Scattering Amplitudes and Feynman integrals

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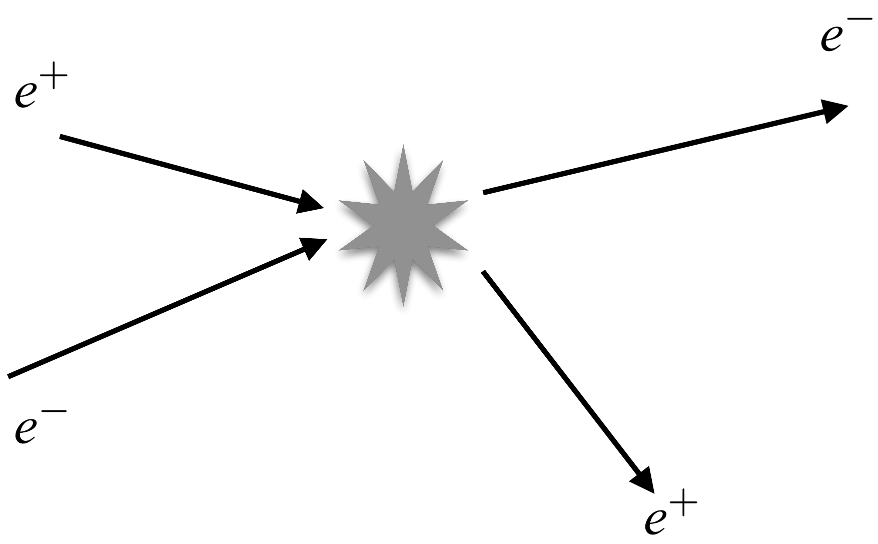
Outline

1. Scattering amplitudes, an formal overview
2. Feynman integral representation and reduction
3. Feynman integral evaluation

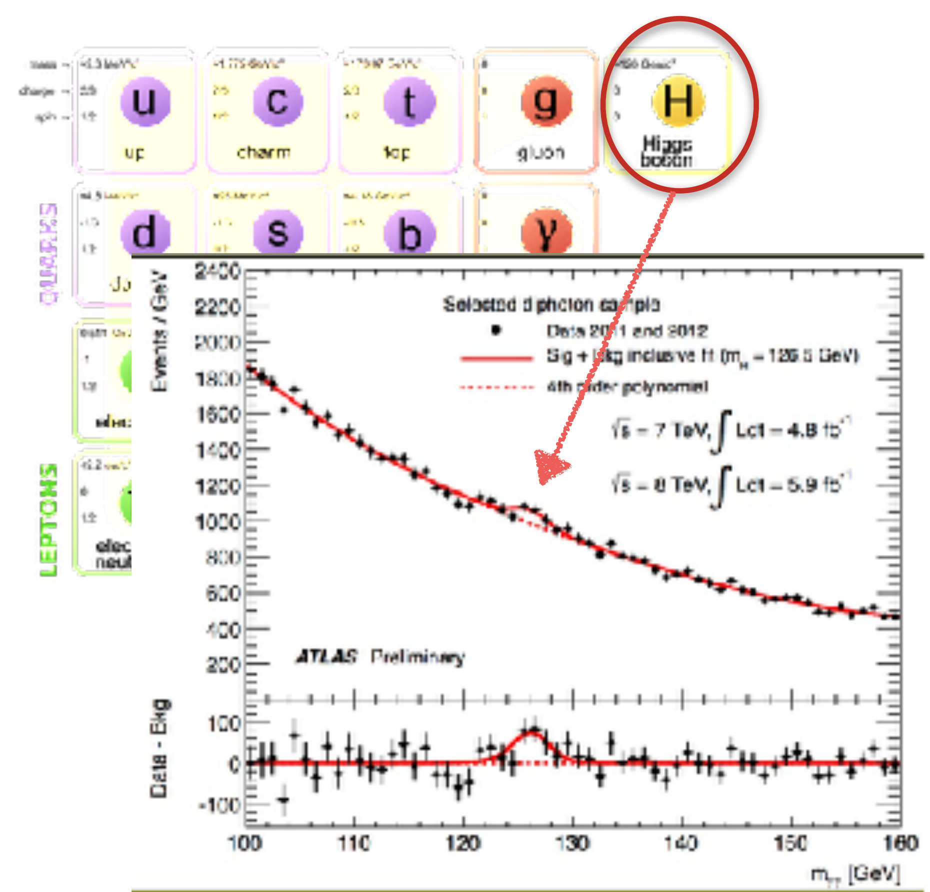
1. Scattering amplitudes

Quantum transit rate for particle scattering



calculable by perturbation theory
in quantum field theory

comparable with the measured
cross sections from experiments



How can scattering amplitude become a branch of theoretical physics ?

A. Surprisingly simple structures

Britto-Cachazo-Feng-Witten (BCFW)
Kawai-Lewellen-Tye (KLT)
Bern-Carassco-Johannson (BCJ) ....

B. Brave exploration of amplitudes in formal theories

amplitudes in N=4 super-Yang-Mills
amplitudes in N=8 supergravity

C. Precision frontier of the LHC physics

multi-loop amplitudes in QCD and the standard model

Surprisingly simple structures in amplitudes

Britto-Cachazo-Feng-Witten (BCFW)

Nucl.Phys. B715 (2005) 499-522
Phys.Rev.Lett. 94 (2005) 181602



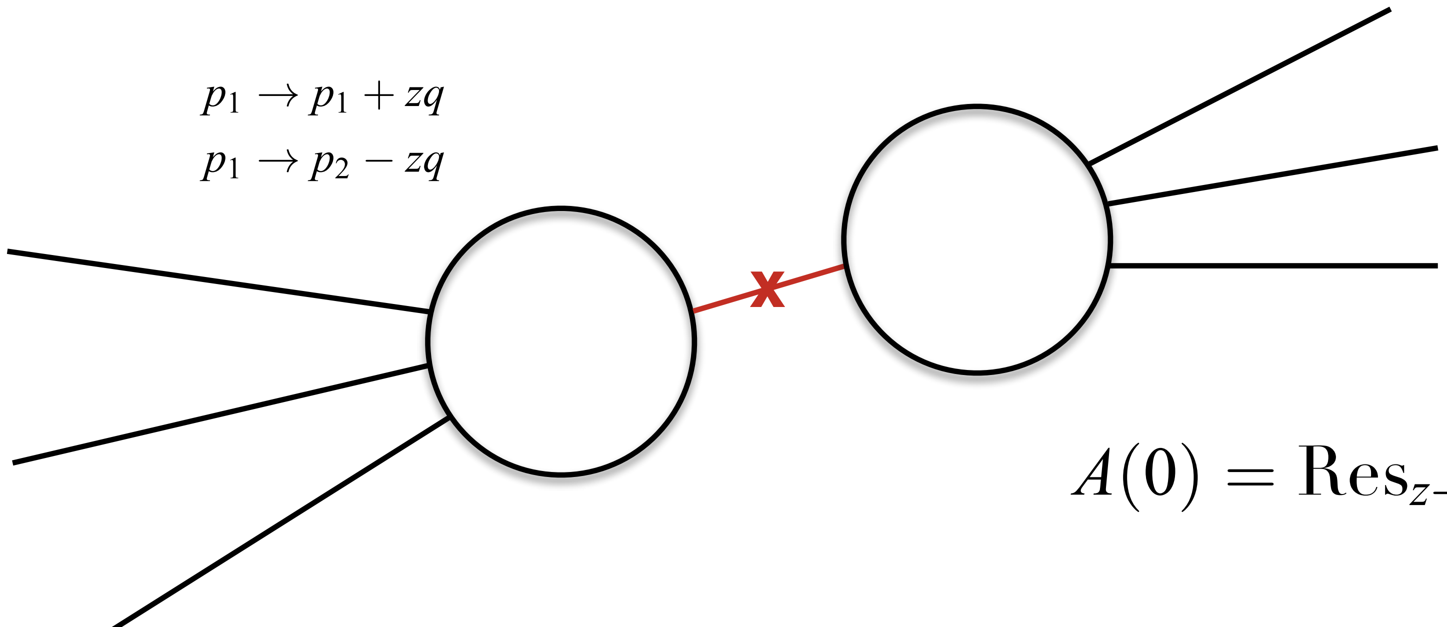

Traditionally scattering amplitudes are calculated by Feynman diagrams

Even for the tree amplitudes, internal off-shell parts exist.

It is not clear how a high-multiplicity tree amplitude can be constructed from low-multiplicity on-shell tree amplitudes.

(BCFW) A high-multiplicity tree amplitude can be constructed from low-multiplicity COMPLEX on-shell tree amplitudes.

$p_1 \to p_1 + zq$
$p_1 \to p_2 - zq$



$$A(0) = \mathrm{Res}_{z\to 0}\frac{A(z)}{z} = -\sum_i \frac{A_L(\xi_i)A_R(\xi_i)}{p_L^2}$$

Surprisingly simple structures in amplitudes

Kawai-Lewellen-Tye (KLT)

Nucl.Phys. B269 (1986) 1-23

(closed string tree amplitude) = (open string tree amplitude) × (open string tree amplitude)

$$A^{(4)}_{\text{closed}}(\text{tachyon}) = -\pi\kappa^2 \sin(\pi k_2 \cdot k_3) A^{(4)}_{\text{open}}(s,t) \bar{A}^{(4)}_{\text{open}}(t,u)\,. \tag{3.11}$$

For the four-point amplitude, this simply means (3.11) can be rewritten as

$$A^{(4)}_{\text{closed}} = -\pi\kappa^2 \sin(\pi k_1 \cdot k_2) A^{(4)}_{\text{open}}(s,t) \bar{A}^{(4)}_{\text{open}}(s,u)\,. \tag{3.29}$$

tree-level graviton amplitude is the "square" of gluon amplitude in Yang-Mills theory

$$\sin(\pi k_2 \cdot k_3) A_{\text{open}}(t,u) = \sin(\pi k_1 \cdot k_2) A_{\text{open}}(s,u)$$

Color-stripped tree amplitudes in Yang-Mills theory are linearly related

Surprisingly simple structures in amplitudes

From KLT to BCJ

$\alpha' \to 0,$

$$\bar{A}^{(4)}_{\text{open}}(s,u) \to -\frac{n_s}{s} + \frac{n_u}{u}, \quad \bar{A}^{(4)}_{\text{open}}(t,u) \to -\frac{n_u}{u} + \frac{n_t}{t}$$

Therefore $n_s + n_t + n_u = 0$ (BCJ, tree-level 4-point).

Bern-Carrasco-Johansson

Physical Review D, 78, 2008, 085011

color kinematics

Tree-level $A_{\text{YM}} = \sum_i \frac{c_i n_i}{P_i}$ Loop-level $A^{L\text{-loop}}_{\text{YM}} = \int d^D l_1 \ldots d^D l_L \sum_i \frac{c_i n_i}{P_i}$

BCJ:

1. We can re-arrange the kinematic terms such that if $c_i + c_j + c_k = 0$, then $n_i + n_j + n_k = 0$ (BCJ numerators).

2. For BCJ numerators, the gravity amplitude is,

Tree-level $A_{\text{grav}} = \sum_i \frac{n_i n_i}{P_i}$ Loop-level $A^{L\text{-loop}}_{\text{grav}} = \int d^D l_1 \ldots d^D l_L \sum_i \frac{n_i n_i}{P_i}$

Brave exploration of amplitudes in formal theories

N=4 super-Yang-Mills

Pro

- maximally supersymmetric gauge theory in 4D
- conformal symmetry
- dual conformal symmetry for the planar N=4 SYM (large N limit)
- Usually its loop integrand is much simpler than the corresponding pure Yang-Mills integrand
- Compass for loop amplitude computation

Con

- N=4 SYM is too far away from the world

N=4 super-Yang-Mills

Integrability

the complete one-loop planar dilatation generator of N = 4 is described by an integrable su(2, 2|4) super spin chain

Beisert, Staudacher   Nucl.Phys. B670 (2003) 439-463

BDS Ansatz

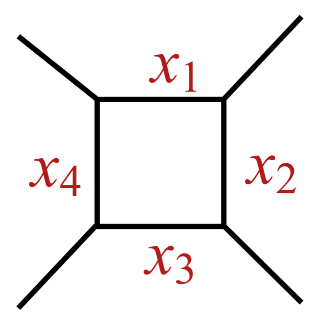
$$\mathcal{M}_n \equiv 1 + \sum_{L=1}^{\infty} a^L M_n^{(L)}(\epsilon) = \exp\left[\sum_{l=1}^{\infty} a^l \left(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + C^{(l)} + E_n^{(l)}(\epsilon)\right)\right].$$

planar MHV

remainder function needed for >=6 gluons

Bern, Dixon, Smirnov   Phys.Rev.D72:085001,2005

Dual conformal symmetry



$p_i = x_{i+1} - x_i$

Drummond, Henn, Korchemsky, Sokatchev   Nucl.Phys. B828 (2010) 317-374

Bootstrap

3 loop 7 point NMHV
4 loop 7 point MHV   planar N=4 determined   Dixon et al, JHEP02(2017)137

2 loop 5 point MHV   non-planar N=4 ???

N=8 supergravity

Is there a UV-finite quantum gravity theory in the framework of field theory?

N=8 supergravity?

"UV-finiteness is not a philosophy question, but an amplitude question"

Loop-level $$A_{\text{grav}}^{L\text{-loop}} = \int d^D l_1 \dots d^D l_L \sum_i \frac{n_i n_i}{P_i}$$ BCJ

$$D_c = \frac{6}{L} + 4\,, \qquad (2 \le L \le 4)$$

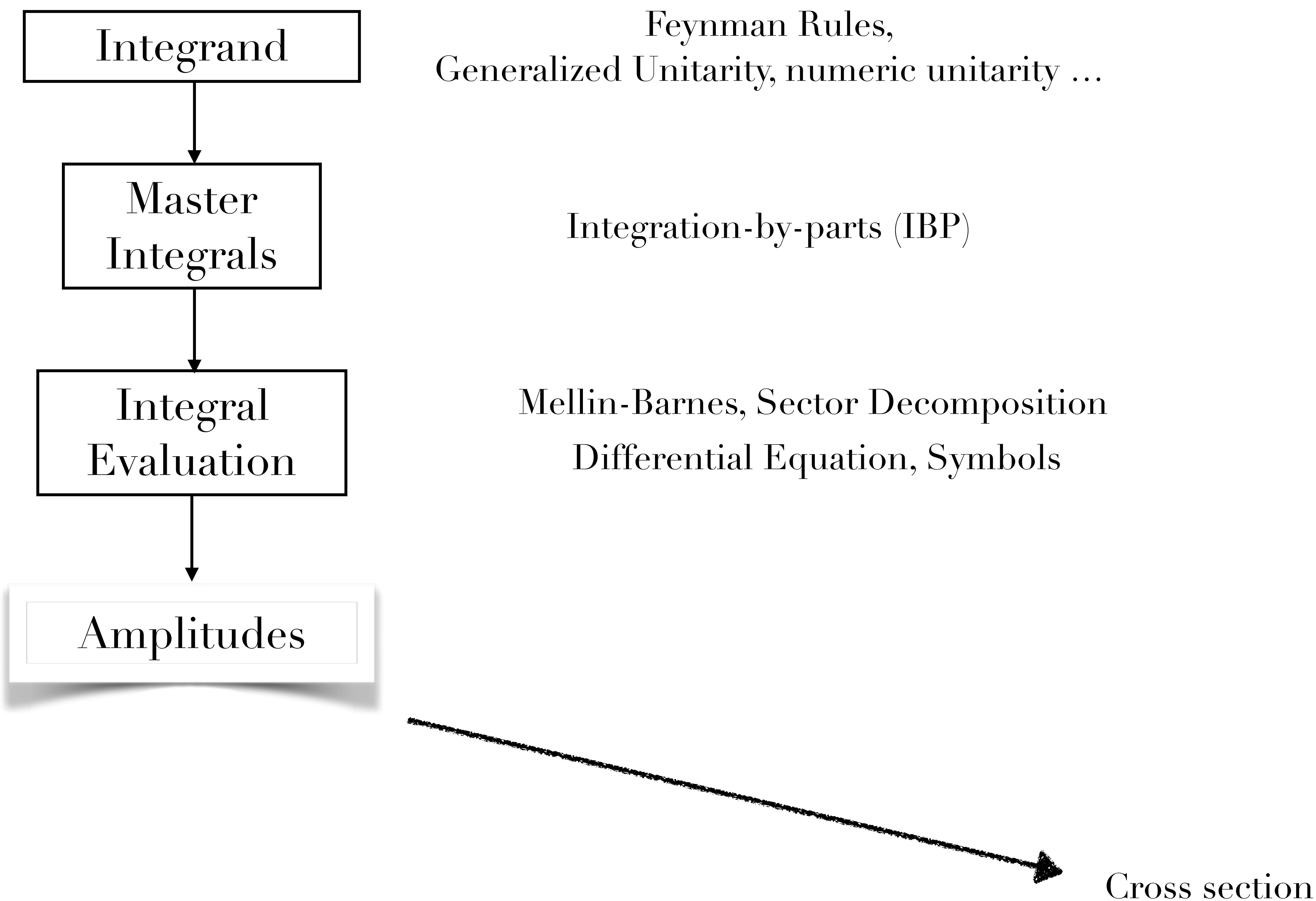
1804.09311 Zvi Bern et al

5-loop $N = 8$ supergravity is UV divergent at $D = 24/5$ which corresponds to a counter term $D^8 R^4$.

… …

7 loop D=4 UV divergent ?

Multi-loop amplitudes for the LHC physics



Multi-loop amplitudes for the LHC physics

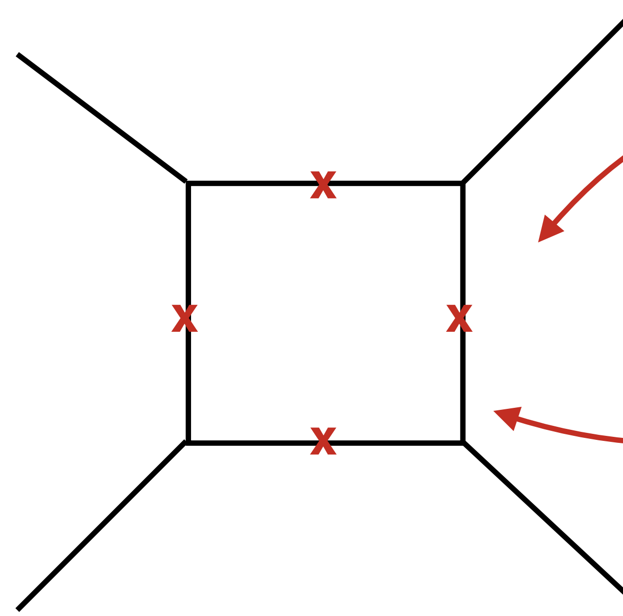
Integrand

$$\int \frac{d^D l_1}{i\pi^{D/2}} \dots \int \frac{d^D l_L}{i\pi^{D/2}} \frac{N}{D_1 \dots D_k}$$

Feynman Rules

Unitarity

Tree amplitudes



OPP method

$$\int \frac{d^D l}{i\pi^{D/2}} \frac{c_0 + c_1 (l \cdot \omega)}{D_1 D_2 D_3 D_4}$$

Ossola et al, Nucl.Phys. B763 (2007) 147-169

High-loop:

integrand reduction via Groebner basis, YZ, JHEP 1209 (2012) 042

2-loop 5-point integrands, Badger et al. Phys.Rev.Lett. 120 (2018) no.9, 092001

numeric unitarity (Ita et al., 1712.03946)

2. Feynman integral representation and reduction

Feynman integral representation

$$I[\alpha_1,\dots\alpha_m] = \int \frac{d^D l_1}{i\pi^{D/2}} \dots \int \frac{d^D l_L}{i\pi^{D/2}} \frac{1}{D_1^{\alpha_1}\dots D_m^{\alpha_m}}$$

Feynman parametric representation

$$I[\alpha_1,\dots\alpha_m] = \frac{(-1)^{|\alpha|}\Gamma(|\alpha| - \frac{DL}{2})}{\Gamma(\alpha_1)\dots\Gamma(\alpha_m)} \int_0^1 \prod_{i=1}^m dz_i \, \delta(1 - \sum_j z_j) z_1^{\alpha_1 - 1} \dots z_m^{\alpha_m - 1} \frac{F^{LD/2 - |\alpha|}}{U^{(L+1)D/2 - |\alpha|}}$$

parametric representation

Roman Lee 1405.5616

$$I[\alpha_1,\dots\alpha_m] = \frac{(-1)^{|\alpha|}\Gamma(D/2)}{\Gamma\big((L+1)D/2 - |\alpha|\big)\Gamma(\alpha_1)\dots\Gamma(\alpha_m)} \int_0^\infty \prod_{i=1}^m dz_i \, z_1^{\alpha_1 - 1} \dots z_m^{\alpha_m - 1} G^{-D/2}$$

$$G = F + U$$

useful for eduction problems

Baikov representation

Phys.Lett. B385 (1996) 404-410

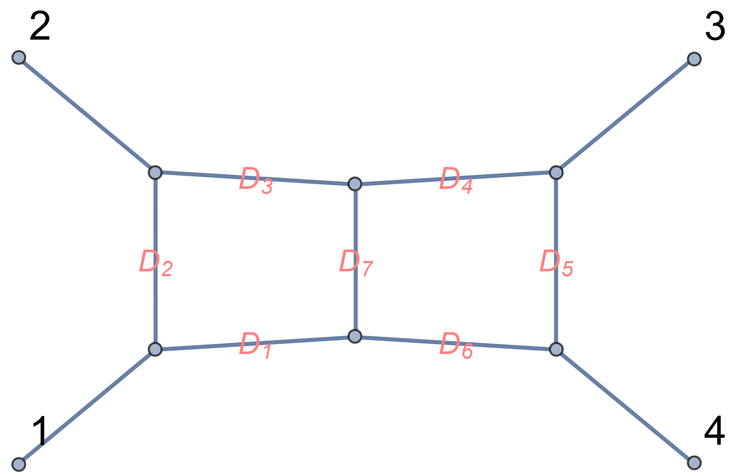
When $k = LE + L(L+1)/2$, (E is the number of independent legs), the Baikov rep. is

$$\int \frac{d^D l_1}{i\pi^{D/2}} \cdots \int \frac{d^D l_L}{i\pi^{D/2}} \frac{1}{D_1^{\alpha_1} \ldots D_k^{\alpha_k}} \propto \prod_{1\le i\le L+E, \max\{i,E+1\}\le j\le L+E} \left(\int dx_{ij} \right) \det(S)^{\frac{D-L-E-1}{2}} \frac{1}{D_1^{\alpha_1} \ldots D_k^{\alpha_k}}$$

$$\propto \prod_{i=1}^{k} \left(\int dz_i \right) \det(S)^{\frac{D-L-E-1}{2}} \frac{1}{z_1^{\alpha_1} \ldots z_k^{\alpha_k}} \qquad z_i \equiv D_i$$

Baikov representation

where $\{v_1, \ldots v_{L+E}\} \equiv \{k_1, \ldots k_E, l_1, \ldots l_L\}$ and $x_{ij} \equiv v_i \cdot v_j$. S is a $(L+E) \times (L+E)$ matrix with $S_{ij} = x_{ij}$ (Gram matrix).



$L = 2, E = 3, m = 7$ and $k = 9$. $\{v_1, \ldots, v_5\} \equiv \{k_1, k_2, k_4, l_1, l_2\}$.

$$S = \begin{pmatrix} 0 & \frac{s}{2} & \frac{t}{2} & x_{11} & x_{21} \\ \frac{s}{2} & 0 & \frac{1}{2}(-s-t) & x_{12} & x_{22} \\ \frac{t}{2} & \frac{1}{2}(-s-t) & 0 & x_{13} & x_{23} \\ x_{11} & x_{12} & x_{13} & x_{44} & x_{45} \\ x_{21} & x_{22} & x_{23} & x_{45} & x_{55} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{s}{2} & \frac{t}{2} & \frac{z_1}{2} - \frac{z_2}{2} & \frac{z_9}{2} - \frac{z_6}{2} \\ \frac{s}{2} & 0 & \frac{1}{2}(-s-t) & \frac{1}{2}(s-z_3) + \frac{z_2}{2} & \frac{1}{2}(z_4 - s) - \frac{z_9}{2} \\ \frac{t}{2} & \frac{1}{2}(-s-t) & 0 & \frac{z_8}{2} - \frac{z_1}{2} & \frac{z_6}{2} - \frac{z_5}{2} \\ \frac{z_1}{2} - \frac{z_2}{2} & \frac{1}{2}(s-z_3) + \frac{z_2}{2} & \frac{z_8}{2} - \frac{z_1}{2} & z_1 & -\frac{z_1}{2} - \frac{z_6}{2} + \frac{z_7}{2} \\ \frac{z_9}{2} - \frac{z_6}{2} & \frac{1}{2}(z_4 - s) - \frac{z_9}{2} & \frac{z_6}{2} - \frac{z_5}{2} & -\frac{z_1}{2} - \frac{z_6}{2} + \frac{z_7}{2} & z_6 \end{pmatrix}$$

Feynman integral reduction

Integration-by-parts identities

$$0 = \int \frac{d^D\ell_1}{\mathrm{i}\pi^{D/2}} \cdots \frac{d^D\ell_L}{\mathrm{i}\pi^{D/2}} \sum_{j=1}^{L} \frac{\partial}{\partial \ell_j^\mu} \frac{v_j^\mu}{D_1^{\nu_1} \cdots D_m^{\nu_m}}$$

- By IBPs, for any Feynman diagram + sub diagrams, the number of master integrals is finite (Smirnov)
- IBP reduces the integrals to a basis of master integrals
- IBP reduction is difficult with multi-loop multi-scale diagrams (bottleneck)

Air Anastasiou, Lazopoulos
FIRE Smirnov
Kira Maierhoefer, Usovitsch, and Uwer
Reduze von Manteuffel and C. Studerus

IBPs in Baikov representation

$$R = \mathbb{Q}(\text{parameters})[z_1, \ldots z_k]$$

$$\int \frac{d^D l_1}{i\pi^{D/2}} \cdots \int \frac{d^D l_L}{i\pi^{D/2}} \frac{1}{D_1^{\alpha_1} \ldots D_m^{\alpha_m} D_{m+1}^{\alpha_{m+1}} \ldots D_k^{\alpha_k}}, \quad \begin{cases} \alpha_i \leq 1, & 1 \leq i \leq m \\ \alpha_i \leq 0, & m < i \leq k \end{cases}$$

Just consider IBPs

$$0 = \left(\prod_{i=1}^{k} \int dz_i\right) \sum_{j=1}^{k} \frac{\partial}{\partial z_j} \left(a_j(z) \det(S)^{\frac{D-L-E-1}{2}} \frac{1}{z_1 \ldots z_m} \right)$$

Polynomials!

Further require $\quad F \equiv \det(S)$

"Affine varieties and Lie algebras of vector fields"
Hauser, Müller 1993

1. no shifted exponent: $\displaystyle\sum_{j=1}^{k} a_j(z) \frac{\partial F}{\partial z_j} + \beta(z) F = 0$ These $\left(a_1(z), \ldots a_k(z)\right)$ form a module $M_1 \subset R^k$.

2. no doubled propagator: $a_i(z) \in \langle z_i \rangle, \quad 1 \leq i \leq m$ These $\left(a_1(z), \ldots a_k(z)\right)$ form a module $M_2 \subset R^k$.

Both M_1 and M_2 are pretty simple ...

Y.Z. 1612.02249 $\boxed{M_1 \cap M_2}$ Intersection of two modules

Computation of the first module M_1

In principle, the syzygy can be
computed by Schreyer's theorem
but the computation is heavy …

$$\sum_{j=1}^{k} a_j(z)\frac{\partial F}{\partial z_j} + \beta(z)F = 0$$

- syzygy for the $\{\frac{\partial F}{\partial z_1}, \ldots, \frac{\partial F}{\partial z_k}, F\}$
- $\mathrm{Ann}(F^s)$, annihilator of F^s in Weyl algebra.

If F is a determinant matrix whose elements are free variables, this kind of syzygy module is simple.

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}$$

Roman Lee's idea

No computation is needed

syzygy generators (Laplace expansion)

$$\sum_j a_{k,j}\frac{\partial(\det A)}{\partial a_{i,j}} - \delta_{k,i}\cdot \det A = 0$$ provides all syzygy generators

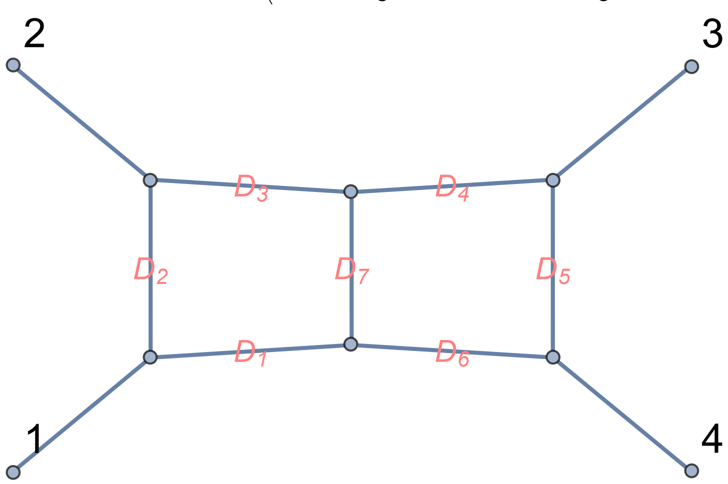
The Gram matrix in Baikov rep. is symmetric and not all entries are free variables

"weighted" Laplace expansion

Example, massless double box

$\mathbb{Q}(s,t)[z_1,\ldots z_9]$: 2 parameters, 9 variables (Each row is a module generator)

$$M_1 = \begin{pmatrix}
z_1-z_2 & z_1-z_2 & -s+z_1-z_2 & 0 & 0 & 0 & z_1-z_2-z_6+z_9 & t+z_1-z_2 & 0 \\
0 & 0 & 0 & s-z_6+z_9 & -t-z_6+z_9 & -z_6+z_9 & z_1-z_2-z_6+z_9 & 0 & -z_6+z_9 \\
s+z_2-z_3 & z_2-z_3 & z_2-z_3 & 0 & 0 & 0 & z_2-z_3+z_4-z_9 & -t+z_2-z_3 & 0 \\
0 & 0 & 0 & z_4-z_9 & t+z_4-z_9 & -s+z_4-z_9 & z_2-z_3+z_4-z_9 & 0 & z_4-z_9 \\
-z_1+z_8 & -t-z_1+z_8 & s-z_1+z_8 & 0 & 0 & 0 & -z_1-z_5+z_6+z_8 & -z_1+z_8 & 0 \\
0 & 0 & 0 & -s-z_5+z_6 & -z_5+z_6 & -z_5+z_6 & -z_1-z_5+z_6+z_8 & 0 & t-z_5+z_6 \\
2\,z_1 & z_1+z_2 & -s+z_1+z_3 & 0 & 0 & 0 & z_1-z_6+z_7 & z_1+z_8 & 0 \\
0 & 0 & 0 & s-z_3-z_6+z_7 & -z_6+z_7-z_8 & -z_1-z_6+z_7 & z_1-z_6+z_7 & 0 & -z_2-z_6+z_7 \\
-z_1-z_6+z_7 & -z_1+z_7-z_9 & s-z_1-z_4+z_7 & 0 & 0 & 0 & -z_1+z_6+z_7 & -z_1-z_5+z_7 & 0 \\
0 & 0 & 0 & -s+z_4+z_6 & z_5+z_6 & 2\,z_6 & -z_1+z_6+z_7 & 0 & z_6+z_9
\end{pmatrix}$$



$$M_2 = \begin{pmatrix}
z_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & z_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & z_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & z_4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & z_5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & z_6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & z_7 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

$M_1 \cap M_2$ (even without cut) is found in ~4 seconds, with Singular 4.1.0,

intersect(M1,M2,”std”)

nonplanar hexagon-box

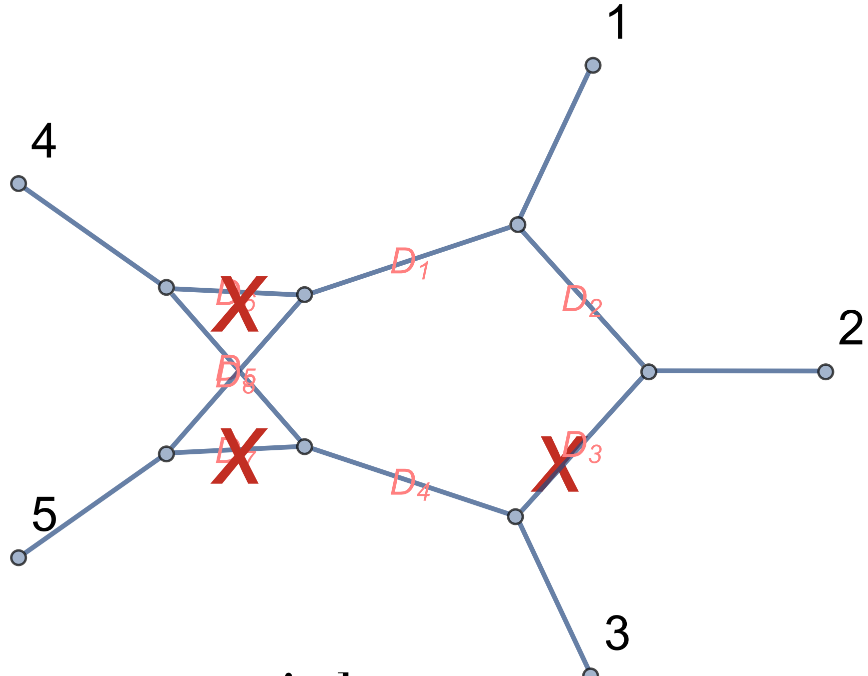
10 different cuts needed (8 triple cuts, 2 quadr

(3,6,7) cut

$\mathbb{Q}(s_{12}, s_{13}, s_{14}, s_{23}, s_{24})[z_1, z_2, z_4, z_5, z_8, z_9, z_{10}, z_{11}]$: 5 parameters, 11-3=8 variables

$z_3 \to 0, z_6 \to 0, z_7 \to 0$

$$M_1' = \left(\begin{matrix} z_1 - z_2 \\ 0 \\ s12 + z_2 \\ 0 \\ s13 + s23 - z_4 \\ 0 \\ -s12 - s13 - s23 + z_4 - z_9 \\ 0 \\ 2 z_1 \\ 0 \\ -z_1 - z_8 \\ 0 \end{matrix} \quad \begin{matrix} z_1 - z_2 \\ 0 \\ z_2 \\ 0 \\ s23 - z_4 \\ 0 \\ -s12 - s13 - s14 - s23 + z_4 - z_9 \\ 0 \\ z_1 + z_2 \\ 0 \\ -z_1 - z_{10} \\ 0 \end{matrix} \quad \begin{matrix} -s12 + z_1 - z_2 \\ 0 \\ z_2 \\ 0 \\ -z_4 \\ 0 \\ -s12 - s13 - s14 - s23 - s24 + z_4 - z_9 \\ 0 \\ -s12 + z_1 \\ 0 \\ -z_1 + z_8 - z_{10} - z_{11} \\ 0 \end{matrix} \quad \begin{matrix} -s12 - s13 + z_1 - z_2 \\ 0 \\ -s23 + z_2 \\ 0 \\ -z_4 \\ 0 \\ z_4 - z_9 \\ 0 \\ -s12 - s13 - s23 + z_1 + z_4 \\ 0 \\ s12 + s13 + s23 - z_1 - z_4 + z_5 + z_9 \\ 0 \end{matrix} \quad \begin{matrix} s14 + z_1 - z_2 - z_8 + z_{10} \\ s14 + z_1 - z_2 - z_8 + z_{10} \\ s12 + s24 + z_2 - z_8 + z_{11} \\ s12 + s24 + z_2 - z_8 + z_{11} \\ -2 s12 - s13 - s14 - s23 - s24 - z_5 + z_8 - z_9 - z_{10} - z_{11} \\ -2 s12 - s13 - s14 - s23 - s24 - z_5 + z_8 - z_9 - z_{10} - z_{11} \\ z_5 \\ z_5 \\ -s12 - s13 - s23 + z_1 + z_4 - z_8 - z_9 \\ -s12 - s13 - s23 + z_1 + z_4 - z_8 - z_9 \\ s12 + s13 + s23 - z_1 - z_4 + z_5 + z_8 + z_9 \\ s12 + s13 + s23 - z_1 - z_4 + z_5 + z_8 + z_9 \end{matrix} \quad \begin{matrix} z_1 - z_2 - z_8 + z_{10} \\ z_1 - z_2 - z_8 + z_{10} \\ s12 + z_2 - z_8 + z_{11} \\ s12 + z_2 - z_8 + z_{11} \\ -s12 - z_5 + z_8 - z_9 - z_{10} - z_{11} \\ -s12 - z_5 + z_8 - z_9 - z_{10} - z_{11} \\ z_5 \\ z_5 \\ z_1 - z_8 \\ z_1 - z_8 \\ -z_1 + z_8 \\ -z_1 + z_8 \end{matrix} \quad \begin{matrix} 0 \\ s12 + s13 + s14 - z_8 + z_{10} \\ 0 \\ s12 + s23 + s24 - z_8 + z_{11} \\ 0 \\ -2 s12 - s13 - s14 - s23 - s24 + z_4 - z_5 + z_8 - z_9 - z_{10} - z_{11} \\ 0 \\ -z_4 + z_5 + z_9 \\ 0 \\ -z_8 - z_9 \\ 0 \\ z_8 \end{matrix} \quad \begin{matrix} 0 \\ -z_8 + z_{10} \\ 0 \\ -z_8 + z_{11} \\ 0 \\ -s12 - s13 - s23 + z_4 - z_5 + z_8 - z_9 - z_{10} - z_{11} \\ 0 \\ s12 + s13 + s23 - z_4 + z_5 + z_9 \\ 0 \\ -z_1 - z_8 \\ 0 \\ 2 z_8 \end{matrix} \quad \begin{matrix} -s12 - s13 - s14 + z_1 - z_2 \\ 0 \\ -s23 - s24 + z_2 \\ 0 \\ s12 + s13 + s14 + s23 + s24 - z_4 \\ 0 \\ z_4 - z_9 \\ 0 \\ z_1 + z_9 \\ 0 \\ -z_1 \\ 0 \end{matrix} \quad \begin{matrix} 0 \\ -z_8 + z_{10} \\ 0 \\ s12 - z_8 + z_{11} \\ 0 \\ -s12 - s23 + z_4 - z_5 + z_8 - z_9 - z_{10} - z_{11} \\ 0 \\ s12 + s13 + s14 + s23 - z_4 + z_5 + z_9 \\ 0 \\ -z_2 - z_8 \\ 0 \\ z_8 + z_{10} \end{matrix} \quad \begin{matrix} 0 \\ s12 - z_8 + z_{10} \\ 0 \\ -z_8 + z_{11} \\ 0 \\ -s12 - s13 + z_4 - z_5 + z_8 - z_9 - z_{10} - z_{11} \\ 0 \\ s12 + s13 + s23 + s24 - z_4 + z_5 + z_9 \\ 0 \\ s12 - z_1 + z_2 - z_8 \\ 0 \\ z_8 + z_{11} \end{matrix} \right)$$



triple cut

$$M_2' = \begin{pmatrix} z_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & z_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & z_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & z_8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$M_1' \cap M_2' =?$

Numerically (and over finite field) the intersection can be found in sever

Analytically the intersection seems very difficult

A trick for computational algebraic geometry

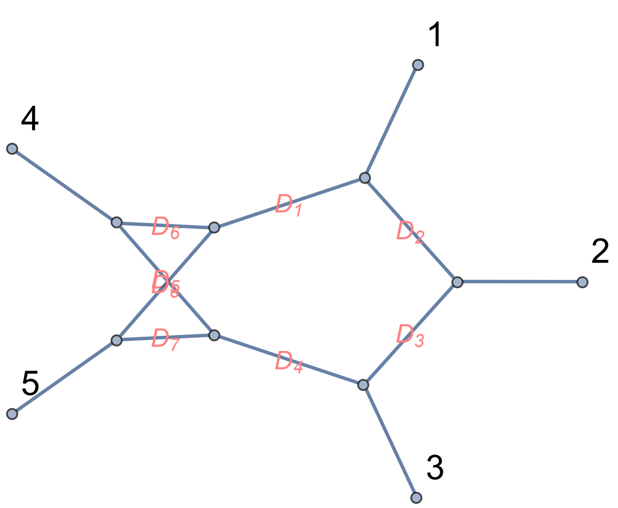
Janko Boehm, Hans Schoeneman, University of Kaisers

Analytic Mandelstam variables (parameters) slow down the computation,

Treat parameters as variables, and compute in a particular monomial ordering

[variables] > [parameters]

this trick makes all polynomials homogeneous



Module intersection for (1,4,6,7)

$\mathbb{Q}(s_{12}, s_{13}, s_{14}, s_{23}, s_{24})[z_2, z_3, z_5, z_8, z_9, z_{10}, z_{11}]$ with
$z_2 > z_3 > z_5 > z_8 > z_9 > z_{10} > z_{11}$

? (does not finish)

$\mathbb{Q}[z_2, z_3, z_5, z_8, z_9, z_{10}, z_{11}, s_{12}, s_{13}, s_{14}, s_{23}, s_{24}]$ with
$[z_2, z_3, z_5, z_8, z_9, z_{10}, z_{11}] > [s_{12}, s_{13}, s_{14}, s_{23}, s_{24}]$

4 seconds with Singu

All module intersections for 10 necessary cuts found analyt

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Complete integration-by-parts reductions of the non-planar hexagon-box via module intersections

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Abstract: We present the powerful module-intersection integration-by-parts (IBP) method, suitable for multi-loop and multi-scale Feynman integral reduction. Utilizing modern computational algebraic geometry techniques, this new method successfully trims traditional IBP systems dramatically to much simpler integral-relation systems on unitarity cuts. We demonstrate the power of this method by explicitly carrying out the complete analytic reduction of two-loop five-point non-planar hexagon-box integrals, with degree-four numerators, to a basis of 73 master integrals.

3. Feynman Integral evaluation

Differential equation (DE)

I: a list of N master integrals in dimensional regularization (DimReg)

$$\frac{\partial}{\partial x_i} I = M_i(\epsilon) I$$

Kotikov 1991

Bern, Dixon and Kosower 1994

Gehrmann and Remiddi 2000

x_i's are kinematic variables

Matrices can be generated by IBPs

Gauge Transformation

$$I \mapsto TI, \quad M_i \mapsto TM_iT^{-1} + (\partial_i T)T^{-1}$$

M_i's form the connection of the N-dim vector bundle

Integrability

$$\partial_i M_j - \partial_j M_i + [M_i, M_j] = 0$$

The connection has zero curvature

$$dI = \omega I, \quad \omega \equiv \sum_i M_i dx_i$$

flat meromorphic connection

A smart basis choice, Henn, Phys.Rev.Lett. 110 (2013) 251601

Canonical basis I

- In general, only first several orders in ϵ are needed.
- Some integrals are known to have constant leading singularities or *dlog*.
- In differential geometry, zero curvature $\Rightarrow$ constant metric after a change of coordinates. Here for flat connection …

$$\boxed{\tilde{I} = T(\epsilon) I, \quad \frac{\partial}{\partial x_i}\tilde{I} = \epsilon\, A_i \tilde{I}}$$

Henn 2013

$$\partial_i A_j = \partial_j A_i, \quad [A_i, A_j] = 0, \quad \omega = dA$$

connection trivialized

Feynman integrals become iterated integral

$$\tilde{I}(x) = \underbrace{P\exp\left(\epsilon \int_{\mathcal{C}} dA\right)}\tilde{I}(x_0)$$

Boundary condition given by kinematic limit/ traditional Feynman parameterization

path ordered integral,
expressible as Chen's (陳國才) iterated integrals

Canonical basis II

$$\tilde{I}(x) = P\exp\left(\epsilon\int_{\mathcal{C}} dA\right)\tilde{I}(x_0)$$ Henn 2013

When dA has the d-logarithm form,

$$A = \sum_k C_k \log \alpha_k(x)$$ → alphabets, indicate the special functions in amplitudes

it is clear that $\tilde{I}(x)$ has uniform transcendentality.

$$\mathcal{T}(\log) = 1, \mathcal{T}(\pi) = 1, \mathcal{T}(\zeta_n) = n, \mathcal{T}(\mathrm{Li}_n) = n, \ldots, \mathcal{T}(f_1 f_2) = \mathcal{T}(f_1) + \mathcal{T}(f_2)$$

- makes the DE solving simple
- the canonical basis integrals have simple expression (UT)
- also makes numeric integration much easier

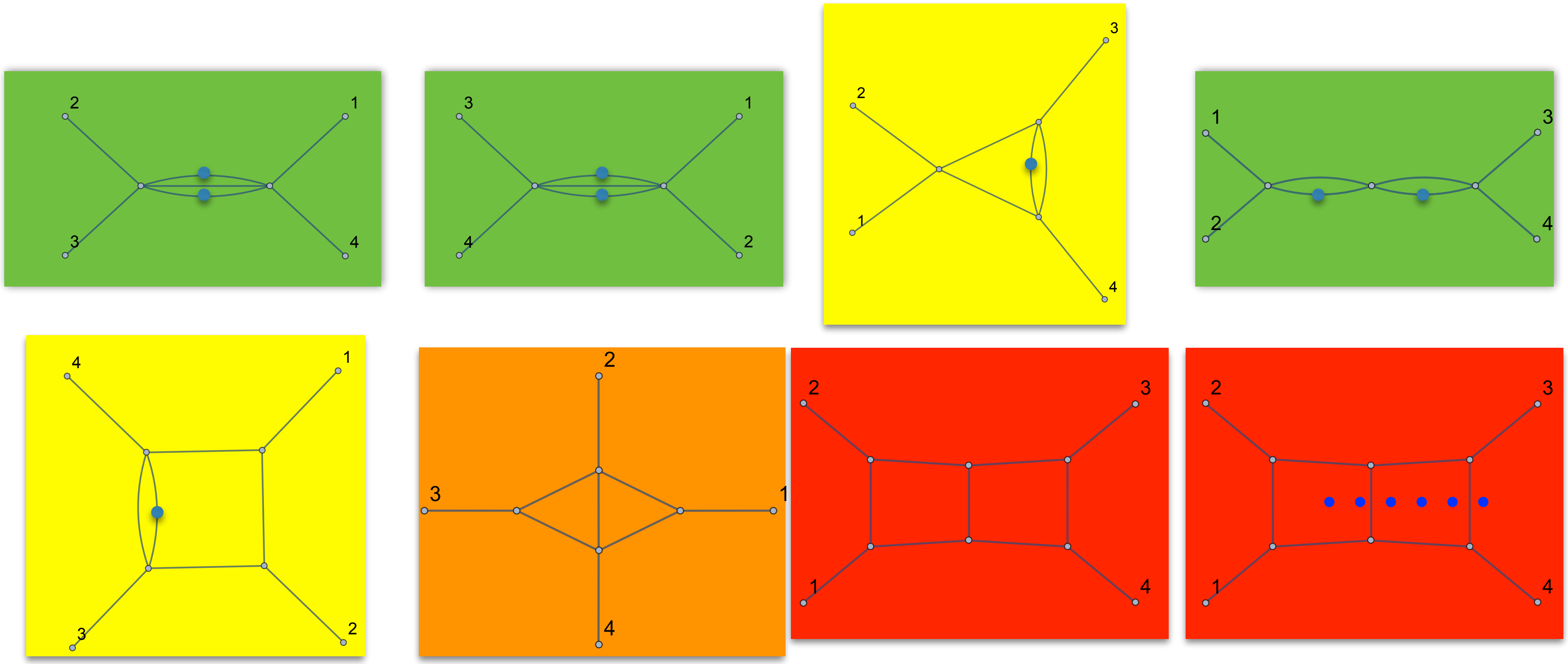
Find canonical basis

- Leading singularity/ d-log integrand form (Henn 2013)
- Feynman parametrization (Henn 2013, 2014)
- Eigenvalue analysis for residue matrix and balance transformation
  One-variable case (Lee 2014)
- Packages: Fuchsia (Gituliar, Magerya), epsilon (Prausa)

Elliptic case, transformation using modular form (Adam, Weinzierl 2017)

dbox example

Henn 2013



$$f_1 = -\epsilon^2 (-s)^{2\epsilon}\, t\, I_{0,2,0,0,0,0,0,1,2}\,,$$
$$f_2 = \epsilon^2 (-s)^{1+2\epsilon}\, I_{0,0,2,0,1,0,0,0,2}\,,$$
$$f_3 = \epsilon^3 (-s)^{1+2\epsilon}\, I_{0,1,0,0,1,0,1,0,2}\,,$$
$$f_4 = -\epsilon^2 (-s)^{2+2\epsilon}\, I_{2,0,1,0,2,0,1,0,0}\,,$$
$$f_5 = \epsilon^3 (-s)^{1+2\epsilon}\, t\, I_{1,1,1,0,0,0,0,1,2}\,,$$
$$f_6 = -\epsilon^4 (-s)^{2\epsilon}\, (s+t)\, I_{0,1,1,0,1,0,0,1,1}\,,$$
$$f_7 = -\epsilon^4 (-s)^{2+2\epsilon}\, t\, I_{1,1,1,0,1,0,1,1,1}\,,$$
$$f_8 = -\epsilon^4 (-s)^{2+2\epsilon}\, I_{1,1,1,0,1,-1,1,1,1}\,.$$

Simple known integrals …

Integrate out the bubble …

Feynman parametrization

d-log form

HPLs

$$f_1 = -\epsilon^2 (-s)^{2\epsilon} \, t \, I_{0,2,0,0,0,0,0,1,2} \, ,$$
$$f_2 = \epsilon^2 (-s)^{1+2\epsilon} \, I_{0,0,2,0,1,0,0,0,2} \, ,$$
$$f_3 = \epsilon^3 (-s)^{1+2\epsilon} \, I_{0,1,0,0,1,0,1,0,2} \, ,$$
$$f_4 = -\epsilon^2 (-s)^{2+2\epsilon} \, I_{2,0,1,0,2,0,1,0,0} \, ,$$
$$f_5 = \epsilon^3 (-s)^{1+2\epsilon} \, t \, I_{1,1,1,0,0,0,0,1,2} \, ,$$
$$f_6 = -\epsilon^4 (-s)^{2\epsilon} (s+t) \, I_{0,1,1,0,1,0,0,1,1} \, ,$$
$$f_7 = -\epsilon^4 (-s)^{2+2\epsilon} \, t \, I_{1,1,1,0,1,0,1,1,1} \, ,$$
$$f_8 = -\epsilon^4 (-s)^{2+2\epsilon} \, I_{1,1,1,0,1,-1,1,1,1} \, .$$

$$\partial_x f = \epsilon \left[\frac{a}{x} + \frac{b}{1+x} \right] f \, ,$$

$$a = \begin{pmatrix} -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{2} & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & -2 & 0 & 0 \\ -3 & -3 & 0 & 0 & 4 & 12 & -2 & 0 \\ \frac{9}{2} & 3 & -3 & -1 & -4 & -18 & 1 & 1 \end{pmatrix} ,$$

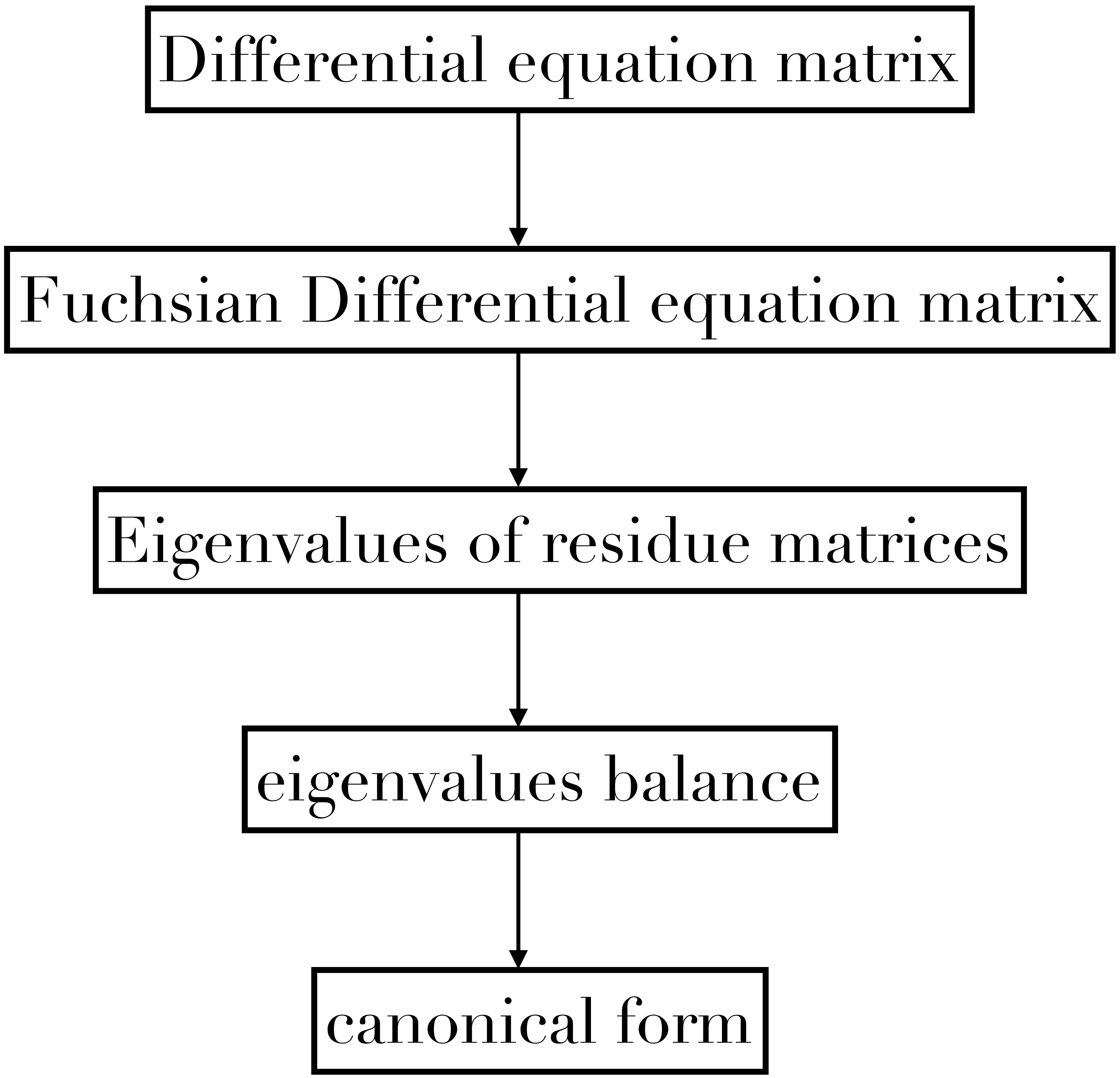
and

$$b = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{3}{2} & 0 & 3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 3 & 6 & 6 & 2 & -4 & -12 & 2 & 2 \\ -\frac{9}{2} & -3 & 3 & -1 & 4 & 18 & -1 & -1 \end{pmatrix} .$$

Henn 2013

Roman Lee's algorithm

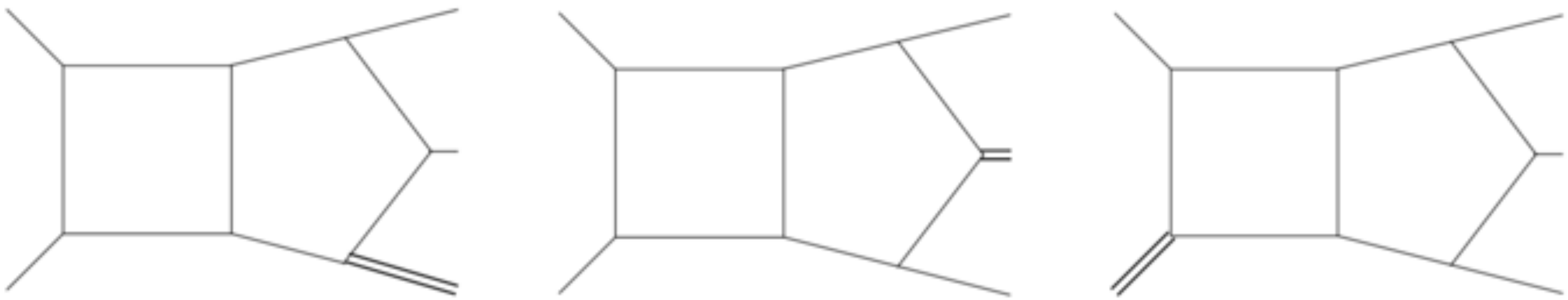
JHEP 1504 (2015) 108



works very well for ODEs

Other modern differential equation techniques

SDE



$$p_i(x) = p_i + (1 - x)q_i$$

Papadopoulos, JHEP 1604 (2016) 078

DE from vacuum

$$I(D;\{\nu_\alpha\};\eta) \equiv \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \prod_{\alpha=1}^{N} \frac{1}{(\mathcal{D}_\alpha + \mathrm{i}\eta)^{\nu_\alpha}},$$

Xiao Liu, Yan-Qing Ma and Chen-Yu Wang
Phys.Lett. B779 (2018) 353-357