Outline

- Introduction to perturbative QCD:
  asymptotic freedom, IR-safety, factorization, S/W jets
- Jet physics
  algorithms, jet functions, factorization
- Jet substructure: hadrons inside a jet, jet mass, grooming
- Jets in HI collisions

Bibliography

more references later

pQCD & factorization

- G. Sterman TASI '95 hep-ph/9606312
- Dave Soper hep-ph/9702203
- Collins, Soper, Sterman hep-ph/0409313
- CTEQ cteq.org
- Richard Field: Applications of perturbative QCD
- Vogelsang / Stratmann

SCET

- C. Bauer / I. Stewart lecture notes & videos
- Becher, Broggio, Ferroglia: arXiv: 1410.1892

Jets & substructure

- Salam: PhD level lectures
- Larkoski, Moult, Nachman: arXiv 1709.04464

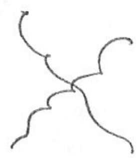
The QCD Lagrangian

$$\mathcal{L} = \bar\psi (i\partial^\mu \gamma_\mu - m)\psi - g_s \bar\psi \gamma^\mu T_a \psi A_\mu^a - \frac{1}{4} F_a^{\mu\nu} F^a_{\mu\nu}$$

$$F_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu - g f_{abc} A_b^\mu A_c^\nu$$

- SU(3) non-abelian gauge group
  $\psi(x) \to e^{i\alpha_a(x) T_a} \psi(x)$, T_a : generators : $8 = N_c^2 - 1$ 3x3 matrices ; $[T_a, T_b] = i f_{abc} T_c$
- like QED $g_s T_a$

$g f_{abc}$



- experimentally
  10 % 3-jet events

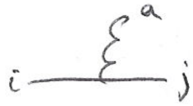
$\boxed{\text{LEP}}$

→ later quark/gluon discrimination

Color algebra ... $C_F, C_A, T_F N_f$:

$i, j = 1, N_c (=3)$
$a = 1, 8 (= N_c^2 - 1)$

$i \longrightarrow j = \delta_{ij}$



$= T^a_{ij}$

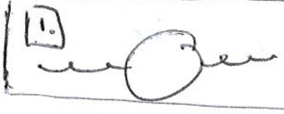
a ~~~~ $b = \delta_{ab}$

$= i f^{abc}$

e.g.

$= \mathrm{tr}[T^a] = 0$

$\boxed{1.}$



$= T_F$ ~~~~

($\& N_f \ldots$) $\mathrm{tr}[T^a T^b] = T_F \delta^{ab}$

$[T^a, T^b] = i f^{abc} T^c$

$(T^a)^\dagger = T^a$, $\mathrm{tr}[T^a] = 0$

hermitian & traceless

normalization

$\mathrm{tr}[T^a T^b] = T_F \delta^{ab}$

with $T_F = 1/2$

i) show $i f^{abc} = 2\,\mathrm{tr}([T^a, T^b] T^c)$

$$[T^a, T^b] = i f^{abc} T^c \quad | \cdot T^d$$

$$[T^a, T^b] T^d = i f^{abc} T^c T^d \quad | \,\mathrm{tr}$$

$$\mathrm{tr}([T^a, T^b] T^d) = i f^{abc} \underbrace{\mathrm{tr}[T^c T^d]}_{= \delta^{cd}/2}$$

$$\Rightarrow 2\,\mathrm{tr}([T^a, T^b] T^d) = i f^{abd} \quad ✓ \quad \Rightarrow f \text{ anti-sym. in all indices}$$

ii) Fierz / completeness relation $T^a_{il}\, T^a_{jk} = \frac{1}{2}\delta_{ik}\delta_{jl} - \frac{1}{2N}\delta_{il}\delta_{jk}$

use T^a, $\mathbb{1}$ as basis for hermitian $N\times N$ matrices:

Ansatz: $A_{ij} = c_0\,\delta_{ij} + \sum_{a=1}^{N^2-1} c_a\, T^a_{ij}$ (*) ⇒ determine coefficients

(*)$\cdot\delta_{ij}$ ⇒ $A_{ij}\delta_{ij} = A_{ii} = c_0\delta_{ii} + \sum_a c_a \underbrace{T^a_{ii}}_{=0} = c_0 N$

(*)$\cdot T^b_{ji}$ ⇒ $A_{ij}T^b_{ji} = 0 + \sum_a c_a \underbrace{T^a_{ij}T^b_{ji}}_{=\mathrm{tr}[T^aT^b] = \delta^{ab}/2} = c_b/2 \Rightarrow c_a = A_{ij}T^a_{ji}$

⇒ back to (*): $A_{ij} = \frac{A_{ll}}{N}\delta_{ij} + 2A_{lm}T^b_{ml}T^b_{ij}$ ← [implicit sum over b]

now factor out A_{lm}: $A_{lm}\left(\delta_{li}\delta_{jm} - \frac{1}{N}\delta_{ij}\delta_{lm} - 2T^b_{ml}T^b_{ij}\right) = 0$

↑ arbitr. ⇒ $(\dots) = 0$ ✓

12.  $= C_F$ ——— $T^a_{ik}T^a_{kj} = C_F\,\delta_{ij}$ | & is Casimir op. [...]

use Fierz: $T^a_{ik}T^a_{kj} = \frac{1}{2}\delta_{ij}\delta_{kk} - \frac{1}{2N}\delta_{ik}\delta_{kj} = \frac{1}{2}\delta_{ij}N - \frac{1}{2N}\delta_{ij}$

$= \delta_{ij}\left(\frac{N}{2} - \frac{1}{2N}\right) = \delta_{ij}\frac{N^2-1}{2N} \equiv \delta_{ij}C_F$ ✓

13.  $= C_A$ ~~~ $f^{acd}f^{bcd} = C_A\,\delta^{ab}$

(sketch): $= -4\,\mathrm{tr}(T^a[T^c,T^d])\,\mathrm{tr}(T^b[T^c,T^d])$

$= -8\left(\mathrm{tr}(T^aT^cT^d)\cdot\mathrm{tr}(T^bT^cT^d) - \mathrm{tr}(T^aT^cT^d)\,\mathrm{tr}(T^bT^dT^c)\right)$

use Fierz to compute product of traces e.g. $(T^a_{ij}T^c_{jl}T^d_{li})(T^b_{\alpha\beta}T^c_{\beta\gamma}T^d_{\gamma\alpha})$

$= T^a_{ij}T^b_{\alpha\beta}\underbrace{T^c_{jl}T^c_{\beta\gamma}}_{\text{Fierz}}\underbrace{T^d_{li}T^d_{\gamma\alpha}}$

$= \dots = -\frac{1}{2N}\mathrm{tr}(T^aT^b) = \frac{-1}{4N}\delta^{ab}$

find ⇒ $f^{abc}f^{bcd} = -8\left(-\cancel{\frac{1}{4N}}\delta^{ab} - \frac{1}{8}\left(N - \cancel{\frac{2}{N}}\right)\delta^{ab}\right) = N\delta^{ab} \equiv C_A\delta^{ab}$ ✓

later e…

→ know th…
there is a…
color factor…
C_F relativ…
to LO 
[...]

Asymptotic freedom

$\alpha_s = \frac{g_s^2}{4\pi}$

LO solution for the running coupling constant

$\cancel{\frac{1}{\alpha_s(\mu^2)} = \frac{1}{\alpha_s(\mu_0^2)} + \frac{\beta_0}{2\pi}\ln(\mu/\mu_0)}$

Nobel prize '04
Gross, Wilczek, Politzer

Renormalization introduces the dependence on $\mu_{(R)}$. By demanding invariance of phys. observables on μ, we can derive the RGE for α_s:

$$\mu^2 \frac{\partial \alpha_s}{\partial \mu^2} = \frac{\partial \alpha_s}{\partial \ln \mu^2} = -\alpha_s^2 (b_0 + b_1 \alpha_s + \dots) \equiv \beta(\alpha_s) \leftarrow \text{QCD beta function}$$

where $b_0 = \frac{1}{12\pi}(11 C_A - 2N_f)$ | $\beta_i = b_i (4\pi)^{i+1}$

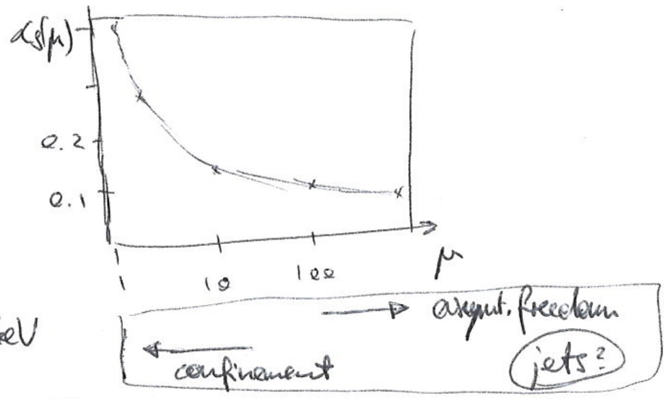
$b_1 = \dots$

- negative QCD beta function means asymptotic freedom: decreasing coupling strength at high energies / short distances $\rightarrow$ pQCD. Satisfied for $N_f \le 16$.
- A closed form for the analytical solution for $\alpha_s(\mu^2)$ exists only @ LO:

$\mu^2 \frac{\partial \alpha_s}{\partial \mu^2} = -b_0 \alpha_s^2$ rewrite as $\frac{\partial \alpha_s}{\alpha_s^2} = -b_0 \frac{\partial \mu^2}{\mu^2}$

integrate between initial scale μ_0 & μ: $\frac{1}{\alpha_s(\mu_0^2)} - \frac{1}{\alpha_s(\mu^2)} = -b_0 \log(\mu^2/\mu_0^2)$

$$\Rightarrow \alpha_s(\mu^2) = \frac{\alpha_s(\mu_0^2)}{1 + b_0 \alpha_s(\mu_0^2) \ln(\mu^2/\mu_0^2)}$$



- pQCD doesn't predict the absolute value but the change once we know $\alpha_s(\mu_0^2)$
- note: μ_0, μ both in pert. regime, say, 1 GeV
- extracted nowadays from data typically together with PDF fits, DIS, e^+e^- event shapes, $pp \rightarrow jet + X$, ...

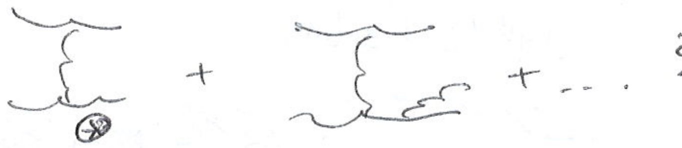
$\alpha_s(M_Z) = 0.1181 \pm 0.0011$ [for pp precision pheno one of the main sources of uncertain (2016)

- First example of resummation: $\alpha_s(\mu^2) = \alpha_s(\mu_0^2) - b_0 \ln(\mu^2/\mu_0^2)\, \alpha_s^2(\mu_0^2) + (b_0 \ln(\mu^2/\mu_0^2))^2 \alpha_s^3(\mu_0^2) + \dots$

1st expand in α_s 2nd sum up dominant logs to all orders

~~pQCD? i) colliding protons~~

Jets @ high energy $\rightarrow$ pQCD



- colliding protons
- detectors observe hadrons

$\Big\}$ confinement, NP physics

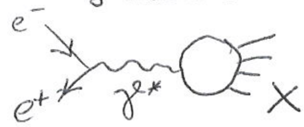
$\Rightarrow$ need to separate from hard interaction that is producing the jet

$\Rightarrow$ $\boxed{\text{factorization}}$, $\boxed{\text{infrared safety}}$

(*) Cutler, Sivers Phys. Rev D 17 (1978) Appendix

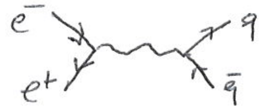
$e^+e^- \to$ hadrons / $e^+e^- \to X$

s-channel → relevant propagator $\frac{1}{(p_1+p_2)^2}$



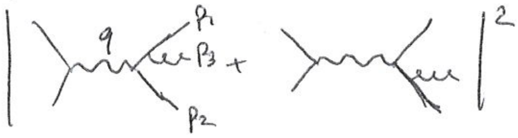
- totally inclusive cross section $e^+e^- \to X$
  (nothing observed about X besides total rate)

- Born / LO approximation $\sigma_0 = \frac{4\pi\alpha^2}{3s} N_c \sum_q e_q^2$



$s = (p_1+p_2)^2 = Q^2 = c.m.\ \text{energy}^2$, α = QED fine structure constant

- NLO $e^+e^- \to q\bar{q}g$



real emission diagrams

$q = (\sqrt{s}, 0, 0, 0)$ cms frame

$\boxed{\text{everywhere } (+,-,-,-) \text{ metric}}$

detailed NLO derivation in R. Field using dim. re[g]

- define energy fractions $x_i = \frac{2p_i \cdot q}{s} = \frac{2E_i \cdot \sqrt{s}}{s} = \frac{E_i}{\sqrt{s}/2} \Rightarrow x_i \geq 0$

- energy conservation: $\sum_i x_i \overset{\text{def}}{=} \frac{2(\overbrace{p_1+p_2+p_3}^{=q})\cdot q}{s} = 2\, q^2/s = 2 \Rightarrow$ only 2 x_i are independent

- angles:

e.g. $2p_1 \cdot p_2 = (p_1+p_2)^2 = (q-p_3)^2 = \underbrace{q^2}_{=s} - \underbrace{2q\cdot p_3}_{=x_3 s} = s(1-x_3)$

$\quad\ \ \| $

$2E_1E_2(1-\cos\theta_{12})$

divide by $s/2 \Rightarrow x_1 x_2 (1-\cos\theta_{12}) = 2(1-x_3)$

similarly: $x_2 x_3(1-\cos\theta_{23}) = 2(1-x_1)$

$x_1 x_3 \underbrace{(1-\cos\theta_{13})} = 2(1-x_2)$

$\cos \in [-1,1] \Rightarrow (..) \in [0,2] \wedge x_i \geq 0 \Rightarrow 0 \leq x_1 x_3(1-\cos\theta_{13}) = 2(1-x_2)$

$\Rightarrow x_2 \leq 1$

energy fractions $\Rightarrow 0 \leq x_i \leq 1$

- real emission contribution gives $d=4$

$$\sigma(e^+e^- \to q\bar{q}g) = \sigma_0 \frac{\alpha_s}{2\pi} C_F \int_0^1 dx_1 \int_{1-x_1}^1 dx_2 \frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)}$$

$\uparrow$ $\uparrow$

1. $g \parallel \bar{q}$ $\quad g \parallel q$ $\quad \boxed{\text{collinear divergences!}}$

2. $x_1 = x_2 = 1$ $\quad \boxed{\text{soft-collinear}}$
   $\Rightarrow x_3 = 2 - x_1 - x_2 = 0$

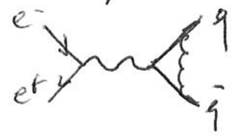
double pole: partial fractioning:

$$\frac{1}{(1-x_1)(1-x_2)} = \frac{1}{x_3}\left(\frac{1}{1-x_1} + \frac{1}{1-x_2}\right)$$ is combination of collinear (x_1 or $x_2 \to 1$) & soft divergence $x_3 \to 0$

$e^+e^- \to$ hadrons / $e^+e^- \to X$

... how do we get the correct result that doesn't diverge?

use e.g. dim. reg. & include virtual correction



$\Rightarrow \sigma(e^+e^- \to X) = \sigma_0 \left(1 + \frac{\alpha_s}{\pi}\right)$ (**)

- moderate perturbative correction
- production cross section for strongly interacting particles. It is insensitive to the details of hadronic effects. Sufficiently inclusive.
- most basic example of an <u>Infrared (IR) safe observable</u>
- want to calculate less inclusive observables: systematize more clearly later.
- Kinoshita-Lee-Nauenberg (KLN) theorem guarantees cancellation to all orders. IR cancellation of divergences for IR safe observables ... unitarity, UV, renormalized

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QCD scale dependence

- what is $\alpha_s(\mu)$ in (**)? || μ needs to be i) perturbative
  ii) not involve parton level variables like $\hat{s}$ ...

- higher orders actually give (LO is essentially QED)

$$\sigma = \sigma_0 \left(1 + \sum_{n=1}^{\infty} \alpha_s^n(\mu) C_n(\mu)\right) = \sigma_0 \left(1 + \frac{\alpha_s(\mu)}{\pi} + \left[1.409 + 1.917 \ln(\mu^2/s)\right]\left(\frac{\alpha_s(\mu)}{\pi}\right)^2 + \dots\right)$$

$\Rightarrow$ choose $\mu \sim s$ to avoid large logarithms at fixed order. $\oplus$
and we would have to worry about the convergence of the pert. expansion.

- QCD scale uncertainties estimates: ...

phys. cross section needs to be indep. of the choice of μ

$\Rightarrow \frac{d}{d\ln\mu^2}\sigma = 0$. This holds true if all orders are known.

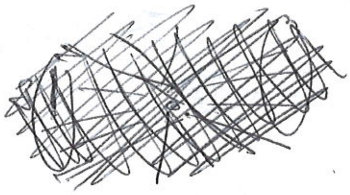
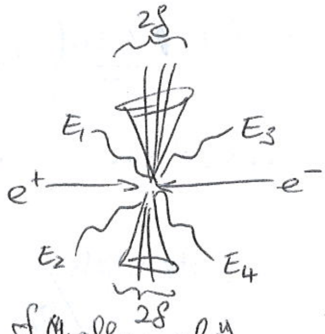
For given fixed order expansion: $\frac{d}{d\ln\mu^2}\left(\sum_{n=0}^{N} \alpha_s^n(\mu) C_n(\mu)\right) = -\frac{d}{d\ln\mu^2}\left(\sum_{n=N+1}^{\infty} \alpha_s^n(\mu) C_n(\mu)\right)$

$\sim \mathcal{O}(\alpha_s^{N+1})$

& is reduced when higher orders are included.

Sterman / Weinberg jets 1977 PRL [~~[illegible]~~] – an "exclusive" jet algorithm

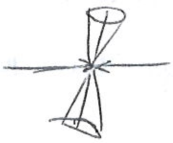
"Jets from Quantum Chromodynamics"

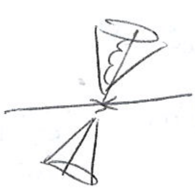
typical e^+e^- event  or, 

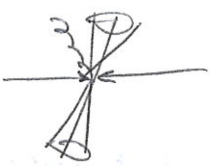
$\sqrt{s_{e^+e^-}} = Q$

- draw 2 oppositely directed cones with radius δ (half-angle)
- require out-of-jet radiation to be less than εQ ⟹ here $E_1+E_2+E_3+E_4 < \varepsilon Q$ & $\varepsilon \ll 1$
  → count when satisfied → 3,4 jets not allowed.
- 2-jet rate, "exclusive" measurement imposed on out-of-jet region to enforce 2-jet configuration. (≠ inclusive jets discussed later)

calculation up to NLO: 

LO:  2-jets $q\bar{q}$ contribute

NLO:  all virt. (interference with LO) contribute to 2 jet rate

real:  out-of-cone $k^0 < \varepsilon Q$ (anywhere)

 in cone contribution $k^0 > \varepsilon E$ allowed

⟹ Born: σ_0

NLO: virtual: $$\approx -\sigma_0 \frac{4\alpha_s C_F}{2\pi} \int_0^Q \frac{dk^0}{k^0} \int_{\theta=0}^{\theta=\pi} \frac{d\cos\theta}{1-\cos^2\theta}$$ ← usual factor, cf. IR-safety (propagator & gluon phase space)

all separately divergent $\theta = 0, \pi$ → $\cos^2\theta = 1$

real-out: $$= \sigma_0 \frac{4\alpha_s C_F}{2\pi} \int_0^{\varepsilon Q} \frac{dk^0}{k^0} \int_{\theta=0}^{\theta=\pi} \frac{d\cos\theta}{1-\cos^2\theta}$$

real-in: $$= \sigma_0 \frac{4\alpha_s C_F}{2\pi} \int_{\varepsilon Q}^{Q} \frac{dk^0}{k^0} \left[\int_{\theta=0}^{\theta=\delta} \frac{d\cos\theta}{1-\cos^2\theta} + \int_{\theta=\pi-\delta}^{\theta=\pi} \frac{d\cos\theta}{1-\cos^2\theta} \right]$$

I

- original paper: IR divergences regulated using a non-zero gluon mass (4d)
  → regulator cancels for final result. We'll later discuss dim-reg.
  (breaks gauge invariance but ok if ghost doesn't show up)

↳

- the sum of all contributions is IR-safe:

III $\quad \sigma^{NLO}_{e^+e^-\to 2jets} = \sigma_0 \left(1 - \frac{\alpha_s C_F}{\pi}\left(\underbrace{2\ln\varepsilon \ln\delta^2}_{\text{soft \& collinear "double logarithmic" (*)}} + \underbrace{2\ln\delta^2 \ln 2 + \frac{3}{2}\ln\delta^2 + \frac{\pi^2}{3} - 5/2}_{\text{less singular}}\right.\right.$

- interpretation $\sigma^{e^+e^-\to 2jets} / \sigma_{e^+e^-\to X} = 1 - \frac{\alpha_s C_F}{\pi}\left(2\ln\varepsilon\ln\delta^2 + 2\ln\delta^2\ln 2 + \frac{3}{2}\ln\delta^2 + \frac{\pi^2}{3} - 7/4\right)$

IV $\quad \sigma_{e^+e^-\to X} = \sigma_0(1+\alpha_s/\pi)$

for small α_s, high energies, 2-jet events dominates as ratio $\simeq 1$.
(for reasonable min. values of ε, δ)

- what happens for ε &/or $\delta \to 0$? ($\alpha_s \ln^2 ...$) not small $\to$ higher order terms become relevant $(\alpha_s \ln^2 ...)^n$ $\to$ breakdown of perturbation theory?

  $\Rightarrow$ requires resummation: $\underset{\text{leading-logs (LL)}}{\alpha_s^n \ln^{2n} ...}$ vs. $\underset{\text{next-to-leading (NLL)}}{\alpha_s^n \ln^{2n-1} ...}$ of logarithmic terms to all orders

- active research interests e.g. within SCET see e.g. Chien, Hornig, Lee 1509.0428?
  + NGLs ...

- (*) take soft-collinear limit to see that all divergences are indeed regulated by ε, δ:

III' $\quad \sigma \sim 1 - 4\frac{\alpha_s C_F}{\pi}\int_{\varepsilon Q}^{Q}\frac{dk^0}{k^0}\int_{\delta}^{1}\frac{d\theta}{\theta}$

$\theta \to \pi$: factor 2

$\leftarrow$ 1. $\int_0^{\varepsilon Q} dk^0 \left(-\int_0^1 \frac{d\theta}{\theta} + \int_0^1\frac{d\theta}{\theta}\right) = 0$

2. $\int_{\varepsilon Q}^{Q}\frac{dk^0}{k^0}\left(-\int_0^1\frac{d\theta}{\theta} + \int_0^{\delta}\frac{d\theta}{\theta}\right) = -\int_{\varepsilon Q}^{Q}\frac{dk^0}{k^0}\int_{\delta}^{1}\frac{d\theta}{\theta}$

$\hookrightarrow$ double logarithmic contribution from soft-collinear limit

<u>note:</u> $\neq$ inclusive jet production: only collinear div.

IR - safety

- divergences found for $e^+e^- \to X$ are generic when using pert. methods in QFTs
- singularities arise when internal propagators go on-shell

$$\frac{1}{(k_1+k_3)^2} = \frac{1}{2E_1E_3(1-\cos\theta_{13})}$$

$$\to \frac{1}{2k_1 k_3} = \frac{1}{2E_1E_3 - |\vec{k}_1||\vec{k}_3|\cos\theta_{13}}$$

$$k_1^2 = k_1^{0\,2} - \vec{k}_1^2 = E_1^2 - \vec{k}_1^2 = 0$$

- double logarithmic structure to the cross section $\sigma \sim \int \frac{dE_3}{E_3}\frac{d\theta_{13}}{\theta_{13}} \Rightarrow E_3 \to 0$ or $\theta_{13} \to 0$ (gluon PS.)

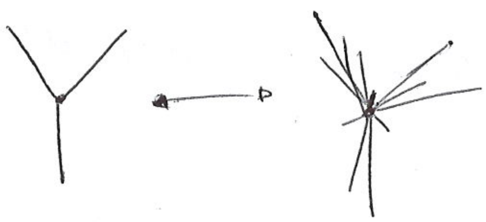
  → use later for S/W jet cross section

- Def. of IR safe observables:

  IR-safe observables are insensitive to soft radiation & collinear splittings.

  (2) observable has to be
  (3) • insensitive to long-dist. physics
  • call for a factorization of long & short dist. phys. to all orders
  (1) so that it can be calculated from first principles in QCD

  Require approx equivalence of



→ jet algorithms act as projection operators gathering all coll & soft splittings projecting onto LO:

Formally for an inclusive observable for e^+e^-:

$$I = \frac{1}{2!}\int d\Omega_2 \frac{d\sigma^2}{d\Omega_2} S_2(p_1^\mu, p_2^\mu) + \frac{1}{3!}\int d\Omega_3 \frac{d\sigma^3}{d\Omega_3} S_3(p_1^\mu, p_2^\mu, p_3^\mu) + \dots$$

n-particle cross sections — measurement functions

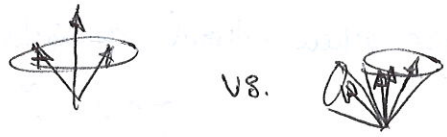
require: $S_{n+1}(p_1^\mu, \dots, (1-\lambda)p_n^\mu, \lambda p_n^\mu) = S_n(p_1^\mu, \dots, p_n^\mu)$ for $0 \le \lambda \le 1$, ($\lambda = 0$ soft, collin als

i.e. the observable is insensitive to whether we have n+1 or n particles in the final state in the limit that the n+1 particles have n-particle kinematics.

- Examples: $e^+e^- \to X$, event shapes like thrust ($S=1$), n-jet rates (e.g. S/W jets... see later)
- will have to keep those requirements in mind when constructing cone & recursive clustering algorithms: these 2 situations should not change the # of jets found in an event

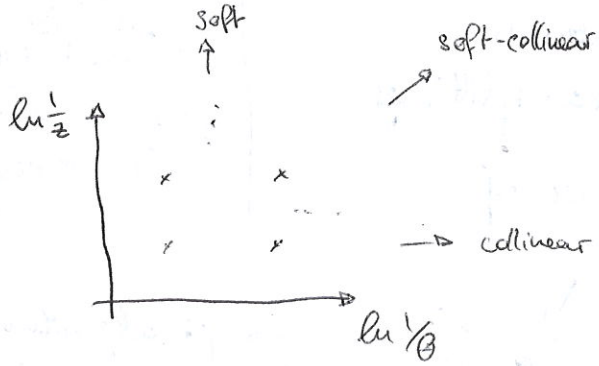
i). add a soft particle

ii) replace one particle with 2 collinear ones



⇒ jet definitions have to be independent of such alterations.
<u>Pert. level</u>: find divergences that do not cancel e.g. traditional cone algorithm at NNLO where cones are only drawn around each particle as seeds.

• Lund plane



$\boxed{\frac{dz}{z}\ \frac{d\theta}{\theta}}$

... see jet substructure later.

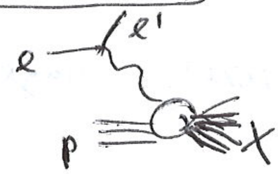
in soft, collinear limit are degenerate states. ~~An observable~~ can't distinguish them. Finite answer only if summed over all those states.

Deep Inelastic Scattering (DIS) [Vogelsang, Stratmann]

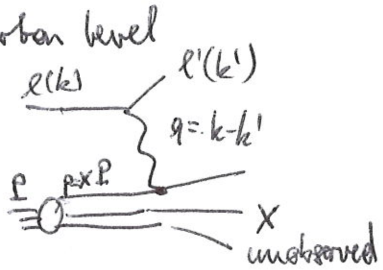
So far only e^+e^- as initial state. But eventually we want to discuss jets in pp & AA $\Rightarrow$ need to understand the initial state. Here asympt. freedom & IR safety are not going to be enough. We'll have to introduce a new concept, $\boxed{\text{"factorization"}}$.

cf. IR "unsafe" but factorizable
also $e^+e^- \to hX$
$\to$ fragmentation functions
$\to$ additional input beyond pert. theory
$\to$ can still make powerful predictions from 1st principles

__Simplest process:__ $lp \to l' + X$ __DIS__



parton level

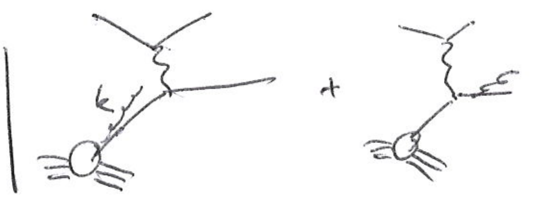


but sensitivity to long-distance physics to emissions along the initial proton direction.

__Factorization__: separate phys. observable into calculable IR safe piece & non-pert. but universal piece, here: PDFs.

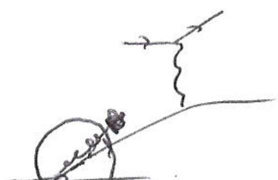
__LO DIS__: $x = \frac{Q^2}{2P\cdot q}$ $\to$ structure function at LO: $F_1(x) = \frac{1}{2}\sum_q e_q^2 [q(x) + \bar{q}(x)]$ ← PDFs

__NLO__:

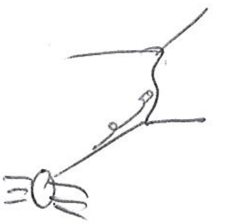


$\Big|\ \ldots + \ldots \Big|^2 + \ldots \Rightarrow \alpha_s \int_0^Q \frac{dk_\perp}{k_\perp} = \infty$

$\Rightarrow$ introduce factorization scale μ:



part of quark distribution $\quad \alpha_s \int_\Lambda^\mu \frac{dk_\perp}{k_\perp} = \alpha_s \ln\left(\frac{\mu}{\Lambda}\right)$ for $k_\perp < \mu$ ← determine from data



part of hard scattering $\quad \alpha_s \int_\mu^Q \frac{dk_\perp}{k_\perp} = \alpha_s \ln\left(\frac{Q}{\mu}\right)$ for $k_\perp > \mu$

↖ introduces cutoff (or dim.-reg.) in pert. calculation $\Rightarrow$ finite but depends on μ

- It is possible to show that this is possible to all orders in α_s:

→ "DIS factorizes" $F_{DIS}(x, Q^2) = q(x,\mu) \otimes C_q(x, \mu/Q)$

$\delta(1-x) + \frac{\alpha_s}{\pi} \ldots$

↳ back to LO

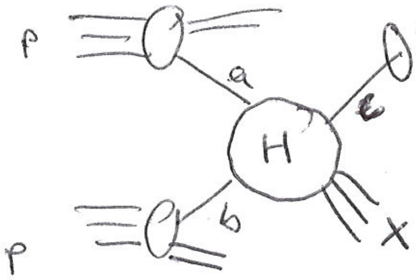
NP object with a perturbative RGE ($\neq$ TMDs...)

"$\otimes$" denotes a convolution product in x

$$\int_x^1 \frac{dx'}{x'} \, q\left(\frac{x}{x'}\right) C_q(x') = q(x) \otimes C_q(x)$$

... continue "solving DGLAP..." page

Factorization in proton-proton collisions



Sterman, Libby
Ellis et al,
Collins, Soper, Sterman

cross section: $\boxed{\text{collinear factorization}}$

universal, same as in DIS

partonic hard scattering

universal

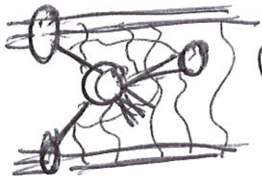
$$\frac{d\sigma}{dp_T\,d\eta} = \sum_{abc} \int dx_a\, dx_b\, dz_c\, f_a(x_a,\mu)\, f(x_b,\mu)\, H_{ab}^{c}(x_a, x_b, z_c, p_T, \eta, \mu)\, D_c(z_c,\mu) + \text{p.c.}$$

Collins, Sivers $\boxed{\neq \text{ exclusive jets}}$ $\boxed{\neq \text{ TMD factorization}}$

NP → fragmentation or
pert. → jet functions

- $H_{ab}^{c(0)}$  $+\ldots$
- $H_{ab}^{c(1)}$  $+\ldots$

- detailed calculation of DY NLO : CTEQ, Potter.
- Proof of factorization is <u>highly non-trivial</u> in pp. Only rigorously shown for DIS, DY, SIDIS
  → established phenomenologically by universality / consistency (e.g. PDF fits)



? → need to show all those cancel at leading power

- Factorization in AA ? ... !
  only nPDFs ...
- e.g. no factorization for pp → dijets in back-to-back limit

---

- <u>Now</u>: what ~~exc~~ jet functions we need here ?

---

p.c. $\mathcal{O}\left(\frac{\Lambda}{p_T}\right)$ or $\mathcal{O}\left(\frac{\Lambda}{z p_T}\right)$

Jet substructure modification

$$\boxed{\frac{d\sigma}{dp_T\,d\eta\,dm_J^2} = \ldots \otimes \mathcal{G}(m_J^2)}$$

replace $\mathcal{J}$

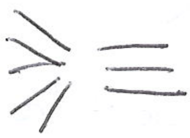
$\mathcal{J}$: only recently clear that there is an actual factorization like that.

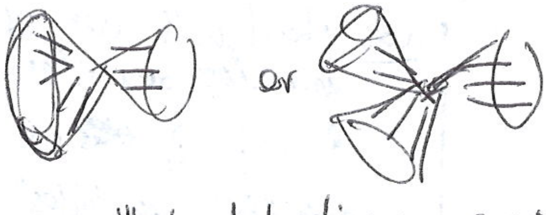
most important ingredient
→ show detailed calculation

Jet algorithms overview

recursive clustering algorithms (cone type later)

- Different than exclusive n-jet rates (S/W e.g.) we now want to find any jet in a given (p_T, η) interval ⇒ inclusive jets.

- 
  how many jets do we find?
  There is no unique answer! We'll have to define what we want to call a jet → jet algorithm.

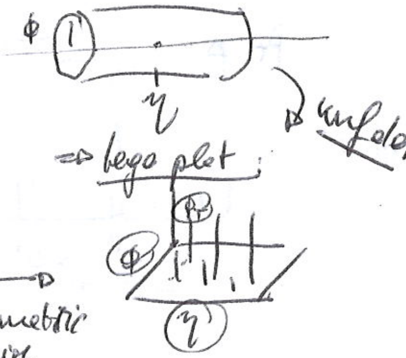


This ambiguity reflects the fact that q & g are not phys. objects (no asympt. sta

- q, g will lead to divergences ⇒ requirement is that the jet algorithm gives an IR safe observable & the algorithm acts as a regulator cf. S/W jets.

- Choice of size (R) depends on what you want to achieve: e.g minimize hadronization or UE contribution (opposite scalings with R), capture all the decay products of a decaying particle, minimize background fluctuations . . .

3 recursive algorithms



Define a distance measure/metric

- between every pair of particles ij

$$d_{ij} = \min\left(p_{Ti}^{2p}, p_{Tj}^{2p}\right) \frac{\Delta R_{ij}^2}{R^2} \quad \text{where } \Delta R_{ij}^2 = \Delta\eta^2 + \Delta\phi^2 \longrightarrow$$

geometric dist in $\eta\phi$-plane

- particle i and beam

$$d_{iB} = p_{Ti}^2$$

then:

1. find the smallest of d_{ij} & d_{iB}
2. if it is d_{ij} → $p_{(ij)} = p_i + p_j$ (recombination scheme) add (ij) to list / remove i, j
3. if d_{iB}: call i a jet & remove from list of particles
4. repeat until no particles are left
5. discard all jets with $p_T < p_T^{min}$

⇒ 2 parameters p_T^{min} & R like ε, δ (S/W jets)

R radius: i & j not merged when $d_{iB} < d_{ij}$ ⇒ if $\Delta R_{ij} > R$

p = 1 : k_T-alg. Ellis, Soper, ~~Dokshitzer~~ ~~Webber~~ Catani

p = 0 : C/A Dokshitzer, Webber

p = −1 : anti-k_T Cacciari, Salam, Soyez

← 1977–2008

jet radius R, esp. for p = −1

Cο

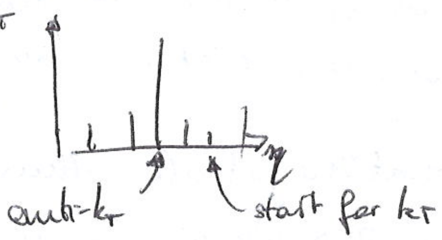
1. k_T-jets: idea is to project onto LO, i.e. go backwards and undo all the splittings that happened during the showering process:

metric requires to start with softest particles

k_T-ordered clustering tree

"problem" fuzzy jets / non-linearity

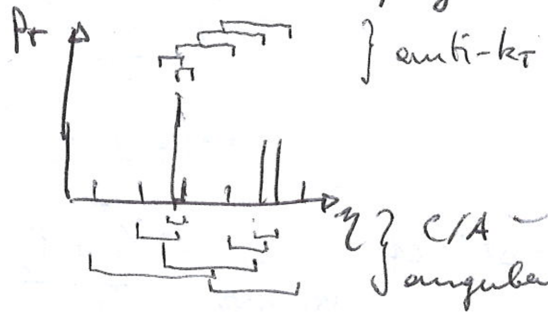
2. anti-k_T: start with hardest / most energetic ones & cluster particles around them. E.g.



→ & work outwards until you hit R

note: start with list of massless particles. After $p_{(ij)} = p_i + p_j$ we get massive "particles" or "protojets"!

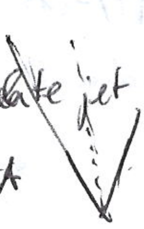
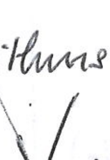
3. C/A: clustering purely based on geometric distance of particles in η-ϕ plane. Can be advantageous for jet substructure. E.g. observe 2-prong structure of a decaying W:



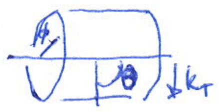
$$\boxed{d_{ij} = \Delta R_{ij}^2 / R^2 \,, \quad d_{iB} = 1}$$

When you recursively go backward through the clustering history, one expects a more significant drop (in energy/p_T) when undoing a clustering step with C/A, should better contain information about 2-prong structure. See for example soft drop mMDT.

- efficient algorithm to apply to given event → FastJet

- How do we relate jet algorithms & factorization? → jet functions (pQCD / SCET)
- E-scheme, WTA  vs.  ⇒ tests different physics! (soft sensitivity)

Kinematics in the narrow jet approximation



ϕ azimuth angle
θ polar angle $\to$ use rapidity $\eta = -\ln\tan\theta/2$
k_T trans. mtm.

where $\left[\begin{array}{l} \theta = \pi/2 \Rightarrow \eta = -\ln 1 = 0 \text{ central} \\ \theta \to 0, \pi \Rightarrow \tan 0 = 0, \tan\pi/2 \to \infty \Rightarrow \eta = +\infty, - \end{array}\right.$
forward/backw

2 massless partons $k_{1,2}$:

- $(k_1+k_2)^2 = +2k_1\cdot k_2 \overset{(*)}{=} 2E_1E_2(1-\cos\theta_{12}) \simeq E_1E_2\theta_{ij}^2$

- parametrize ~~as~~ in terms of k_T, ϕ, θ using $k_{1,2}^2 = 0$:

$$k^\mu_{1,2} = k_{T,12}(\cosh\eta_{12}, \cos\phi_{12}, \sin\phi_{12}, \sinh\eta_{12})$$

$$k_i^2 = k_{T_i}^2(\cosh^2\eta - \cos^2\phi - \sin^2\phi + \sinh^2\eta) = 0$$

$\cosh\eta = \frac{1}{2}(e^{\eta}+e^{-\eta})$
$\sinh\eta = \frac{1}{2}(e^{\eta}-e^{-\eta})$

- $(*)$ becomes: $2k_1\cdot k_2 = 2k_{T_1}k_{T_2}(\cosh\eta_1\cosh\eta_2 - \sinh\eta_1\sinh\eta_2 - \cos\phi_1\cos\phi_2 - \sin\phi_1\sin\phi_2)$

$$= 2k_{T_1}k_{T_2}\big(\cosh\underbrace{(\eta_1-\eta_2)}_{[\,=\Delta\eta} - \cos\underbrace{(\phi_1-\phi_2)}_{=\Delta\phi\,]}\big)$$

$$\simeq 2k_{T_1}k_{T_2}\left(1+\frac{\Delta\eta^2}{2} - \left(1-\frac{\Delta\phi^2}{2}\right)\right) = k_{T_1}k_{T_2}(\Delta\eta^2+\Delta\phi^2)$$

$\Rightarrow$ using $E_i = k_{Ti}\cosh\eta_i$ $\Rightarrow$ $\boxed{\theta_{ij}^2\cosh^2\eta = \Delta\eta_{ij}^2 + \Delta\phi_{ij}^2 = \Delta R_{ij}}$

$\Rightarrow$ write as $\theta_{ij} < \frac{R}{\cosh\eta} \equiv \mathcal{R}$ & w_J has another $\cosh\eta$ $\Rightarrow$ always cancels

---

~~lightcone coordinates~~

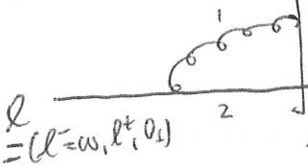
Jet algorithm constraints arXiv: 1001.0014 & 1606.06732

- We'll perform the calculation using lightcone coordinate. Write any 4-vector as $k^\mu = (k^0, \vec{k})$ as $k^\mu = (k^-, k^+, \vec{k}_\perp)$ where $k^+ = k^0 + k^3$, $k^- = k^0 - k^3$

  $\Rightarrow k^2 = k^+ k^- - \vec{k}_\perp^2$

$\boxed{\text{$\omega$: energy/large l.c. of parton initiating the jet}}$

- Consider a collinear splitting of a quark producing a jet:



$q = (q^-, q^+, q_\perp)$

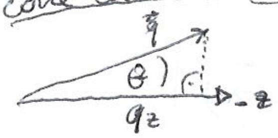
$\ell - q = (\omega - q^-, \ell^+ - q^+, -q_\perp)$ choose a frame where $\ell_\perp = 0$

$\rightarrow \ell^2 = \ell^+ \ell^- \equiv \omega \ell^+$

↑ final state cut $\rightarrow q^2 = 0 = q^+ q^- - \vec{q}_\perp^{\,2} = (q^0)^2 - \vec{q}^{\,2}$ (same for $\ell - q$) $\rightarrow$ on-shell

- cone jets: require $\theta_{1J} < R$ & $\theta_{2J} < R$ s.t. both particles get clustered into the same jet. Here: θ_{iJ} is the angle between particle i & the jet axis
- k_T-type jets: require $\theta_{12} < R$, where θ_{12} is the angle between particles 1,2
- consider a highly energetic jet moving into the "$-z$" direction: $\ell^z < 0$, $\ell^- = \ell^0 - \ell^z \gg \ell^+$
  ↑ large lightcone compon

$\Rightarrow$ cone constraint: particle 1 in the jet:



$$\cos\theta = \frac{|q_z|}{|\vec{q}|} \overset{q \text{ on-shell}}{=} \frac{|q_z|}{q^0} = \frac{\frac{1}{2}(q^- - q^+)}{\frac{1}{2}(q^- + q^+)} = \left[\frac{q^- - q^+}{q^- + q^+}\right] = \frac{1 - q^+/q^-}{1 + q^+/q^-}$$

$$\boxed{\text{check}} \quad = \frac{1 - \tan^2\theta/2}{1 + \tan^2\theta/2} \quad \Rightarrow q^+/q^- = \tan^2\theta/2$$

from $\theta_{1J} < R \Rightarrow$

$$\cos\theta > \cos R$$

$$\frac{1 - \tan^2\theta/2}{1 + \tan^2\theta/2} > \frac{1 - \tan^2\theta/2}{1 + \tan^2\theta/2}$$

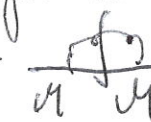
$$\tan^2\theta/2 < \tan^2 R/2$$

$$\boxed{\frac{q^+}{q^-} < \tan^2 R/2}$$ similarly for particle 2:

$$\boxed{\frac{\ell^+ - q^+}{\omega - q^-} < \tan^2 R/2}$$

"$\tan^2 R/2$" often used in SCET literature

cut-diagram notation:

$|\mathcal{E}|^2 =$  real

$\mathcal{M}$ $\mathcal{M}^*$

e.g.  virtual

· Born

$$\Rightarrow \Theta_{cone}^{in} = \Theta(\tan^2 R/2 - q^+/q^-) \quad \Theta\left(\tan^2 R/2 - \frac{\ell^+ - q^+}{\omega - q^-}\right)$$

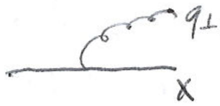
- k_T-type jets: [...] $\Theta_{k_T} = \Theta\left(\tan^2 R/2 - \dfrac{q^+\omega^2}{q^-(\omega-q^-)^2}\right)$

---

note: when both particles are in the jet, we have $\omega = \omega_J \leftarrow$ "observed" jet energy
otherwise only a fraction $z = \omega_J/\omega$ ends up in the jet.

---

now rewrite constraints i.e. $\boxed{q_\perp, x}$



$q_\perp$, $\boxed{1-x = q^-/\omega}$ (long. mtm fraction
[different convention than many energy loss calculations $x \leftrightarrow 1-x$]

$\boxed{1-x = q^-/\omega}$

- Θ_{cone} becomes with $q^2 = 0$ & $q^- = \omega(1-x)$

$$\Theta\left(\tan^2 R/2 - \frac{q_\perp^2}{(q^-)^2}\right) \Rightarrow \Theta((1-x)\omega \tan R/2 - q_\perp)$$

$$\Rightarrow \boxed{\Theta_{cone}^{in} = \Theta((1-x)\omega \tan R/2 - q_\perp) \ \Theta(x\omega \tan R/2 - q_\perp)}$$

$\Rightarrow$ for both particles to be in the jet, the $q_\perp$ wrt. jet axis is required to be smaller than a certain value

The out-of-jet contribution is a little subtle & we'll discuss it later.

- k_T: $\Theta_{k_T} = \Theta\left(\tan^2 R/2 - \dfrac{q_\perp^2}{(q^-(\omega-q^-))^2}\right) \Rightarrow \boxed{\Theta(x(1-x)\tan R/2 - q_\perp)}$

The plus distribution

We want to write $(1-z)^{-1-2\varepsilon}$ in terms of distributions:

I. $\int_0^1 dz \frac{1}{(1-z)^{1+2\varepsilon}} = \frac{\Gamma(1)\Gamma(-2\varepsilon)}{\Gamma(1-2\varepsilon)} \stackrel{\Gamma(x+1)=x\Gamma(x)}{=} \frac{-1}{2\varepsilon}$

$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 dt\, t^{x-1}(1-t)^{y-1}$

II. convolution of $(1-z)^{-1-2\varepsilon}$ with a test function is:

$$\int_0^1 dz \frac{f(z)}{(1-z)^{1+2\varepsilon}} = \int_0^1 dz \frac{f(z)-f(1)+f(1)}{(1-z)^{1+2\varepsilon}} \stackrel{\mathrm{I}}{=} \int_0^1 dz \frac{f(z)-f(1)}{(1-z)^{1+2\varepsilon}} - \frac{f(1)}{2\varepsilon} \quad (*)$$

next: expand $(1-z)^{-1-2\varepsilon}$ in powers of ε:

$$(1-z)^{-1-2\varepsilon} = \frac{1}{1-z} e^{-2\varepsilon \ln(1-z)} = \frac{1}{1-z}(1-2\varepsilon \ln(1-z)+\dots)$$

$\Rightarrow$ (*) becomes:
$$\int_0^1 dz \frac{f(z)}{(1-z)^{1-2\varepsilon}} = \int_0^1 dz \frac{f(z)-f(1)}{(1-z)} - 2\varepsilon \int_0^1 dz \,(f(z)-f(1)) \frac{\ln(1-z)}{1-z}$$
$$- \frac{1}{2\varepsilon}\int_0^1 dz\, f(z)\,\delta(1-z)$$

abstracting from the integrals, we find the identity:

$$\boxed{(1-z)^{-1-\varepsilon} = -\frac{1}{2\varepsilon}\delta(1-z) + \frac{1}{(1-z)_+} - 2\varepsilon\left(\frac{\ln(1-z)}{1-z}\right)_+ + \mathcal{O}(\varepsilon^2)}$$

<u>Note</u>: The "plus distributions" are always defined for $z \in [0,1]$,
For e.g. $\int_A^1 dz \frac{f(z)}{(1-z)_+}$, additional terms $\sim \ln(1-A)\dots$ appear: [e.g. conduct formula

$$\Rightarrow \int_A^1 dz \frac{f(z)}{(1-z)_+} = \int_0^1 dz \frac{f(z)}{(1-z)_+} - \int_0^A dz \frac{f(z)}{1-z} = \int_0^1 dz \frac{f(z)-f(1)}{1-z} - \int_0^A dz \frac{f(z)}{1-z}$$

$\leftarrow$ can drop "+" as $A<1$

$$= \int_A^1 dz \frac{f(z)-f(1)}{1-z} \underbrace{- f(1)\int_0^A \frac{dz}{1-z}}_{= +f(1)\ln(1-A)}$$

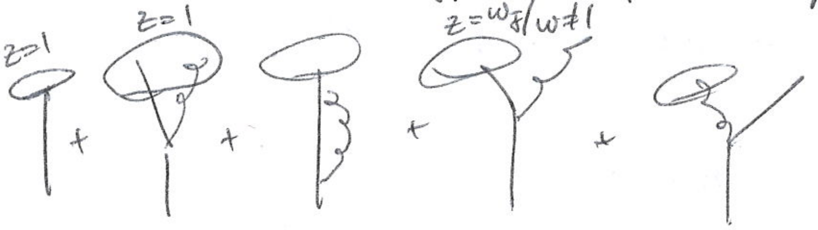
$\hookrightarrow$ <u>schematically</u>: $\frac{1}{(1-z)_+} = \frac{1}{(1-z)_{+A}} + \ln(1-A)\,\delta(1-z)$

& $\left(\frac{\ln(1-z)}{1-z}\right)_+ = \left(\frac{\ln(1-z)}{1-z}\right)_{+A} + \frac{1}{2}\ln^2(1-A)\,\delta(1-z)$

The semi-inclusive jet fct. for k_T jets   arXiv: 1606.06732

see later why

How much energy ends up inside a jet from an initial parton?

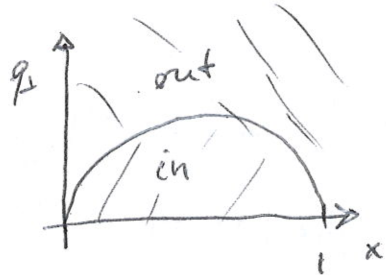


← medium: also additional medium induced out-of-jet energy loss

⇒ need jet alg. constraints

earlier TODAY

- recall: $\Theta_{k_T} = \Theta\left(x(1-x)\omega \tan R/2 - q_\perp\right)$



→ write i.e. $J_{q,g}(z, \omega_J R, \mu)$

- note: momentum conservation requires $\int_0^1 dz\, z J_{q,g}(z,\omega_J R,\mu) = 1.$ $\boxed{\text{check}}$
- we'll work within dimensional regularization & SCET & $\overline{MS}$.
- in-jet contrib.: $J_{q,g}^{(0)} = \delta(1-z)$

$$= C_F\left(\frac{1+x^2}{1-x} - \varepsilon(1-x)\right)$$

NLO: 

$$J_q^{(1)} = \delta(1-z)\frac{\alpha_s}{\pi}\frac{(\mu^2 e^{\gamma_E})^\varepsilon}{\Gamma(1-\varepsilon)}\int_0^1 dx\, \hat{P}_{qq}(x,\varepsilon)\int\frac{dq_\perp}{q_\perp^{1+2\varepsilon}}\Theta_{k_T}$$

from $d=4-2\varepsilon$ phase space

$$\Rightarrow \int\frac{dq_\perp}{q_\perp^{1+2\varepsilon}}\Theta_{k_T} = \int_0^{x(1-x)\omega_J\tan R/2}\frac{dq_\perp}{q_\perp^{1+2\varepsilon}} = -\frac{1}{2\varepsilon}\left(\omega_J\tan R/2\right)^{-2\varepsilon}\left(x(1-x)\right)^{-2\varepsilon}$$

regulate x-integral

$$\Rightarrow J_q^{(1)\,in} = \delta(1-z)\frac{\alpha_s}{2\pi}C_F\left[\frac{1}{\varepsilon^2} + \frac{3}{2\varepsilon} + \frac{L^2}{2} + \frac{3}{2}L + \frac{13}{2} - \frac{3\pi^2}{4}\right]$$

$\boxed{\text{check e.g. in mathematica}}$

where $L = \ln\left(\frac{\mu^2}{\omega_J^2\tan^2 R/2}\right)$

- out-of-jet: quark in the jet,  fraction $z = \omega_J/\omega$ in the jet

$$\Rightarrow \Theta(q_\perp - z(1-z)\omega\tan R/2) = \Theta(q_\perp - (1-z)\omega_J\tan R/2)$$

$$\Rightarrow J_q^{(1)\,out} = \frac{\alpha_s}{\pi}\frac{(\mu^2 e^{\gamma_E})^\varepsilon}{\Gamma(1-\varepsilon)}\hat{P}_{qq}(z,\varepsilon)\int\frac{dq_\perp}{q_\perp^{1+2\varepsilon}}\Theta_{k_T}$$

$$\Rightarrow \int_{(1-z)\omega_J\tan R/2}^{\infty}\frac{dq_\perp}{q_\perp^{1+2\varepsilon}} = \frac{1}{2\varepsilon}\left(\omega_J\tan R/2\right)^{-2\varepsilon}(1-z)^{-2\varepsilon}$$

$(1-z)^{-1-2\varepsilon}$ → introduce plus distributions & expand to $O(\varepsilon^0)$

$$\Rightarrow J^{(1)}_{q\to q(g)}(z, \omega_J R, \mu) = \frac{\alpha_s}{2\pi} C_F \delta(1-z)\left[\frac{-1}{\varepsilon^2} - \frac{L}{\varepsilon} - \frac{L^2}{2} + \cancel{\ldots} + \frac{\pi^2}{12}\right]$$

$$+ \frac{\alpha_s}{2\pi} C_F\left[\left(\frac{1}{\varepsilon} + L\right)\frac{1+z^2}{(1-z)_+} - 2(1+z^2)\left(\frac{\ln(1-z)}{1-z}\right)_+ - (1-z)\right]$$

gluon-in-jet analogously ...

SCET UV pole

single logarithmic!

AP

$$\Rightarrow J_q^{(\ldots)}(z, \omega_J R, \mu) = \delta(1-z) + \frac{\alpha_s}{2\pi}\left(\frac{1}{\varepsilon} + \ln\left(\frac{\mu^2}{\omega_J^2 \tan^2 R/2}\right)\right)\left[P_{qq}(z) + P_{gq}(z)\right]$$

$$- \frac{\alpha_s}{2\pi}\left\{C_F\left[2(1+z^2)\left(\frac{\ln(1-z)}{1-z}\right)_+ + (1-z)\right] - \delta(1-z)\, C_F\left(\frac{13}{2} - \frac{2\pi^2}{3}\right)\right.$$

$$\left. + P_{gq}(z)\, 2\ln(1-z) + C_F z\right\}$$

- gluon similarly
- Renormalization & RG-evolution

multiplicative $\quad J_{i\,\text{bare}}(z, \omega_J R, \mu) = \sum_j \int_z^1 \frac{dz'}{z'} Z_{ij}(z/z', \mu)\, J_j(z', \omega_J R, \mu)$

$\hookrightarrow Z_{ij} = \delta_{ij}\delta(1-z) + \frac{\alpha_s(\mu)}{2\pi}\frac{1}{\varepsilon}P_{ji}(z)$

anomalous dim.: AP splitting fcts. $\Rightarrow$ DGLAP

$$\mu\frac{d}{d\mu} J_i = \frac{\alpha_s(\mu)}{\pi}\sum_j P_{ji} \otimes J_j \quad \Rightarrow \text{solve in Mellin space as before}$$

- scale setting in SCET: choose μ such that large logs are eliminated at fixe order which then sets the initial scale for the evolution

$\mu \sim p_T$

$\uparrow$

$\mu_J \sim p_T R$

Also note: $\mu \sim \omega_J \tan R/2 \equiv \mu_J = (2p_T\cosh\eta)\tan\left(\frac{R}{2\cosh\eta}\right) \approx p_T R$

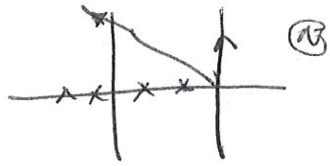
- Note: there are actually 3 large logs: $\ln R$, $\frac{\ln(1-z)}{1-z}$, $\ln z$ ...

Solving DGLAP evolution equations in Mellin space

Andreas Vogt
"Pegasus"
hep-ph/0408244

Mellin transformation $\quad f(N) = \int_0^1 dz\, z^{N-1} f(z)$

$$f(z) = \int_{C_N} \frac{dN}{2\pi i} z^{-N} f(N) \quad \leftarrow \text{ contour integral in cplx. plane}$$



Take Mellin moments of a convolution integral:

$$\int_0^1 d\xi\, \xi^{N-1} \int_\xi^1 \frac{dx}{x} f(\xi/x)\, g(x) = [f \otimes g(\xi)](N)$$

$$= \int_0^1 d\xi\, \xi^{N-1} \int_0^1 dx \int_0^1 dy\, f(y) g(x)\, \delta(xy-\xi) = \left(\int_0^1 dx\, x^{N-1} g(x)\right)\left(\int_0^1 dy\, y^{N-1} f(y)\right)$$

$= g(N)\cdot f(N)$ simple product in Mellin space.

$\Rightarrow$ back to DGLAP: $\quad F_{DIS}(x,Q^2) = q(x,\mu) \otimes C_q(x, \mu/Q) \sim \sigma$ cross section

$\Rightarrow$ Mellin space $\quad F_{DIS}(N,Q^2) = q(N,\mu)\cdot C_q(N,\mu/Q)$

Now the final phys. observable needs to be independent of μ $\rightsquigarrow$ RG eq.: DGLAP

$$\frac{d}{d\ln\mu} F(x,Q^2) = 0 \quad \Rightarrow \quad \frac{dq(N,\mu)}{d\ln\mu}\cdot C_q(N,\mu/Q) + q(N,\mu)\frac{dC_q(N,\mu/Q)}{d\ln\mu} = 0$$

write as: $\quad \frac{d\ln C_q(N,\mu/Q)}{d\ln\mu} = \boxed{\frac{d\ln q(N,\mu)}{d\ln\mu} = \frac{\alpha_s}{2\pi} P_{qq}(N)}$

DGLAP

$$\int_0^1 dz\, z^{N-1}\left(\frac{1+z^2}{1-z}\right)_+ = P_{qq}(N)$$

$\Rightarrow$ solve: $\quad q(N,\mu) = q(N,\mu_0)\exp\left[\frac{\alpha_s}{2\pi} P_{qq}(N) \ln\left(\frac{\mu}{\mu_0}\right)\right]$ $\Big\}$ for fixed coupling.

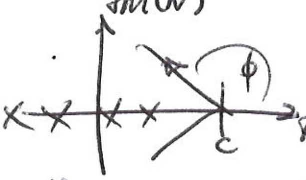
$\Rightarrow$ once we know $q(N,\mu_0)$ we can predict them at any $\mu > \mu_0$.

[Note: also jet functions give rise to DGLAP evolution equations.]

similar for the jet function

Resummation:

$$\exp[\ldots] = 1 + \frac{\alpha_s}{2\pi} P_{qq}(N) \ln\frac{\mu}{\mu_0} + \left(\frac{\alpha_s}{2\pi} P_{qq}(N)\ln\frac{\mu}{\mu_0}\right)^2 + \ldots$$



parametrize contour & evaluate numerically

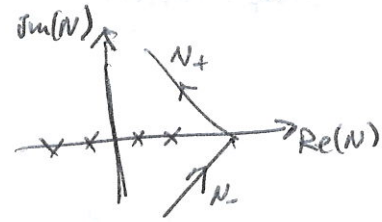
- Mellin inverse: $\quad f(z) = \int \frac{dN}{2\pi i} z^{-N} f(N)$

$$\circledast \quad = \frac{1}{\pi}\int_0^\infty dz\, \mathrm{Im}\left\{ e^{i\phi} z^{-c - z e^{i\phi}} f(N = c + z^{i\phi})\right\}$$

(*) $\frac{1}{2\pi i}\int_{C_N} dN\, z^{-N} f(N)$ choose $N_+ = c + z e^{i\phi}$

$N_- = c + z e^{-i\phi} = N_+^*$

$dN = e^{\pm i\phi} dz$



$$= \frac{1}{2\pi i}\left[\int_0^\infty dz\, e^{i\phi} z^{-N_+} f(N_+) + \int_\infty^0 dz\, e^{-i\phi} z^{-N_-} f(N_-)\right]$$

$$= \frac{1}{2\pi i}\left[\text{—"—} - \int_0^\infty dz\, e^{-i\phi} z^{-N_+^*} f^*(N_+)\right]$$

$f(N_+^*) = f^*(N_+)$

$$= \frac{1}{\pi}\int_0^\infty dz\, \mathrm{Im}\left(e^{i\phi} z^{-N} f(N)\right)$$

$N \equiv N_+$

---

di-gamma function

Example: $\int dz\, z^{N-1} \frac{1+z^2}{(1-z)_+} = -2(\psi(N)+\gamma_E) - \frac{1+2N}{N(N+1)}$ (check)

analytic continuation to the cplx. plane (see inverse trafo above)

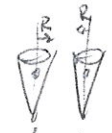
---

cone jets: the siJF arxiv: 1705.05375

[check $\int_0^1 dz\, z\, J_i = 1$ for both]

1. The traditional cone algorithm: find cones around a stable axis of dominant momentum flow with all particles as seeds.

e.g. 2-particle final state:

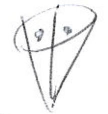


2 jets for $d > 2R$

overlap but identified as two jets

$R < d < 2R$

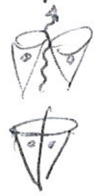
} 2 jets



$d < R$: original cones get "merged" because contain 2 particles → find new stable cone/axis that contains both

⇒ procedure only depends on dist. of particles 1&2 : θ_{ij} ⇒ same as k_T-type alg!

⇒ only at NLO IR safe. NNLO & beyond: $R < d < 2R$

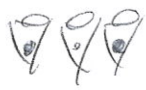


adding infinitely soft particle leads to

⇒ IR unsafe

2. midpoint / SISCone (seedless cone alg.)

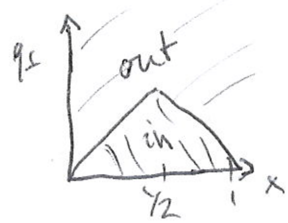
midpoint: add additional seed in the middle



⇒ for $R < d < 2R$: 1 jet

SISCone: same @ NLO, different @ NNLO & beyond: look for any stable cone that you can possibly find (→ w/o seeds)

⇒ siJF for 2nd case: recall in-jet: $\theta((1-x)\omega \tan R/2 - q_\perp)\, \theta(x\omega \tan R/2 - q_\perp)$



out-of-jet region: ① $\theta(z<1/2)\, \theta(q_\perp > \omega_J \tan R/2)$

② $\theta(z>1/2)\, \theta(q_\perp > \omega_J \frac{(1-z)}{z} \tan R/2)$

final result:
$$J_q^{\text{cone}}(z,\omega_J R,\mu) = \delta(1-z) + \frac{\alpha_s}{2\pi}\Big\{ \ln\Big(\frac{\mu^2}{\omega_J^2 \tan^2 R/2}\Big)\big[P_{qq}(z) + P_{gq}(z)\big] - 2C_F(1+z^2)\Big(\frac{\ln(1-z)}{1-z}\Big)_+$$
$$- C_F z - 2P_{gq}\ln(1-z) + C_F\Big(\frac{7}{2} + 3\ln 2 - \frac{\pi^2}{3}\Big)\delta(1-z)$$
$$+ 2\big[P_{qq}(z) + P_{gq}(z)\big]\big[\theta(z>1/2)\ln z + \theta(z<1/2)\ln(1-z)\big]$$

NOW: ~~complete~~ factorization?

Factorization for ~~inclusive~~ & ~~exclusive~~ jets and their substructure

1\. inclusive : $\boxed{pp \to \text{jet} + X}$ or $pp \to j_1 j_2 + X$ -- sum over everything else X
here: or $pp \to Z + \text{jet} + X$ in the final state besides the observed jets.
→ heavy ion!

- $$\frac{d\sigma}{d\eta\, dp_T} = \sum_{abc} f_a(x_a,\mu) \otimes f_b(x_b,\mu) \otimes H_{ab}^c(x_a, x_b, \eta, p_T, z_c, \mu) \otimes \underbrace{J_c(z_c, p_T R, \mu)}_{\text{si-JF}}$$

Same as for example $pp \to h + X$. All RG equations DGLAP.

NLO structure: $A + B \ln R + \underbrace{C R^2}_{\text{generally small}}$ ⇐ "lucky" that it is small (more stric... intuitively: soft-collinear nature of ...

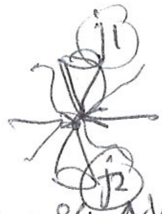
all order structure: single logs $\alpha_s^n \ln^n R$ → DGLAP

- jet substructure

$$\frac{d\sigma}{d\eta\, dp_T\, dv} = \dots\; \mathcal{G}_c(z_c, p_T R, v, \mu)$$

2\. exclusive e.g. $e^+e^- \to$ di-jet.

For example Sterman/Weinberg jets ; energy cut ε on the out-of-jet radiation.



$J_1(v, R, \mu) \otimes J_2(v, R, \mu) \otimes S(v, \Lambda, R) \otimes S_{sc}(v, R)$

⇒ Factorization: $\dfrac{d\sigma}{dv} = J_1(v)\, J_2(v)\, \underline{S(v,\Lambda)}\, H$

soft

hard functions ($\neq$ above)
→ virtual corrections

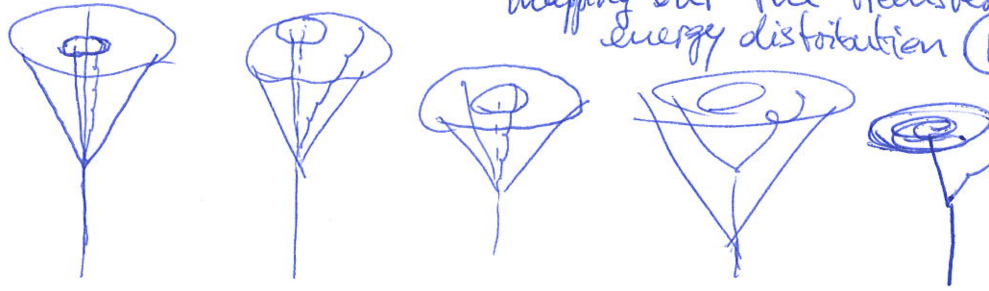
in-jet
combination of si-JF: $\alpha_s^n \ln^{2n} R$

$\boxed{\text{soft function!}}$

pp: typically p_T^{cut}, η^{cut} instead of ε, Λ

The jet shape

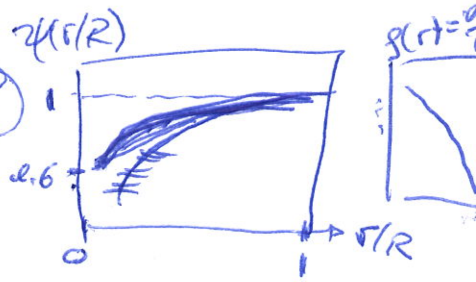
mapping out the transverse energy distribution (...



(1) • draw cone around jet axis of size r
what's the energy/p_T in it?
• outside jet R

[arXiv: 1705.05375, 1405.4293 ...]
incl., k_T fact. excl.

(3)



actually k_T factorization, SCET$_{II}$
(any transverse soft sensitive...)

"differential jet shape"
$\rho = \frac{d\psi(r)}{dr}$ → slope

(2) what are the jet algorithm constraints?

i) both in outside jet
$q_\perp < z(1-z)\omega\tan R/2$

ii) both in inner cone (like cone jets)
$q_\perp < z\omega\tan r/2$ & $q_\perp < (1-z)\omega\tan r/2$

or only one of them & the other one with ">"

• resummation $\ln R$, $\ln r/R$ | hard-coll.-soft | track based... → additional NP sensitivity | Yan-Jie earlier today

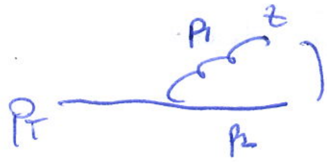
... jet fragmentation tomorrow. [hard-coll.]
... need factorization to understand to what scales a particular observable is sensitive to.

The jet mass   arxiv: 1709.06195 & 1801.00790 & 1803.03645

& quark/gluon tagging

Let's consider a jet with $R=1$ for simplicity

Let's consider a quark jet where one gluon gets emitted inside the jet



) θ opening angle, energy fraction z, initial parton/jet with p_T

$\to$ determine the jet mass:

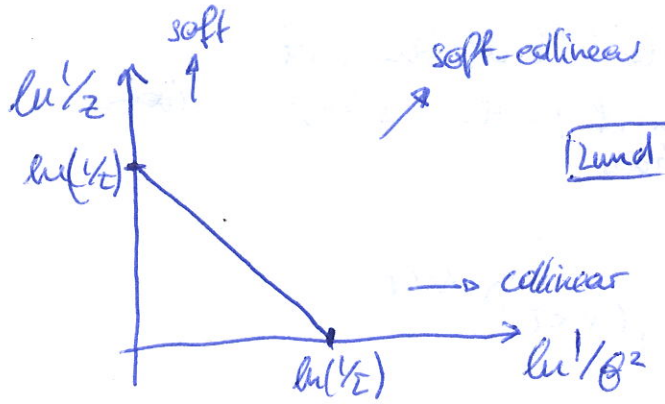
$$m_J^2 = (p_1+p_2)^2 = 2p_1\cdot p_2 = 2E_1E_2(1-\cos^2\theta)$$

we consider the soft collinear limit to get the LL contribution:

$$m_J^2 \approx z(1-z)\,p_T^2\theta^2 \simeq z p_T^2\theta^2$$

& let's introduce τ (dimensionless): $\tau = \frac{m_J^2}{p_T^2} = z\theta^2$

- What are the allowed emissions? Recall $\int \frac{dz}{z}\frac{d\theta}{\theta}$



$\boxed{\text{Lund}}$

$\tau = z\theta^2$

fixed $\tau \to$ line in $(\ln 1/z, \ln 1/\theta)$ plane

$\ln(1/\tau) = \ln(1/z) + \ln(1/\theta^2)$

cumulative: anything above this line $\triangle$ is allowed.

- what is the cross section? $\frac{d\sigma}{d\tau}$

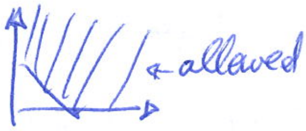
$$\frac{2\alpha_s}{\pi} C_F \int_0^1 \frac{d\theta}{\theta}\int_0^1 \frac{dz}{z}\,\underbrace{\delta(\tau - z\theta^2)}_{=\frac{1}{\theta^2}\delta(z-\tau/\theta^2)} = \frac{2\alpha_s C_F}{\pi}\int_{\sqrt{\tau}}^1 \frac{d\theta}{\theta}\frac{1}{\tau} = \frac{\alpha_s C_F}{\pi}\frac{\ln(1/\tau)}{\tau} \approx \frac{d\sigma^q}{d\tau}$$

& $z<1 \Rightarrow \tau/\theta^2 < 1$

& $\frac{d\sigma^g}{d\tau} \sim \frac{\alpha_s C_A}{\pi}\frac{\ln(1/\tau)}{\tau}$ ... limit $\tau \to 0$? resummation

+ complete factorization

Cumulative distribution: cross section to be less than τ

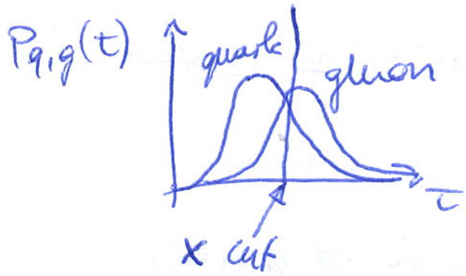


$$P(\tau' < \tau) = \exp\left[-\frac{\alpha_s}{\pi}\frac{C_F}{2}\ln^2(1/\tau)\right]$$ & similarly for gluons

$$\Rightarrow p_q(\tau) = \frac{d}{d\tau}P_q(\tau'<\tau) = \frac{\alpha_s C_F}{\pi}\frac{\ln(1/\tau)}{\tau}\exp\left[-\frac{\alpha_s}{\pi}C_F\ln^2(1/\tau)\right]$$

$\longrightarrow 0$ for $\tau \to 0$

Qualitatively, we find:



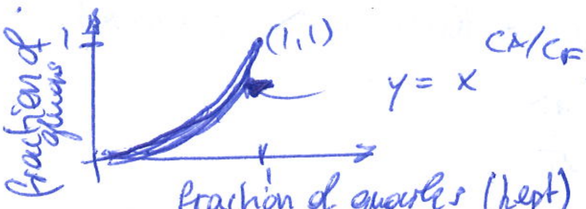
Quark-gluon discrimination

make a cut on τ and keep events only left of the cut
$\Rightarrow$ quark enhanced jet sample. The fraction of q, g that is kept is given by the respective cumulative distributions:

$$P_q(x<\tau) = \exp\left(-\frac{\alpha_s}{\pi}\frac{C_F}{2}\ln^2(1/\tau)\right)$$

$$P_g(x<\tau) = \exp\left(-\frac{\alpha_s}{\pi}\frac{C_A}{2}\ln^2(1/\tau)\right) = \left(P_q(x<\tau)\right)^{C_A/C_F}$$

$\Rightarrow$ ROC-curve analytically



$$y = x^{C_A/C_F} = x^{9/4}$$

$\Rightarrow$ study energy loss for q & g separately ... M1...

Grooming e.g.

1. **trimming**: recluster a given jet with a smaller jet radius $r < R$ & remove all subjets that have an energy fraction $z_r < z_{cut}$

arXiv: 0912.1342

→ take all remaining particles in the subjets with $z_r > z_{cut}$ & determine the ~~jet~~ groomed jet mass for them.
(assemble remaining subjets into the "trimmed jet")

2. **soft drop declustering** arXiv: 1402.2657

1. recluster an anti-k_T jet with C/A → angular ordered clustering tree
2. step backward through clustering history & remove branches until the soft drop criterion is met

example 2-prong structure

$$\frac{\min(p_{Ti}, p_{Tj})}{p_{Ti} + p_{Tj}} > z_{cut} \left(\frac{\Delta R_{ij}}{R}\right)^{\beta}$$

- $\Delta R_{ij} = R$ → just z_{cut} → "hard" threshold
- softer criterion if ΔR_{ij} is small

z_{cut}: soft threshold

β: angular exponent → target specifically wide angle soft radiation

1-prong: QCD jet
2-prong: W, Z, H } boosted decay products
3-prong: top

- no grooming for $\beta \to \infty$

3. measure any observable (e.g. jet mass) on remaining constituents.

- $z_g = \frac{\min(p_{Ti}, p_{Tj})}{p_{Ti}+p_{Tj}}$ when grooming stops ... Sudakov safe, ~ splitting fct.

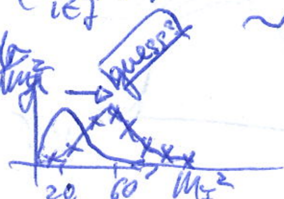
advantage:
- can include soft drop declustering in QCD factorization theorems.
- reduces sensitivity to NGLs
- largely remove NP component from jets: hadronization, MPI, pile up

Note: only relevant for "soft sensitive" observables.

why grooming? (pp) $m_J^2 = \left(\sum_{i \in J} p_i\right)^2$

why? $\sim \frac{1}{z}$ soft at any angle

factor of 3
2012



... NP correction!

(3) pruning   S. Ellis et. al.

- rerun alg. for a given jet.
- At each clustering step $i,j \to p$ check:

$$z = \frac{\min(p_{Ti}, p_{Tj})}{p_{Tp}} < z_{cut} \quad \& \quad \Delta R_{ij} > D_{cut}$$

| soft | | wide angle |

- Do not merge if conditions are met. Discard softer branch & continue clustering.
- depends on 2 parameters like soft drop. (Trimming only 1)

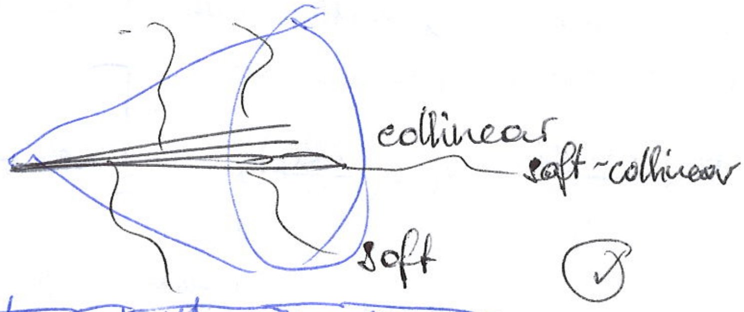
z_g, R_g:

$$z_g = \frac{\min(p_{T1}, p_{T2})}{p_{T1} + p_{T2}}$$

$$R_g = \frac{\Delta R_{12}}{R}$$

$\frac{d\sigma}{dz_g} \sim P(z_g)$ AP (see yesterday)

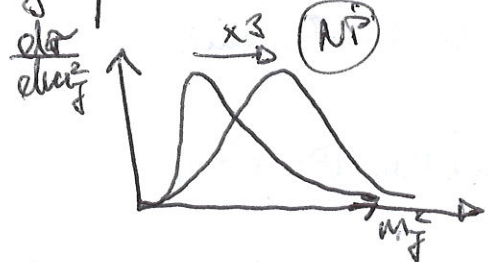
Sudakov safety $\sqrt{\alpha_s}$ | P_{ji}^{med} needed for all processes |

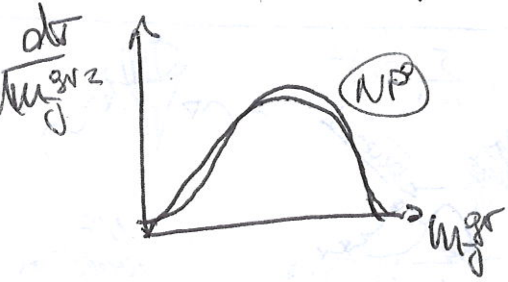


only soft drop used in HI

why grooming? soft sensitivity
↳ reduce NP contribution
- hadronization
- MPI, UE
- pile up

e.g. jet mass:





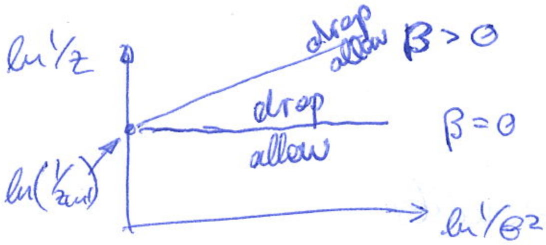
much less than above

⇒ soft physics enhances jet mass

The soft drop groomed jet mass   arXiv: 1803.03645, 1402.2657

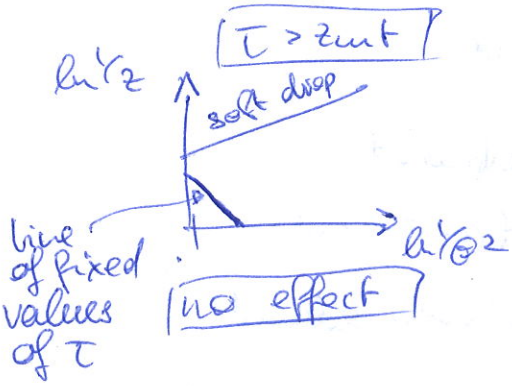
soft drop grooming condition: $z > z_{cut}\,\theta^{\beta}$

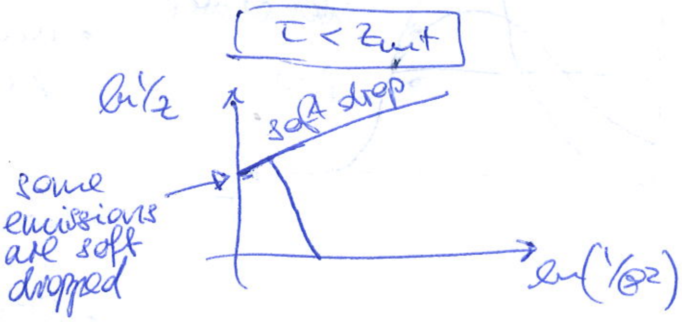
$\rightarrow$ write as $\ln 1/z < \ln\left(\frac{1}{z_{cut}}\right) + \beta/2 \ln 1/\theta^2$



any emissions above those lines (depending on β) will be soft dropped.

$\Rightarrow$ what does it mean for the groomed jet mass? ... depends on τ





$\Rightarrow$ redo calculation: soft-collinear, LL, quark-jet:

$$\frac{\alpha_s}{\pi} C_F \int_0^1 \frac{d\theta}{\theta} \int_0^1 \frac{dz}{z}\, \delta(\tau - z\theta^2)\, \Theta(z - z_{cut}\theta^{\beta})$$

$\delta \rightarrow \theta > \sqrt{\tau}$

$\Theta \rightarrow$ $z > z_{cut}\,\theta^{\beta}$; $\tau/\theta > z_{cut}\,\theta^{\beta}$; $\tau/z_{cut} > \theta^{2+\beta}$; $\theta < (\tau/z_{cut})^{\frac{1}{2+\beta}}$

$$= \frac{\alpha_s C_F}{\pi} \int_{\sqrt{\tau}}^{(1,(\tau/z_{cut})^{\frac{1}{2+\beta}})} \frac{d\theta}{\theta}$$

$$= \frac{\alpha_s C_F}{\pi}\left(\frac{\beta}{2+\beta}\frac{\ln(1/\tau)}{\tau} + \frac{2}{2+\beta}\frac{\ln(1/z_{cut})}{\tau}\right)$$

recall ungroomed: $\frac{\alpha_s C_F}{\pi}\frac{\ln(1/\tau)}{\tau}$

$\tau < z_{cut}$ $\leftarrow$ transition point at $\tau = z_{cut}$ $\bullet$ $\tau \geq z_{cut}$

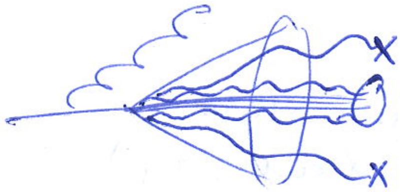
consider limits: $\bullet$ $\beta = 0$ mMDT $\frac{\alpha_s C_F}{\pi}\frac{2}{2+\beta}\frac{\ln(1/z_{cut})}{\tau}$ $\rightarrow$ LL changed!

- $\beta \to \infty$: remove the groomer $\to$ ungroomed case.

"Exponentiation" : resummation generally requires $\ln R$, $\ln(1/\tau)$, $\ln(1/z_{cut})$

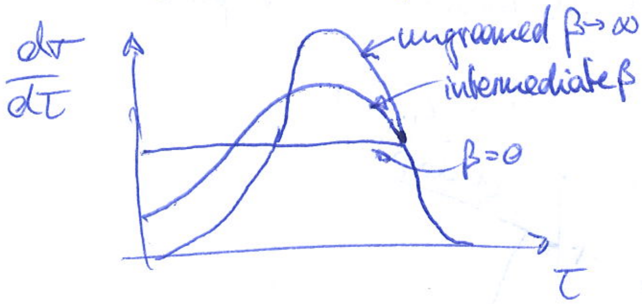
... see Ref. above using RG evolution equations.

sketch:



- grooming is only relevant for "soft sensitive" observables

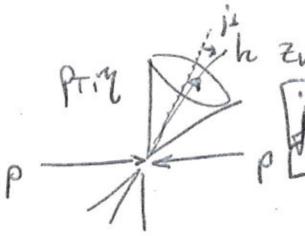
$\Rightarrow$ e.g. hadron-in-jet collinear for intermediate z_h : grooming is irrelevant



- more aggressive grooming
  $\to$ more significant
- transition point @ $\tau = z_{cut}$
- NP contrib. significantly reduced.

Hadron-in-jet distributions

arXiv: 1606.07063, 1705.05375, 1506.01415

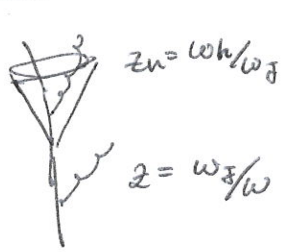


$z_h = p_T^h / p_T$ longitudinal mtm distribution inside jets

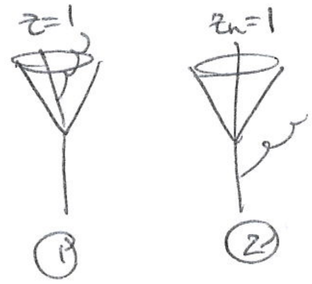
[$j_\perp$ transverse mtm distribution —"— w.r.t. a predetermined axis]

} longit. & transverse ⇓ map of jet energy dist.

Longitudinal case: 2-step process $\mathcal{G}$:



→ @ NLO:



Hard-collinear factorization

$$\frac{d\sigma}{dp_T d\eta dz_h} = f_a \otimes f_b \otimes H_{ab} \otimes \mathcal{G}_c^h$$

LO ~ $\delta(1-z)\,\delta(1-z_h)$

NLO: same set of diagrams as for siJF before

(1) $$\mathcal{G}_{i\to jk}(z, z_h, \omega_J R, \mu) = \delta(1-z)\,\frac{\alpha_s}{\pi}\,\frac{(e^{\gamma_E}\mu^2)^\varepsilon}{\Gamma(1-\varepsilon)}\,\hat{P}_{ji}(z_h,\varepsilon)\int\frac{dq_\perp}{q_\perp^{1+2\varepsilon}}\,\Theta_{alg.}$$

(2) $$\mathcal{G}_{i\to j(k)}(z, z_h, \omega_J R, \mu) = \delta(1-z_h)\,\frac{\alpha_s}{\pi}\,\frac{(e^{\gamma_E}\mu^2)^\varepsilon}{\Gamma(1-\varepsilon)}\,\hat{P}_{ji}(z,\varepsilon)\int\frac{dq_\perp}{q_\perp^{1+2\varepsilon}}\,\Theta_{alg.}$$

⇒ find $$\mathcal{G}_i(z, z_h, \omega_J R, \mu) = \sum_j \mathcal{J}_{ij}(z, z_h, \omega_J R, \mu)\otimes D_j(z_h,\mu)$$

& RG: DGLAP again: $$\mu\frac{d}{d\mu}\mathcal{G}_i = \frac{\alpha_s}{\pi}\sum_j P_{ji}\otimes\mathcal{G}_j$$

emerging pattern for inclusive jet production

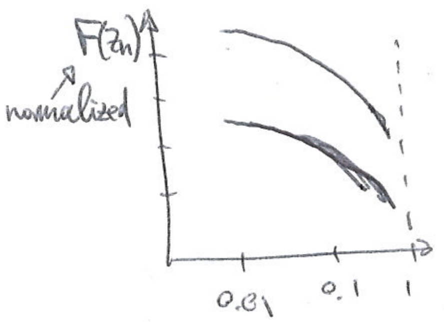
NP-fragmentation function which "absorbs" remaining IR divergence (also DGLAP)

2x DGLAP

$\mu \sim p_T$

$\mu_J \sim p_T R$

1 GeV



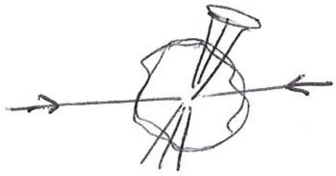
⚠ hard-collinear factorization (✓) for a jet substructure observable

→ important benchmark for HI & consistency checks

Jets in heavy-ion collisions

arXiv: 1701.03939

Jets & their substructure are unique probes of the QGP because the baseline pp is under control within pQCD



factorization: $f_a \otimes f_b \otimes H_{ab}^c \otimes J_c$ (pp)

requires an in-medium jet function $J_c^{vac} \to J_c^{med} + J_c^{v}$
(does factorization hold in AA? At least approximately?)

$\uparrow$ medium modified jet fct.

model independent & completely general for any jet / jet substructure observable that allows for a hard-collinear factorization

$\to$ test universality, consistency, factorization

$\Rightarrow$ medium part: $|A_{SB}|^2 + 2\,\mathrm{Re}\{A_{DB}^{*} A_{Born}\}$

- recall that the siJF is given in terms of integrals over splitting functions:

$$P_{qq}^{vac}(z, q_\perp) = \frac{\alpha_s}{\pi} C_F \frac{1+z^2}{1-z} \frac{1}{q_\perp}$$

- in the medium we generally have: $P_{ji}^{med} = P_{ji}(z, q_\perp)\, \underbrace{f_{ji}(z, q_\perp; \beta)}_{\text{e.g. } SCET_G \text{ or similar or fit.}}$ $\leftarrow$ medium modified splitting fct

- need to evaluate numerically & use cut off scheme rather than dim-reg. Only one UV sing. $\frac{1}{\varepsilon}$ or use μ:

- write the different contributions as:

here: $\hat{p}_T = p_T / z$

1.  $= \delta(1-z) \int_0^1 dx \int_{\mu_{IR}}^{x(1-x)\hat{p}_T R} dq_\perp\, P_{qq}(x, q_\perp)$

2.  $\overset{virt}{=} -\delta(1-z) \int_0^1 dx \int_{\mu_{IR}}^{\mu_{UV}} dq_\perp\, P_{qq}(x, q_\perp)$

(1.) + (2.) $= -\delta(1-z) \int_0^1 dx \int_{x(1-x)\hat{p}_T R}^{\mu_{UV}} dq_\perp\, P_{qq}(x, q_\perp)$

3.  $= \int_{z(1-z)\hat{p}_T R}^{\mu_{UV}} dq_\perp\, P_{qq}(z, q_\perp)$

$\Rightarrow$ $(1.) + (2.) + (3.) = \left[\int_{z(1-z)\hat{p}_T R}^{\mu_{UV}} dq_\perp\, P_{qq}(z, q_\perp) \right]_+$ where $\int_0^1 dz\, f(z) [g(z)]_+ = \int_0^1 dz\, [f(z) - f(1)]\, g$

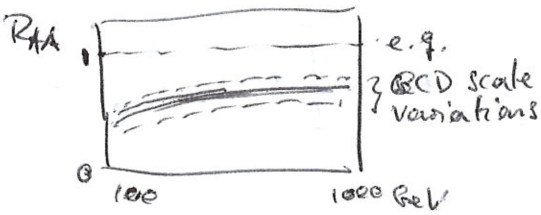
$\nwarrow$ $z \to 1$ divergence regularized.

4.  $= \int_{z(1-z)\hat{p}_T R}^{\mu_{UV}} dq_\perp \, P_{gq}(z,q_\perp)$ ← integrable for $z \to 1$ (cf. vacuum)

$$\Rightarrow \; J_q^{\mathrm{med}\,(1)}(z,\hat{p}_T R,\mu) = \left[\int_{z(1-z)\hat{p}_T R}^{\mu} dq_\perp \, P_{qq}(z,q_\perp) \right]_+ + \int_{z(1-z)\hat{p}_T R}^{\mu_{UV}} dq_\perp \, P_{gq}(z,q_\perp)$$

Then calculate convolution with $d\sigma^{(0)}_{q,g}$, i.e. [gluon similarly]

$$d\sigma_{AA}^{\mathrm{med}} = \sum_{i=q,\bar{q},g} d\sigma_i^{(0)} \otimes J_i^{\mathrm{med.}}$$



→ $\frac{pp+AA}{pp} = R_{AA}$

Note: jet substructure more sensitive to P_{ji}^{med} in general → better probe

→ collinear hadron/subjet dist ✓ jet mass, shape, grooming ⊗