

Physics & Simulation

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e^+e^- Colliders

Shao-Feng Ge

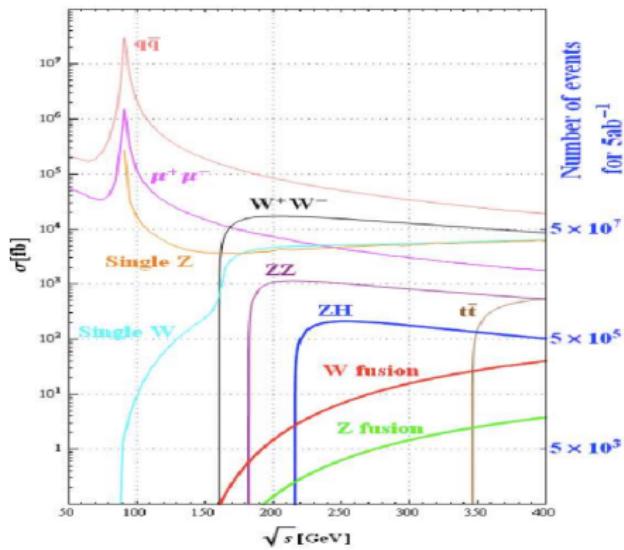
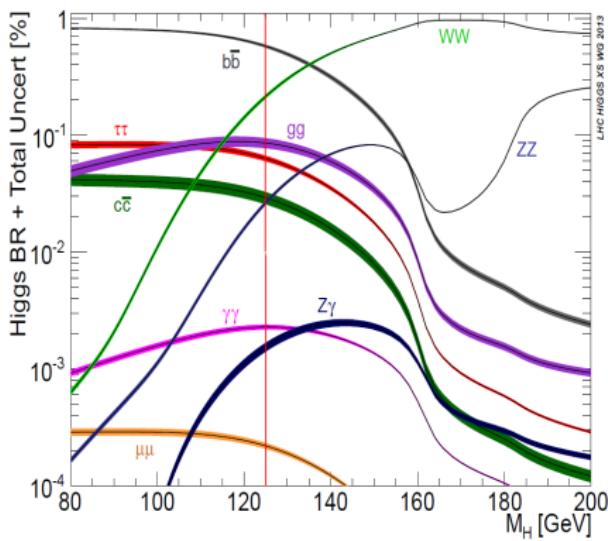
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Higgs Factory @ 250 GeV

- LHC tells us: $h(125)$ is **SM-like** → Dream Case for Experiments!
- ILC250 & CEPC produces $h(125)$ via $e^+e^- \rightarrow Zh, \nu\bar{\nu}h, e^+e^-h$
- Indirect Probe to New Physics. 5/ ab with 2 detectors in 10y → 10^6 Higgs → Relative Error $\sim 10^{-3}$.



Mo, Li, Ruan & Lou, Chin.Phys.C 2015

New Physics Scales

Ge, He, Xiao, 1603.03385

- New physics appears @ high energy scale & can only be probed **Indirectly**

$$\mathcal{L} = \mathcal{L}_{\text{SM}} + \sum_{ij} \frac{\mathbf{y}_{ij} \sim \mathcal{O}(1)}{\Lambda \sim 10^{14} \text{GeV}} (\bar{L}_i \tilde{\mathbf{H}})(\tilde{\mathbf{H}}^\dagger L_j) + \sum_i \frac{\mathbf{c}_i}{\Lambda^2} \mathcal{O}_i .$$

- **SM Gauge Invariance** is respected

Higgs	EW Gauge Bosons	Fermions
$\mathcal{O}_H = \frac{1}{2}(\partial_\mu \mathbf{H} ^2)^2$	$\mathcal{O}_{WW} = g^2 \mathbf{H} ^2 W_{\mu\nu}^a W^{a\mu\nu}$	$\mathcal{O}_{LL}^{(3)} = (\bar{\Psi}_L \gamma_\mu \sigma^a \Psi_L)(\bar{\Psi}_L \gamma^\mu \sigma^a \Psi_L)$
$\mathcal{O}_T = \frac{1}{2}(\mathbf{H}^\dagger \overset{\leftrightarrow}{D}_\mu \mathbf{H})^2$	$\mathcal{O}_{BB} = g^2 \mathbf{H} ^2 B_{\mu\nu} B^{\mu\nu}$	$\mathcal{O}_L^{(3)} = (i \mathbf{H}^\dagger \sigma^a \overset{\leftrightarrow}{D}_\mu \mathbf{H})(\bar{\Psi}_L \gamma^\mu \sigma^a \Psi_L)$
	$\mathcal{O}_{WB} = gg' \mathbf{H}^\dagger \sigma^a \mathbf{H} W_{\mu\nu}^a B^{\mu\nu}$	$\mathcal{O}_L = (i \mathbf{H}^\dagger \overset{\leftrightarrow}{D}_\mu \mathbf{H})(\bar{\Psi}_L \gamma^\mu \Psi_L)$
Gluon	$\mathcal{O}_{HW} = ig(D^\mu \mathbf{H})^\dagger \sigma^a (D^\nu \mathbf{H}) W_{\mu\nu}^a$	$\mathcal{O}_R = (i \mathbf{H}^\dagger \overset{\leftrightarrow}{D}_\mu \mathbf{H})(\bar{\psi}_R \gamma^\mu \psi_R)$
$\mathcal{O}_g = g_s^2 \mathbf{H} ^2 G_{\mu\nu}^a G^{a\mu\nu}$	$\mathcal{O}_{HB} = ig' (D^\mu \mathbf{H})^\dagger (D^\nu \mathbf{H}) B_{\mu\nu}$	$\mathcal{O}_f = \mathbf{H} ^2 \overline{F}_L H f$

Existing EWPO & Future HO

- Observables: **EWPO** (PDG14) + **HO** (preCDR)

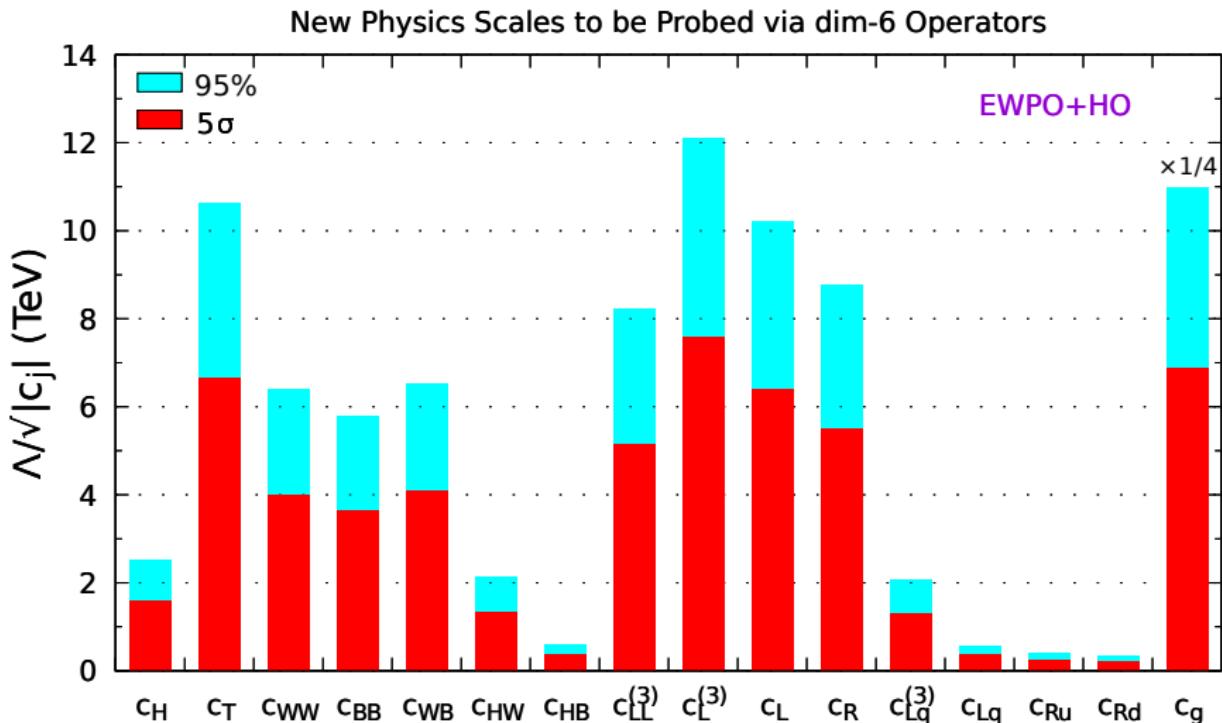
Observables	Central Value	Relative Error	SM Prediction
α	$7.2973525698 \times 10^{-3}$	3.29×10^{-10}	—
G_F	$1.1663787 \times 10^{-5} \text{GeV}^{-2}$	5.14×10^{-7}	—
M_Z	91.1876 GeV	2.3×10^{-5}	—
M_W	80.385 GeV	1.87×10^{-4}	—
$\sigma[Zh]$	—	0.50%	—
$\sigma[\nu\bar{\nu}h]$	—	2.86%	—
$\sigma[\nu\bar{\nu}h]_{350 \text{GeV}}$	—	0.75%	—
$\text{Br}[WW]$	—	1.2%	22.5%
$\text{Br}[ZZ]$	—	4.3%	2.77%
$\text{Br}[bb]$	—	0.54%	58.1%
$\text{Br}[cc]$	—	2.5%	2.10%
$\text{Br}[gg]$	—	1.4%	7.40%
$\text{Br}[\tau\tau]$	—	1.1%	6.64%
$\text{Br}[\gamma\gamma]$	—	9.0%	0.243%
$\text{Br}[\mu\mu]$	—	17%	0.023%

- Exclusion (95%) & Discovery (5 σ) Reach

Ge, He, Xiao, [1603.03385](#)

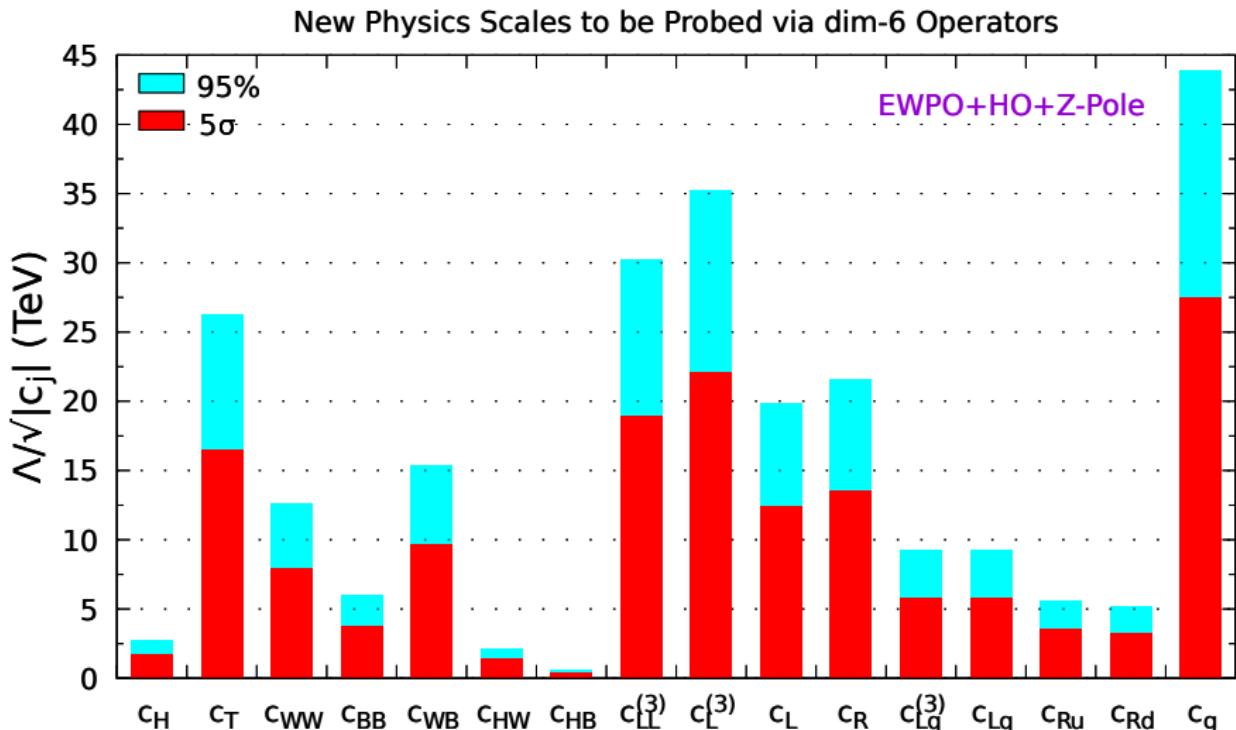
	\mathcal{O}_H	\mathcal{O}_T	\mathcal{O}_{WW}	\mathcal{O}_{BB}	\mathcal{O}_{WB}	\mathcal{O}_{HW}	\mathcal{O}_{HB}	$\mathcal{O}_{LL}^{(3)}$	$\mathcal{O}_L^{(3)}$	\mathcal{O}_L	\mathcal{O}_R	$\mathcal{O}_{L,q}^{(3)}$	$\mathcal{O}_{L,q}$	$\mathcal{O}_{R,u}$	$\mathcal{O}_{R,d}$	\mathcal{O}_g
95%	2.50	10.6	6.38	5.78	6.53	2.12	0.604	8.23	12.1	10.2	8.78	2.06	0.568	0.393	0.339	43.8
5 σ	1.57	6.65	4.00	3.62	4.09	1.33	0.378	5.15	7.57	6.39	5.49	1.29	0.356	0.246	0.212	27.4

Sensitivities from Existing EWPO & Future HO



Ge, He, Xiao, 1603.03385

Sensitivity from EWPO+HO+Z-Pole



SFG, He, Xiao, 1603.03385

Enhancement from M_Z & M_W @ CEPC

Observables	Relative Error	
	Current	CEPC
M_Z	2.3×10^{-5}	$5.5 \times 10^{-6} \sim 1.1 \times 10^{-5}$
M_W	1.9×10^{-4}	$3.7 \times 10^{-5} \sim 6.2 \times 10^{-5}$

Table: The M_Z & M_W @ CEPC [Z.Liang, "Z & W Physics @ CEPC" & preCDR].

Scheme-Independent Analysis

$\frac{\Lambda}{\sqrt{c_j}} [\text{TeV}]$	\mathcal{O}_H	\mathcal{O}_T	\mathcal{O}_{WW}	\mathcal{O}_{BB}	\mathcal{O}_{WB}	\mathcal{O}_{HW}	\mathcal{O}_{HB}	$\mathcal{O}_{LL}^{(3)}$	$\mathcal{O}_L^{(3)}$	\mathcal{O}_L	\mathcal{O}_R	$\mathcal{O}_{L,q}^{(3)}$	$\mathcal{O}_{L,q}$	$\mathcal{O}_{R,u}$	$\mathcal{O}_{R,d}$	\mathcal{O}_g
HO+EWPO	2.74	10.6	6.38	5.78	6.53	2.16	0.604	8.58	12.1	10.2	8.78	2.06	0.568	0.393	0.339	43.8
+M _Z	2.74	10.7	6.38	5.78	6.54	2.16	0.604	8.62	12.1	10.2	8.78	2.06	0.568	0.393	0.339	43.8
+M _W	2.74	21.0	6.38	5.78	10.4	2.16	0.604	15.5	16.4	10.2	8.78	2.06	0.568	0.393	0.339	43.8
+M _{Z,W}	2.74	23.7	6.38	5.78	11.6	2.16	0.604	17.4	18.1	10.2	8.78	2.06	0.568	0.393	0.339	43.8

Table: Impacts of the projected M_Z and M_W measurements at CEPC on the reach of new physics scale $\Lambda/\sqrt{|c_j|}$ (in TeV) at 95% C.L. The Higgs observables (including $\sigma(\nu\bar{\nu}h)$ at 350 GeV) and the existing electroweak precision observables are always included in each row. The differences among the four rows arise from whether taking into account the measurements of M_Z and M_W or not. The second (third) row contains the measurement of M_Z (M_W) alone, while the first (last) row contains none (both) of them. We mark the entries of the most significant improvements from M_Z/M_W measurements in red color.

Ge, He, Xiao, 1603.03385

Enhancement from Z-Pole Observables @ CEPC

N_ν	$A_{FB}(b)$	R^b	R^μ	R^τ	$\sin^2 \theta_w$
1.8×10^{-3}	1.5×10^{-3}	8×10^{-4}	5×10^{-4}	5×10^{-4}	1×10^{-4}

Table: The Z-pole measurements at CEPC [Z.Liang, "Z & W Physics @ CEPC" & preCDR].

Ge, He, Xiao, [1603.03385](#)

Z-Pole Observables are **IMPORTANT** for New Physics Scale Probe

\mathcal{O}_H	\mathcal{O}_T	\mathcal{O}_{WW}	\mathcal{O}_{BB}	\mathcal{O}_{WB}	\mathcal{O}_{HW}	\mathcal{O}_{HB}	$\mathcal{O}_{LL}^{(3)}$	$\mathcal{O}_L^{(3)}$	\mathcal{O}_L	\mathcal{O}_R	$\mathcal{O}_{L,q}^{(3)}$	$\mathcal{O}_{L,q}$	$\mathcal{O}_{R,u}$	$\mathcal{O}_{R,d}$	\mathcal{O}_g
2.74	23.7	6.38	5.78	11.6	2.16	0.604	17.4	18.1	10.2	8.78	2.06	0.568	0.393	0.339	43.8
2.74	23.7	6.38	5.78	11.6	2.16	0.604	17.5	18.3	10.5	8.78	2.06	0.568	0.393	0.339	43.8
2.74	24.0	8.32	5.80	12.2	2.16	0.604	20.7	23.0	12.5	13.0	2.23	1.62	0.393	3.97	43.8
2.74	24.0	8.33	5.80	12.2	2.16	0.604	20.7	23.0	12.5	13.0	7.90	7.89	3.55	4.05	43.8
2.74	24.0	8.54	5.80	12.2	2.16	0.604	20.7	23.4	14.4	14.0	8.63	8.62	4.88	4.71	43.8
2.74	24.0	8.75	5.81	12.3	2.16	0.604	20.7	23.7	15.8	14.9	9.21	9.21	5.59	5.17	43.8
2.74	26.3	12.6	5.93	15.3	2.16	0.604	30.2	35.2	19.8	21.6	9.21	9.21	5.59	5.17	43.8

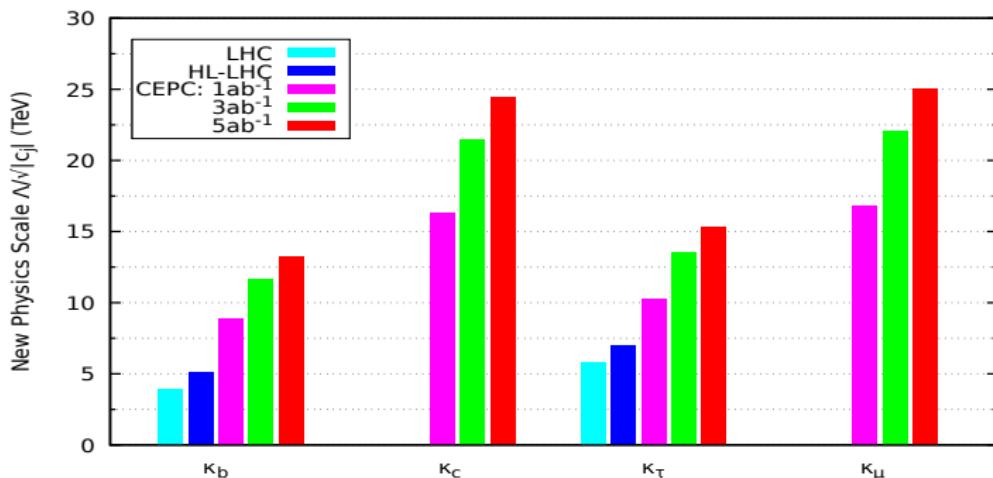
Table: Impacts of the projected Z-pole measurements at the CEPC on the reach of new physics scale $\Lambda/\sqrt{|c_j|}$ (in TeV)

at 95% C.L. For comparison, the first row of this table repeats the last row of Table ??, as our starting point of this table. For the $(n+1)$ -th row, the first n observables are taken into account. In addition, the estimated M_Z and M_W measurements at the CEPC, the Higgs observables (HO), and the existing electroweak precision observables (EWPO) are always included for each row. The entries with major enhancements of the new physics scale limit are marked in red color.

Another factor of 2 enhancement from Z-Pole Observables

Yukawa-like Operators: $\mathcal{O}_f \equiv |H|^2 \overline{F}_L H f_R$

$\Lambda/\sqrt{ c_j }$ (TeV)	σ	CEPC	LHC	HL-LHC	ILC-250	ILC-500
b quark	1.27%	13.2	3.87	5.12	6.89	15.2
τ lepton	1.33%	15.4	5.74	6.95	12.8	20.0
c quark	1.75%	24.4	—	—	7.76	12.5
μ lepton	8.59%	25.1	—	—	—	—



SFG, He, Xiao, 1603.03385

Yukawa-like Operators

- Dim-6 Yukawa-like Operators

$$\mathcal{O}_f \equiv |H|^2 \bar{F}_L H f_R$$

- Shifting Yukawa Couplings

$$y_f \rightarrow y_f + \frac{3c_f v^2}{2\Lambda^2} = \frac{\sqrt{2}m_f}{v} + \frac{c_f v^3}{\sqrt{2}m_f \Lambda^2}$$

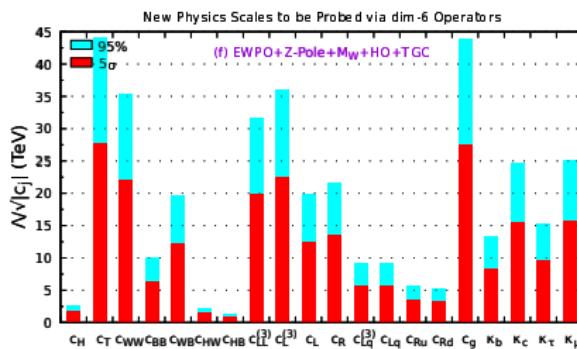
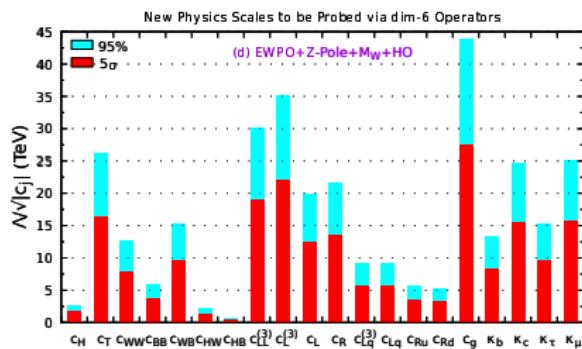
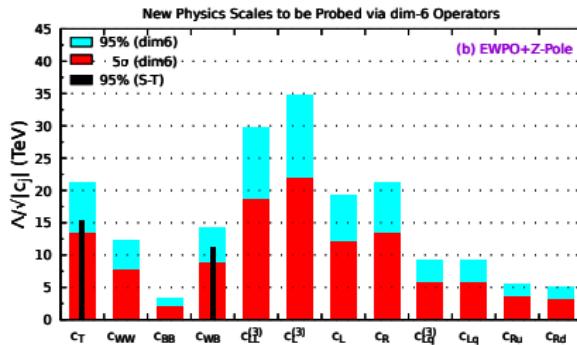
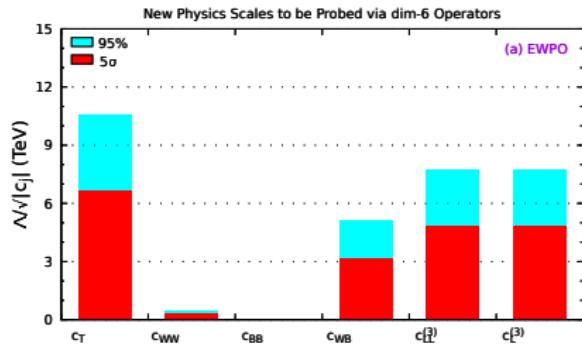
- Constraining New Physics Scales

$$\frac{\Lambda}{\sqrt{c_f}} \leq \sqrt{\frac{v^3}{\sqrt{2}m_f \Delta \kappa_f}}.$$

Naive Expectations

$$N_f \propto y_f^2 \quad \Rightarrow \quad \Delta \kappa_f \propto y_f^{-1} \quad \Rightarrow \quad \Lambda \propto y_f^0$$

TGC Constraints



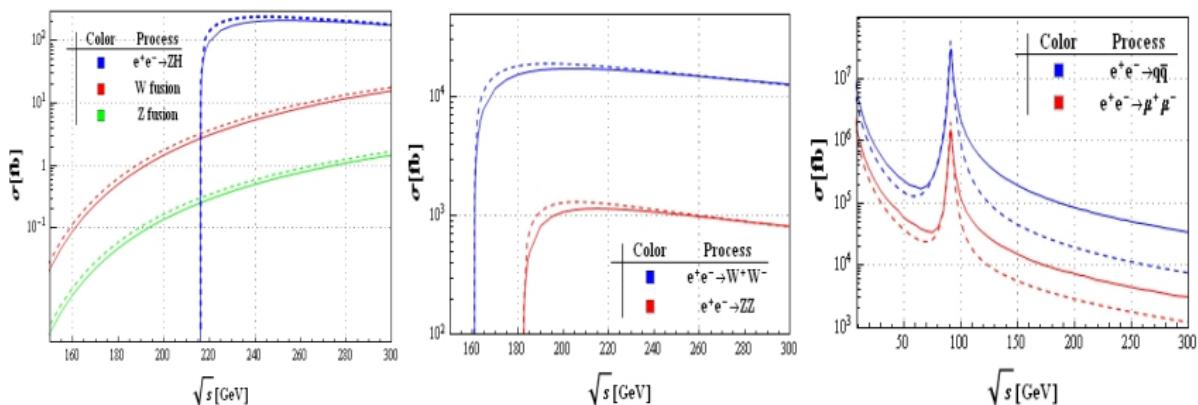
$$\frac{\delta\sigma_{WW}}{\sigma_{WW}} = 1.94 \frac{\delta G_F}{G_F} + 20.8 \frac{\delta M_Z}{M_Z} + 0.246 \frac{\delta\alpha}{\alpha} + 0.0956 \frac{C_T}{\Lambda_{\text{TeV}}^2} - 0.0214 \frac{C_{WB}}{\Lambda_{\text{TeV}}^2} + 0.000922 \frac{C_{HW}}{\Lambda_{\text{TeV}}^2} + 0.000611 \frac{C_{HB}}{\Lambda_{\text{TeV}}^2}$$

Key Feature 1 of Lepton Colliders: Precision

Many events due to large statistics

- **Higgsstrahlung:** $e^+e^- \rightarrow Zh \sim 10^6$ events (**0.1%**)
- **TGC:** $e^+e^- \rightarrow W^+W^- \sim 10^8$ events (**0.01%**)
- **Z-Pole:** $e^+e^- \rightarrow Z \rightarrow f\bar{f} \sim 10^{11}$ events (**0.001%**)

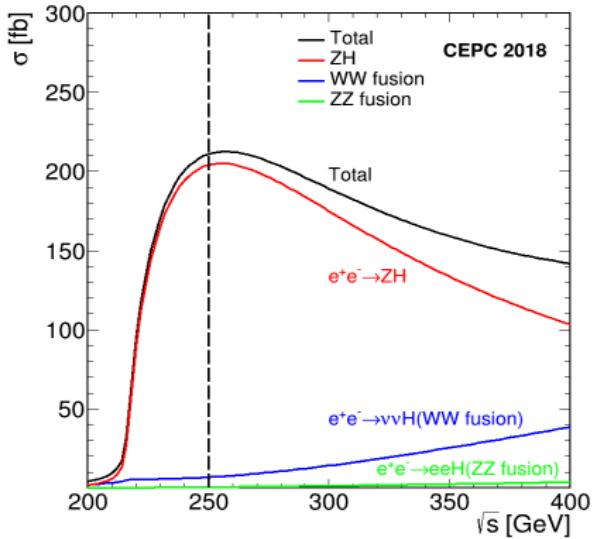
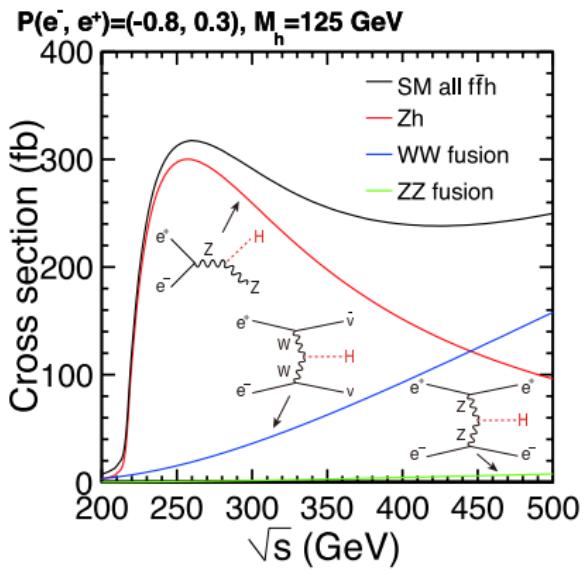
Needs to do simulation precisely!



1505.01008, Mo et al

Key Feature 2 of Lepton Colliders: Polarization

CEPC Higgs White Paper [1810.09037]



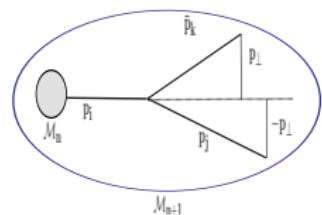
- **Polarization affects almost all processes @ lepton colliders!**
- Even without polarized beams, polarization effect appear in angular distribution!

Parton Shower vs Particle Decay

- For $i \rightarrow jk$ splitting

$$\frac{dN_i}{N_i} = -\frac{d\sigma_{n+1}}{d\sigma_n} = -\frac{dt}{t} P_{ij}(z, \phi) dz \frac{d\phi}{2\pi}$$

is the **branching probability**.



- Particle decay

$$\frac{dN}{N} = -\Gamma dt, \quad \frac{dN}{N} = -dt \frac{d^2\Gamma}{dE d\phi} dE \frac{d\phi}{2\pi}$$

- Parton Shower vs Particle Decay

$$\frac{dt}{t} \leftrightarrow dt$$

$$P_{ij}(z, \phi) \leftrightarrow \frac{d^2\Gamma}{dE d\phi}$$
$$dz \leftrightarrow dE$$

Markov Chain MC of Decays

- Multiple decays + Each decay has **multiple variables** (t_i, E_i, ϕ_i);
- First decide at what time t_i the next splitting happens

$$\frac{dN}{N} = -\Gamma dt \quad \Rightarrow \quad \text{PDF}(\mathbf{t}) \equiv \frac{N(t)}{N_0} = e^{-\Gamma t} \in (0, 1]$$

- Once t_i determined, sampling E according to 1D PDF

$$\text{PDF}(\mathbf{E})|_{t=t_i} \equiv \frac{d\Gamma(t_i)}{dE} = \int \frac{d^2\Gamma}{dEd\phi} \frac{d\phi}{2\pi}.$$

- Once (t_i, E_i) determined, sampling ϕ according to 1D PDF

$$\text{PDF}(\phi) \equiv \left. \frac{d^2\Gamma}{dEd\phi} \right|_{t=t_i, E=E_i}$$

Markov Chain MC of Parton Shower

- Multiple decays + Each decay has multiple variables (t_i, z_i, ϕ_i) ;
- First decide at what time t_i the next splitting happens

$$\frac{dN_i}{N_i} = -\frac{dt}{t} \int \mathbf{P}_{ij}(z, \phi) dz \quad \Rightarrow \quad \Delta_i(\mathbf{t}) = e^{-\int \frac{dt}{t} P_{ij}(z) dz} \in (0, 1]$$

according to Sudakov Factor $\Delta_i(t)$.

- Once t_i determined, sampling z according to 1D PDF

$$\mathbf{PDF}(\mathbf{z})|_{t=t_i} \equiv P_{ij}(z)$$

- If there are several species

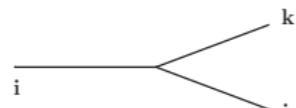
$$\Delta_i(\mathbf{t}) = e^{-\int \frac{dt}{t} \sum_j P_{ij}(z) dz}$$

$$\Gamma_{ij} = \int P_{ij}(z) dz$$

$$\mathbf{PDF}(\mathbf{z})|_{t=t_i} \equiv P_{ij}(z)$$

Backward Evolution for ISPS

- We start from the hard process with x_1 & x_2
- Cross section ratio for $i \rightarrow j$ evolution



$$\frac{dN_i}{N_i} = -\frac{d\sigma_{n+1}}{d\sigma_n} = -\frac{dt}{t} P_{ij}(z, \phi) dz \frac{d\phi}{2\pi}$$

- Event ($N_i = f_i \sigma_i$) ratio for backward $j \rightarrow i$ evolution

$$\frac{dN_j}{N_j} = -\frac{d\sigma_{n+1}}{d\sigma_n} \frac{f_i}{f_j} = -\frac{dt}{t} \frac{f_i(t, x/z)}{f_j(t, x)} P_{ij}(z, \phi) dz \frac{d\phi}{2\pi}$$

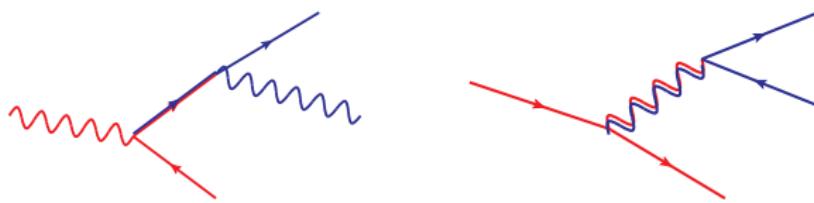
- Modified Sudakov Factor

$$\Pi_j(t, x) = \exp \left[- \int_{t_0}^t \frac{dt'}{t'} \int_{z_{min}(t')}^{z_{max}(t')} dz \sum_i \frac{f_i(t', x/z)}{f_j(t', x)} P_{ij}(t', z) \right].$$

Very intuitive picture

Polarization

- Factorization of amplitudes



$$\mathcal{M}_{\gamma \rightarrow f\bar{f}\gamma} = \bar{u} \not{e} u \frac{1}{q^2} \bar{u} \not{e} v \quad \mathcal{M}_{f \rightarrow ff\bar{f}} = \bar{u} \not{e} u \frac{1}{q^2} \bar{u} \not{e} v$$

- Basic ingredients with spin/helicity

$$\mathcal{M}_{f_i \rightarrow f_j \gamma_k} = \bar{u}_j \not{e}_k u_i \quad \mathcal{M}_{\gamma_i \rightarrow f_j \bar{f}_k} = \bar{u}_j \not{e}_i v_k \quad \mathcal{M}_{\bar{f}_i \rightarrow \bar{f}_j \gamma_k} = \bar{v}_i \not{e}_k v_j$$

Polarized Wave Functions

• Fermions

$$u_{\pm}(p) \equiv \begin{pmatrix} \omega_{\mp}(p)\chi_{\pm}(\vec{p}) \\ \omega_{\pm}(p)\chi_{\pm}(\vec{p}) \end{pmatrix}, \quad v_{\pm}(p) \equiv \begin{pmatrix} \mp\omega_{\pm}(p)\chi_{\mp}(\vec{p}) \\ \pm\omega_{\mp}(p)\chi_{\mp}(\vec{p}) \end{pmatrix},$$

$$\chi_+(\mathbf{p}) = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right)e^{i\phi} \end{pmatrix}, \quad \chi_-(\mathbf{p}) = \begin{pmatrix} -\sin\left(\frac{\theta}{2}\right)e^{-i\phi} \\ \cos\left(\frac{\theta}{2}\right) \end{pmatrix}.$$

with $\omega_{\pm}(p) \equiv \sqrt{E \pm |\mathbf{p}|}$ & $\mathbf{p} = |\mathbf{p}|(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

• Photon

$$\epsilon_{\pm}^{\mu} = \frac{\mp 1}{\sqrt{2}} (0, \cos\theta\cos\phi \mp i\sin\phi, \cos\theta\sin\phi \pm i\cos\phi, -\sin\theta).$$

Home Work 1

Polarized Amplitudes

h_0	h_1	h_2	$V_{f \rightarrow f\gamma}$	$V_{f \rightarrow \gamma f}$	$V_{\gamma \rightarrow f\bar{f}}$	$V_{g \rightarrow gg}$
+	+	+	$-1/\sqrt{1-z}$	$-1/\sqrt{z}$	0	$\frac{-1}{\sqrt{z(1-z)}}$
+	+	-	$z/\sqrt{1-z}$	0	$-z$	$\frac{z^{3/2}}{\sqrt{1-z}}$
+	-	+	0	$(1-z)/\sqrt{z}$	$+(1-z)$	$\frac{(1-z)^{3/2}}{\sqrt{z}}$
+	-	-	0	0	0	0
-	+	+	0	0	0	0
-	+	-	0	$-(1-z)/\sqrt{z}$	$-(1-z)$	$-\frac{(1-z)^{3/2}}{\sqrt{z}}$
-	-	+	$-z/\sqrt{1-z}$	0	$+z$	$-\frac{z^{3/2}}{\sqrt{1-z}}$
-	-	-	$1/\sqrt{1-z}$	$1/\sqrt{z}$	0	$\frac{1}{\sqrt{z(1-z)}}$

Helicity is conserved along the fermion line in massless limit.

$$P_{ij} = \frac{1}{2} \sum_{h_i h_j h_k} \left| V_{i \rightarrow j}^{h_i h_j h_k} \right|^2$$

Retrieve Polarization in PS

- Sudakov Factor

$$\Delta_i(t) = \exp \left[- \int_{t_0}^t \frac{dt'}{t'} \int_{z_{min}(t')}^{z_{max}(t')} dz \sum_j P_{ij}(t', z) \right]$$

where (i, j) represents **flavors & helicity is summed over**.

- Divide particles to helicity eigenstates

$$\Delta_i^{h_i}(t) = \exp \left[- \int_{t_0}^t \frac{dt'}{t'} \int_{z_{min}(t')}^{z_{max}(t')} dz \sum_j^{h_j h_k} P_{ij}^{h_i h_j h_k}(t', z) \right]$$

The summation over helicity is only for h_j & h_k but not h_i .

$$P_{ij}^{h_i h_j h_k} \equiv \left| V_{i \rightarrow j}^{h_i h_j h_k} \right|^2$$

Polarized Splitting Functions

- $e^\pm \rightarrow e^\pm \gamma$

$$P_{ee} = \frac{\alpha}{2\pi} \frac{1+z^2}{1-z} \rightarrow \begin{cases} P_{e_L e_L} = P_{e_R e_R} = \frac{\alpha}{2\pi} \frac{1+z^2}{1-z} \\ P_{e_R e_L} = P_{e_L e_R} = \mathcal{O}\left(\frac{m_e}{E_e}\right) \end{cases}$$

- $\gamma \rightarrow e^\pm e^\mp$

$$P_{\gamma e} = \frac{\alpha}{2\pi} \frac{1+(1-z)^2}{z} \rightarrow \begin{cases} P_{\gamma_L e_L} = P_{\gamma_R e_R} = \frac{\alpha}{2\pi} \frac{1}{z} \\ P_{\gamma_L e_R} = P_{\gamma_R e_L} = \frac{\alpha}{2\pi} \frac{(1-z)^2}{z} \end{cases}$$

- $e^\pm \rightarrow e^\mp \gamma$

$$P_{e_L e_L^+} = P_{e_R e_R^+} = \left(\frac{\alpha}{2\pi}\right)^2 \left(\ln \frac{Q^2}{m_e^2} - 1\right) \left[\frac{2}{3z} + 3 - 3z - \frac{2}{3}z^2 + 2(1+z)\log z\right],$$

$$P_{e_L e_R^+} = P_{e_R e_L^+} = \left(\frac{\alpha}{2\pi}\right)^2 \left(\ln \frac{Q^2}{m_e^2} - 1\right) \frac{2(1-z)^3}{3z},$$

Note that $P_{e_L e_R}$ & $P_{e_R e_L}$ also receives $\mathcal{O}(\alpha^2)$ contribution.

Unpolarized → Polarized Parton Shower

- Particles labeled by (**flavor** × **helicity**) as different particles

$$(e^-, e^+, \gamma) \rightarrow (e_L^-, e_R^-, e_L^+, e_R^+, \gamma_L, \gamma_R)$$

- Divide particles to helicity eigenstates

$$\Delta_i^{h_i}(t) = \exp \left[- \int_{t_0}^t \frac{dt'}{t'} \int_{z_{min}(t')}^{z_{max}(t')} dz \sum_j^{h_j h_k} \frac{f_j^{h_j}(t', x/z)}{f_i^{h_i}(t', x)} P_{ji}^{h_j h_i h_k}(t', z) \right]$$

For ISPS, we also need to input **polarized PDF**.

Polarized PDF (1)

$f_i(t, x)$ = parton momentum (x) distribution @ scale $\sqrt{t} = Q$

$$f_i(t, x) = \int_{t_0}^t P_{bi}(t, z=x) \frac{dt}{t}$$

By this naive analogy we can derive $f_{\gamma_{h'}/e_h^\pm}$ & $f_{e_{h'}^\pm/\gamma_h}$

- f_{γ/e^\pm}

$$f_{\gamma_L/e_L^\pm} = f_{\gamma_R/e_R^\pm} = \frac{\alpha}{2\pi} \ln \left(\frac{Q^2}{m_e^2} \right) \frac{1}{x} \quad f_{\gamma_R/e_L^\pm} = f_{\gamma_L/e_R^\pm} = \frac{\alpha}{2\pi} \ln \left(\frac{Q^2}{m_e^2} \right) \frac{(1-x)^2}{x}$$

- $f_{e^\pm/\gamma}$

$$f_{e_L^\pm/\gamma_L} = f_{e_R^\pm/\gamma_R} = \frac{\alpha}{2\pi} \ln \frac{Q^2}{m_e^2} x^2 \quad f_{e_L^\pm/\gamma_R} = f_{e_R^\pm/\gamma_L} = \frac{\alpha}{2\pi} \ln \frac{Q^2}{m_e^2} (1-x)^2$$

Polarized PDF(2)

- e^\pm in e^\mp arrives from double splittings: $e^\pm \rightarrow e^\pm \gamma \rightarrow e^\pm e^\pm e^\mp$.

$$f_{j/i}(Q^2, x) = \sum_k \int_t^{Q^2} \frac{dt_2}{t_2} \int_{\Lambda^2}^t \frac{dt_1}{t_1} \int_x^1 \frac{dz}{z} P_{kj}(t_2, z) P_{ik}\left(t_1, \frac{x}{z}\right).$$

After integrating over (t_1, t_2, z)

$$f_{e_{L,R}^+ / e_{L,R}} = \frac{1}{2} \left\{ \frac{\alpha}{2\pi} \left(\ln \frac{Q^2}{m_e^2} - 1 \right) \right\}^2 \left[\frac{2}{3x} + 3 - 3x - \frac{2}{3}x^2 + 2(1+x)\log x \right]$$

$$f_{e_{L,R}^+ / e_{R,L}} = \frac{1}{2} \left\{ \frac{\alpha}{2\pi} \left(\ln \frac{Q^2}{m_e^2} - 1 \right) \right\}^2 \frac{2(1-x)^3}{3x}$$

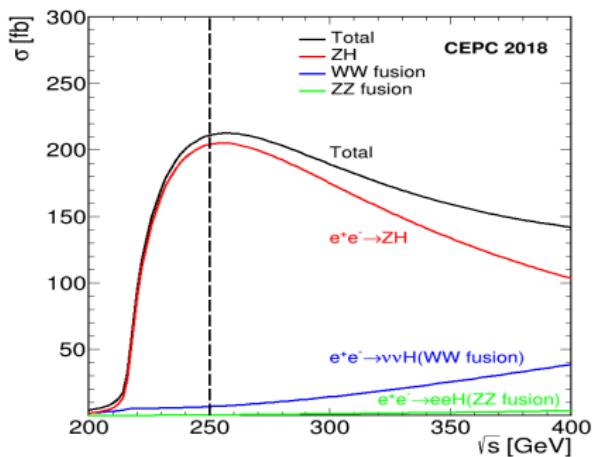
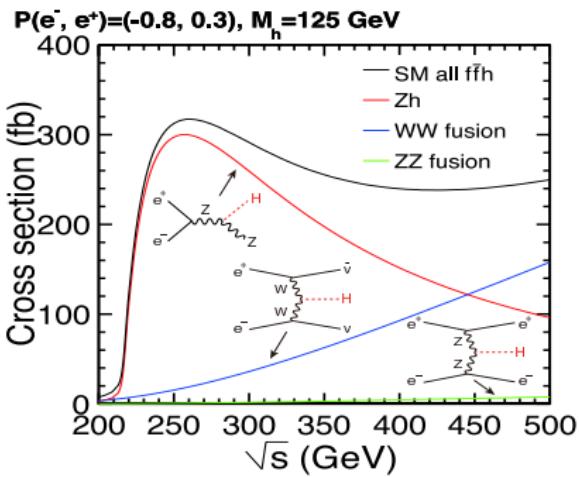
- The most intricate part is for e^\pm in e^\pm

Home Work 3

$$f_{e_{LR}^\pm / e_{LR}^\pm} = f_{e^\pm / e^\pm} \approx \beta(1-x)^{\beta-1}, \quad \text{with} \quad \beta = \frac{\alpha}{\pi} \left(\ln \frac{Q^2}{m_e^2} - 1 \right)$$

Parton Shower with Spin Correlation

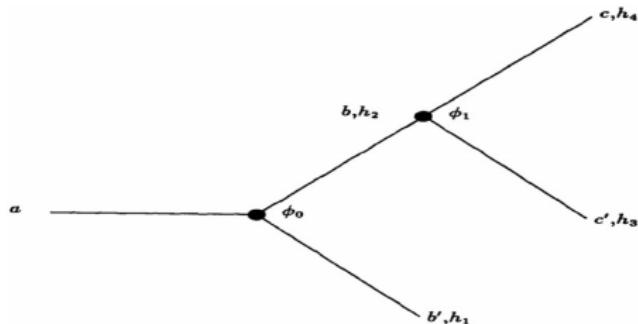
- If only spin correlation, the PS algorithm needs no change.
- More species by splitting a particle into Left & Right.
- Polarized splitting kernels & PDFs



$$\sigma(r_L, r_R) = \frac{1 - r_L}{2} \frac{1 + r_R}{2} g_L^2 + \frac{1 + r_L}{2} \frac{1 - r_R}{2} g_R^2 \Rightarrow \frac{\sigma(-0.8, 0.3)}{\sigma(0, 0)} \approx 1.5$$

Azimuthal Angle Dependence

- Each branching plane is characterized by ϕ_i



$$\chi_+(\mathbf{p}) = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) e^{i\phi} \end{pmatrix}, \quad \chi_-(\mathbf{p}) = \begin{pmatrix} -\sin\left(\frac{\theta}{2}\right) e^{-i\phi} \\ \cos\left(\frac{\theta}{2}\right) \end{pmatrix}$$

$$\epsilon_{\pm}^{\mu} = \frac{\mp 1}{\sqrt{2}} (0, \cos \theta \cos \phi \mp i \sin \phi, \cos \theta \sin \phi \pm i \cos \phi, -\sin \theta)$$

- Azimuthal angle is always associated with photon

$$V_{f \rightarrow f\gamma} e^{-i \mathbf{h}_{\gamma} \phi}, \quad V_{\gamma \rightarrow f\bar{f}} e^{i \mathbf{h}_{\gamma} \phi}$$

Polarized Amplitudes

h_0	h_1	h_2	$V_{f \rightarrow f\gamma}$	$V_{f \rightarrow \gamma f}$	$V_{\gamma \rightarrow f\bar{f}}$	$V_{g \rightarrow gg}$
+	+	+	$-1/\sqrt{1-z}$	$-1/\sqrt{z}$	0	$\frac{-1}{\sqrt{z(1-z)}}$
+	+	-	$z/\sqrt{1-z}$	0	$-z$	$\frac{z^{3/2}}{\sqrt{1-z}}$
+	-	+	0	$(1-z)/\sqrt{z}$	$+(1-z)$	$\frac{(1-z)^{3/2}}{\sqrt{z}}$
+	-	-	0	0	0	0
-	+	+	0	0	0	0
-	+	-	0	$-(1-z)/\sqrt{z}$	$-(1-z)$	$-\frac{(1-z)^{3/2}}{\sqrt{z}}$
-	-	+	$-z/\sqrt{1-z}$	0	$+z$	$-\frac{z^{3/2}}{\sqrt{1-z}}$
-	-	-	$1/\sqrt{1-z}$	$1/\sqrt{z}$	0	$\frac{1}{\sqrt{z(1-z)}}$

Table: The polarized amplitude $\mathcal{M}_{h_0 h_1 h_2} \propto V_{h_0 h_1 h_2} e^{i(h_0 - h_1 - h_2)\phi}$ where $h_i = \pm 1$ for photon or $\pm \frac{1}{2}$ for fermions.

Home Work 4

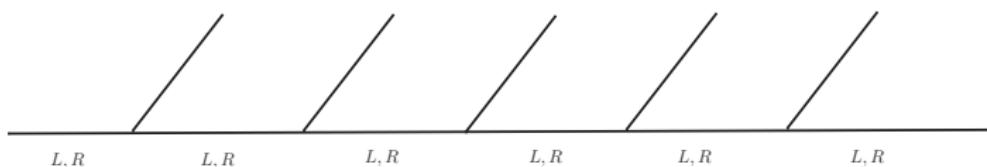
In massless limit, only the ϕ associated with photon can survive!

Azimuthal Correlation

- The azimuthal angle dependence can be retrieved:

$$\Delta_i^{h_i}(t) = \exp \left[- \int_{t_0}^t \frac{dt'}{t'} \int_{z_{min}(t')}^{z_{max}(t')} dz \sum_j h_j h_k P_{ij}^{h_i h_j h_k}(t', z, \phi) \frac{d\phi}{2\pi} \right]$$

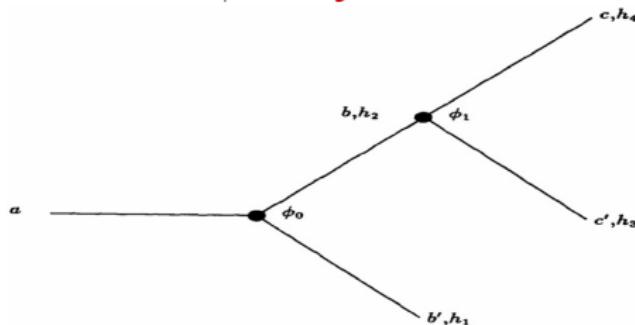
- Spin correlation propagate as flavor



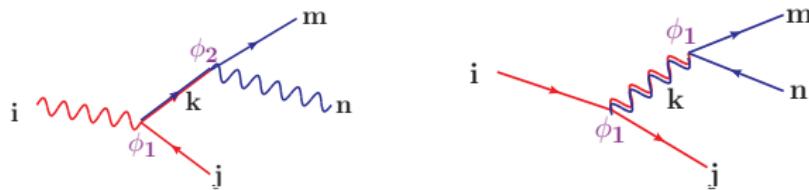
- Azimuthal Angle Correlation is quite non-trivial!
 - Continuous variable ϕ_i cannot propagate as flavor;
 - Azimuthal angle ϕ_i causes interference among helicity states;
 - Sampling of ϕ_i has dependence on z_i & helicity
 - Value of ϕ_i decides helicity composition for next splitting

Correlation among splittings

- Although each branch has ϕ_i , **only the differences matters**



- Azimuthal angle propagation involves **at least 2 vertices**



$$\mathcal{M}_{\gamma \rightarrow f\bar{f}\gamma} = \sum_k \bar{\mathbf{u}}_m \not{\epsilon}_n \mathbf{u}_k \frac{1}{q^2} \bar{\mathbf{u}}_k \not{\epsilon}_i \mathbf{v}_j$$

$$\mathcal{M}_{f \rightarrow f\bar{f}} = \sum_k \bar{\mathbf{u}}_j \not{\epsilon}_k \mathbf{u}_i \frac{1}{q^2} \bar{\mathbf{u}}_m \not{\epsilon}_k \mathbf{v}_n$$

Spin & Decay Matrices by Amplitudes

- All information in the factorized amplitudes

$$\mathcal{M}_{i \rightarrow jmn} = \sum_k \mathcal{M}_{ijk} \frac{1}{q^2} \mathcal{M}_{kmn}$$

- Interference due to internal helicity states

$$|\mathcal{M}|^2 = \sum_{kk'} \left(\sum_{ij} \mathcal{M}_{ijk} \mathcal{M}_{ijk'}^* \right) \frac{1}{(q^2)^2} \left(\sum_{mn} \mathcal{M}_{kmn} \mathcal{M}_{k'mn}^* \right)$$

- Spin ($\rho_{kk'}$) & Decay ($D_{kk'}$) Matrices

$$\rho_{kk'}(\phi) \propto \sum_{ij} \mathcal{M}_{ijk} \mathcal{M}_{ijk'}^* \quad D_{kk'}(\phi') \propto \sum_{mn} \mathcal{M}_{kmn} \mathcal{M}_{k'mn}^*$$

with $\sum_k \rho_{kk} = \sum_k D_{kk} = 1$. Then $|\mathcal{M}|^2 \propto \text{Tr}(\rho D)$.

Physical Meaning of Spin & Decay Matrices

- Density Matrix in Quantum Mechanics

$$|\psi\rangle = \sum_k c_k |\psi_k\rangle \quad \Rightarrow \quad |\psi\rangle\langle\psi| = \sum_{kk'} c_k c_{k'}^* |\psi_k\rangle\langle\psi_{k'}|$$

In the basis of eigenstates $|\psi_k\rangle$, define an operator

$$\rho \equiv c_k c_{k'}^* = \begin{pmatrix} |c_1|^2 & c_1 c_2^* \\ c_1^* c_2 & |c_2|^2 \end{pmatrix} \quad \langle \hat{\mathcal{O}} \rangle = \text{Tr}(\rho \hat{\mathcal{O}})$$

ρ_{ii} = probability for eigenstate $|\psi_i\rangle$ & $\rho_{i \neq j}$ = interference

- Spin Matrix for polarization along some direction

$$\rho = \frac{1}{2} \begin{pmatrix} 1 + P_{||} & P_{\perp} e^{i\phi} \\ P_{\perp} e^{-i\phi} & 1 - P_{||} \end{pmatrix},$$

$P_{||}$ = degree of polarization along the direction

P_{\perp} = degree of polarization perpendicular to the direction

Home Work 5

Spin Matrices in Parton Shower

- Fermion Splitting $f \rightarrow f\gamma$

$$\rho_{f/f_L} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \rho_{f/f_R} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\rho_{\gamma/e_L} = \begin{pmatrix} \frac{1}{1+z^2} & \frac{-z e^{2i\phi}}{1+z^2} \\ \frac{-z e^{-2i\phi}}{1+z^2} & \frac{z^2}{1+z^2} \end{pmatrix} \quad \rho_{\gamma/e_R} = \begin{pmatrix} \frac{z^2}{1+z^2} & \frac{-z e^{2i\phi}}{1+z^2} \\ \frac{-z e^{-2i\phi}}{1+z^2} & \frac{1}{1+z^2} \end{pmatrix}$$

Spin is preserved along fermion line in the massless limit.

- Photon Splitting $\gamma \rightarrow f\bar{f}$

$$\rho_{f/\gamma_L} = \begin{pmatrix} \frac{z^2}{z^2+(1-z)^2} & 0 \\ 0 & \frac{(1-z)^2}{z^2+(1-z)^2} \end{pmatrix} \quad \rho_{f/\gamma_R} = \begin{pmatrix} \frac{(1-z)^2}{z^2+(1-z)^2} & 0 \\ 0 & \frac{z^2}{z^2+(1-z)^2} \end{pmatrix}$$

No interference among fermion spins in the massless limit.

Decay Matrices in Parton Shower

• Fermion Decay

$$D_f^L(z) = \begin{pmatrix} \frac{1}{\sqrt{1-z}} & 0 \\ 0 & \frac{z}{\sqrt{1-z}} \end{pmatrix} \quad D_f^R(z) = \begin{pmatrix} 0 & \frac{z}{\sqrt{1-z}} \\ \frac{z}{\sqrt{1-z}} & \frac{1}{\sqrt{1-z}} \end{pmatrix}$$

- Helicity index (L, R) is for final-state photon;
- **No interference between e_L^\pm and e_R^\pm .**

• Photon Decay

$$D_\gamma^R = \frac{1}{z^2 + (1-z)^2} \begin{pmatrix} z^2 & z(1-z)e^{2i\phi} \\ z(1-z)e^{-2i\phi} & (1-z)^2 \end{pmatrix}$$

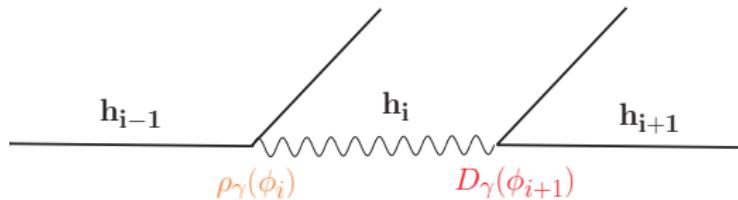
$$D_\gamma^L = \frac{1}{z^2 + (1-z)^2} \begin{pmatrix} (1-z)^2 & z(1-z)e^{2i\phi} \\ z(1-z)e^{-2i\phi} & z^2 \end{pmatrix}$$

- Helicity index (L, R) is for final-state fermion;

Home Work 6

Azimuthal Unit

- Azimuthal angle dependence is associated with photon!



- The probability form $h_{i-1} \rightarrow h_{i+1}$ transition is

$$P_{h_{i-1} \rightarrow h_{i+1}} = \text{Tr} \left[\rho_{\gamma/e_{h_{i-1}}}(\phi_i) D_\gamma^{h_{i+1}}(\phi_{i+1}) \right]$$

- Take $e_L \rightarrow X \rightarrow e_L$ for illustration

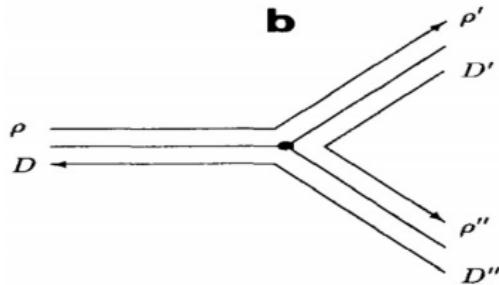
$$P_{LL} = P_{RR} = \frac{z_i^2 \bar{z}_{i+1}^2 + z_{i+1}^2 - 2z_i z_{i+1} \bar{z}_{i+1} \cos(2\Delta\phi)}{(1+z_i^2)(z_{i+1}^2 + \bar{z}_{i+1}^2)}$$

$$P_{LR} = P_{RL} = \frac{\bar{z}_{i+1}^2 + z_i^2 z_{i+1}^2 - 2z_i z_{i+1} \bar{z}_{i+1} \cos(2\Delta\phi)}{(1+z_i^2)(z_{i+1}^2 + \bar{z}_{i+1}^2)}$$

Home Work 7

Azimuthal angle dependence propagates to further splittings!

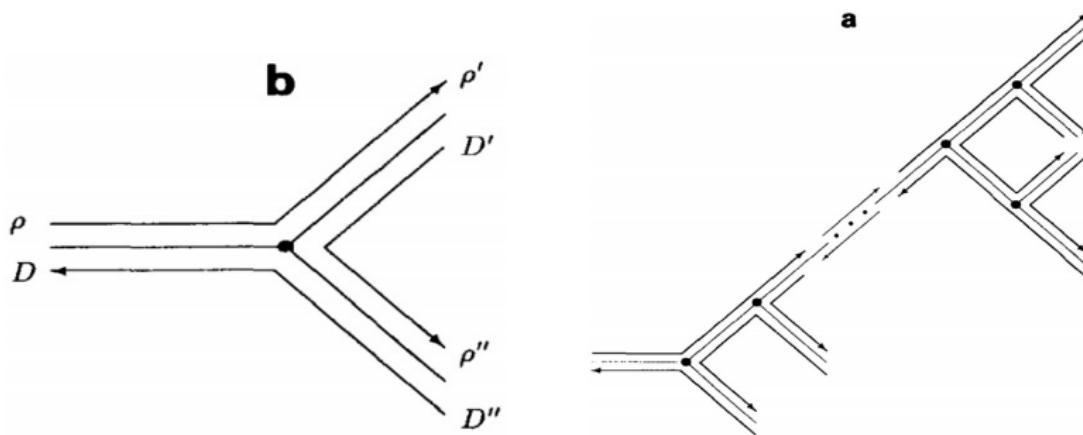
Density Matrices @ Vertex



$$\sum \rho_{ii'} V_{ijk} V_{i'j'k'}^* D_{jj'} D_{jj'} + \rho, D = \mathbb{I} \text{ whenever incomplete info}$$

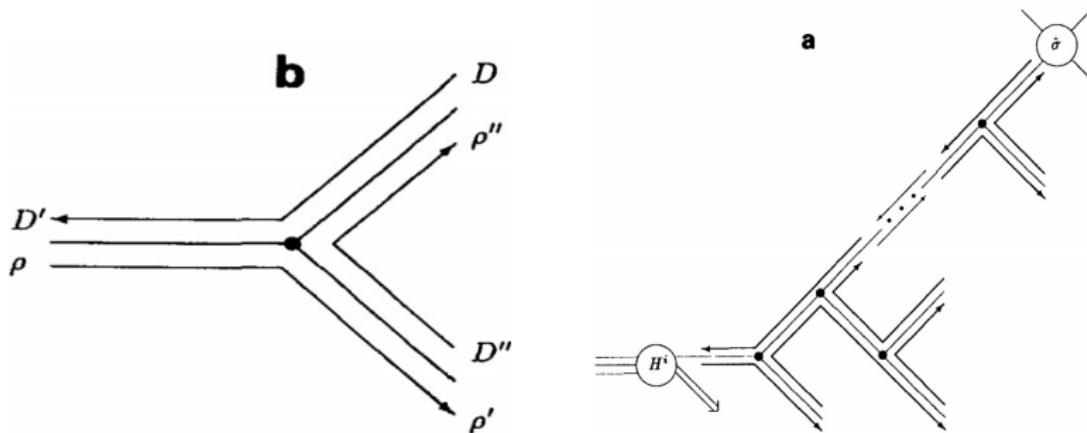
- First sample vertex with $\sum \rho_{ii'} V_{ijk} V_{i'j'k'}^*(t, z, \phi)$
- $\rho'_{jj'} = \sum \rho_{ii'} V_{ijk} V_{i'j'k'}^*(t, z, \phi)$ for upper branch, return $D'_{jj'}$
- $\rho''_{kk'} = \sum \rho_{ii'} V_{ijk} V_{i'j'k'}^* D'_{jj'}$ for lower branch, return $D''_{kk'}$
- $D_{ii'} = \sum V_{ijk} V_{i'j'k'}^* D'_{jj'} D''_{kk'}$

Propagation of Spin & Decay Matrices



- Density matrix manipulation at individual vertices;
- Each vertex starts with spin matrix ρ , returns decay matrix D ;
- Finally all final-state partons are correlated with each other.

Polarized Parton Shower for Initial State



- Backward evolution;
- Each vertex starts with decay matrix D , returns spin matrix ρ'' ;
- **Finally all partons are correlated with each other.**
 - Final-State Parton Shower partons
 - Initial-State Parton Shower partons
 - Hard scattering process
 - Beam particles

FSPS vs ISPS

- FSPS

$$\Delta_i^{h_i}(t) = \exp \left[- \int_{t_0}^t \frac{dt'}{t'} \int_{z_{min}(t')}^{z_{max}(t')} dz \sum_j^{h_j h_k} P_{ij}^{h_i h_j h_k}(t', z, \phi) \frac{d\phi}{2\pi} \right]$$

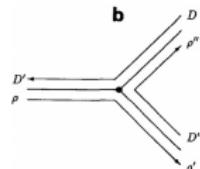
- ISPS

$$\Delta_i^{h_i}(t) = \exp \left[- \int_{t_0}^t \frac{dt'}{t'} \int_{z_{min}(t')}^{z_{max}(t')} dz \sum_j^{h_j h_k} \frac{f_j^{h_j}(x/z)}{f_i^{h_i}(x)} P_{ji}^{h_j h_i h_k}(t', z, \phi) \frac{d\phi}{2\pi} \right]$$

- For ISPS, the splitting kernel is weighted by $f_j^{h_j}(x/z)/f_i^{h_i}(x)$;
- **Predefined spin matrix** even when information is incomplete

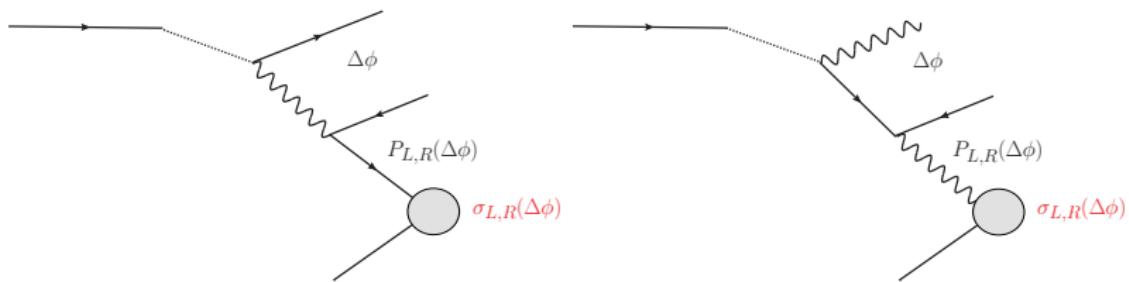
$$\rho = \begin{pmatrix} f_L & \\ & f_R \end{pmatrix}$$

To guide the backward evolution!



Spin & Azimuthal Correlations @ Circular Colliders

- Even without beam polarization, correlation can appear!



- Azimuthal angle in previous splittings affects spin;
- The hard scattering usually also has spin dependence;
- The cross section then has spin & angular dependences;
- Azimuthal correlation is inevitable.

Status of Parton Shower Softwares

- Inclusive Initial-State Radiation (PDF)
 - Whizard
 - MG-ISR
- Spin Correlation
 - VINCIA
- Spin & Azimuthal Correlation (in progress)
 - Herwig++
 - Mad-ee

Coming Soon!

Thank You!

Home Works

- ① Construct wave functions
- ② Polarized splitting kernels with spin indices
- ③ Polarized PDF
- ④ Polarized splitting kernels with azimuthal angle
- ⑤ Density matrix in Quantum Mechanics
- ⑥ Spin & Decay Matrices
- ⑦ Azimuthal Unit