Testing the Relativistic effects of the semileptonic Bc decays to charmonium in the Bethe-Salpeter method

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Motivation

Motivation

- After the discovery of Higgs, Standard Model needs precise study and test, relativistic study is important.
- Usually, we pay attention to the relativistic effects of light hadrons, ignore ones of heavy hadrons because of heavy mass, especially double-heavy mesons.
- We know little about the relativistic effects of heavy excited meson which has higher mass than the corresponding ground heavy meson.

Motivation

- After the year 2003, more and more heavy excited states are discovered. Non-relativistic and semi-relativistic models will give large errors.

- We will study the relativistic corrections of Bc semileptonic decays to charmonium by the instantaneous Bethe-Salpeter method.

Bethe-Salper equation and Salpeter equation

Bethe-Salpter and Salpeter equation

- Bethe-Salpeter Equation:

$$(\not{p}_1 - m_1)\chi_P(q)(\not{p}_2 + m_2) = \int \frac{d^4k}{(2\pi)^4} V(P,q,k)\chi_P(k)$$

- Total momentum P and relative momentum q:

$$p_1 = \alpha_1 P + q, \quad \alpha_1 = \frac{m_1}{m_1 + m_2}$$
$$p_2 = \alpha_2 P - q, \quad \alpha_2 = \frac{m_2}{m_1 + m_2}$$

- Normalization:

$$\int \frac{d^4k d^4q}{(2\pi)^4} Tr\left\{\overline{\chi}(k)\frac{\partial}{\partial P_0}\left[S_1^{-1}(p_1)S_2^{-1}(p_2)\delta^4(k-q) + V(P,k,q)\right]\chi(q)\right\} = 2iP_0$$

- There is difficulty about the kernel, the time-inspired interaction?

- Salpeter suggested the instantaneous version

- The relative momentum q is divided into parallel and vertical parts to the momentum P:

$$q_{||}^{\mu} \equiv (P \cdot q/M^2)P^{\mu} \; , \quad q_{\perp}^{\mu} \equiv q^{\mu} - q_{||}^{\mu} \; .$$

$$q^{\mu} = q_{||}^{\mu} + q_{\perp}^{\mu} \; ,$$

the momenta :

$$q_P = \frac{(P \cdot q)}{M} \; , \qquad q_T = \sqrt{q_P^2 - q^2} = \sqrt{-q_{\perp}^2} \; .$$

will become to the q_0 and $|\vec{q}|$ in center-mass-system of the meson

Salpeter Eequation

The reduced Bethe-Salpeter wave function:

$$\varphi_P(q_\perp^\mu) \equiv i \int \frac{dq_P}{2\pi} \chi(q_\parallel^\mu, q_\perp^\mu) \ ,$$

$$\eta(q_\perp^\mu) \equiv \int \frac{k_T^2 dk_T ds}{(2\pi)^2} V(k_\perp, q_\perp, s) \varphi_p(k_\perp^\mu) \ .$$

and the useful notations

$$\omega_i = \sqrt{m_i^2 + q_T^2} \ , \quad \Lambda_i^\pm(q_\perp) = \frac{1}{2\omega_i}\left[\frac{\not{P}}{M}\omega_i \pm J(i)(m_i + \not{q}_\perp)\right]$$

$$S_i(p_i) = \frac{\Lambda_{ip}^+(q_\perp)}{J(i)q_p + \alpha_i M_H - \omega_{ip} + i\epsilon} + \frac{\Lambda_{ip}^-(q_\perp)}{J(i)q_p + \alpha_i M_H + \omega_{ip} - i\epsilon}$$

so we define

$$\varphi_P^{\pm\pm}(q_\perp) \equiv \Lambda_1^\pm(q_\perp) \frac{\not{P}}{M} \varphi_P(q_\perp) \frac{\not{P}}{M} \Lambda_2^\pm(q_\perp)$$

then Salpeter wave function include four parts

$$\varphi_P(q_\perp) = \varphi_P^{++}(q_\perp) + \varphi_P^{+-}(q_\perp) + \varphi_P^{-+}(q_\perp) + \varphi_P^{--}(q_\perp)$$

and the Salpter equation can be written as:

$$\varphi_P(q_\perp) = \frac{\Lambda_1^+(q_\perp)\eta_P(q_\perp)\Lambda_2^+(q_\perp)}{(M-\omega_1-\omega_2)} - \frac{\Lambda_1^-(q_\perp)\eta_P(q_\perp)\Lambda_2^-(q_\perp)}{(M+\omega_1+\omega_2)} \ ,$$

Salpeter equation (positive and negative energy wave function)

$$(M-\omega_1-\omega_2)\varphi_P^{++}(q_\perp) = \Lambda_1^+(q_\perp)\eta_P(q_\perp)\Lambda_2^+(q_\perp) \ ,$$

$$(M+\omega_1+\omega_2)\varphi_P^{--}(q_\perp) = -\Lambda_1^-(q_\perp)\eta_P(q_\perp)\Lambda_2^-(q_\perp) \ ,$$

$$\varphi_P^{+-}(q_\perp) = \varphi_P^{-+}(q_\perp) = 0 \ .$$

Normalization:

$$\int \frac{d^3 q_\perp}{(2\pi)^3} tr \left[\overline{\varphi}_P^{++} \frac{\not{P}}{M} \varphi_P^{++} \frac{\not{P}}{M} - \overline{\varphi}_P^{--} \frac{\not{P}}{M} \varphi_P^{--} \frac{\not{P}}{M} \right] = 2P_0$$

We solve the full Salpeter equation will the simple Cornell potential:

$$I(r) = V_s(r) + V_0 + \gamma_0 \otimes \gamma^0 V_v(r) = \lambda r + V_0 - \gamma_0 \otimes \gamma^0 \frac{4}{3} \frac{\alpha_s}{r}$$

To avoid infrared divergence

$$V_s(r) = \frac{\lambda}{\alpha}(1 - e^{-\alpha' r}) + V_0 \ ,$$

$$V_v(r) = -\frac{4}{3}\frac{\alpha_s}{r} e^{-\alpha r} \ .$$

In momentum space:

$$V_s(\vec{q}) = -(\frac{\lambda}{\alpha} + V_0)\delta^3(\vec{q}) + \frac{\lambda}{\pi^2}\frac{1}{(\vec{q}^{\,2} + \alpha^2)^2} \ ,$$

$$V_v(\vec{q}) = -\frac{2}{3\pi^2}\frac{\alpha_s(\vec{q})}{(\vec{q}^{\,2} + \alpha^2)} \ .$$

$$\alpha_s(\vec{q}) = \frac{12\pi}{27}\frac{1}{\log(a + \frac{\vec{q}^{\,2}}{\Lambda^2_{QCD}})} \ .$$

Relativistic Wave functions and solutions

Wave Fuctions

- The wave functions for S wave meson in a non-ralativistic model are:

$$\varphi_{0^-}(\vec{q}) = M(1+\gamma_0)\gamma_5\phi(\vec{q}), \quad \varphi_{1^-}^{\lambda}(\vec{q}) = M(1+\gamma_0)\vec{\epsilon}^{\lambda}\cdot\vec{\gamma}\phi(\vec{q}),$$

- In our method the relativistic wave function for a pseudoscalar $0^-(0^{-+})$ meson

$$\varphi_{^1S_0}(\vec{q}) = M\left[\gamma_0\varphi_1(\vec{q}) + \varphi_2(\vec{q}) + \frac{\not{q}_\perp}{M}\varphi_3(\vec{q}) + \frac{\gamma_0\not{q}_\perp}{M}\varphi_4(\vec{q})\right]\gamma_5$$

Since we have $\varphi^{+-}_{1S_0}(\vec{q}) = \varphi^{-+}_{1S_0}(\vec{q}) = 0$

The four wave functions are not independent

$$\varphi_3(\vec{q}) = \frac{\varphi_2(\vec{q})M(-\omega_1+\omega_2)}{m_2\omega_1+m_1\omega_2}\,, \quad \varphi_4(\vec{q}) = -\frac{\varphi_1(\vec{q})M(\omega_1+\omega_2)}{m_2\omega_1+m_1\omega_2}\,.$$

And the positive and negative wave function

$$\varphi^{++}_{1S_0}(\vec{q}) = \frac{M}{2}\left(\varphi_1(\vec{q}) + \varphi_2(\vec{q})\frac{m_1+m_2}{\omega_1+\omega_2}\right)\left[\frac{\omega_1+\omega_2}{m_1+m_2} + \gamma_0 - \frac{\not{q}_\perp(m_1-m_2)}{m_2\omega_1+m_1\omega_2} + \frac{\not{q}_\perp\gamma_0(\omega_1+\omega_2)}{(m_2\omega_1+m_1\omega_2)}\right]\gamma_5$$

$$\varphi^{--}_{1S_0}(\vec{q}) = \frac{M}{2}\left(\varphi_1(\vec{q}) - \varphi_2(\vec{q})\frac{m_1+m_2}{\omega_1+\omega_2}\right)\left[-\frac{\omega_1+\omega_2}{m_1+m_2} + \gamma_0 + \frac{\not{q}_\perp(m_1-m_2)}{m_2\omega_1+m_1\omega_2} + \frac{\not{q}_\perp\gamma_0(\omega_1+\omega_2)}{(m_2\omega_1+m_1\omega_2)}\right]\gamma_5\,.$$

And the normalization

$$\int\frac{d\vec{q}}{(2\pi)^3}4\varphi_1(\vec{q})\varphi_2(\vec{q})M^2\left\{\frac{\omega_1+\omega_2}{m_1+m_2} + \frac{m_1+m_2}{\omega_1+\omega_2} + \frac{2\vec{q}^{\,2}(\omega_1 m_1+\omega_2 m_2)}{(\omega_1 m_2+\omega_2 m_1)^2}\right\} = 2M\,.$$

- For equal mass system 0^{-+} state:

$$\varphi_{P,0^{-+}}(q) = \left[\not{P} f_1(q) + M f_2(q) + \not{q}_\perp \frac{\not{P}}{m_1} f_1(q)\right]\gamma_5$$
$$= \left[(1 + \frac{\not{q}_\perp}{m_1})\not{P} f_1(q) + M f_2(q)\right]\gamma_5 \; .$$

- From the first two Salpeter equations, we obtained:

$$(M - 2\omega_1)\left[f_1(q) + f_2(q)\frac{m_1}{\omega_1}\right] = -\int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{\omega_1^2}$$
$$\times \left\{(V_s - V_v)\left[f_1(k)m_1^2 + f_2(k)m_1\omega_1\right] - (V_s + V_v)f_1(k)(\vec{q}\cdot\vec{k})\right\} \; ,$$

$$(M + 2\omega_1)\left[f_1(q) - f_2(q)\frac{m_1}{\omega_1}\right] = \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{\omega_1^2}$$
$$\times \left\{(V_s - V_v)\left[f_1(k)m_1^2 - f_2(k)m_1\omega_1\right] - (V_s + V_v)f_1(k)(\vec{q}\cdot\vec{k})\right\} \; ,$$

- Normalization

$$\int \frac{d^3q}{(2\pi)^3} 4 f_1(q) f_2(q) M^2 \left\{\frac{\omega_1}{m_1} + \frac{m_1}{\omega_1} + \frac{q^2}{\omega_1 m_1}\right\} = 2M$$

Wave function

- Wave function for ${}^3P_0 \quad 0^+(0^{++})$ scalar meson

$$\varphi_{0^+}(q_\perp) = f_1(q_\perp)\not{q}_\perp + f_2(q_\perp)\frac{\not{P}\not{q}_\perp}{M} + f_3(q_\perp)M + f_4(q_\perp)\not{P}$$

where only two of them are independent

$$f_3(q_\perp) = \frac{f_1(q_\perp)q_\perp^2(m_1+m_2)}{M(\omega_1\omega_2+m_1m_2+q_\perp^2)}, \quad f_4(q_\perp) = \frac{f_2(q_\perp)q_\perp^2(\omega_1-\omega_2)}{M(m_1\omega_2+m_2\omega_1)}$$

- Normalization:

$$\int \frac{d\vec{q}}{(2\pi)^3}\frac{16 f_1 f_2 \omega_1 \omega_2 \vec{q}^2}{m_1\omega_2+m_2\omega_1} = 2M.$$

Wave function

- Wave function for ${}^3P_1 \;\; 1^+(1^{++})$ meson

$$\varphi_{1^+}(q_\perp) = i\varepsilon_{\mu\nu\alpha\beta}P^\nu q_\perp^\alpha \epsilon^\beta \left[f_1 M\gamma^\mu + f_2 \not{P}\gamma^\mu + f_3 \not{q}_\perp \gamma^\mu + i f_4 \varepsilon^{\mu\rho\sigma\delta} q_{\perp\rho} P_\sigma \gamma_\delta \gamma_5 / M\right] / M^2.$$

- where only two of them are independent

$$f_3(q_\perp) = \frac{f_1(q_\perp)M(m_1\omega_2 - m_2\omega_1)}{q_\perp^2(\omega_1+\omega_2)}, \quad f_4(q_\perp) = \frac{f_2(q_\perp)M(-\omega_1\omega_2 + m_1 m_2 + q_\perp^2)}{q_\perp^2(m_1+m_2)}$$

- Normalization:

$$\int \frac{d\vec{q}}{(2\pi)^3} \frac{32 f_1 f_2 \omega_1\omega_2(\omega_1\omega_2 - m_1 m_2 + \vec{q}^2)}{3(m_1+m_2)(\omega_1+\omega_2)} = 2M.$$

Wave function

- Wave function for $^1P_1$ $1^+(1^{+-})$ meson

$$\varphi_{1^+}(q_\perp) = q_\perp \cdot \epsilon_\perp^\lambda \left[f_1(q_\perp) + f_2(q_\perp)\frac{\not P}{M} + f_3(q_\perp)\frac{\not q_\perp}{M} + f_4(q_\perp)\frac{\not P \not q_\perp}{M^2} \right] \gamma_5$$

and
$$f_3(q_\perp) = -\frac{f_1(q_\perp) M (m_1 - m_2)}{(\omega_1\omega_2 + m_1 m_2 - q_\perp^2)}, \quad f_4(q_\perp) = -\frac{f_2(q_\perp) M (\omega_1 + \omega_2)}{(m_1\omega_2 + m_2\omega_1)}$$

- Normalization:

$$\int \frac{d\vec{q}}{(2\pi)^3} \frac{16 f_1 f_2 \omega_1 \omega_2 \vec{q}^2}{3(m_1\omega_2 + m_2\omega_1)} = 2M.$$

Wave function

Wave function for $^3S_1\ 1^-(1^{--})$ vector meson

$$\varphi^{\lambda}_{1^-}(q_\perp) = q_\perp \cdot \epsilon^{\lambda}_\perp \left[f_1(q_\perp) + \frac{\not{P}}{M} f_2(q_\perp) + \frac{\not{q}_\perp}{M} f_3(q_\perp) + \frac{\not{P}\not{q}_\perp}{M^2} f_4(q_\perp) \right] + M \not{\epsilon}^{\lambda}_\perp f_5(q_\perp)$$

$$+ \not{\epsilon}^{\lambda}_\perp \not{P} f_6(q_\perp) + (\not{q}_\perp \not{\epsilon}^{\lambda}_\perp - q_\perp \cdot \epsilon^{\lambda}_\perp) f_7(q_\perp) + \frac{1}{M}(\not{P}\not{\epsilon}^{\lambda}_\perp \not{q}_\perp - \not{P} q_\perp \cdot \epsilon^{\lambda}_\perp) f_8(q_\perp),$$

There are only four independent wave functions

$$f_1(q_\perp) = \frac{\left[q_\perp^2 f_3(q_\perp) + M^2 f_5(q_\perp)\right](m_1 m_2 - \omega_1\omega_2 + q_\perp^2)}{M(m_1+m_2)q_\perp^2}, \quad f_7(q_\perp) = \frac{f_5(q_\perp) M(-m_1 m_2 + \omega_1\omega_2 + q_\perp^2)}{(m_1 - m_2)q_\perp^2},$$

$$f_2(q_\perp) = \frac{\left[-q_\perp^2 f_4(q_\perp) + M^2 f_6(q_\perp)\right](m_1\omega_2 - m_2\omega_1)}{M(\omega_1+\omega_2)q_\perp^2}, \quad f_8(q_\perp) = \frac{f_6(q_\perp) M(m_1\omega_2 - m_2\omega_1)}{(\omega_1 - \omega_2)q_\perp^2}.$$

normalization

$$\int \frac{d\vec{q}}{(2\pi)^3} \frac{16\omega_1\omega_2}{3} \left\{ 3 f_5 f_6 \frac{M^2}{m_1\omega_2 + m_2\omega_1} + \frac{\omega_1\omega_2 - m_1 m_2 + \vec{q}^2}{(m_1+m_2)(\omega_1+\omega_2)} \left[f_4 f_5 - f_3 \left(f_4 \frac{\vec{q}^2}{M^2} + f_6 \right) \right] \right\} = 2M.$$

Mass spectra

- Parameters:

$$a = e = 2.7183\ , \quad \alpha = 0.06\ \text{GeV}, \quad \lambda = 0.21\ \text{GeV}^2,$$

$$\text{and} \quad m_c = 1.62\ \text{GeV}, \quad m_b = 4.96\ \text{GeV}\ .$$

for charmonium $N_f = 3,\ \Lambda_{QCD} = 0.27\ \text{GeV}$

$$\alpha_s(m_c) = 0.38$$

for bottomonium $N_f = 4,\ \Lambda_{QCD} = 0.20\ \text{GeV}$

$$\alpha_s(m_b) = 0.23$$

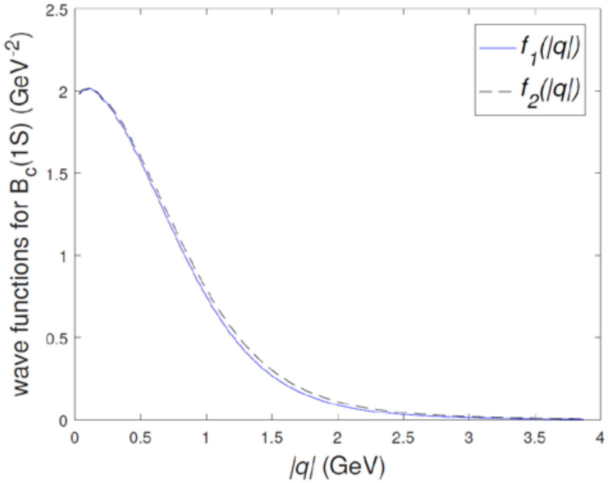
TABLE I: Parameter of V_0 in unit of MeV

	$c\bar{c}$	$b\bar{b}$
$\mathbf{n}J^{PC} = \mathbf{n}\ 0^{-+}(^1S_0)$	-0.314	-0.240
$\mathbf{n}J^{PC} = \mathbf{n}\ 1^{--}(^3S_1)$	-0.176	-0.166
$\mathbf{n}J^{PC} = \mathbf{n}\ 0^{++}(^3P_0)$	-0.282	-0.174
$\mathbf{n}J^{PC} = \mathbf{n}\ 1^{++}(^3P_1)$	-0.162	-0.141
$\mathbf{n}J^{PC} = \mathbf{n}\ 2^{++}(^3P_2)$	-0.110	-0.121
$\mathbf{n}J^{PC} = \mathbf{n}\ 1^{+-}(^1P_1)$	-0.144	-0.135

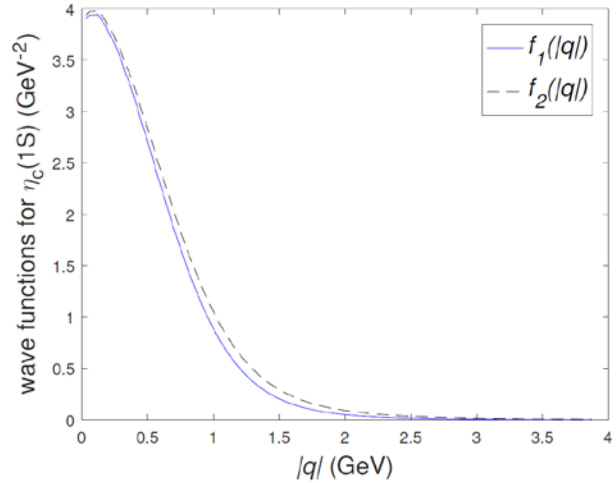
n $J^{PC}(^{(2S+1)}L_J)$	Th($c\bar{c}$)	Ex($c\bar{c}$)	Th($b\bar{b}$)	Ex($b\bar{b}$)
1 $0^{-+}(^1S_0)$	2980.3(input)	2980.3	9390.2(input)	9388.9
2 $0^{-+}(^1S_0)$	3576.4	3637	9950.0	
3 $0^{-+}(^1S_0)$	3948.8		10311.4	
1 $1^{--}(^3S_1)$	3096.9(input)	3096.916	9460.5(input)	9460.30
2 $1^{--}(^3S_1)$	3688.1	3686.09	10023.1	10023.26
3 $1^{--}(^3D_1)$	3778.9	3772.92	10129.5	
4 $1^{--}(^3S_1)$	4056.8	4039	10368.9	10355.2
5 $1^{--}(^3D_1)$	4110.7	4153	10434.7	
6 $1^{--}(^3S_1)$	4329.4	4421	10635.8	10579.4
7 $1^{--}(^3S_1)$	4545.9		10852.1	10865

n $J^{PC}(^{2S+1})L_J$	Th($c\bar{c}$)	Ex($c\bar{c}$)	Th($b\bar{b}$)	Ex($b\bar{b}$)
1 $0^{++}(^3P_0)$	3414.7(input)	3414.75	9859.0	9859.44
2 $0^{++}(^3P_0)$	3836.8		10240.6	10232.5
3 $0^{++}(^3P_0)$	4140.1		10524.7	
1 $1^{++}(^3P_1)$	3510.3(input)	3510.66	9892.2	9892.78
2 $1^{++}(^3P_1)$	3928.7		10272.7	10255.46
3 $1^{++}(^3P_1)$	4228.8		10556.2	

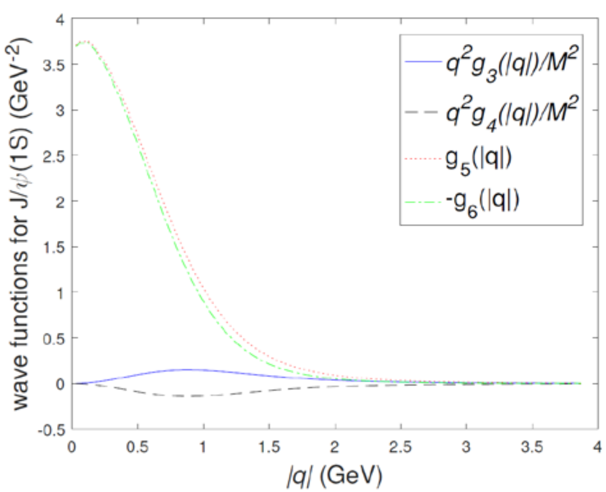
n $J^{PC}(^{2S+1})L_J$	Th($c\bar{c}$)	Ex($c\bar{c}$)	Th($b\bar{b}$)	Ex($b\bar{b}$)
1 $2^{++}(^3P_2)$	3556.1(input)	3556.20	9914.4	9912.21
2 $2^{++}(^3P_2)$	3972.4		10293.6	10268.65
3 $2^{++}(^3F_2)$	4037.9		10374.4	
4 $2^{++}(^3P_2)$	4271.0		10561.5	
1 $1^{+-}(^1P_1)$	3526.0(input)	3525.93	9900.2	
2 $1^{+-}(^1P_1)$	3943.0		10280.4	
3 $1^{+-}(^1P_1)$	4242.4		10562.0	



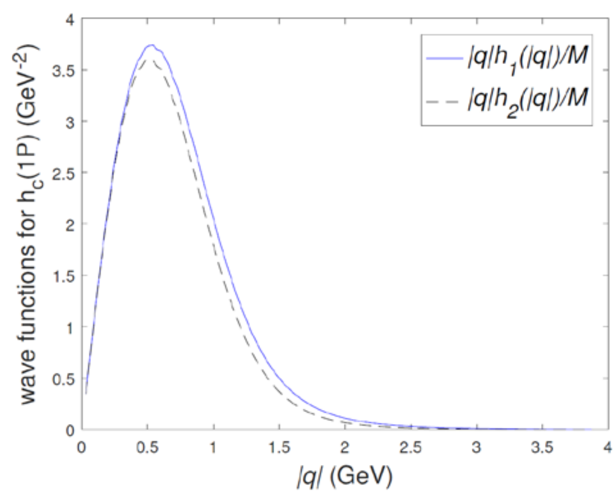
(a)B_c



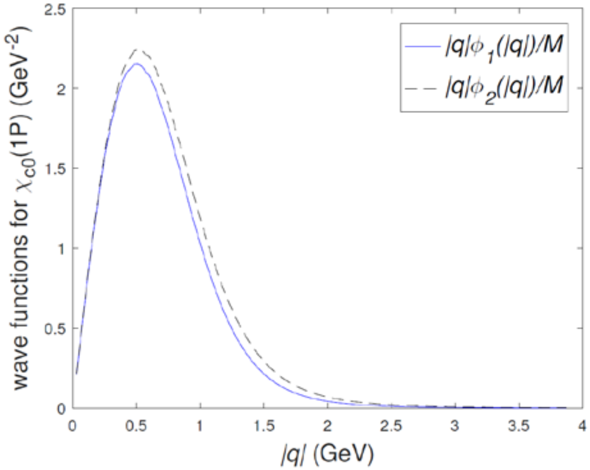
(b)η_c



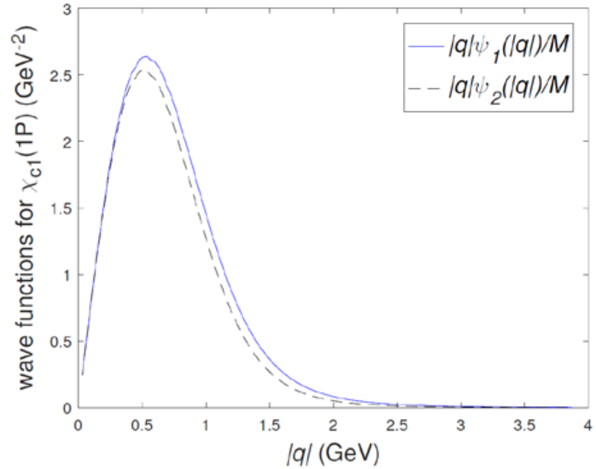
(c)J/ψ



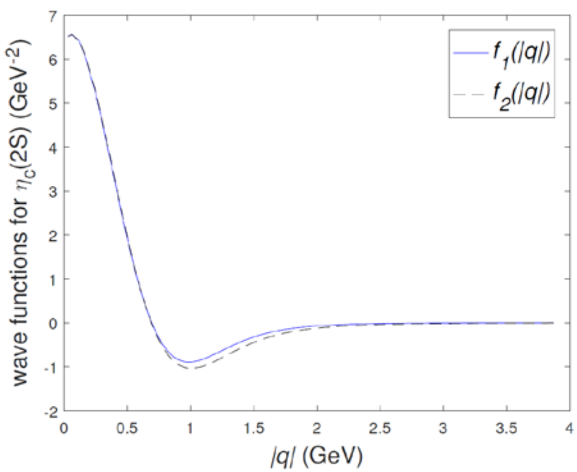
(d)h_c



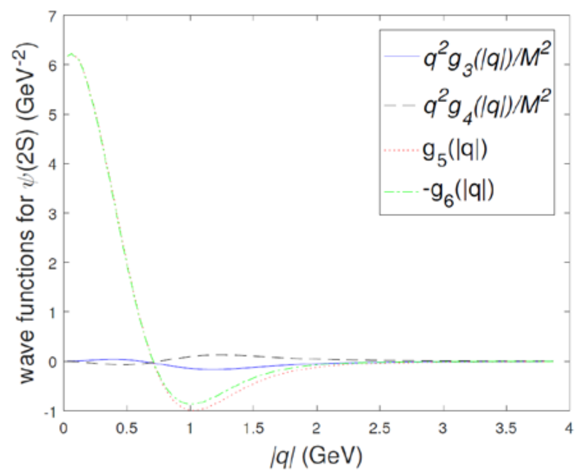
(e)χ_{c0}



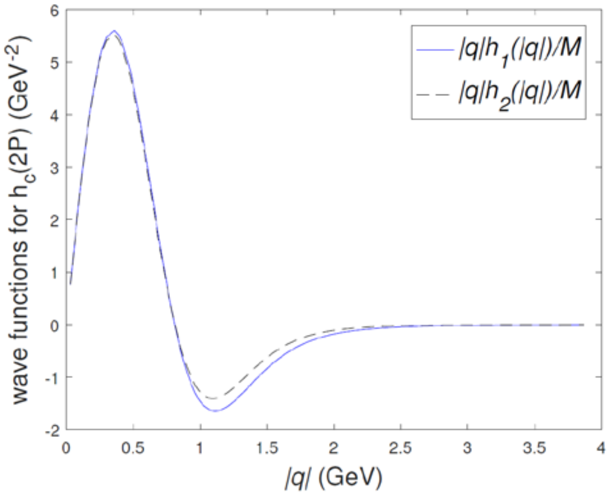
(f)χ_{c1}



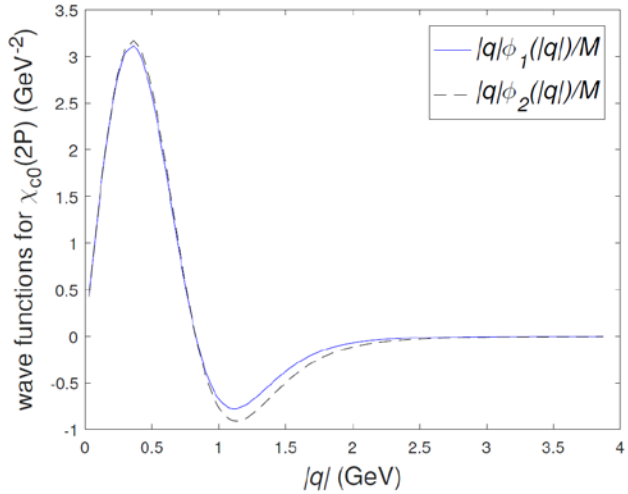
(a)$\eta_c(2S)$



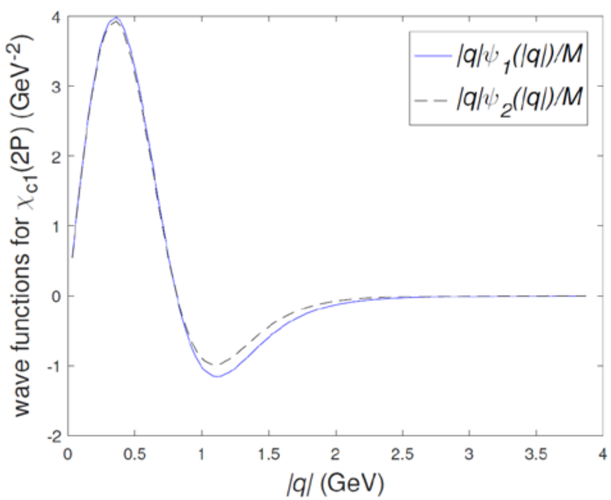
(b)$\psi(2S)$



(c)$h_c(2P)$



(d)$\chi_{c0}(2P)$



(e)$\chi_{c1}(2P)$

Transition amplitude for Semileptonic Bc decays to charmonium

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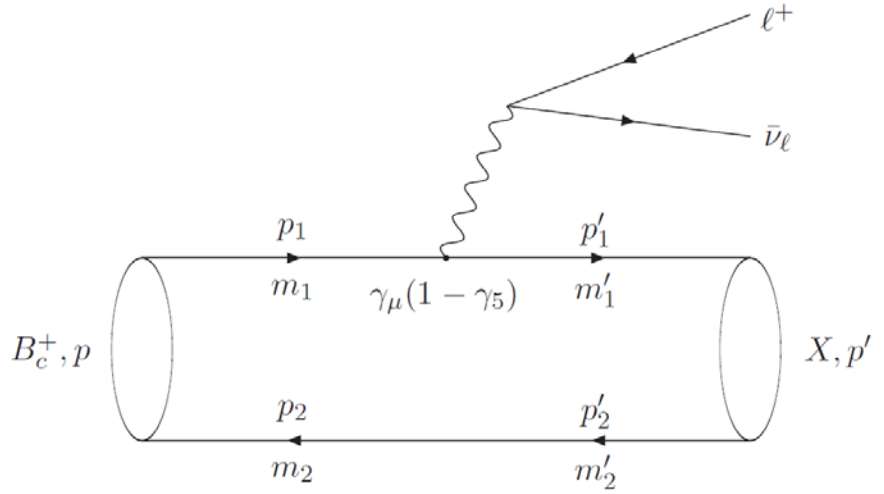


FIG. 1: Feynman diagram corresponding to the semileptonic decays $B_c^+ \to X + \ell^+ + \bar{\nu}_\ell$.

$$T = \frac{G_F}{\sqrt{2}} V_{ij} \bar{u}_{\nu_\ell} \gamma^\mu (1-\gamma_5) v_\ell \langle X(p', \epsilon) | J_\mu | B_c^+(p) \rangle,$$

Hadronic transition matrix element

$$
\begin{aligned}
&\langle \chi_c(h_c)(P_f)|J^\mu|B_c(P)\rangle \\
&\quad = \mathrm{i}\int \frac{\mathrm{d}^4q\mathrm{d}^4q'}{(2\pi)^4}\mathrm{Tr}[\overline{\chi}_{\chi_c(h_c)}(P',q')(\not{p}_1-m_1)\chi_{B_c}(P,q)V_{cb}\gamma^\mu(1-\gamma_5)\delta(p_1-p_1')] \\
&\quad = \mathrm{i}\int \frac{\mathrm{d}^4q}{(2\pi)^4}\mathrm{Tr}\left[\overline{\chi}_{\chi_c(h_c)}(P',q')(\alpha_1\not{P}+\not{q}-m_1)\chi_{B_c}(P,q)V_{cb}\gamma^\mu(1-\gamma_5)\right], \\
&\quad = \int \frac{\mathrm{d}^3q_\perp}{(2\pi)^3}\mathrm{Tr}\left\{\left[\bar{\varphi}'^{++}(q'_\perp)\frac{\not{P}}{M}\varphi^{++}(q_\perp)+\bar{\varphi}'^{++}(q'_\perp)\frac{\not{P}}{M}\psi^{+-}(q_\perp)\right.\right. \\
&\qquad -\bar{\psi}'^{-+}(q'_\perp)\frac{\not{P}}{M}\varphi^{++}(q_\perp)-\bar{\psi}'^{+-}(q'_\perp)\frac{\not{P}}{M}\varphi^{--}(q_\perp)+\bar{\varphi}'^{--}(q'_\perp)\frac{\not{P}}{M}\psi^{-+}(q_\perp) \\
&\qquad \left.\left.-\bar{\varphi}'^{--}(q'_\perp)\frac{\not{P}}{M}\varphi^{--}(q_\perp)\right]\gamma^\mu(1-\gamma_5)\right\},
\end{aligned}
$$

$$
\langle \chi_c(h_c)(P_f)|J^\mu|B_c(P)\rangle = \int \frac{\mathrm{d}^3q_\perp}{(2\pi)^3}\mathrm{Tr}\left\{\bar{\varphi}'^{++}(q'_\perp)\frac{P}{M}\varphi^{++}(q_\perp)\gamma^\mu(1-\gamma_5)\right\}
$$

Transition amplitude and form factors

- If final state is pseudoscalar η_c or scalar χ_{c0}

$$
\begin{aligned}
&\langle P|\bar{c}\gamma^\mu(1-\gamma^5)b|B_c^+\rangle \\
=&\int \frac{\mathrm{d}\vec{q}}{(2\pi)^3}\mathrm{Tr}\left[\frac{\not{P}}{M}\overline{\varphi}^{++}_{P_f}(\vec{q}+\alpha_2\vec{P}_f)\gamma^\mu(1-\gamma^5)\varphi^{++}_P(\vec{q})\right] \\
=&S_+(P+P_f)^\mu+S_-(P-P_f)^\mu,
\end{aligned}
$$

- If final state is vector, J/ψ, h_c or χ_{c1}

$$
\begin{aligned}
&\langle V|\bar{c}\gamma^\mu(1-\gamma^5)b|B_c^+\rangle \\
=&\int \frac{\mathrm{d}\vec{q}}{(2\pi)^3}\mathrm{Tr}\left[\frac{\not{P}}{M}\bar{\varphi}^{++}_{P_f}(\vec{q}+\alpha_2\vec{P}_f)\gamma^\mu(1-\gamma^5)\varphi^{++}_P(\vec{q})\right] \\
=&(t_1P^\mu+t_2P_f^\mu)\frac{\epsilon\cdot P}{M}+t_3(M+M_f)\epsilon^\mu+\frac{2t_4}{M+M_f}\mathrm{i}\varepsilon^{\mu\nu\sigma\delta}\epsilon_\nu P_\sigma P_{f\delta},
\end{aligned}
$$

Relativistic corrections Method I and Method II

Expansion Method I

- Relativistic wave function of pseudoscalar

$$\varphi_{0^-}^{++}(q_\perp) = \left[A_1(q_\perp) + \frac{\not{P}}{M} A_2(q_\perp) + \frac{\not{q}_\perp}{M} A_3(q_\perp) + \frac{\not{P}\not{q}_\perp}{M^2} A_4(q_\perp) \right] \gamma^5,$$

where $A_1 = \frac{M}{2}\left[\frac{\omega_1+\omega_2}{m_1+m_2} f_1 + f_2\right], \qquad A_3 = -\frac{M(\omega_1-\omega_2)}{m_1\omega_2+m_2\omega_1} A_1,$

$$A_2 = \frac{M}{2}\left[f_1 + \frac{m_1+m_2}{\omega_1+\omega_2} f_2\right], \qquad A_4 = -\frac{M(m_1+m_2)}{m_1\omega_2+m_2\omega_1} A_1,$$

and $\omega_i = \sqrt{m_i^2 - q_\perp^2} = \sqrt{m_i^2 + \vec{q}^2} \ (i=1,2)$

- Expansion on $|\vec{q}|/M$ or $|\vec{q}|/m_i \ i=1,\ 2$, when q is small, these quantities are small, when q is large, these contribution will be suppressed by wave functions.
- $\vec{q}$ relate to relative velocity between quarks $\vec{q} = \frac{m_1 m_2}{m_1+m_2}\vec{v}$

Expansion Method II

- Expansion according to the wave functions

$$\varphi^{++}_{0^-}(q_\perp) = \varphi^{++}_0(q_\perp) + \varphi^{++}_1(q_\perp),$$

where

$$\varphi^{++}_0(q_\perp) = \left[A_1(q_\perp) + \frac{\not{P}}{M} A_2(q_\perp)\right]\gamma^5$$

$$\varphi^{++}_1(q_\perp) = \left[\frac{\not{q}_\perp}{M} A_3(q_\perp) + \frac{\not{P}\not{q}_\perp}{M^2} A_4(q_\perp)\right]\gamma^5$$

- Without expansion about ω_i on q, if $\omega_i = m_i + \vec{q}^2/2m_i$

and $f_1 = f_2$, then $\varphi^{++}_0(q_\perp) = M f_1 \left[\left(1 + \frac{\vec{q}^2}{4m_1 m_2}\right) + \frac{\not{P}}{M}\left(1 - \frac{\vec{q}^2}{4m_1 m_2}\right)\right]\gamma^5$

then the difference is leave to the q^2

Expansion Method II

- So we have

$$\langle \eta_c | \bar{c}\gamma^\mu(1-\gamma^5) b | B_c^+ \rangle = T_0 + T_1 + T_2$$

$$= \int \frac{d\vec{q}}{(2\pi)^3} \mathrm{Tr}\left[\frac{\not{P}}{M} \overline{\varphi}_{P_f}^{++}(\vec{q}') \gamma^\mu (1-\gamma^5) \varphi_P^{++}(\vec{q}) \right]$$

$$= \int \frac{d\vec{q}}{(2\pi)^3} \mathrm{Tr}\left[\frac{\not{P}}{M} (\overline{\varphi}_0'^{++} + \overline{\varphi}_1'^{++}) \gamma^\mu (1-\gamma^5) (\varphi_0^{++} + \varphi_1^{++}) \right]$$

$$= \int \frac{d\vec{q}}{(2\pi)^3} \mathrm{Tr}\left[\frac{\not{P}}{M} (\overline{\varphi}_0'^{++}) \gamma^\mu (1-\gamma^5) (\varphi_0^{++}) \right] \Leftrightarrow \textit{the leading order (LO)}$$

$$+ \int \frac{d\vec{q}}{(2\pi)^3} \mathrm{Tr}\left[\frac{\not{P}}{M} (\overline{\varphi}_1'^{++}) \gamma^\mu (1-\gamma^5) (\varphi_0^{++}) + \frac{\not{P}}{M} (\overline{\varphi}_0'^{++}) \gamma^\mu (1-\gamma^5) (\varphi_1^{++}) \right]$$

$$\Leftrightarrow \textit{the first order of relativistic correction (1stRC)}$$

$$+ \int \frac{d\vec{q}}{(2\pi)^3} \mathrm{Tr}\left[\frac{\not{P}}{M} (\overline{\varphi}_1'^{++}) \gamma^\mu (1-\gamma^5) (\varphi_1^{++}) \right]$$

$$\Leftrightarrow \textit{the second order of relativistic correction (2ndRC)}$$

Expansion Method II

- Then

$$
\begin{aligned}
|T|^2 &= (T_0 + T_1 + T_2)(T_0^* + T_1^* + T_2^*) \\
&= |T_0|^2 \Leftrightarrow LO \\
&+ T_0 T_1^* + T_0^* T_1 \Leftrightarrow 1stRC \\
&+ |T_1|^2 + (T_0 T_2^* + T_0^* T_2) \Leftrightarrow 2ndRC \\
&+ T_1 T_2^* + T_1^* T_2 \Leftrightarrow 3rdRC \\
&+ |T_2|^2 \Leftrightarrow 4thRC.
\end{aligned}
$$

Numerical results

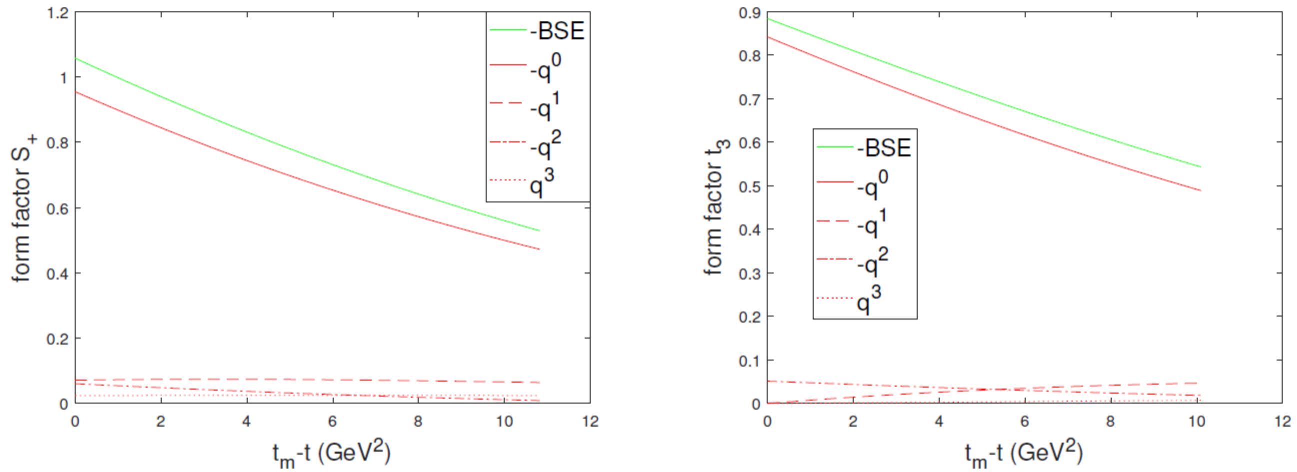
Numerical results

- Parameters

$$m_b = 4.96 \text{ GeV}, \ m_c = 1.62 \text{ GeV}, \ V_{cb} = 40.5 \times 10^{-3},$$

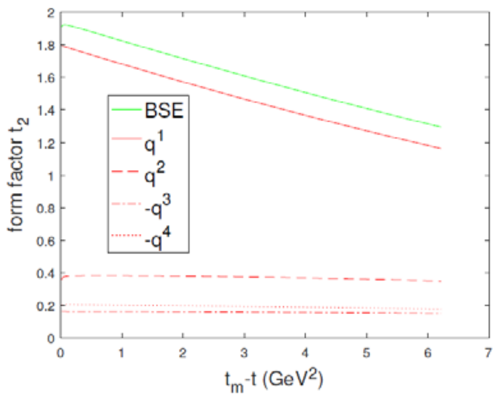
$$M_{h_c(2P)} = 3.887 \text{ GeV}, \ M_{\chi_{c0}(2P)} = 3.862 \text{ GeV}, \ M_{\chi_{c1}(2P)} = 3.872 \text{ GeV}.$$

- Form factors

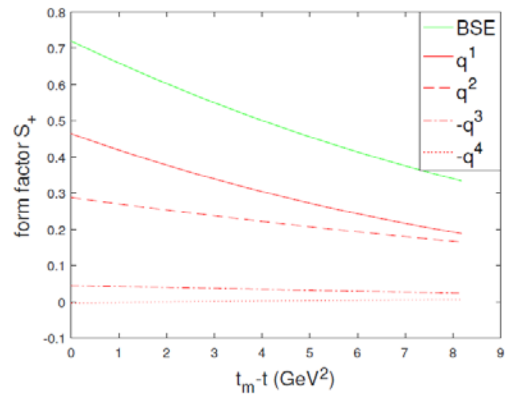


(a) $B_c^+ \to \eta_c e^+ \nu_e$ (b) $B_c^+ \to J/\psi e^+ \nu_e$

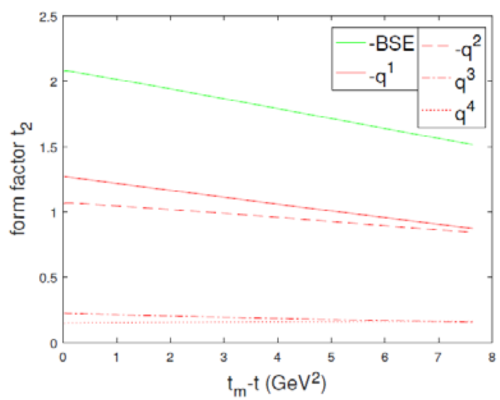
Form factors



(c) $B_c^+ \to h_c e^+ \nu_e$

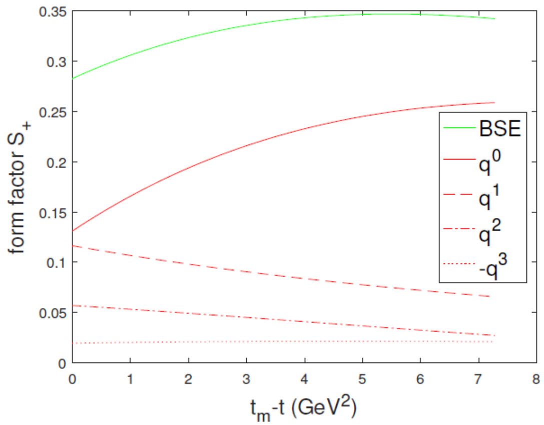


(d) $B_c^+ \to \chi_{c0} e^+ \nu_e$

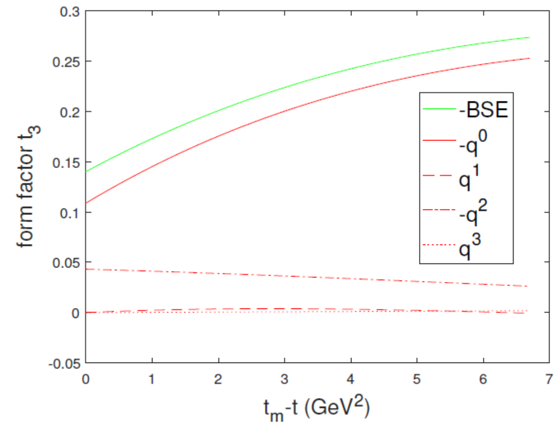


(e) $B_c^+ \to \chi_{c1} e^+ \nu_e$

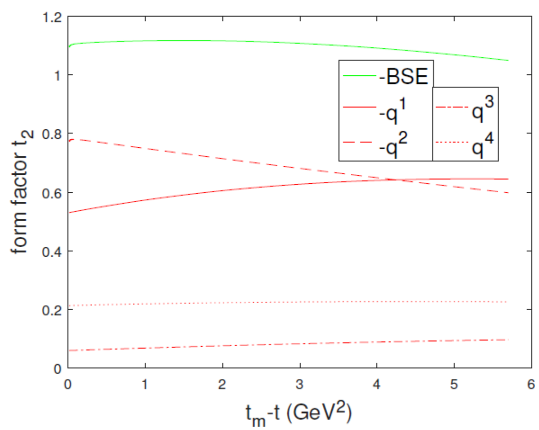
Form factors



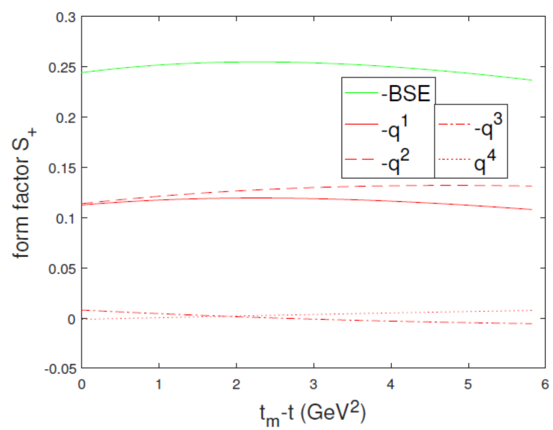
(a) $B_c^+ \to \eta_c(2S)e^+\nu_e$



(b) $B_c^+ \to \psi(2S)e^+\nu_e$



(c) $B_c^+ \to h_c(2P)e^+\nu_e$



(d) $B_c^+ \to \chi_{c0}(2P)e^+\nu_e$

Branching ratios in Method I

Mode	$\vec{q}^{\,0}$	$\vec{q}^{\,1}$	$\vec{q}^{\,2}$	$\vec{q}^{\,3}$	$\vec{q}^{\,4}$	$\vec{q}^{\,5}$	$\vec{q}^{\,6}$
η_c	47.4	11.0	4.10	-3.49	-0.390	-0.135	0.0811
J/ψ	157	20.2	18.8	0.517	0.150	-0.00270	0.0203
$\eta_c(2S)$	3.66	2.29	1.43	-0.294	-0.122	-0.0951	0.0284
$\psi(2S)$	6.88	1.87	2.74	0.362	0.185	0.0247	0.00849
$\eta_c(3S)$	0.408	0.339	0.293	-0.0100	-0.0133	-0.0287	0.00675
$\psi(3S)$	0.652	0.241	0.510	0.108	0.0809	0.0138	0.00282
Mode	$\vec{q}^{\,2}$	$\vec{q}^{\,3}$	$\vec{q}^{\,4}$	$\vec{q}^{\,5}$	$\vec{q}^{\,6}$	$\vec{q}^{\,7}$	$\vec{q}^{\,8}$
h_c	15.4	11.0	4.20	-0.420	-0.142	0.00354	0.0100
χ_{c0}	5.13	7.90	1.85	-1.12	-0.0765	0.0224	0.00224
χ_{c1}	7.82	1.50	2.30	0.218	-0.480	-0.0189	0.0278
$h_c(2P)$	1.75	1.86	1.11	-0.0110	-0.0810	-0.0168	0.00575
$\chi_{c0}(2P)$	0.508	1.16	0.635	-0.0776	-0.0531	0.00158	0.00121
$\chi_{c1}(2P)$	0.666	0.104	0.444	0.0265	-0.157	-0.00197	0.0159
$h_c(3P)$	0.173	0.249	0.207	0.0205	-0.0136	-0.00470	0.00103
$\chi_{c0}(3P)$	0.0731	0.220	0.170	-0.00436	-0.0183	-0.000377	0.000541
$\chi_{c1}(3P)$	0.0580	0.00784	0.0758	0.00580	-0.0379	-0.000542	0.00539

Table 1. The branch ratios of $B_c^+ \to (c\bar{c}) + e^+ + \nu_e$ in Method I according to the power $\vec{q}^{\,n}$ (in 10^{-4}).

Branching ratios in Method II

Mode	LO	1st	2nd	3rd	4th	Total(BS)
η_c	44.1	8.24	7.32	0.650	0.279	60.7
J/ψ	158	18.2	15.2	1.94	0.219	193
$\eta_c(2S)$	3.24	1.81	1.71	0.420	0.166	7.34
$\psi(2S)$	6.96	1.66	2.00	0.353	0.108	11.1
$\eta_c(3S)$	0.355	0.272	0.311	0.101	0.0475	1.09
$\psi(3S)$	0.651	0.201	0.365	0.0834	0.0388	1.34
h_c	14.7	10.2	5.21	0.688	0.0822	30.9
χ_{c0}	5.20	7.88	2.03	-0.736	0.0453	14.4
χ_{c1}	7.75	1.35	2.31	0.307	0.0292	11.8
$h_c(2P)$	1.61	1.67	1.27	0.288	0.0448	4.88
$\chi_{c0}(2P)$	0.523	1.16	0.664	0.0211	0.000490	2.37
$\chi_{c1}(2P)$	0.650	0.0804	0.406	0.0353	0.00421	1.18
$h_c(3P)$	0.159	0.228	0.222	0.0614	0.0111	0.682
$\chi_{c0}(3P)$	0.0760	0.221	0.175	0.0208	0.000717	0.493
$\chi_{c1}(3P)$	0.0562	0.00589	0.0615	0.00427	0.000765	0.129

Mode	$\vec{q}^{\,0}$	sum	BS	NR	$\frac{\text{BS}-\text{sum}}{\text{BS}}$
η_c	47.4	58.6	60.7	56.7	3.4%
J/ψ	157	197	193	188	-1.8%
$\eta_c(2S)$	3.66	6.90	7.34	4.48	6.0%
$\psi(2S)$	6.88	12.1	11.1	8.40	-8.8%
$\eta_c(3S)$	0.408	0.995	1.09	0.509	8.7%
$\psi(3S)$	0.652	1.61	1.34	0.806	-20%
Mode	$\vec{q}^{\,2}$	sum	BS	NR	$\frac{\text{BS}-\text{sum}}{\text{BS}}$
h_c	15.4	30.0	30.9	18.8	2.9%
χ_{c0}	5.13	13.7	14.4	6.28	4.8%
χ_{c1}	7.82	11.4	11.8	9.60	2.8%
$h_c(2P)$	1.75	4.62	4.88	2.18	5.3%
$\chi_{c0}(2P)$	0.508	2.17	2.37	0.633	8.4%
$\chi_{c1}(2P)$	0.666	1.10	1.18	0.853	7.2%
$h_c(3P)$	0.173	0.633	0.682	0.220	7.1%
$\chi_{c0}(3P)$	0.0731	0.440	0.493	0.0923	11%
$\chi_{c1}(3P)$	0.0580	0.114	0.129	0.0735	11%

Table 3. Comparisons of the branch ratios of $B_c^+ \to (c\bar{c}) + e^+ + \nu_e$ obtained by different ways, where $\vec{q}^{\,0}$ means the leading order result; **sum** means the sum of all of expansion orders; **BS** means the result by BS method without expansion, and **NR** means the result by the non-relativistic wave function and the leading order expansion of the amplitude (in 10^{-4} except the last column).

Relativistic effects

Method	η_c	J/ψ	$\eta_c(2S)$	$\psi(2S)$	$\eta_c(3S)$	$\psi(3S)$
I	21.9	18.8	50.2	38.0	62.5	51.3
II	27.3	18.5	55.8	37.2	67.3	51.5

Table 4. The relativistic effects of $B_c^+ \to (c\bar{c})e^+\nu_e$: $\frac{BS-LO}{BS}$ from two methods (in %).

Method	h_c	χ_{c0}	χ_{c1}	$h_c(2P)$	$\chi_{c0}(2P)$	$\chi_{c1}(2P)$	$h_c(3P)$	$\chi_{c0}(3P)$	$\chi_{c1}(3P)$
I	50.2	64.4	33.7	64.1	78.5	43.3	74.6	85.2	54.9
II	52.5	63.9	34.0	67.0	77.9	44.7	76.7	84.6	56.3

Table 5. The relativistic effects of $B_c^+ \to (c\bar{c})e^+\nu_e$: $\frac{BS-LO}{BS}$ from two methods (in %).

Bc to tau

Method	η_c	J/ψ	$\eta_c(2S)$	$\psi(2S)$	$\eta_c(3S)$	$\psi(3S)$
I	18.4	15.8	53.8	39.1	65.6	52.6
II	21.3	15.7	56.7	39.7	66.6	55.3

Table 9. The relativistic effects of $B_c^+ \to (c\bar{c})\tau^+\nu_\tau$: $\frac{BS-LO}{BS}$ from two methods (in %).

Method	h_c	χ_{c0}	χ_{c1}	$h_c(2P)$	$\chi_{c0}(2P)$	$\chi_{c1}(2P)$	$h_c(3P)$	$\chi_{c0}(3P)$	$\chi_{c1}(3P)$
I	58.3	76.0	37.2	78.0	90.0	56.2	92.1	95.9	79.8
II	59.0	75.8	37.6	78.8	89.8	56.8	92.1	95.7	80.3

Table 10. The relativistic effects of $B_c^+ \to (c\bar{c})\tau^+\nu_\tau$: $\frac{BS-LO}{BS}$ from two methods (in %).

Summary

- Relativistic corrections (RC) in the semileptonic Bc decays to charmonium are large.
- RC of 1S are about 19~22%
- RC of 2S are about 38~50%
- RC of 3S are about 51~62%
- RC of 1P are about 34~64%
- RC of 2P are about 43~79%
- RC of 3P are about 55~85%
- Relativistic corrections are very important for excited heavy mesons.

Thank you!