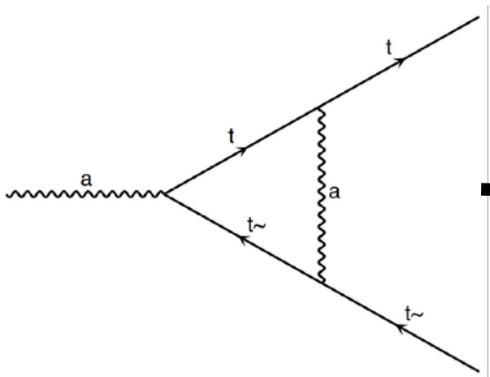
Problem : numerical technique can only evaluate the numerator in 4 dimensions

Solution : isolate the ε-dim part of the numerator: $$\underbrace{\bar{N}(\bar{q})}_{\text{d-dim}}=\underbrace{N(q)}_{\text{4-dim}}+\underbrace{\tilde{N}(\tilde{q},q,\epsilon)}_{\epsilon-dim}$$

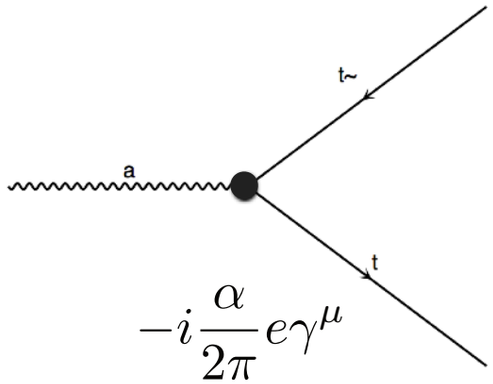
Then : compute analytically the finite set of loops for which its contribution does not vanish, and re-express it in terms of an R2 Feynman rules.

$$R2\equiv\lim_{\epsilon\to 0}\frac{1}{(2\pi)^4}\int d^d\bar{q}\frac{\tilde{N}(\tilde{q},q,\epsilon)}{\bar{D}_0\bar{D}_1\cdots\bar{D}_{m-1}}$$

Ex. :



$$\supset\int d^d\bar{q}\frac{\tilde{q}^2}{\bar{D}_0\bar{D}_1\bar{D}_2}=-\frac{i\pi^2}{2}+\mathcal{O}(\epsilon)$$



$$-i\frac{\alpha}{2\pi}e\gamma^\mu$$

NLOCT

[C. Degrande, arXiv:1412.6955]

- UV counterterms:

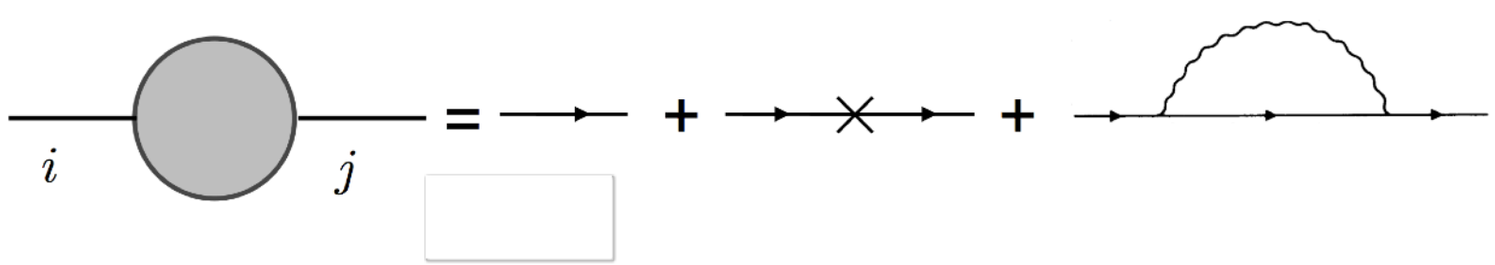
A) Renormalize the Lagrangian

$$\left.\begin{array}{ll} \text{Fields} & \phi_0 \to (1+\frac{1}{2}\delta Z_{\phi\phi}) + \sum_\chi \frac{1}{2}\delta Z_{\phi\chi}\chi \\ \text{ext. params} & x_0 \to x + \delta x \\ \text{int. params} & g(x) \to g(x+\delta x) \end{array}\right\} \quad \mathcal{L}_0 \to \mathcal{L} + \delta\mathcal{L}$$

B) Compute the defining loops

→ Done in FeynArts. Notice that for $\overline{MS}$, only poles are needed.

C) Solve for the counterterms by applying renormalization conditions

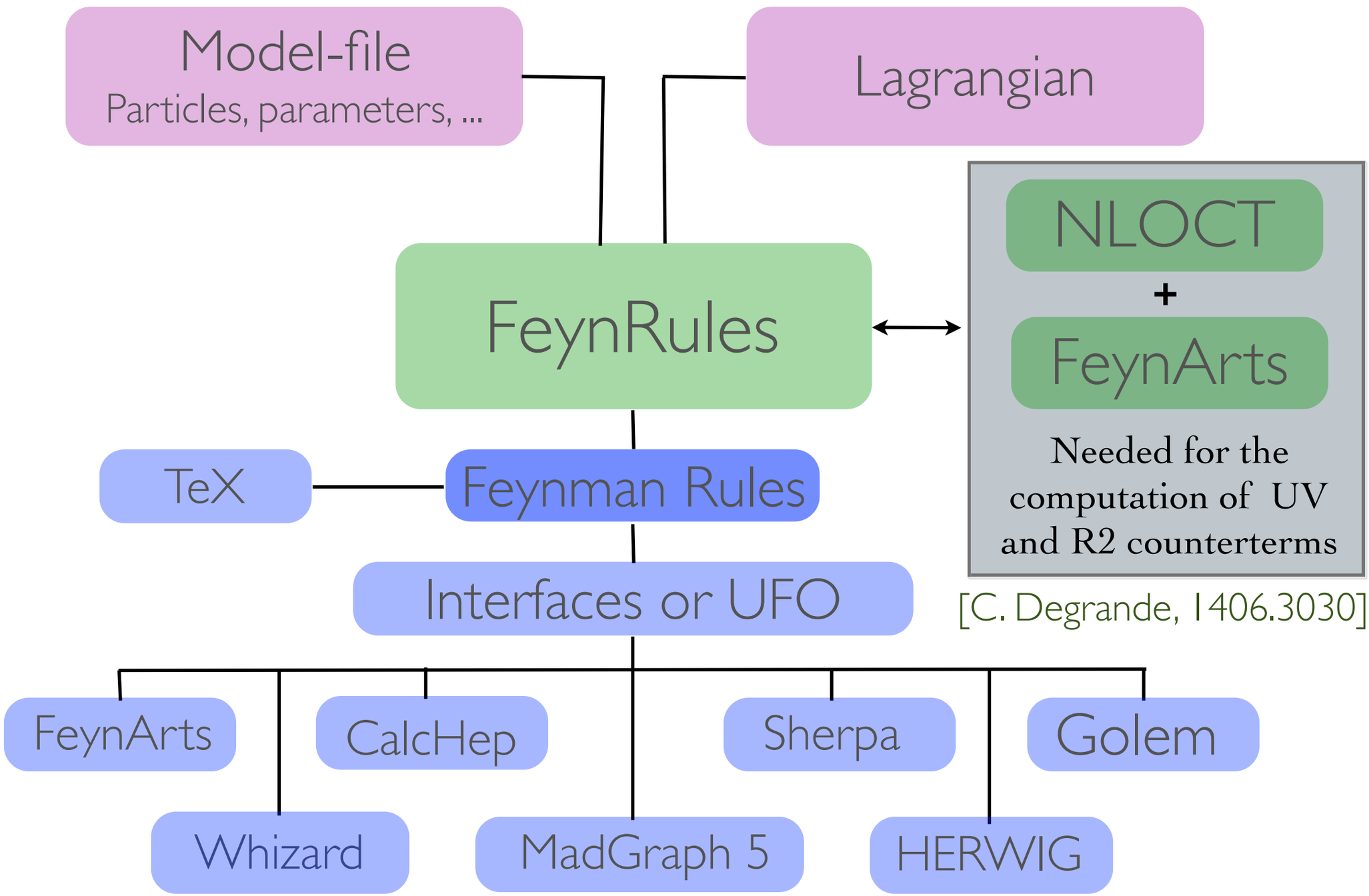


D) Derive and output the corresponding UV counterterms.

- R2 counterterms, computed using FeynArts amplitudes as well.

FeynRules @ NLO (version 2.1)

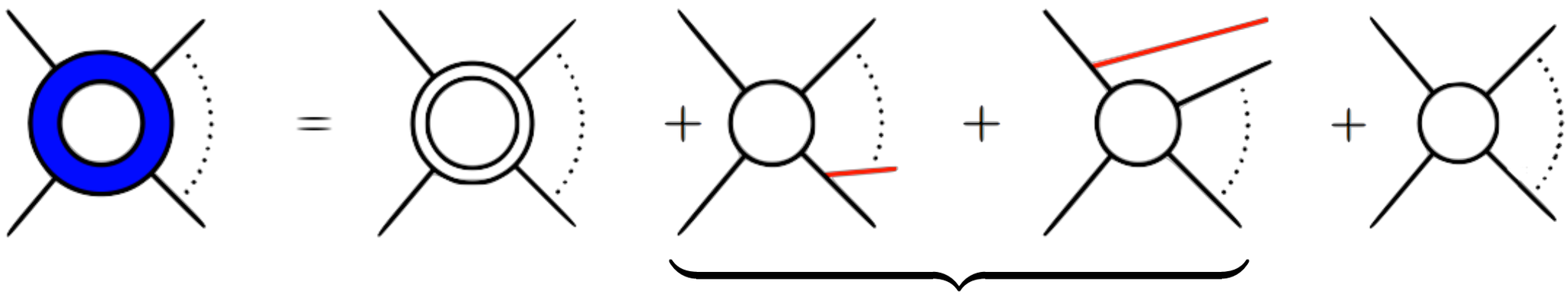
[Alloul, N. Christensen, C. Degrande, C. Duhr, B.Fuks, in 1310.1921]



Plan

- Why and what are **higher order corrections** ?
- Computing **one-loop** Feynman diagrams
- **Renormalisation** and **rational terms**
- **Subtraction** techniques
- **Matching** to Parton showers beyond LO

NLO Anatomy



$$\sigma^{\mathrm{NLO}} = \int_m d^{(d)}\sigma^V + \int_{m+1} d^{(d)}\sigma^R + \int_m d^{(4)}\sigma^B$$

Virtual (V) Real-emission (R) Born (B)

Virutal: computed analytically in dimensional regularisation ($d = 4 - 2\epsilon$):

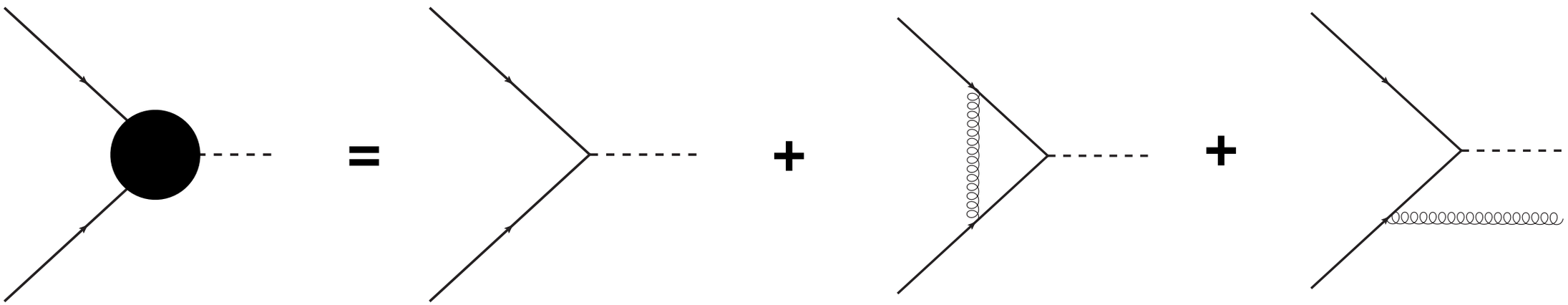
$$\mathrm{Virtual} = \frac{A}{\epsilon^2} + \frac{B}{\epsilon} + V$$

Real: Diverges when unresolved extra emission is integrated over:

$$\int d\phi_1 \ \mathrm{Real} = -\frac{A}{\epsilon^2} - \frac{B}{\epsilon} + R$$

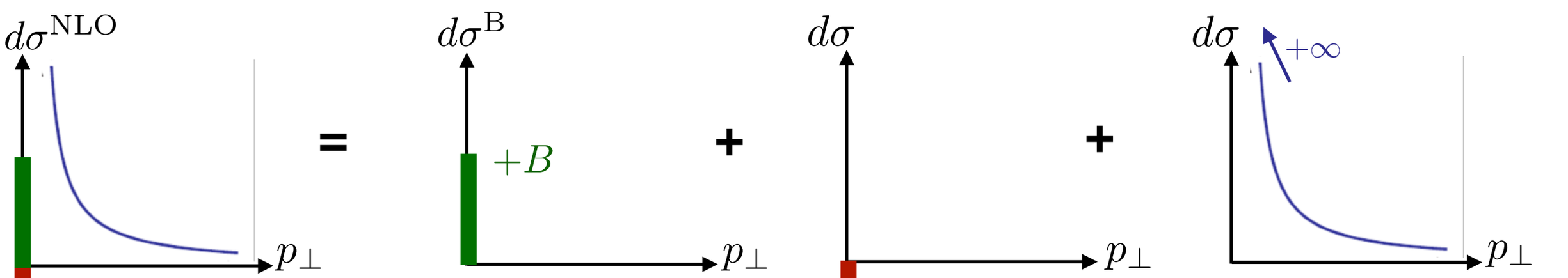
Total: Finite in 4 dimensions, and more accurate: $\sigma^{\mathrm{NLO}} = \underbrace{B}_{\sigma^{\mathrm{LO}}} + \underbrace{R + V}_{\mathrm{NLO\ correction}}$

NLO Anatomy



$$\sigma^{\mathrm{NLO}} = \int_m d^{(4)}\sigma^B + \int_m d^{(d)}\sigma^V + \int_{m+1} d^{(d)}\sigma^R$$

NLO inclusive | Born (B) | Virtual (V) | Real-emission (R)

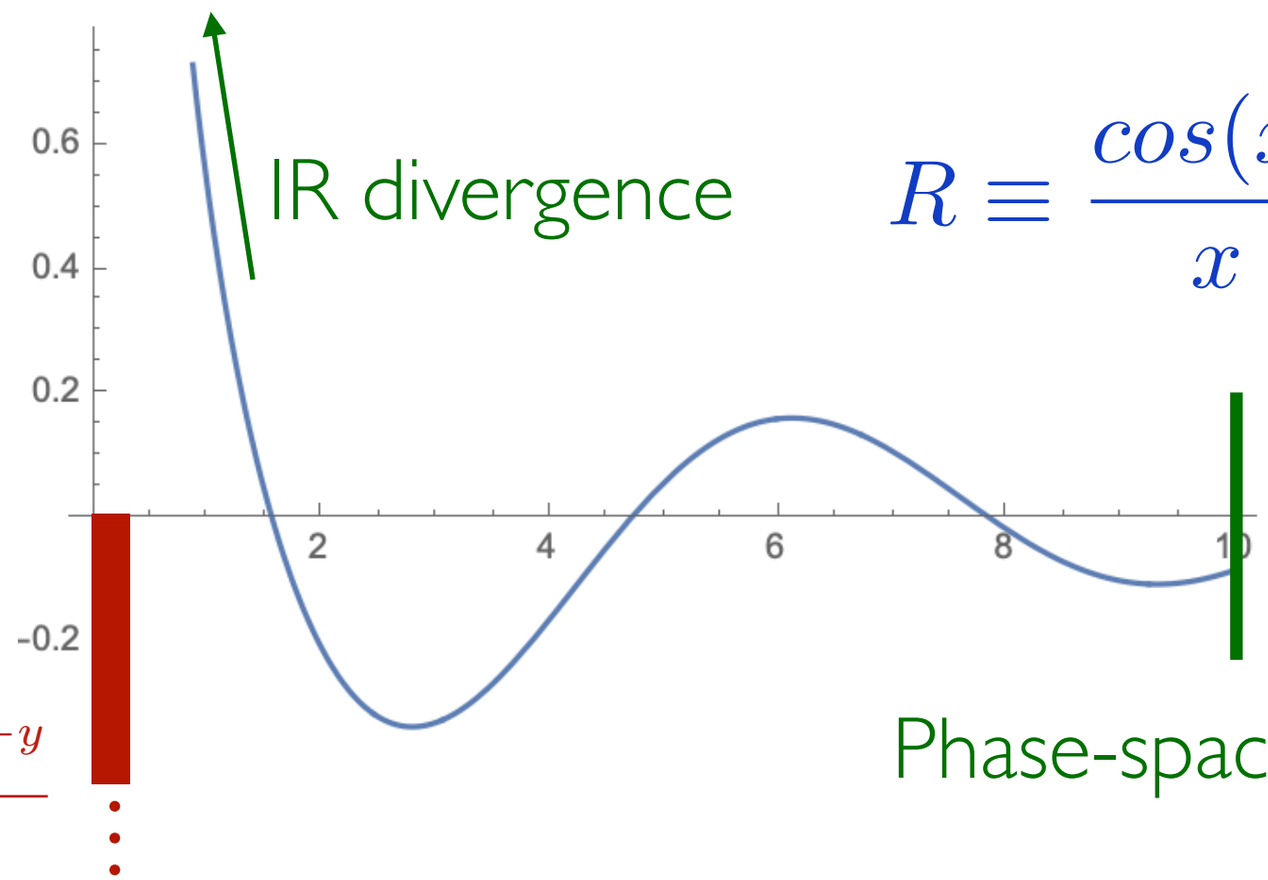


$\underbrace{B}_{\sigma^{\mathrm{LO}}} + \underbrace{R+V}_{\text{NLO correction}}$ | B | $-\frac{A}{\epsilon^2} - \frac{B}{\epsilon^2} + V$ | $\int d^{(d)}\phi_1\, \mathrm{R} = \frac{A}{\epsilon^2} + \frac{B}{\epsilon} + R$

$-\infty$

Toy example

In order simplify the discussion, simplify **V** to some dummy divergent function one a one-dimensional compact volume:



$$R \equiv \frac{cos(x)}{x}$$

$$V \equiv \delta(x) \int_0^1 dy \frac{-e^{-y}}{y}$$

Prediction for "infrared safe" observable $\mathcal{J}(x) \not\propto \delta(x)$:

$$\mathcal{J} = \int dx (R(x) + V\delta(x))\mathcal{J}(x) = V\mathcal{J}(0) + \int dx R(x)\mathcal{J}(x)$$

Toy example

- Toy expression with $\mathcal{J}$ a measurement function, over $x \in [0, 10]$

$$\sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \frac{\cos(x)}{x} \mathcal{J}(x) + \left[\int_0^1 dy \frac{-e^{-y}}{y}\right] \mathcal{J}(0) \boxed{- \left[\int_0^{10} dx \frac{1}{x}\right] \mathcal{J}(0) + \left[\int_0^{10} dx \frac{1}{x} \mathcal{J}(0)\right]} = 0$$

- Distribute the local (in x) **counterterm** over both pieces:

$$\sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x} \mathcal{J}(x) - \frac{1}{x} \mathcal{J}(0)\right] + \left(\left[\int_0^1 dy \frac{-e^{-y}}{y}\right] + \left[\int_0^{10} dx \frac{1}{x}\right]\right) \mathcal{J}(0)$$

- And a **regulator** to evaluate the divergent integrals

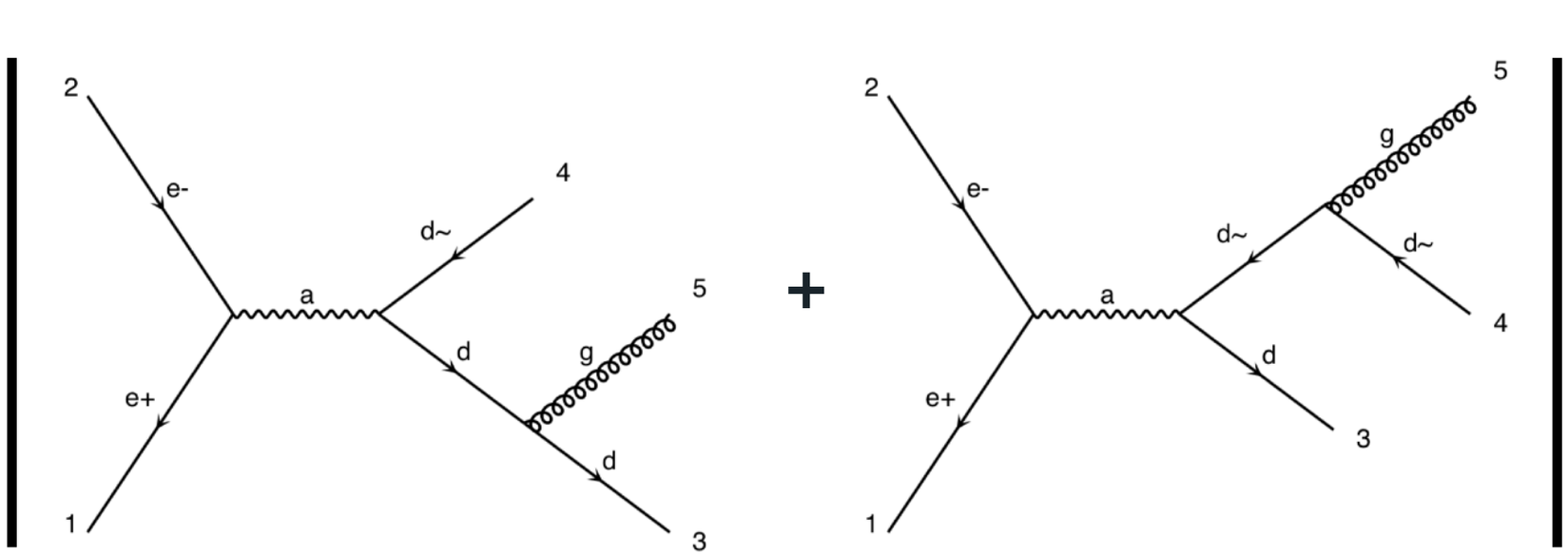
$$\sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x} \mathcal{J}(x) - \frac{1}{x} \mathcal{J}(0)\right] + \lim_{\epsilon \to 0} \left(\left[\int_\epsilon^1 dy \frac{-e^{-y}}{y}\right] + \left[\int_\epsilon^{10} dx \frac{1}{x}\right]\right) \mathcal{J}(0)$$

- To finally arrive at a **finite** result, differential in $x \in [0, 10]$

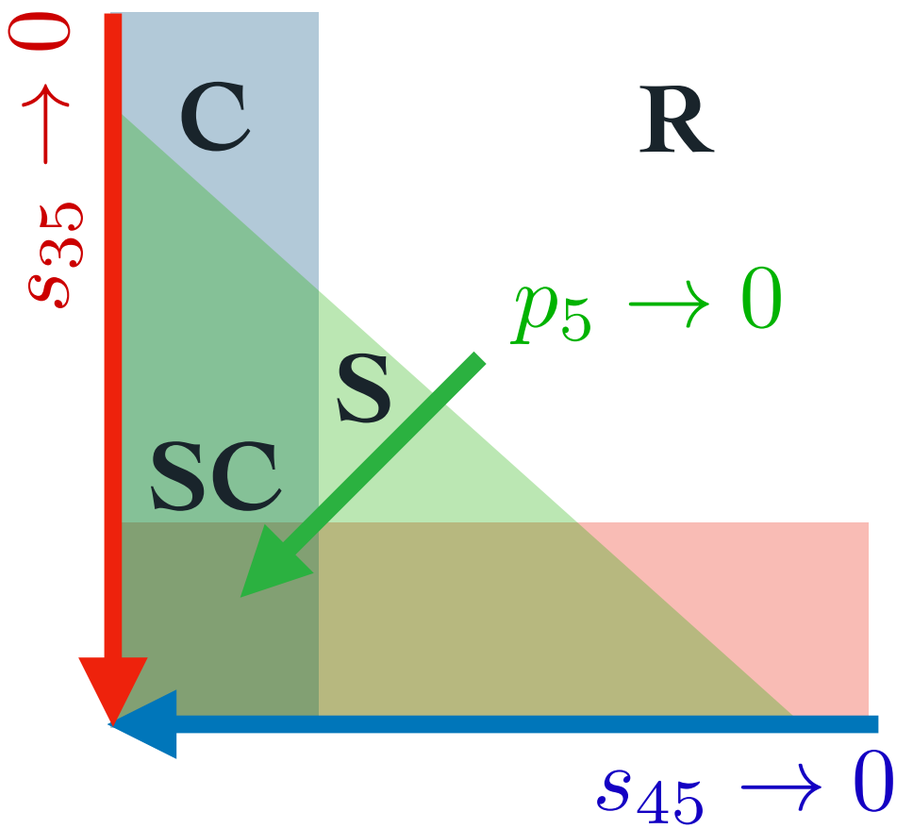
$$= \int_0^{10} dx \left[\frac{\cos(x)}{x} \mathcal{J}(x) - \frac{1}{x} \mathcal{J}(0)\right] + \lim_{\epsilon \to 0} \left(\cancel{\log(\epsilon)} + \gamma - \mathrm{Ei}(-1) + \log(10) - \cancel{\log(\epsilon)}\right) \mathcal{J}(0)$$

A PHYSICS CASE : $e^+e^- \to d\bar{d}g$

R =



$$\sim \frac{s_{12}}{s_{35}s_{45}}$$



R : Resolved region (finite)

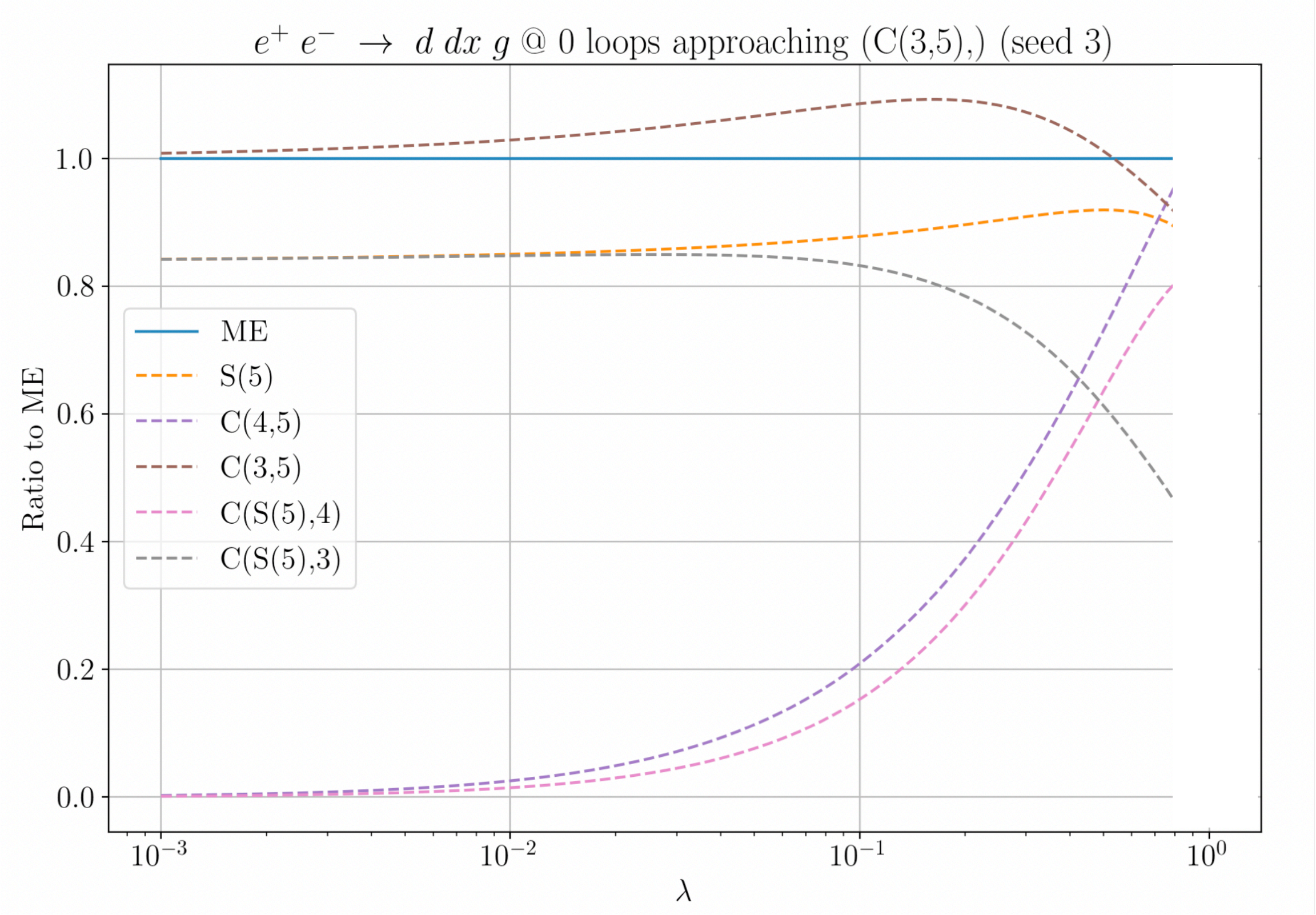
S : Soft gluon region

C : Collinear **4//5** region

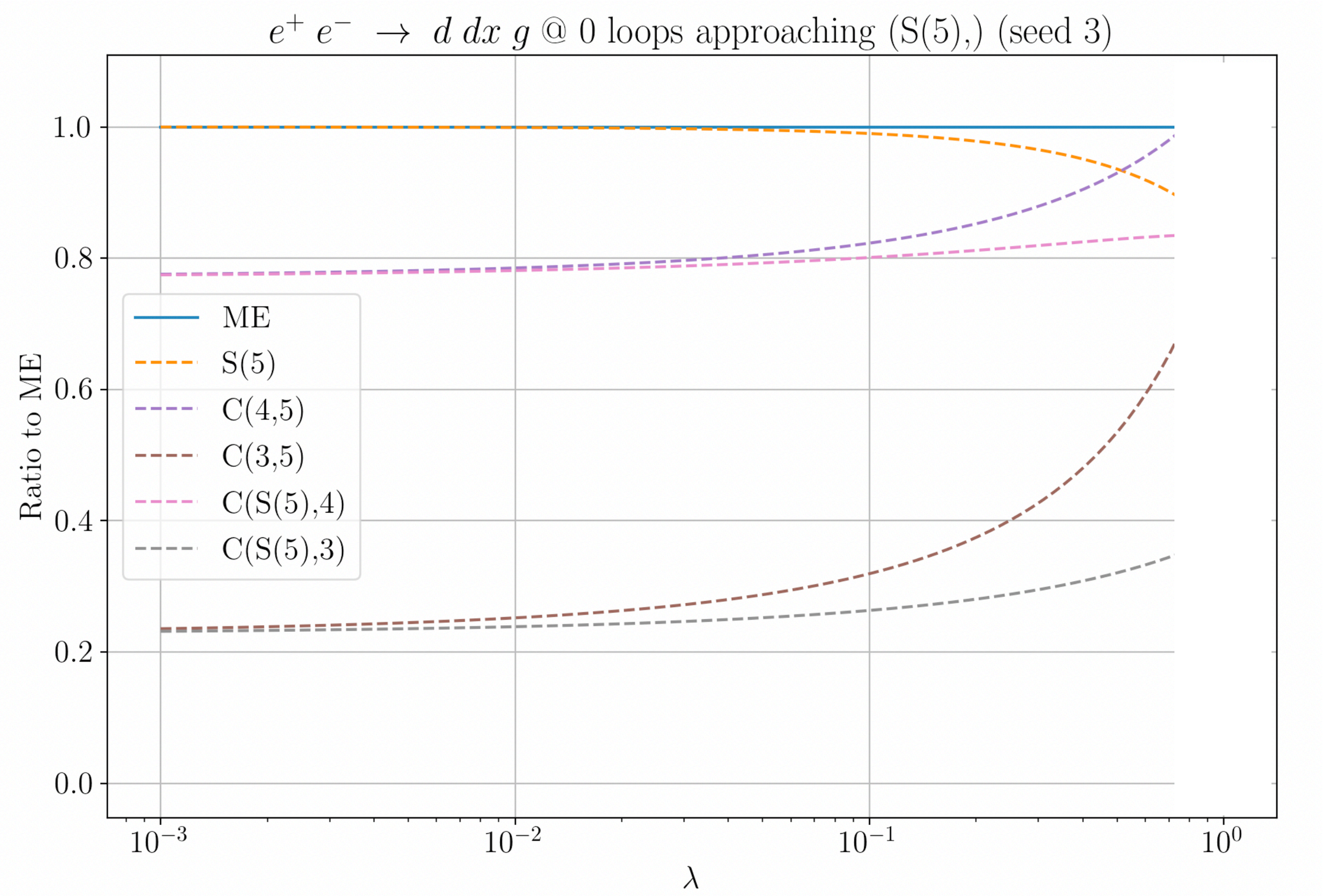
SC : Soft **and** collinear **4//5** region

$$R_{\text{subtracted}} = (1 - \mathcal{C}_{35} - \mathcal{C}_{45} - \mathcal{S}_3 + \mathcal{S}_3\mathcal{C}_{35} + \mathcal{S}_3\mathcal{C}_{45})R$$

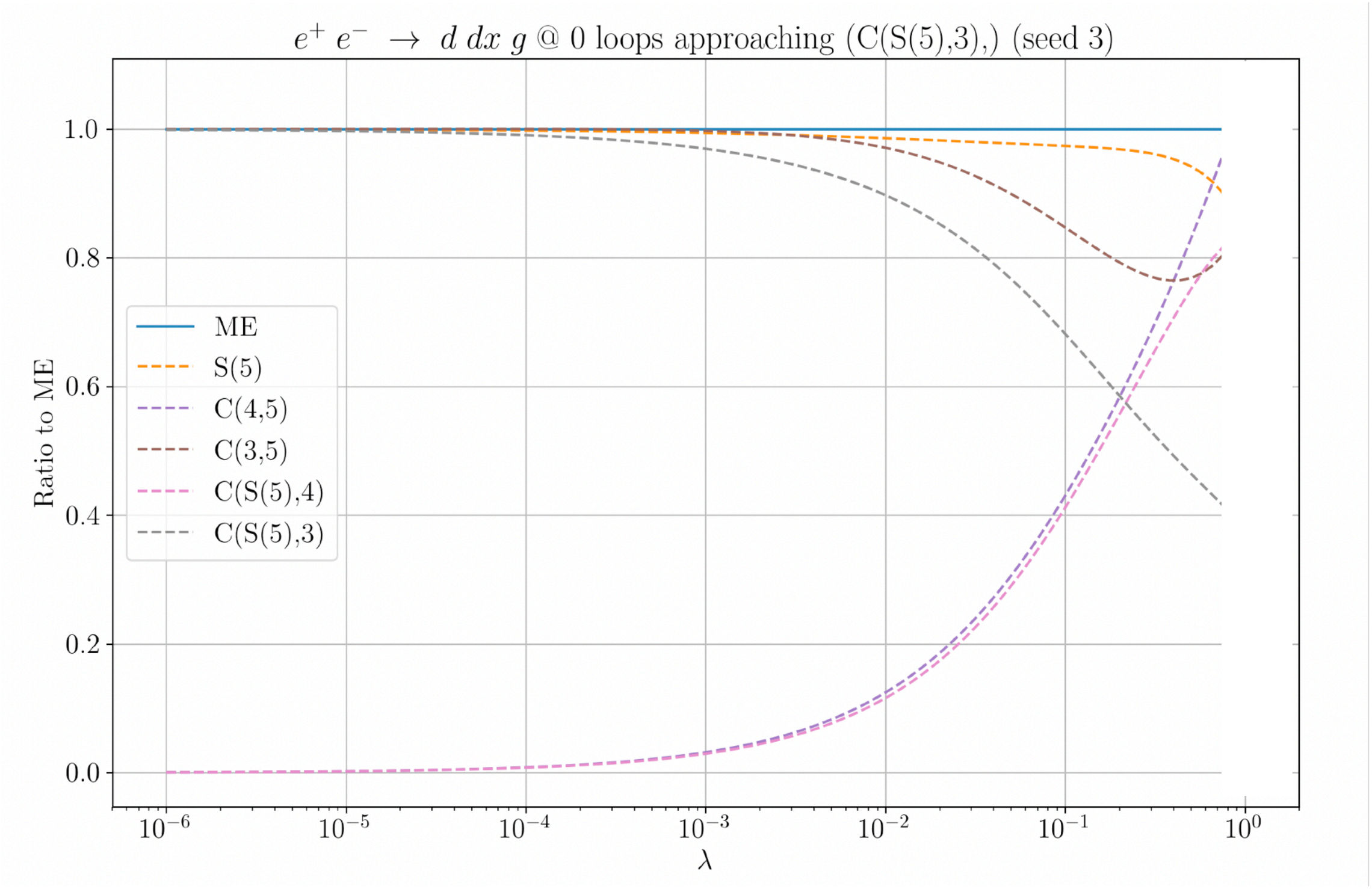
Collinear limit



Soft limit


$e^+ e^- \to d\, dx\, g$ @ 0 loops approaching (S(5),) (seed 3)
Ratio to ME
1.0
0.8
0.6
0.4
0.2
0.0
ME
S(5)
C(4,5)
C(3,5)
C(S(5),4)
C(S(5),3)
10^{-3}
10^{-2}
10^{-1}
10^{0}
λ

Soft-collinear limit



NLO Subtraction

$$\sigma^{\mathrm{NLO}} \sim \int d^4\Phi_m\, B(\Phi_m) + \int d^4\Phi_m \int_{\mathrm{loop}} d^d l\, V(\Phi_m) + \int d^d\Phi_{m+1}\, R(\Phi_{m+1})$$

In order to remain fully differential, one must regularise divergences in **R** using a subtraction method:

$$\begin{aligned}\sigma^{\mathrm{NLO}} \sim & \int d^4\Phi_m\, B(\Phi_m) \\ & + \int d^4\Phi_m \left[\int_{\mathrm{loop}} d^d l\, V(\Phi_m) + \int d^d\Phi_1 G(\overline{\Phi}_{m+1}) \right]_{\epsilon\to 0} \\ & + \int d^4\Phi_{m+1} \left[R(\Phi_{m+1}) - G(\overline{\Phi}_{m+1}) \right]\end{aligned}$$

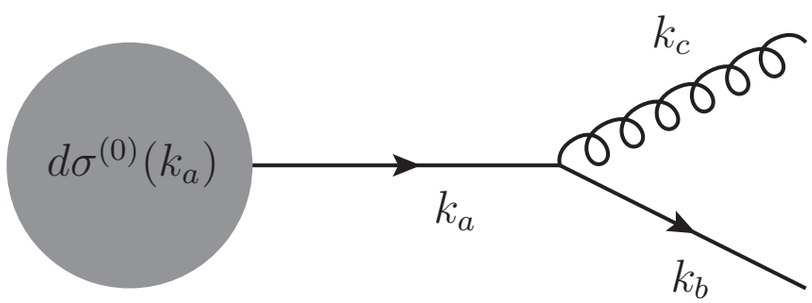
Terms in brackets are now both **finite and fully differential** in the real-emission degrees of freedom.

Subtraction collinear CT

Required characteristics of the counterterms G :

- ➡ Reproduce singularities of R, allowing numerical integration in 4D
- ➡ Analytically integrable , $\int d^d\Phi_1 G(\overline{\Phi}_{m+1})$ must be "simple enough"
- ➡ Universal, that is: process-independent

Factorised universality of collinear (and soft) radiation:



$$k_b = zk_a + k_T + \beta_b \hat{n}$$

$$k_c = (1-z)k_a - k_T + \beta_c \hat{n}$$

$$d\sigma^{(1,R)} = \frac{\alpha_s}{2\pi} \int dk_T^2 \int_0^1 dz C_F \frac{1+z^2}{1-z} \frac{1}{k_T^2} d\sigma^{(0)}(k_a) + \mathcal{R}$$

Allows to schematically write :

$$G(\phi_{m+1}) \sim \underbrace{B(\overline{\phi}_m)}_{\text{process dep.}} \otimes \underbrace{P(z, k_T)}_{\text{universal}}$$

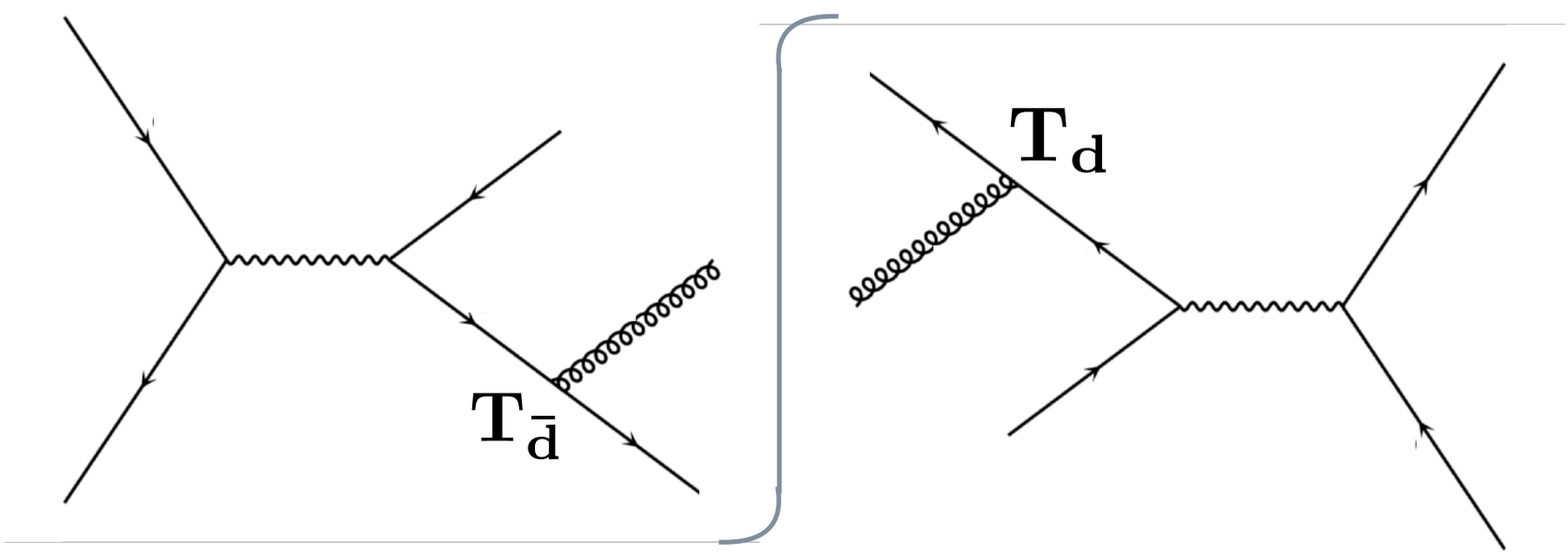
Subtraction soft CT

Similarly for the soft limit, know as the Eikonal approximation:

$$\mathcal{S}_3|\mathcal{M}(p_d,p_{\bar{d}},p_g)|^2 \sim$$

$$\frac{s_{d\bar{d}}}{s_{dg}s_{\bar{d}g}}\langle\mathcal{M}(p_d,p_{\bar{d}})\underbrace{|_{i_d}\quad t^a_{i_dk}t^a_{ki_{\bar{d}}}\quad {}_{i_{\bar{d}}}|}_{\mathbf{T_d}\cdot\mathbf{T_{\bar{d}}}}\mathcal{M}(p_d,p_{\bar{d}})\rangle$$

The origin of the colour correlation is the interference nature of the soft limit:

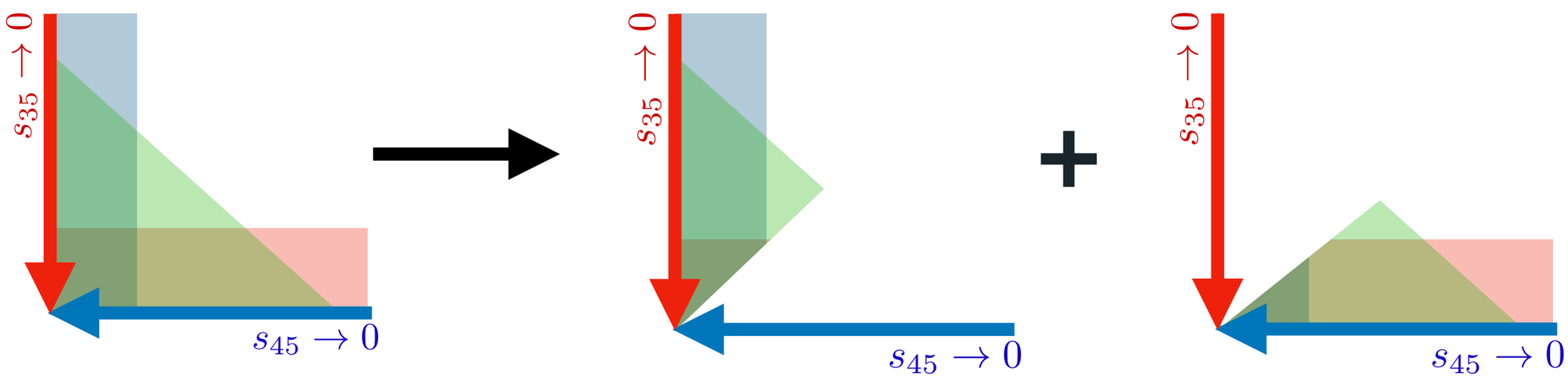


FKS implementation

Divide and conquer, partition the phase-space into sectors:

$$d\sigma_{d\bar{d}g} = \underbrace{\left(S_{gd} + S_{g\bar{d}}\right)}_{=1} d\sigma_{d\bar{d}g} = \underbrace{S_{gd}\, d\sigma_{d\bar{d}g}}_{:=d\sigma^{(gd)}_{d\bar{d}g}} + \underbrace{S_{g\bar{d}}\, d\sigma_{d\bar{d}g}}_{:=d\sigma^{(g\bar{d})}_{d\bar{d}g}}$$

Design the partition functions to isolate collinear singularities



Possible choice here: $S_{gx}(p_d, p_{\bar{d}}, p_g) = \dfrac{s_{g\bar{x}}}{s_{gd} + s_{g\bar{d}}} \quad x \in \{d, \bar{d}\}$

FKS : Parametrisation

Choose a wise parametrisation for each sector :

$$d\sigma^{(gd)}_{d\bar{d}g} = (S_{gd}\mathcal{M}_{d\bar{d}g})\mathbf{d\Phi_{d\bar{d}g}} = (S_{gd}\mathcal{M}_{d\bar{d}g})\; E_g dE_g d\cos(\theta_{gd}) d\phi_g \mathbf{d\tilde{\Phi}^{(gd)}_{d\bar{d}}}$$

Now that singularities are factorised, introduce twice the identity:

$$1 \equiv \frac{1-\delta(x)}{x} + \frac{\delta(x)}{x} = \left(\frac{1}{x}\right)_+ + \frac{\delta(x)}{x} \qquad \left(\text{ i.e : } \int dx \left(\frac{1}{x}\right)_+ f(x) := \int dx \frac{f(x)-f(0)}{x} \right)$$

Thereby formally obtaining a subtraction scheme ($y_{gq} := 1 - \cos(\theta_{gq})$)

$$d\sigma^{(gd)}_{d\bar{d}g} = \left[\left(\frac{1}{E_g}\right)_+ + \frac{\delta(E_g)}{E_g}\right]\left[\left(\frac{1}{y_{gd}}\right)_+ + \frac{\delta(y_{gd})}{y_{gd}}\right] \times$$

$$(E_g^2\; y_{gd} S_{gd}\mathcal{M}_{d\bar{d}g})\; dE_g dy_{gd} d\phi_g \mathbf{d\tilde{\Phi}^{(gd)}_{d\bar{d}}}$$

Local CT $(d=4)$

Integrated CT $(d=4-2\epsilon)$

FKS : "Residue CT"

Last step is to expand the deltas and invoke QCD factorisation:

Collinear :

$$\delta(y_{gd}) S_{gd} \mathcal{M}_{d\bar{d}g} \stackrel{!}{=} \delta(y_{gd}) C_{gd}(z_{gd}) \mathcal{M}_{d\bar{d}}$$

Soft :

$$\delta(E_g)(S_{gd} + S_{g\bar{d}}) \mathcal{M}_{d\bar{d}g}) \stackrel{!}{=} \delta(E_g) \mathbf{S_g} \otimes \mathcal{M}_{\mathbf{d\bar{d}}}$$

Soft-Collinear :

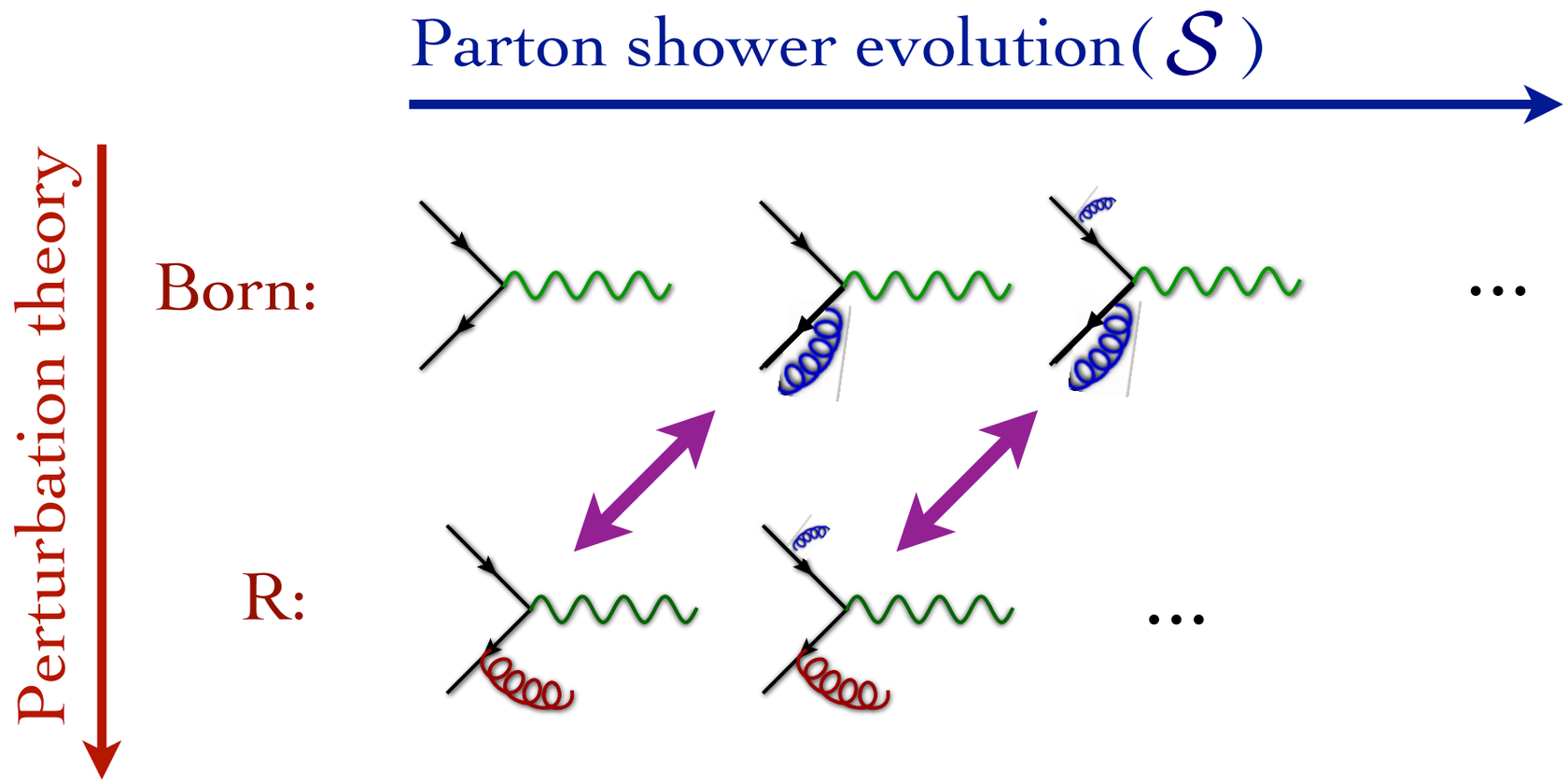
$$\delta(E_g)\delta(y_{gd}) S_{gd} \mathcal{M}_{d\bar{d}g} \stackrel{!}{=} \delta(E_g)\delta(y_{gd}) S C_{gd}(z_{gd}) \mathcal{M}_{d\bar{d}}$$

Plan

- Why and what are **higher order corrections** ?
- Computing **one-loop** Feynman diagrams
- **Renormalisation** and **rational terms**
- **Subtraction** techniques
- **Matching** to Parton showers beyond LO

Double-counting : aMC@NLO

Double counting between real-emission contributions R and $\mathcal{S}_{\{P\}\to\{H\}}$



Similar more subtle double-counting also between **Virtual** and $\mathcal{S}$

This issue can again be solved by constructing additional MC counterterms

MC counterterms

One can loosely write $\mathcal{S}$ as follows:

$$\mathcal{S} \sim B - \left[\int d\phi_1 MC\right] + d\phi_1 MC + \mathcal{O}(\alpha_s^2)$$

MC is constructed as the $\mathcal{O}(\alpha_s)$ term of $\mathcal{S}$ and can be subtracted:

$$\begin{aligned} d\sigma^{\text{NLOwPS}} \sim & \left[\int d^4\Phi_m \left(B + \int d^d l\, V + \int d^d\Phi_1 G + \int d^4\Phi_1 (MC - G)\right)\right] \mathcal{S}^{(m)} \\ + & \left[\int d\Phi_{m+1} (R - MC)\right] \mathcal{S}^{(m+1)} \end{aligned}$$

At NLO the **MC** counterterms are universal and hand-crafted analytically for each implementation of

The term **R-MC** is now bounded from above, so that one can produce unweighted events (though possibly negative)

MC counterterms

$$d\sigma^{\mathrm{NLOwPS}} \sim \left[\int d^4\Phi_m \left(B + \int d^d l\, V + \int d^d\Phi_1 G + \int d^4\Phi_1 (MC - G)\right)\right] \mathcal{S}^{(m)}$$
$$+ \left[\int d\Phi_{m+1} (R - MC)\right] \mathcal{S}^{(m+1)}$$

In the **soft/collinear** limit, $R - MC \simeq 0$ and the shower dictates the shape of the spectrum emission:

$$d\sigma^{\mathrm{NLOwPS}}_{\text{soft or coll.}} \sim \left[\int d^4\Phi_m \left(B + \int d^d l\, V + \int d^d\Phi_1 G + \cancel{\int d^4\Phi_1 (MC - G)}\right)\right]$$
$$\times \left(1 - \left[\int d\phi_1 \cancel{\frac{MC}{B}}\right] + \boxed{d\phi_1 \frac{MC}{B}} + \mathcal{O}(\alpha_s^2)\right)$$

Note that **fixed-order NLO** normalisation is maintained thanks to the unitarity of the shower operator (unlike in POWHEG)

In the **hard** limit, $MC \simeq 0,\ \mathcal{S}^{(m)} \simeq 1,\ (B+V)\mathcal{J}^{(m)} = 0$ and the real-emission ME dictates the shape:

$$d\sigma^{\mathrm{NLOwPS}}_{hard} \sim \left[\int d\Phi_{m+1} (R)\right] \mathcal{S}^{(m+1)}$$

MC@NLO

Main features of this matching scheme:

- ➡ Specific to the Parton Shower MC and its configuration
- ➡ Yields events with negative weights
- ➡ Does not exponentiate matrix element corrections
- ➡ Maintains the fixed-order NLO inclusive normalisation
- ➡ Matching uncertainty introduced via shower starting scale definition (equiv. to ***h_fact*** in POWHEG)

TL;DL

Take-home messages:

- ➡ One-loop ME can be computed fully automatically to build the virtual
- ➡ NLO computations are automated but demand a tailored UFO model
- ➡ Real-emission contributions are IR divergent and require subtraction
- ➡ Matching to PSMC with MC@NLO is shower specific but does not exponentiate the real-emission matrix element.