Analytic calculation of energy-energy correlations in e^+e^- annihilation at next-to-leading order

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in collaboration with
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Outline

- Event shape observables: IR- and collinear safe quantities that can be directly measured in experiments.
- Characterize the geometric structure of the final states (event topology) produced in strong interaction processes.
- Sensitivity to QCD radiation through soft gluon emissions influencing the shape of the energy flow.
- Examples:
  - Thrust [Brandt et al., 1964; Farhi, 1977].
  - C-parameter [Parisi, 1978; Donoghue et al., 1979; Ellis et al., 1981].
  - Jet broadening [Rakow & Webber, 1981; Ellis & Webber, 1986; Catani et al., 1992].
  - Jet masses [Clavelli, 1979].
  - Energy-energy correlations [Basham et al., 1978].
- Historically of crucial importance to verify the early predictions of QCD.
- Currently still very useful for many purposes:
  - Determination of α_s.
  - Tuning of Monte Carlo event generators.
  - New physics searches.
- Precise measurements done by the LEP experiments: ALEPH [Heister et al., 2004], DELPHI [Abdallah et al., 2004], L3 [Achard et al., 2004], OPAL [Abbiendi et al., 2005].
- Worthwhile task for the future e^+e^- colliders (CEPC in China, ILC in Japan and FCC-ee or CLIC in Europe).

- The mathematical structure of the event shapes observables is still poorly understood.
- Full analytic results (fixed-order) are scarce, in most cases available only at LO.

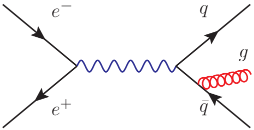
Observable	Full analytic result at LO	Full analytic result at NLO
C-parameter	No[1]	No
Thrust	Yes [De Rujula et al., 1978]	No
Heavy jet mass	Yes[2]	No
EEC	Yes [Basham et al., 1978]	Yes [THIS WORK]

- NLO and often also NNLO results can be obtained using numerical methods.
- Nevertheless, full analytic results beyond LO are still useful and hence desirable.

[1] Can be written as a single integral in a compact form [Ellis et al., 1981].
[2] Trivially follows from the result for Thrust.

- The original definition of the energy-energy correlation function (EEC) [Basham et al., 1978] uses e^+e^- annihilation in QCD at high energies

$$e^+e^- \to a + b + X.$$



- Two calorimeters at relative angle χ measure the energies of the hadrons a and b.
- Formal definition

$$\frac{1}{\sigma_{\text{tot}}} \frac{d\Sigma(\chi)}{d\cos\chi} = \sum_{a,b} \int \frac{E_a E_b}{Q^2} \delta(\cos\theta_{ab} - \cos\chi)\, d\sigma_{a+b+X}, \quad \cos\theta_{ab} = \hat{\boldsymbol{p}}_a \cdot \hat{\boldsymbol{p}}_b.$$

- EEC is a differential angular distribution of the energy flowing through the calorimeters.
- Can be computed in pQCD by the virtue of the momentum sum rule

$$\sum_h \int_0^1 dx\, x\, D_{h/q}(x, \mu_F^2) = 1.$$

- Nonperturbative corrections [Dokshitzer et al., 1999] are beyond the scope of this work.
- A good probe of QCD, as already the LO contribution starts with α_s.
- Also interesting for the determination of α_s [Kardos et al., 2018].

- The analytic form of the EEC at LO is known since 40 years [Basham et al., 1978]

$$\frac{1}{\sigma_{\text{tot}}}\frac{d\Sigma(\chi)}{d\cos\chi} = \frac{\alpha_s(\mu)}{2\pi} C_F \frac{3-2z}{4(1-z)z^5}\left[3z(2-3z) + 2(2z^2-6z+3)\log(1-z)\right] + \mathcal{O}(\alpha_s^2), \quad \text{with } z = (1-\cos\chi)/2.$$

- Emission of soft and collinear particles: The limits $z \to 0$ and $z \to 1$ require resummation of logarithms.
- Subsequent studies of higher order corrections relied on numerical methods.
- Different approaches to handle soft and collinear singularities from real radiation at NLO in the 80s and 90s:
  - Phase-space slicing [Schneider et al., 1984; Falck & Kramer, 1989; Glover & Sutton, 1995; Kramer & Spiesberger, 1997].
  - Subtraction methods [Ali & Barreiro, 1982, 1984; Richards et al., 1982, 1983; Kunszt et al., 1989; Glover & Sutton, 1995; Catani & Seymour, 1996, 1997].
  - Hybrid schemes [Glover & Sutton, 1995; Clay & Ellis, 1995; Kramer & Spiesberger, 1997].
- Some numerical discrepancies between the results obtained by different groups.

- Results of [Kunszt et al., 1989] (**N**), [Glover & Sutton, 1995] (**G**), [Catani & Seymour, 1996] (**S**) and [Clay & Ellis, 1995] (**C**) collected by P. Nason [Nason et al., 1996].
- The comparison is done between the moments of EEC

$$\int \sin^{2+m} \chi \cos^n \chi \, d\Sigma(\chi) \, d\cos\chi =$$

$$+ \frac{\alpha_s}{2\pi} A^{(m,n)} + C_F \left(C_A B_{C_A}^{(m,n)} + C_F B_{C_F}^{(m,n)} + T_F N_F B_{T_F}^{(m,n)} \right) + \mathcal{O}(\alpha_s^3).$$

Comparison of different computations of the $B_{C_A}^{(m,n)}$ coefficients

m	n	**N**	**G**	**S**	**C**
0	0	50.82 ± 0.05	50.54 ± 0.03	50.72 ± 0.02	46.4 ± 0.2
1	0	35.76 ± 0.04	35.53 ± 0.02	35.64 ± 0.02	32.09 ± 0.06
2	0	28.94 ± 0.03	28.75 ± 0.02	28.82 ± 0.02	25.73 ± 0.04
3	0	24.92 ± 0.03	24.75 ± 0.02	24.80 ± 0.02	22.03 ± 0.04
4	0	22.20 ± 0.03	22.05 ± 0.02	22.09 ± 0.02	19.54 ± 0.04
5	0	20.21 ± 0.03	20.07 ± 0.02	20.10 ± 0.02	17.74 ± 0.03
0	1	-6.468 ± 0.006	-6.50 ± 0.01	-6.455 ± 0.005	-6.0 ± 0.15
1	1	-2.356 ± 0.004	-2.365 ± 0.009	-2.344 ± 0.003	-2.15 ± 0.03
2	1	-1.189 ± 0.003	-1.194 ± 0.008	-1.177 ± 0.003	-1.06 ± 0.02
3	1	-0.714 ± 0.003	-0.718 ± 0.007	-0.702 ± 0.003	-0.62 ± 0.01
4	1	-0.478 ± 0.003	-0.479 ± 0.007	-0.466 ± 0.003	-0.41 ± 0.01
5	1	-0.344 ± 0.003	-0.344 ± 0.006	-0.331 ± 0.003	-0.28 ± 0.01

- Results of [Kunszt et al., 1989] (**N**), [Glover & Sutton, 1995] (**G**), [Catani & Seymour, 1996] (**S**) and [Clay & Ellis, 1995] (**C**) collected by P. Nason [Nason et al., 1996].
- The comparison is done between the moments of EEC

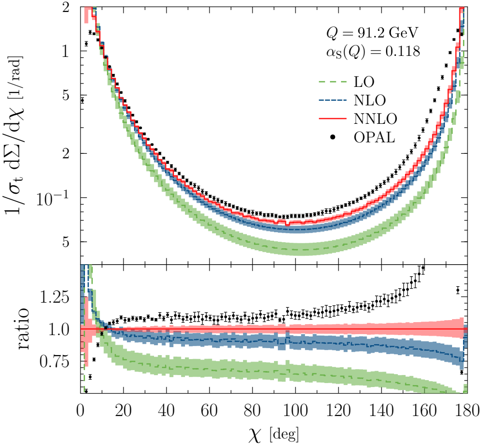
$$\int \sin^{2+m} \chi \cos^n \chi \, d\Sigma(\chi) \, d\cos\chi =$$

$$+ \frac{\alpha_s}{2\pi} A^{(m,n)} + C_F \left(C_A B^{(m,n)}_{C_A} + C_F B^{(m,n)}_{C_F} + T_F N_F B^{(m,n)}_{T_F} \right) + \mathcal{O}(\alpha_s^3).$$

Comparison of different computations of the $B^{(m,n)}_{C_F}$ coefficients

m	n	**N**	**G**	**S**	**C**
0	0	-13.29 ± 0.01	-13.94 ± 0.05	-13.40 ± 0.05	7.2 ± 0.2
1	0	-5.09 ± 0.01	-5.38 ± 0.04	-5.14 ± 0.04	9.98 ± 0.04
2	0	-2.98 ± 0.01	-3.20 ± 0.03	-3.02 ± 0.03	9.55 ± 0.03
3	0	-2.11 ± 0.01	-2.29 ± 0.03	-2.14 ± 0.03	8.86 ± 0.02
4	0	-1.65 ± 0.01	-1.81 ± 0.03	-1.67 ± 0.03	8.22 ± 0.02
5	0	-1.36 ± 0.01	-1.51 ± 0.03	-1.39 ± 0.03	7.69 ± 0.02
0	1	4.906 ± 0.002	4.92 ± 0.01	4.892 ± 0.006	2.6 ± 0.2
1	1	0.240 ± 0.002	0.259 ± 0.008	0.232 ± 0.004	-0.58 ± 0.02
2	1	-0.383 ± 0.002	-0.367 ± 0.006	-0.386 ± 0.004	-0.80 ± 0.01
3	1	-0.458 ± 0.002	-0.445 ± 0.005	-0.459 ± 0.003	-0.72 ± 0.01
4	1	-0.428 ± 0.002	-0.417 ± 0.005	-0.429 ± 0.003	-0.606 ± 0.007
5	1	-0.381 ± 0.002	-0.371 ± 0.004	-0.381 ± 0.003	-0.510 ± 0.006

- Dipole subtraction method is implemented in the publicly available code **EVENT2** [Catani & Seymour, 1996, 1997].
- The NNLO results using CoLoRfulNNLO subtraction are now also available [Del Duca et al., 2016; Tulipánt et al., 2017].
- NLO and NNLO corrections for the fixed-order predictions are not negligible.



Source: [Tulipánt et al., 2017].

- Impressive progress in the numerical calculations of EEC.
- Fully analytic fixed-order results: LO state of the art for almost 40 years!
- Important milestone: analytic NLO calculation in $\mathcal{N}=4$ SYM [Belitsky et al., 2014]

$$\Sigma_{\mathcal{N}=4}(z) = \frac{1}{4z^2(1-z)} \left(a F_1(z) + a^2 [(1-z) F_2(z) + F_3(z)] \right), \quad a = \frac{g_{\mathrm{YM}}^2 N}{4\pi^2}.$$

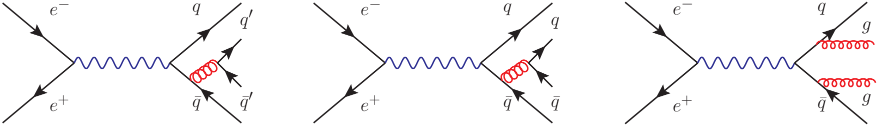
- Very simple and compact.
- Only classical polylogarithms.
- Highest transcendental weight 3.
- Alphabet contains $\sqrt{z}$.
- Maximal transcendentality principle [Kotikov & Lipatov, 2003; Kotikov et al., 2004]: QCD result also simple?

$$F_1(z) = -\ln(1-z),$$

$$F_2(z) = 4\sqrt{z}\left[\mathrm{Li}_2\left(-\sqrt{z}\right) - \mathrm{Li}_2\left(\sqrt{z}\right) + \frac{\ln z}{2}\ln\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right]$$
$$+(1+z)\left[2\mathrm{Li}_2(z) + \ln^2(1-z)\right] + 2\ln(1-z)\ln\left(\frac{z}{1-z}\right) + z\frac{\pi^2}{3},$$

$$F_3(z) = \frac{1}{4}\left\{(1-z)(1+2z)\left[\ln^2\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\ln\left(\frac{1-z}{z}\right) - 8\mathrm{Li}_3\left(\frac{\sqrt{z}}{\sqrt{z}-1}\right)\right.\right.$$
$$\left. -8\mathrm{Li}_3\left(\frac{\sqrt{z}}{\sqrt{z}+1}\right)\right] - 4(z-4)\mathrm{Li}_3(z) + 6(3+3z-4z^2)\mathrm{Li}_3\left(\frac{z}{z-1}\right)$$
$$-2z(1+4z)\zeta_3 + 2\left[2(2z^2-z-2)\ln(1-z) + (3-4z)z\ln z\right]\mathrm{Li}_2(z)$$
$$+\frac{1}{3}\ln^2(1-z)\left[4(3z^2-2z-1)\ln(1-z) + 3(3-4z)z\ln z\right]$$
$$\left.+\frac{\pi^2}{3}\left[2z^2\ln z - (2z^2+z-2)\ln(1-z)\right]\right\}.$$

- Unfortunately, the Mellin-space techniques used in [Belitsky et al., 2014] are not easily applicable to QCD.
- Hardest part of the QCD calculation: Real corrections.



- The IBP-reduction [Chetyrkin & Tkachov, 1981] applied to the on-shell amplitudes appears more tractable.

- Use the reverse unitarity method [Anastasiou & Melnikov, 2002; Anastasiou et al., 2003]

$$\frac{d^D p}{(2\pi)^D} 2\pi\delta_+(p^2 - m^2) \to \frac{1}{i}\frac{d^D p}{(2\pi)^D}\left(\frac{1}{p^2 - m^2 - i\eta} - \frac{1}{p^2 - m^2 + i\eta}\right)$$

to rewrite the measurement function $\delta(\cos\theta_{ab} - \cos\chi)$ as

$$\left.\frac{1}{(1-\cos\chi)(p_a\cdot Q)(p_b\cdot Q) - Q^2(p_a\cdot p_b)}\right|_{\text{cut}}.$$

- Nonstandard propagator, but no fundamental obstacles to use the IBPs.
- Important ingredient: Need to augment our set of IBP equations with

$$\int\left[\left((1-\cos\chi)(p_a\cdot Q)(p_b\cdot Q) - Q^2(p_a\cdot p_b)\right) J_{ab}(n_1,\ldots,n_{10}) - J_{ab}(n_1,\ldots,n_{10}-1)\right] = 0.$$

- Alternative approach [Gituliar & Moch, 2017]: Introduce an auxiliary variable to linearize the propagator

$$\delta\left[(1-\cos\chi)(p_a\cdot Q)(p_b\cdot Q) - Q^2(p_a\cdot p_b)\right] = \int_0^1 dx\,\delta[x-(p_a\cdot Q)]\,\delta[x(1-\cos\chi)(p_b\cdot Q) - Q^2(p_a\cdot p_b)].$$

- The full result using the method of [Gituliar & Moch, 2017] is not available yet.

- Loop integrals with propagators raised to integer powers are ubiquitous in multi-loop calculations.
- Reduction to a smaller set of *master integrals* using integration-by-parts (IBP) identities [Chetyrkin & Tkachov, 1981].
- Trivial example (massive 1-loop tadpole):

$$0 = \int \frac{d^D q}{(2\pi)^D} \frac{\partial}{\partial q^\mu} \frac{q^\mu}{q^2 - m^2} = \int \frac{d^D q}{(2\pi)^D} \left(\frac{D-2}{q^2 - m^2} - \frac{2m^2}{(q^2 - m^2)^2} \right)$$

$$\Rightarrow \int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2 - m^2)^2} = \frac{D-2}{2m^2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 - m^2}$$

- IBPs generate a large number of relations between the integrals with different powers of the propagators.
- Laporta algorithm [Laporta, 2000] provides a systematic way to solve these equations.
- Publicly available software packages: **AIR** [Anastasiou & Lazopoulos, 2004], **LiteRed** [Lee, 2012], **FIRE** [Smirnov & Smirnov, 2013], **Reduze** [Studerus, 2010], **Kira** [Maierhoefer et al., 2017], **Azurite** [Georgoudis et al., 2017], ...

After the IBP reduction we still need to compute the master integrals:

- Feynman parameters (including Sector Decomposition [Hepp, 1966; Speer, 1977; Binoth & Heinrich, 2000, 2004])

$$\frac{1}{A_1^{m_1} A_2^{m_2} \ldots A_N^{m_N}} = \frac{\Gamma(N_m)}{\prod_{j=1}^N \Gamma(m_j)} \prod_{j=1}^N \int_0^1 dx_j x_j^{m_j-1} \delta\left(1 - \sum_{i=1}^N x_i\right) \left[\sum_{i=1}^N x_i A_i\right]^{-N_m}, \quad N_m = \sum_{i=1}^N m_i$$

$$G = \int \prod_{l=1}^L \frac{d^d k_l}{i\pi^{d/2}} \frac{1}{A_1^{m_1} A_2^{m_2} \ldots A_N^{m_N}},$$

$$G = (-1)^{N_m} \frac{\Gamma(N_m - \frac{Ld}{2})}{\prod_{j=1}^N \Gamma(m_j)} \int_0^1 \prod_{j=1}^N dx_j x_j^{m_j-1} \delta\left(1 - \sum_{i=1}^N x_i\right) \frac{\mathcal{U}^{N_m - \frac{(L+1)d}{2}}}{\mathcal{F}^{N_m - \frac{Ld}{2}}},$$

- Mellin-Barnes representation [Bergere & Lam, 1974; Usyukina, 1975; Gluza et al., 2007]
- Difference equations [Laporta, 2000, 2003]
- Differential equations [Kotikov, 1991b, 1991c, 1991a; Bern et al., 1994; Remiddi, 1997; Gehrmann & Remiddi, 2000]
- ...

The main idea behind the method of differential equations:

- Differentiate the master integrals w.r.t one of their kinematic invariants or masses.
- Taking derivatives changes the powers of propagators i. e. generates new loop integrals.
- Use IBP reduction to rewrite those integrals in terms of the master integrals.
- We obtain a system of differential equations for our master integrals.
- To calculate the master integrals we need to
  - Solve the system.
  - Determine the integration constants (i. e. specify suitable boundary conditions).

- Schematically, the resulting system of differential equations can be written as

$$\frac{dF}{dx} = M(x, \epsilon)F,$$

  where x is some kinematic invariant, F is a vector of the master integrals and M is a matrix that depends on x and ϵ (from $D = 4 - 2\epsilon$)
- In general, it is not so simple to determine F.

- Suppose that we can find a basis transformation $T(x, \epsilon)$ that converts our system to the *canonical form* or ϵ*-form*

$$\frac{dF}{dx} = M(x,\epsilon)F \to \frac{dG}{dx} = \epsilon L(x) G,$$

where $F = T(x,\epsilon)G$ and $L(x)$ does not depend on ϵ.

- In this case we can trivially solve the new system order by order in ϵ

$$G(x,\epsilon) = G_0(x) + G_1(x)\epsilon + G_2(x)\epsilon^2 + \dots,$$
$$G_0 = C_0, \quad G_1 = C_1 + \int dx L(x) C_0, \dots$$

- How can we find such a $T(x, \epsilon)$?

In the last years a lot of progress occurred in the field of differential equation:

- In 2013 J. Henn conjectured that for many loop integrals the system can be indeed transformed to the ϵ*-form* [Henn, 2013]
- In 2014 R. Lee presented an algorithm for finding the explicit transformation [Lee, 2015]
- Tools that can automatically find the ϵ*-form* are now available **Fuchsia** [Gituliar & Magerya, 2016], **Epsilon** [Prausa, 2017] **CANONICA** [Meyer, 2018]

- The calculation of loop integrals is difficult, but we have a lot of machinery (theory, algorithms, software) to simplify this task.
- For *phase-space integrals*, numerical evaluation is the preferred method.
- Multiparticle phase-space integrals are very complicated, a brute force analytic calculation is usually not possible.
- Since many years it was observed [Anastasiou & Melnikov, 2002; Gehrmann-De Ridder et al., 2004] that a phase space integral can be converted into a loop integral by exploiting Cutkosky rules

$$\frac{d^{D-1}p}{(2\pi)^{D-1}2p^0} \to \frac{d^D p}{(2\pi)^D} 2\pi\delta_+(p^2 - m^2)$$
$$\to \frac{1}{i}\frac{d^D p}{(2\pi)^4}\left(\frac{1}{p^2 - m^2 - i\eta} - \frac{1}{p^2 - m^2 + i\eta}\right)$$

- *In principle*, we can analytically calculate phase-space integrals using the same tools that we use to handle the loop integrals.
- *In practice*, things are not so simple:
  - Arguments of the Dirac delta may generate non-standard propagators.
  - Limited knowledge in dealing with difficulties of phase-space integrals.

- The actual calculation of EEC at NLO consists of mostly standard steps:
  - Generate LO and NLO Feynman diagrams with QGRAF [Nogueira, 1993].
  - Calculate $|\mathcal{M}|^2$ using FORM [Vermaseren, 2000] and COLOR [van Ritbergen et al., 1999].
  - Self-written *Mathematica* code to identify the topologies.
  - Derive the IBP-equations with LiteRed [Lee, 2012, 2014].
  - Perform the IBP-reduction with FIRE [Smirnov, 2008, 2015].
- Some steps are, however, nonstandard:
  - FIRE and LiteRed cannot work with our nonlinear propagator out-of-the box.
  - Minimal modifications in LiteRed to be able to derive the IBP equations.
  - The equations are then imported into FIRE.
  - Very useful feature of FIRE: IBP reduction by employing an external (i. e. not derived within FIRE) set of IBP equations.
  - Since FIRE does not need to know to the exact form of our propagators, the nonlinear propagator does not cause any issues.
- The total number of loop integrals at NLO before the IBP reduction is 12375.
- The IBP-reduction with FIRE requires 3 days on a server equipped with Xeon E5-2696 (18 cores) and 128 GB RAM.

- To calculate the resulting 40 master integrals (3-loop) we employ the method of differential equations [Kotikov, 1991a, 1991b, 1991c; Bern et al., 1994; Remiddi, 1997; Gehrmann & Remiddi, 2000]
  - The differential equations are derived using LiteRed and FIRE.
  - Fuchsia [Gituliar & Magerya, 2017] provides a fast and convenient way to convert all systems of differential equations into a canonical form [Henn, 2013] using Lee's algorithm [Lee, 2015].
  - Caveat: Some systems require a nonlinear change of variables

$$z \to \sqrt{z}, \qquad z \to i\sqrt{z}/\sqrt{1-z}$$

  - The transformations readily follow from the $\mathcal{N} = 4$ SYM result.
  - Once in the canonical form, the differential equations are trivial to solve.
  - Fixing the boundary conditions was the last, but also the most time consuming step:
    - Scaling of the master integrals in z in the limits $z \to 0$ and $z \to \infty$.
    - Integrate over z and match to inclusive 4-particle phase-space master integrals [Gehrmann-De Ridder et al., 2004], similar to the approach of [Gituliar & Moch, 2017].

- ✓ The single master integrals are divergent, but all the ε-poles cancel in $\Sigma(\chi)$.

- Example: Differential equation for one of the integral families contributing to the N_f piece of the full NLO result

$$\frac{d}{dy}F(y,\varepsilon) = \mathbb{A}(y,\varepsilon)F(y,\varepsilon), \quad y \equiv \cos\chi$$

$$\mathbb{A}(y,\varepsilon) = \begin{pmatrix} \frac{3-4\varepsilon}{1-y} - \frac{\varepsilon}{1+y} & -\frac{1-\varepsilon}{1-y} & 0 \\ -\frac{(2-3\varepsilon)^2}{(1-\varepsilon)(1+y)} & \frac{1-2\varepsilon}{1+y} + \frac{\varepsilon}{1-y} & 0 \\ \frac{(2-3\varepsilon)^2}{(1-\varepsilon)(1+y)} - \frac{2(2-3\varepsilon)^2}{(1-\varepsilon)(1+y)^2} & \frac{2(1-\varepsilon)}{(1+y)^2} - \frac{3(1-2\varepsilon)}{1+y} & \frac{\varepsilon}{1-y} - \frac{2\varepsilon}{1+y} \end{pmatrix}$$

- Canonical form easily obtained with Fuchsia

$$\frac{d}{dy}G(y,\varepsilon) = \varepsilon\mathbb{B}(y)G(y,\varepsilon),$$

$$\mathbb{B}(y) = \begin{pmatrix} \frac{346}{25(1+y)} - \frac{577}{25(1-y)} & \frac{2709}{200\,(1-y)} - \frac{583}{50(1+y)} & 0 \\ \frac{504}{25(1+y)} - \frac{848}{25(1-y)} & \frac{502}{25\,(1-y)} - \frac{421}{25(1+y)} & 0 \\ -\frac{4840}{351(1+y)} - \frac{4840}{351(1-y)} & \frac{605}{78\,(1+y)} + \frac{605}{78(1-y)} & \frac{1}{1-y} - \frac{2}{1+y} \end{pmatrix}$$

- The transformation matrix $\mathbb{T}(y,\varepsilon) = \left(\mathbb{T}^1, \mathbb{T}^2, \mathbb{T}^3\right)$

$$\mathbb{T}^1(y,\varepsilon) = \begin{pmatrix} -\frac{64(1-5\varepsilon)(2-5\varepsilon)}{(6-11\varepsilon)(2-\varepsilon)(1-y)^3} - \frac{896\varepsilon^2-57\varepsilon+4}{(6-11\varepsilon)(2-\varepsilon)(1-y)} + \frac{16137\varepsilon^2-78\varepsilon+8}{(6-11\varepsilon)(2-\varepsilon)(1-y)^2} \\ \frac{32(1-5\varepsilon)(2-3\varepsilon)^2}{(6-11\varepsilon)(1-\varepsilon)(2-\varepsilon)(1-y)^2} - \frac{8(2-19\varepsilon)(2-3\varepsilon)^2}{(6-11\varepsilon)(1-\varepsilon)(2-\varepsilon)(1-y)} \\ 0 \end{pmatrix}$$

$$\mathbb{T}^2(y,\varepsilon) = \begin{pmatrix} \frac{36(1-5\varepsilon)(2-5\varepsilon)}{(6-11\varepsilon)(2-\varepsilon)(1-y)^3} + \frac{3169\varepsilon^2-98\varepsilon+6}{(6-11\varepsilon)(2-\varepsilon)(1-y)} - \frac{2679\varepsilon^2-376\varepsilon+36}{(6-11\varepsilon)(2-\varepsilon)(1-y)^2} \\ \frac{(9-98\varepsilon)(2-3\varepsilon)^2}{(6-11\varepsilon)(1-\varepsilon)(2-\varepsilon)(1-y)} - \frac{18(1-5\varepsilon)(2-3\varepsilon)^2}{(6-11\varepsilon)(1-\varepsilon)(2-\varepsilon)(1-y)^2} \\ 0 \end{pmatrix}$$

$$\mathbb{T}^3(y,\varepsilon) = \begin{pmatrix} 0 \\ 0 \\ \frac{351(3\varepsilon-2)^2(3\varepsilon-1)^2}{605(\varepsilon-2)(\varepsilon-1)\varepsilon(11\varepsilon-6)} \end{pmatrix}$$

- In the most complicated case $\mathbb{A}(y,\varepsilon)$ is given by a 19×19 matrix.

- Convenient notation for the full NLO result ($\beta_0 = 11C_A/3 - 4N_f T_f/3$)

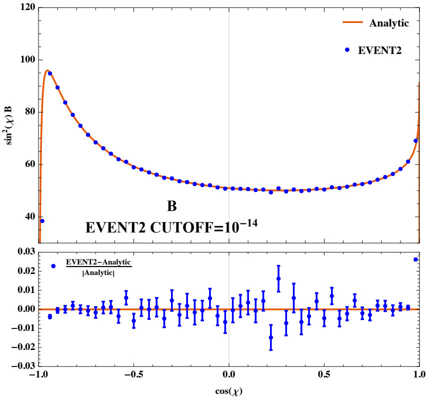
$$\frac{1}{\sigma_{\rm tot}} \frac{d\Sigma(\chi)}{d\cos\chi} = \frac{\alpha_s(\mu)}{2\pi} A(z) + \left(\frac{\alpha_s(\mu)}{2\pi}\right)^2 \left(\beta_0 \log\frac{\mu}{Q} A(z) + B(z)\right) + \mathcal{O}(\alpha_s^3).$$

- Color decomposition of the NLO coefficient $B(z)$

$$B(z) = C_F^2 B_{\rm lc}(z) + C_F(C_A - 2C_F) B_{\rm nlc}(z) + C_F N_f T_f B_{N_f}(z).$$

- Our result for $B(z)$ can be compared to the output of EVENT2.

- Sampling over 10^9 points.
- Perfect agreement over a large range of χ values.
- Rightmost bin discrepancy caused by the finite bin width in EVENT2.



- The analytic NLO result has a remarkably simple structure.
- Basis of pure functions $g_i^{(n)}$ of uniform transcendental weight $n \leq 3$

$$
\begin{aligned}
g_1^{(1)} &= \log(1-z)\,, \qquad g_2^{(1)} = \log(z)\,, \qquad g_1^{(2)} = 2(\mathrm{Li}_2(z) + \zeta_2) + \log^2(1-z)\,,\\
g_2^{(2)} &= \mathrm{Li}_2(1-z) - \mathrm{Li}_2(z)\,,\\
g_3^{(2)} &= -2\,\mathrm{Li}_2\left(-\sqrt{z}\right) + 2\,\mathrm{Li}_2\left(\sqrt{z}\right) + \log\left(\frac{1-\sqrt{z}}{1+\sqrt{z}}\right)\log(z)\,, \qquad g_4^{(2)} = \zeta_2\,,\\
g_1^{(3)} &= -6\left[\mathrm{Li}_3\left(-\frac{z}{1-z}\right) - \zeta_3\right] - \log\left(\frac{z}{1-z}\right)\left(2(\mathrm{Li}_2(z) + \zeta_2) + \log^2(1-z)\right)\,,\\
g_2^{(3)} &= -12\left[\mathrm{Li}_3(z) + \mathrm{Li}_3\left(-\frac{z}{1-z}\right)\right] + 6\,\mathrm{Li}_2(z)\log(1-z) + \log^3(1-z)\,,\\
g_3^{(3)} &= 6\log(1-z)\left(\mathrm{Li}_2(z) - \zeta_2\right) - 12\,\mathrm{Li}_3(z) + \log^3(1-z)\,,\\
g_4^{(3)} &= \mathrm{Li}_3\left(-\frac{z}{1-z}\right) - 3\,\zeta_2\log(z) + 8\,\zeta_3\,,\\
g_5^{(3)} &= -8\left[\mathrm{Li}_3\left(-\frac{\sqrt{z}}{1-\sqrt{z}}\right) + \mathrm{Li}_3\left(\frac{\sqrt{z}}{1+\sqrt{z}}\right)\right] + 2\mathrm{Li}_3\left(-\frac{z}{1-z}\right)\\
&\quad + 4\zeta_2\log(1-z) + \log\left(\frac{1-z}{z}\right)\log^2\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\,.
\end{aligned}
$$

$$B(z) = C_F^2 B_{\mathsf{lc}}(z) + C_F(C_A - 2C_F) B_{\mathsf{nlc}}(z) + C_F N_f T_f B_{N_f}(z)$$

- Leading color coefficient $B_{\mathsf{lc}}(z)$

$$+\frac{122400z^7 - 244800z^6 + 157060z^5 - 31000z^4 + 2064z^3 + 72305z^2 - 143577z + 63298}{1440(1-z)z^4}$$
$$-\frac{-244800z^9 + 673200z^8 - 667280z^7 + 283140z^6 - 48122z^5 + 2716z^4 - 6201z^3 + 11309z^2 - 9329z + 3007}{720(1-z)z^5} g_1^{(1)}$$
$$-\frac{244800z^8 - 550800z^7 + 422480z^6 - 126900z^5 + 13052z^4 - 336z^3 + 17261z^2 - 38295z + 19938}{720(1-z)z^4} g_2^{(1)}$$
$$+\frac{4z^7 + 10z^6 - 17z^5 + 25z^4 - 96z^3 + 296z^2 - 211z + 87}{24(1-z)z^5} g_1^{(2)}$$
$$+\frac{-40800z^8 + 61200z^7 - 28480z^6 + 4040z^5 - 320z^4 - 160z^3 + 1126z^2 - 4726z + 3323}{120z^5} g_2^{(2)}$$
$$-\frac{1 - 11z}{48z^{7/2}} g_3^{(2)} - \frac{120z^6 + 60z^5 + 160z^4 - 2246z^3 + 8812z^2 - 10159z + 4193}{120(1-z)z^5} g_4^{(2)}$$
$$-2\left(85z^4 - 170z^3 + 116z^2 - 31z + 3\right) g_1^{(3)} + \frac{-4z^3 + 18z^2 - 21z + 5}{6(1-z)z^5} g_2^{(3)} + \frac{z^2 + 1}{12(1-z)} g_3^{(3)}.$$

$$B(z) = C_F^2 B_{\mathsf{lc}}(z) + C_F(C_A - 2C_F)B_{\mathsf{nlc}}(z) + C_F N_f T_f B_{N_f}(z)$$

- Subleading color coefficient $B_{\mathsf{nlc}}(z)$

$$+\frac{57600z^7 - 115200z^6 + 75748z^5 - 17359z^4 + 902z^3 + 14966z^2 - 27552z + 9320}{720(1-z)z^4}$$
$$-\frac{-115200z^9 + 316800z^8 - 321680z^7 + 147846z^6 - 31035z^5 + 3225z^4 - 3571z^3 + 11322z^2 - 12412z + 4880}{360(1-z)z^5} g_1^{(1)}$$
$$-\frac{230400z^8 - 518400z^7 + 412960z^6 - 138600z^5 + 18696z^4 - 742z^3 + 10971z^2 - 25029z + 11424}{720(1-z)z^4} g_2^{(1)}$$
$$+\frac{-91z^7 + 235z^6 - 184z^5 + 15z^4 - 140z^3 + 721z^2 - 760z + 314}{120(1-z)z^5} g_1^{(2)}$$
$$+\frac{-19200z^8 + 28800z^7 - 14680z^6 + 2660z^5 - 340z^4 - 40z^3 + 315z^2 - 1431z + 952}{60z^5} g_2^{(2)}$$
$$+\frac{960z^4 - 160z^3 + 992z^2 + 547z + 1435}{480z^{7/2}} g_3^{(2)} - \frac{-120z^6 + 120z^5 - 130z^4 - 585z^3 + 2647z^2 - 3143z + 1266}{60(1-z)z^5} g_4^{(2)}$$
$$+\frac{640z^6 - 1920z^5 + 2196z^4 - 1196z^3 + 318z^2 - 42z + 3}{4(1-z)z} g_1^{(3)} + \frac{2z^7 - 3z^6 + 3z^5 - z^4 - z^3 + 9z^2 - 9z + 1}{12(1-z)z^5} g_2^{(3)}$$
$$-\frac{(1-2z)\left(z^2 - z + 1\right)}{2(1-z)z} g_4^{(3)} - \frac{2z^5 - z^4 + 2z^3 + z^2 + 3}{4z^4} g_5^{(3)}.$$

$$B(z) = C_F^2 B_{\text{lc}}(z) + C_F(C_A - 2C_F)B_{\text{nlc}}(z) + C_F N_f T_f B_{N_f}(z)$$

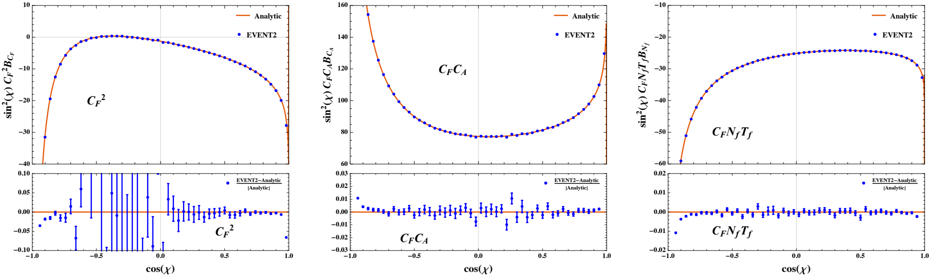
- N_f-piece $B_{N_f}(z)$

$$\begin{aligned}
&- \frac{7200z^7 - 14400z^6 + 8852z^5 - 1568z^4 + 48z^3 + 1825z^2 - 4115z + 2050}{144(1-z)z^4} \\
&- \frac{72000z^9 - 198000z^8 + 193040z^7 - 77700z^6 + 10960z^5 - 100z^4 - 489z^3 + 3269z^2 - 4801z + 1801}{360(1-z)z^5} g_1^{(1)} \\
&+ \frac{36000z^8 - 81000z^7 + 60520z^6 - 16650z^5 + 1190z^4 + 10z^3 + 428z^2 - 939z + 561}{180(1-z)z^4} g_2^{(1)} \\
&+ \frac{-z^7 - 4z^3 + 18z^2 - 24z + 9}{6(1-z)z^5} g_1^{(2)} - \frac{-12000z^8 + 18000z^7 - 7840z^6 + 920z^5 + 72z^2 - 222z + 187}{60z^5} g_2^{(2)} \\
&+ \frac{1-3z}{48z^{7/2}} g_3^{(2)} + \frac{8z^3 - 66z^2 + 71z + 7}{60(1-z)z^5} g_4^{(2)} + 2\left(50z^4 - 100z^3 + 66z^2 - 16z + 1\right) g_1^{(3)}.
\end{aligned}$$

- This contribution was computed by Lance Dixon and Marc Schreiber already in 2004!

- We can also validate the individual color components of $B(z)$ against **EVENT2**.
- More convenient notation:

$$B(z) = C_F^2 B_{C_F}(z) + C_F C_A B_{C_A}(z) + C_F N_f T_f B_{N_f}(z).$$



- We can compare the asymptotic behavior of $B(z)$ to the literature.
- In the collinear limit ($z \to 0$), the $\log(z)/z$ term agrees with the jet calculus prediction [Konishi et al., 1978, 1979; Richards et al., 1983].

$$
\begin{aligned}
B(z) = & C_F \left\{ \frac{1}{z} \left[\log(z) \left(-\frac{107 C_A}{120} + \frac{25 C_F}{32} + \frac{53 N_f T_f}{240} \right) \right. \right. \\
& + C_A \left(-\frac{25\zeta_2}{12} + \frac{\zeta_3}{2} + \frac{17683}{2700} \right) \\
& \left. + C_F \left(\frac{43\zeta_2}{12} - \zeta_3 - \frac{8263}{1728} \right) - \frac{4913 N_f T_f}{3600} \right] + \log(z) \left[C_A \left(\frac{33\zeta_2}{2} - \frac{703439}{25200} \right) \right. \\
& \left. + C_F \left(\frac{42109}{1200} - 21\zeta_2 \right) + N_f T_f \left(\frac{86501}{12600} - 4\zeta_2 \right) \right] \\
& + C_A \left(\frac{213\zeta_2}{5} - \frac{101\zeta_3}{2} - \frac{26986007}{5292000} \right) + C_F \left(-\frac{1541\zeta_2}{30} + 65\zeta_3 + \frac{18563}{2700} \right) \\
& \left. + N_f T_f \left(-\frac{46\zeta_2}{3} + 12\zeta_3 + \frac{2987627}{330750} \right) \right\} + \mathcal{O}(z).
\end{aligned}
$$

- In the back-to-back limit ($z \to 1$) terms enhanced by $1/(1-z)$ agree with the results from the NNLL soft gluon resummation [de Florian & Grazzini, 2005].
- N^3LL factorization formula is already available [Moult & Zhu, 2018].

$$
\begin{aligned}
B(z) = & C_F \left\{ \frac{1}{1-z} \left[\frac{1}{2} C_F \log^3(1-z) + \log^2(1-z) \left(\frac{11 C_A}{12} + \frac{9 C_F}{4} - \frac{N_f T_f}{3} \right) \right. \right. \\
& + \log(1-z) \left(C_A \left(\frac{\zeta_2}{2} - \frac{35}{72} \right) + C_F \left(\zeta_2 + \frac{17}{4} \right) + \frac{N_f T_f}{18} \right) \\
& \left. + C_A \left(\frac{11\zeta_2}{4} + \frac{3\zeta_3}{2} - \frac{35}{16} \right) + C_F \left(3\zeta_2 - \zeta_3 + \frac{45}{16} \right) + N_f T_f \left(\frac{3}{4} - \zeta_2 \right) \right] \\
& + \left(\frac{C_A}{2} + C_F \right) \log^3(1-z) + \log^2(1-z) \left(\frac{27 C_A}{8} + \frac{13 C_F}{2} - \frac{N_f T_f}{2} \right) \\
& + \log(1-z) \left[C_A \left(22\zeta_2 - \frac{2011}{72} \right) + C_F (47 - 19\zeta_2) + N_f T_f \left(\frac{361}{36} - 4\zeta_2 \right) \right] \\
& + C_A \left(\frac{6347\zeta_2}{80} - 21\zeta_2 \log(2) - \frac{137\zeta_3}{4} - \frac{3305}{72} \right) \\
& + C_F \left(-\frac{1727\zeta_2}{20} + 42\zeta_2 \log(2) + \frac{121\zeta_3}{2} + \frac{3437}{96} \right) \\
& \left. + N_f T_f \left(-\frac{1747\zeta_2}{120} + 12\zeta_3 + \frac{2099}{144} \right) \right\} + \mathcal{O}(1-z).
\end{aligned}
$$

- Notice that our σ_{tot} is LO, while de Florian and Grazzini used σ_{tot} at NLO.
- The relation $\sigma_{\text{tot, NLO}} = \sigma_{\text{tot, LO}} \left(1 + \frac{3}{4} C_F \frac{\alpha_s}{\pi}\right)$ shifts coefficients of C_F^2.

- The physical values of $z = (1 - \cos\chi)/2$ are between 0 and 1.
- Analytic continuation: extend $\Sigma(\chi)$ beyond the physical region and consider the limit $z \to \infty$

$$A(z) = \frac{C_F}{z^3} \left[2\,\log(-z) - \frac{9}{2} \right] + \mathcal{O}(1/z^4),$$

$$B_{\mathrm{lc}}(z) = \frac{1}{z^3} \left[\left(4\,\zeta_2 + \frac{4699}{288} \right) \log(-z) - 8\,\zeta_3 + \frac{991}{84}\,\zeta_2 - \frac{85595}{1728} \right] + \frac{i\,\mathrm{sign}(\mathrm{Im}(z))}{z^3} \left[\frac{11}{8}\,\zeta_2 \sqrt{-z} + \pi \left(-\frac{1459}{140} \log(-z) + \frac{466259}{19600} \right) \right] + \mathcal{O}(1/z^{7/2}),$$

$$B_{\mathrm{nlc}}(z) = \frac{1}{z^3} \left[\left(\frac{3}{2}\,\zeta_2 + \frac{473}{72} \right) \log(-z) - \frac{9}{2}\,\zeta_3 + \frac{521}{70}\,\zeta_2 - \frac{32713}{1728} \right] + \frac{i\,\mathrm{sign}(\mathrm{Im}(z))}{z^3} \left[-\frac{2059}{560}\,\zeta_2 \sqrt{-z} + \pi \left(-\frac{2407}{420} \log(-z) + \frac{3}{2}\,\zeta_2 + \frac{20518}{1225} \right) \right] + \mathcal{O}(1/z^{7/2}),$$

$$B_{N_f}(z) = \frac{1}{z^3} \left[-\frac{133}{36} \log(-z) - \frac{404}{105}\,\zeta_2 + \frac{51}{4} \right] + \frac{i\,\mathrm{sign}(\mathrm{Im}(z))}{z^3} \left[-\frac{3}{8}\,\zeta_2 \sqrt{-z} + \pi \left(\frac{26}{21} \log(-z) - \frac{196003}{88200} \right) \right] + \mathcal{O}(1/z^{7/2}).$$

- The strong suppression in $1/z$ is remarkable.

Summary

- After almost 40 years since the EEC has been computed at LO, we now finally have the fully analytic NLO result.
- The result is amazingly simple and can be written in terms of functions with maximal transcendental weight 3.
- New insights into the mathematical structure of event shape observables.
- Testbench for new resummation and subtraction techniques.
- This calculation would hardly be possible without recent advances in our understanding of phase space integrals, IBP-reduction and differential equations.
- The availability of public software tools for dealing with multi-loop integrals was enormously helpful.

Outlook

- Most event shape observables are lacking full analytic results beyond LO. Can this situation be improved?
- Fully analytic result for EEC at NNLO would be highly desirable but also much more difficult to obtain. More powerful techniques and tools?

$$\cos\theta_{ab} = \frac{(\boldsymbol{p}_a \cdot \boldsymbol{p}_b)}{|\boldsymbol{p}_a||\boldsymbol{p}_b|} = \frac{E_a E_b - (p_a \cdot p_b)}{E_a E_b} = 1 - \frac{(p_a \cdot p_b)}{E_a E_b}.$$

In the CM frame with $Q^\mu = (Q, 0, 0, 0)^T$ we have

$$\cos\theta_{ab} = 1 - \frac{Q^2 (p_a \cdot p_b)}{(Q \cdot p_a)(Q \cdot p_b)}.$$

Hence,

$$\begin{aligned}\delta(\cos\theta_{ab} - \cos\chi) &= \delta\left(1 - \cos\chi - \frac{Q^2 (p_a \cdot p_b)}{(Q \cdot p_a)(Q \cdot p_b)}\right) \\ &= \frac{(Q \cdot p_a)(Q \cdot p_b)}{Q^2} \delta\left((1 - \cos\chi)(Q \cdot p_a)(Q \cdot p_b) - Q^2 (p_a \cdot p_b)\right)\end{aligned}$$

Six event shape observables studied at LEP in great details

- Thrust T
- Normalized heavy jet mass M_H^2/s
- Wide and total jet broadening B_W and B_T
- C-Parameter
- The transition from 3-jet to 2-jet final states in the Durham jet algorithm Y_3

Thrust

- Definition of Thrust T ($T \in [0.5; 1.0]$)

$$T = \max_{\boldsymbol{n}} \left(\frac{\sum_i |\boldsymbol{p}_i \cdot \boldsymbol{n}_i|}{\sum_i |\boldsymbol{p}_i|} \right),$$

  where $\boldsymbol{p}_i$ is the 3-momentum of the particle i in the given event.
- Thrust axis $\boldsymbol{n}_T$ is a unit vector $\boldsymbol{n}$ that maximizes T.
- Meaning of $\boldsymbol{n}_T$: direction of the maximum 3-momentum flow.
- Meaning of T: maximum directed momentum in the collision.
  - $T = 0.5$ describes a spherically symmetric distribution.
  - $T = 1$ describes a pencil-like back-to-back 2-jet event.
  - $T = 2/3$ is the minimal value for a 3-particle event.
- Typical reactions to illustrate T

$$e^+(k_1) + e^-(k_2) \to \gamma^*(q) \to q(p_1) + \bar{q}(p_2) + g(p_3),$$

- The T-distribution is singular for $T \to 1$: soft and collinear singularities.

Major, Minor and Oblateness

- Major (useful for 3-jet events) is defined as

$$\max\left(\frac{\sum_i \boldsymbol{p}_i \cdot \boldsymbol{n}'_i}{\sum_i |\boldsymbol{p}_i|}\right), \quad \text{for } \boldsymbol{n}\cdot\boldsymbol{n}' = 0,\ |\boldsymbol{n}'| = 1.$$

- Minor (useful for 4-jet events) is defined as

$$\left(\frac{\sum_i \boldsymbol{p}_i \cdot \boldsymbol{n}''_i}{\sum_i |\boldsymbol{p}_i|}\right), \quad \text{for } \boldsymbol{n}\cdot\boldsymbol{n}' = 0,\ \boldsymbol{n}'\cdot\boldsymbol{n}'' = 0,\ |\boldsymbol{n}'| = |\boldsymbol{n}''| = 1.$$

- Oblateness (useful for 3-jet events) is just Major - Minor.

Heavy jet mass

- Divide the event into 2 hemispheres H_1 and H_2.
- In each hemisphere compute the hemisphere invariant mass as

$$M_i^2/s = \frac{1}{E_{\text{vis}}^2} \left(\sum_{k \in H_i} p_k \right)^2,$$

  where E_{vis} is the total energy visible in the event.
- Original definition: choose H_i such, that $M_1^2 + M_2^2$ is minimized.
- Modern definition: separate H_1 and H_2 by the plane orthogonal to the thrust axis $\boldsymbol{n}_T$.
- Heavy jet mass: the larger of the two hemisphere invariant masses

$$\rho \equiv M_H^2/s = \max(M_1^2/s, M_2^2/s)$$

- In the 2-particle limit $\rho \to 0$, while for a 3-particle event $\rho \leq 1/3$.
- The associated light hemisphere mass

$$M_L^2/s = \min(M_1^2/s, M_2^2/s)$$

  is an example of a 4-jet observable and vanishes in the 3-particle limit.
- At lowest order, the heavy jet mass and the $(1-T)$ distribution are identical. This degeneracy is lifted at NLO.

Linearized sphericity, C parameter

- Define linearized sphericity (unlike sphericity it is collinear safe) as

$$L^{ab} = \frac{\sum_i \frac{\boldsymbol{p}_i^a \boldsymbol{p}_i^b}{|\boldsymbol{p}_i|}}{\sum_i |\boldsymbol{p}_i|},$$

- Define C-parameter ($0 \leq C \leq 1$) via the eigenvalues of L^{ab}

$$C = 3(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1)$$

- Useful for measuring the 3-jet structure of the event.
- Perfect 2-jet event: $C = 0$.

Jet broadening

- Divide each event into two hemisphere H_+ and H_- using a plane perpendicular to the thrust axis $\boldsymbol{n}_T$.
- In each hemisphere calculate the hemisphere broadening

$$B_i = \frac{\sum_{k \in H_i} |\boldsymbol{p}_k \times \boldsymbol{n}_T|}{2 \sum_k |\boldsymbol{p}_k|}$$

- Define the wide jet broadening B_W and the total jet broadening B_T

$$B_W = \max(B_1, B_2), \quad B_T = B_1 + B_2$$

- 2-particle limit: $B_W \to 0$ and $B_T \to 1$.
- The maximum broadening for a 3-particle event: $B_T = B_W = 1/(2\sqrt{3})$
- The narrow jet broadening

$$B_N = \min(B_1, B_2)$$

is a 4-jet observable and vanishes when only 3 particles are in the event.