Self-consistency and covariance of light-front quark models: testing via $f_{P,V,A}$ and $F_{P\to P},_{P\to V},_{V\to V}$ ......

Chang Qin (常钦)

Institute of Particle and Nuclear Physics

Henan Normal University

Collaborated with 李晓楠，李新强，苏芳，杨亚东

arXiv:1810.00296[hep-ph]

第十六届重味物理和CP破坏研讨会(HFPCP-2018)

Oct. 2018. @ 河南工业大学

Outline

1. Motivation

- **Standard light-front quark model** (SLF QM);
  M. V. Terentev, SJNP 24 (1976) 106; P. L. Chung *et al.*, PLB 205 (1988) 545.

  The SLF QM is a relativistic constituent QM based on the LF formalism, which provides a conceptually simple and phenomenologically feasible framework for calculating the non-perturbative quantities of hadrons.

  Form factor: **W. Jaus**, PRD 44 (1991) 2851; **H. Y. Cheng** *et al.*, PRD 55 (1997) 1559;.......
  Decay constant: **C. Q. Geng**, EPJC 76 (2016) 313; **H. Choi** *et al.*, PRD 89 (2014) 033011; ....
  Distribution amplitude: **C. W. Hwang**, Phys. Rev. D 81 (2010) 114024 ....
  Baryon weak decays (LFQM+diquark model) : **W. Wang, F. S. Yu** *et al.*, EPJC 77 (2017) 781. ......

  two problems: non-manifestation of covariance; zero-mode issue

- **Covariant light-front quark model** (CLF QM);
  H. Y. Cheng *et al.*, Phys. Rev. D 57 (1998) 5598; W. Jaus, Phys. Rev. D 60 (1999) 054026;
  It provides a systematic way to explore the zero-mode effects; the results are guaranteed to be covariant after the spurious contribution proportional to $\omega = (0, 2, 0_\perp)$ is canceled by the inclusion of zero-mode contributions.

  Applications: W. Wang and Y. L. Shen, PRD 78 (2008) 054002; **Y. L. Shen** and **Y. M. Wang**, PRD 78 (2008) 074012. X. X. Wang, W. Wang and **C. D. Lu**, PRD 79 (2009) 114018; H. Y. Cheng and X. W. Kang, Eur. Phys. J. C 77 (2017) no.9, 587. **X. W. Kang**, **T. Luo**, Y. Zhang, L. Y. Dai and C. Wang, arXiv:1808.02432 [hep-ph]. W. Wang and **R. Zhu**, arXiv:1808.10830 [hep-ph]. H. W. Ke and X. Q. Li, Eur. Phys. J. C **71** (2011) 1776. **H. W. Ke**, **X. Q. Li** and **Z. T. Wei**, Phys. Rev. D **80** (2009) 074030.......

Two problems:

- **Self-consistency problem of CLF QM**

$[f_V]^{\lambda=0}_{\rm CLF} \neq [f_V]^{\lambda=\pm}_{\rm CLF}$

due to the additional contribution characterized by the $B_1^{(2)}$ function to $[f_V]^{\lambda=0}_{\rm CLF}$.

Possible solution: H. M. Choi and C. R. Ji, Phys. Rev. D 89 (2014) no. 3, 033011.

PHYSICAL REVIEW D 69, 074025 (2004)

Covariant light-front approach for s-wave and p-wave mesons: Its application to decay constants and form factors

Hai-Yang Cheng and Chun-Khiang Chua
Institute of Physics, Academia Sinica, Taipei, Taiwan 115, Republic of China

Chien-Wen Hwang
Department of Physics, National Kaohsiung Normal University, Kaohsiung, Taiwan 802, Republic of China

[2]When $\mathcal{A}_\mu^V$ is contracted with the longitudinal polarization vector $\varepsilon^\mu(0)$, f_V will receive additional contributions characterized by the B functions defined in Appendix B [see Eq. (3.5) of [14]] which give about 10% corrections to f_V for the vertex function h'_V used in Eq. (2.11). It is not clear to us why the result of f_V depends on the polarization vector. Note that the new residual contributions are

$$\sqrt{2N_c}\frac{\chi(x,k_\perp)}{1-x} \to \frac{\psi(x,k_\perp)}{\sqrt{x(1-x)}\hat{M}_0}\,, \qquad D_{V,\rm con} \to D_{V,\rm LF}\,, \qquad (\textbf{type-I})$$

$\chi(x,k_\perp)$: CLF expressions $\longleftrightarrow$ SLF ones via Z.M. independent f_P or $f^+_{P\to P}$.
D: $D_{V,\rm con} = M + m_1 + m_2$ and $D_{V,\rm LF} = M_0 + m_1 + m_2$

$$\sqrt{2N_c}\frac{\chi(x,k_\perp)}{1-x} \to \frac{\psi(x,k_\perp)}{\sqrt{x(1-x)}\hat{M}_0}\,, \qquad M \to M_0\,. \qquad (\textbf{type-II})$$

$$\Longrightarrow \qquad [f_V]^{\lambda=0}_{\rm CLF} = [f_V]^{\lambda=\pm}_{\rm CLF} = [f_V]_{\rm SLF}$$

Questions: (i) f_A, $F_{P\to V}$ ...... ?

(ii) $[f_V]^{\lambda=0}_{\rm SLF} = [f_V]^{\lambda=\pm}_{\rm SLF}$? self-consistency of SLF QM ?

(iii) zero-mode contribution ?

Covariance problem of CLF QM

The manifest covariance is a remarkable feature of CLF QM relative to SLF QM. However,
the covariance is in fact violated when the LF vertex function and operator are used (especially for spin-1 system).

PHYSICAL REVIEW D, VOLUME 60, 054026

Covariant analysis of the light-front quark model

Wolfgang Jaus
Institut für Theoretische Physik der Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland
(Received 21 December 1998; published 9 August 1999)

The formulas for coupling constants and form factors have been derived in a manifestly covariant framework. However, if these formulas are evaluated with the symmetric light-front vertex function (5.2), the covariance conditions (3.32) are violated, i.e., the integrals of Eq. (3.32) are nonzero. Consequently, some residual ω dependence is introduced into these expressions if Eqs. (5.2) and (5.3) are used for the vertex function. This remaining ω dependence is minimal in the sense that only the B coefficients $B_n^{(m)}$ in the

Taking $\mathcal{A} \equiv \langle 0|\bar{q}_2\Gamma q_1|M(p)\rangle$ as an example

$$\hat{\mathcal{A}}_V^\mu = M_V(\epsilon^\mu f_V + \omega^\mu g_V),$$

$\hat{\mathcal{A}}_V^\mu$ is obviously not covariant unless the unphysical decay constant $g_V = 0$ since ω^μ is a fixed vector.

(i) Is the covariance violation minimal?

(ii) Can the strict covariance be recovered ?

2. Brief review of theoretical framework

The main task:

$$\mathcal{A} \equiv \langle 0|\bar{q}_2\Gamma q_1|M(p)\rangle; \qquad \mathcal{B} \equiv \langle M''(p'')|\bar{q}_1''\Gamma q_1'|M'(p')\rangle$$

2.1 The SLF QM

$$|M\rangle = \sum_{h_1,h_2}\int \frac{dk^+ d^2k_\perp}{(2\pi)^3 2\sqrt{k_1^+k_2^+}}\Psi_{h_1,h_2}(k^+,k_\perp)|q_1:k_1^+,k_{1\perp},h_1\rangle|\bar{q}_2:k_2^+,k_{2\perp},h_2\rangle\,,$$

one-particle states: $|q_1\rangle = \sqrt{2k_1^+}b^\dagger|0\rangle$ with $\{b_h^\dagger(k),b_{h'}(k')\} = (2\pi)^3\delta(k^+-k'^+)\delta^2(k_\perp-k'_\perp)\delta_{hh'}$.

Wavefunction:

$$\Psi_{h_1,h_2}(x,k_\perp) = S_{h_1,h_2}(x,k_\perp)\,\psi(x,k_\perp)\,,$$

Radial WF $\quad \psi_s(x,k_\perp) = 4\frac{\pi^{\frac{3}{4}}}{\beta^{\frac{3}{2}}}\sqrt{\frac{\partial k_z}{\partial x}}\exp\left[-\frac{k_z^2+k_\perp^2}{2\beta^2}\right]\,, \qquad$ s-wave

$\psi_p(x,k_\perp) = \frac{\sqrt{2}}{\beta}\psi_s(x,k_\perp)\,. \qquad$ p-wave

Spin-orbital WF $\quad S_{h_1,h_2} = \frac{\bar{u}_{h_1}(k_1)\,\Gamma'\,v_{h_2}(k_2)}{\sqrt{2}\hat{M}_0}\,,$

obtained by the interaction-independent Melosh transformation, where

$$\Gamma'_{P,V,{}^1A,{}^3A} = \gamma_5\,,\ -\hat{\not\epsilon} + \frac{\hat{\epsilon}\cdot(k_1-k_2)}{D_{V,\rm LF}}\,,\ -\frac{\hat{\epsilon}\cdot(k_1-k_2)}{D_{{}^1A,\rm LF}}\gamma_5\,,\ -\frac{\hat{M}_0^2}{2\sqrt{2}M_0}\left[\hat{\not\epsilon} + \frac{\hat{\epsilon}\cdot(k_1-k_2)}{D_{{}^3A,\rm LF}}\right]\gamma_5$$

Equipped with the formulae given above, one can obtain

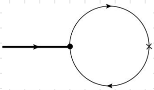
$$\mathcal{A} = \sqrt{N_c}\sum_{h_1,h_2}\int \frac{\mathrm{d}x\mathrm{d}^2k_\perp}{(2\pi)^3 2\sqrt{x\bar{x}}}\psi(x,k_\perp)\,S_{h_1,h_2}(x,k_\perp)\,C_{h_1,h_2}(x,k_\perp)\,,$$

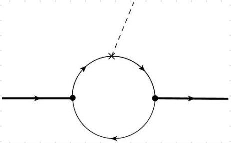
$$\mathcal{B} = \sum_{h_1',h_1'',h_2}\int \frac{\mathrm{d}k'^+\mathrm{d}^2k'_\perp}{(2\pi)^3\,2\sqrt{k'^+\,k''^+}}\psi''^*(k''^+,\bar{k}''_\perp)\,\psi'(k'^+,k'_\perp)$$
$$\times\, S''^\dagger_{h_1'',h_2}(k''^+,k''_\perp)\,C_{h_1'',h_1'}(k''^+,k''_\perp,k'^+,k'_\perp)\,S'_{h_1',h_2}(k'^+,k'_\perp)\,,$$

where $C_{h_1,h_2} \equiv \bar{v}_{h_2}\Gamma u_{h_1}$ and $C_{h_1'',h_1'} \equiv \bar{u}_{h_1''}\Gamma u_{h_1'}$

2.2 The CLF QM

Manifestly covariant one-loop integrals:





$$\mathcal{A} = N_c\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{H_M}{N_1N_2}S_{\mathcal{A}}\,, \qquad \mathcal{B} = N_c\int\frac{\mathrm{d}^4k'}{(2\pi)^4}\frac{H_{M'}H_{M''}}{N_1'\,N_1''\,N_2}iS_{\mathcal{B}}\,,$$

where,
$$S_{\mathcal{A}} = \mathrm{Tr}\left[\Gamma\,(\not{k}_1+m_1)\,(i\Gamma_M)\,(-\not{k}_2+m_2)\right]$$
$$S_{\mathcal{B}} = \mathrm{Tr}\left[\Gamma\,(\not{k}_1'+m_1')\,(i\Gamma_M')\,(-\not{k}_2+m_2)\,(i\gamma^0\Gamma_M''^\dagger\gamma^0)(\not{k}_1''+m_1'')\right]$$

Manifestly covariant expression $\xrightarrow{\text{integrating out } k^-}$ LF expression

Assumption: $H_{M,M',M''}$ are analytic in the upper complex k^- (k'^-) plane.

Consequently, q_2 is on mass-shell, and

$$N_1 \to \hat{N}_1\,, \qquad N_1^{\prime(\prime\prime)} \to \hat{N}_1^{\prime(\prime\prime)}\,, \qquad \mathcal{S} \to \hat{\mathcal{S}}\,,$$
$$\chi_M = H_M/N_1 \to h_M/\hat{N}_1\,, \qquad D_{M,\text{con}} \to D_{M,\text{LF}}\,.$$

Then,

$$\hat{\mathcal{A}} = N_c \int \frac{\mathrm{d}k^+ \mathrm{d}^2 k_\perp}{2(2\pi)^3} \frac{-ih_M}{\bar{x}p^+ \hat{N}_1} \hat{S}_{\mathcal{A}}\,, \qquad \hat{\mathcal{B}} = N_c \int \frac{\mathrm{d}k'^+ \mathrm{d}^2 k'_\perp}{2(2\pi)^3} \frac{h_{M'} h_{M''}}{\bar{x}p'^+ \hat{N}_1' \hat{N}_1''} \hat{S}_{\mathcal{B}}\,. \qquad (1)$$

In order to restore the zero-mode contribution and eliminate ω dependence, we need the following decomposition and replacements

Jaus, Phys. Rev. D 60 (1999) 054026. Phys. Rev. D 69 (2004) 074025

for $\hat{\mathcal{A}}$: $\quad \hat{k}_1^\mu \to x p^\mu + \ldots(\omega, C_i^{(j)})\,,$

$$\hat{k}_1^\mu \hat{k}_1^\nu \to -g^{\mu\nu}\frac{k_\perp^2}{2} + p^\mu p^\nu x^2 + \frac{p^\mu \omega^\nu + p^\nu \omega^\mu}{\omega \cdot p} B_1^{(2)} + \ldots(\omega, C_i^{(j)})\,,$$
$$\hat{N}_2 \to Z_2 = \hat{N}_1 + m_1^2 - m_2^2 + (\bar{x} - x)M^2\,,$$

$$\text{for } \hat{\mathcal{B}}: \quad \hat{k}_1'^{\mu} \to P^{\mu} A_1^{(1)} + q^{\mu} A_2^{(1)} + \ldots (\omega, C_i^{(j)}) \,,$$
$$k_1'^{\mu} \hat{N}_2 \to q^{\mu} \left[A_2^{(1)} Z_2 + \frac{q \cdot P}{q^2} A_1^{(2)} \right] ,$$
$$Z_2 = \hat{N}_1' + m_1'^2 - m_2^2 + (\bar{x} - x) M'^2 + (q^2 + q \cdot P) \frac{k_{1\perp}' \cdot q_\perp}{q^2} \,,$$
$$\ldots\ldots \tag{2}$$

where $P = p' + p''$, $q = p' - p''$ and

$$A_1^{(1)} = \frac{x}{2} \,, \qquad A_2^{(1)} = \frac{x}{2} - \frac{k_{1\perp}' \cdot q_\perp}{q^2} \,, \qquad A_1^{(2)} = -k_{1\perp}'^2 - \frac{(k_{1\perp}' \cdot q_\perp)^2}{q^2} \,, \qquad B_1^{(2)} = \frac{x}{2} Z_2 + \frac{k_\perp^2}{2} \,.$$

For a given quantity, in order to clearly show the zero-mode effect, we have

$$\mathcal{Q}^{\text{full}} = \mathcal{Q}^{\text{val.}} + \mathcal{Q}^{\text{z.m.}}$$

$\mathcal{Q}^{\text{val.}}$: assuming $k_2^+ \neq 0$ and $k_1^+ \neq 0 \Longrightarrow$ poles of N_2 and N_1 are safely located inside and outside, respectively, the contour of k^- (k'^-) integral; zero-mode contributions are absent.

decomposition and replacements $\longrightarrow k_2^2 = m_2^2$ and four-momentum conservation at each vertex.

It is expected that $\mathcal{Q}^{\text{full}}$ $(\mathcal{Q}^{\text{val.}}) = \mathcal{Q}^{\text{SLF}}$ if we believe that zero-mode contribution has (not) been included in $\mathcal{Q}^{\text{SLF}}$,

3. Results: f_P and f_V

Definition: $\langle 0|\bar{q}_2\gamma^\mu\gamma_5 q_1|P(p)\rangle = i f_P p^\mu\,, \qquad \langle 0|\bar{q}_2\gamma^\mu q_1|V(p,\lambda)\rangle = f_V M_V \epsilon^\mu\,.$

3.1 f_P

$$[f_P]_{\rm SLF} = \sqrt{N_c}\int \frac{{\rm d}x\,{\rm d}^2k_\perp}{(2\pi)^3}\frac{\psi_s(x,k_\perp)}{\sqrt{x\bar{x}}}\frac{2}{\sqrt{2}\hat{M}_0}(\bar{x}m_1 + x m_2)\,,$$

$$[f_P]_{\rm full} = [f_P]_{\rm val.} = N_c\int \frac{{\rm d}x\,{\rm d}^2k_\perp}{(2\pi)^3}\frac{\chi_P}{\bar{x}}2(\bar{x}m_1 + x m_2)\,,$$

- no residual ω dependence
- $[f_P]_{\rm full} = [f_P]_{\rm val.}$: f_P is free of the Z.M. contribution
- $[f_P]_{\rm SLF} = [f_P]_{\rm val.} = [f_P]_{\rm full}$ within both type-I and -II schemes.

Fitting to the data of f_P

	$\beta_{q\bar{q}}$	$\beta_{s\bar{q}}$	$\beta_{s\bar{s}}$	$\beta_{c\bar{q}}$	$\beta_{c\bar{s}}$
this work	$314.1^{+0.5}_{-0.5}$	$342.8^{+1.3}_{-1.4}$	$365.8^{+1.2}_{-1.8}$	$464.1^{+11.2}_{-10.8}$	$537.5^{+9.0}_{-8.7}$
PLB 460 (1999) 461	365.9	388.6	412.8	467.9	501.6
	$\beta_{c\bar{c}}$	$\beta_{b\bar{q}}$	$\beta_{b\bar{s}}$	$\beta_{b\bar{c}}$	$\beta_{b\bar{b}}$
this work	$654.5^{+143.3}_{-132.4}$	$547.9^{+9.9}_{-10.2}$	$601.4^{+7.3}_{-7.3}$	$947.0^{+11.2}_{-10.9}$	$1391.2^{+51.6}_{-48.2}$
PLB 460 (1999) 461	650.9	526.6	571.2	936.9	1145.2

3.2 f_V

Theoretical results:

$$[f_V]^{\lambda=0}_{\rm SLF} = \sqrt{N_c}\int \frac{dx d^2k_\perp}{(2\pi)^3}\frac{\psi_s(x,k_\perp)}{\sqrt{x\bar{x}}}\frac{2}{\sqrt{2}\hat{M}_0}\left(\bar{x}m_1 + xm_2 + \frac{2k_\perp^2}{D_{V,{\rm LF}}}\right),$$

$$[f_V]^{\lambda=\pm}_{\rm SLF} = \sqrt{N_c}\int \frac{dx d^2k_\perp}{(2\pi)^3}\frac{\psi_s(x,k_\perp)}{\sqrt{x\bar{x}}}\frac{2}{\sqrt{2}\hat{M}_0}\left(\frac{\hat{M}_0^2}{2M_V} - \frac{k_\perp^2}{D_{V,{\rm LF}}}\frac{M_0}{M_V}\right),$$

$$[f_V]^{\lambda=0}_{\rm full} = N_c\int \frac{dx d^2k_\perp}{(2\pi)^3}\frac{\chi_V}{\bar{x}}\frac{2}{M_V}\left[xM_0^2 - m_1(m_1-m_2) - \left(1-\frac{m_1+m_2}{D_{V,{\rm con}}}\right)(k_\perp^2 \mathbf{-2B_1^{(2)}})\right],$$

$$[f_V]^{\lambda=\pm}_{\rm full} = N_c\int \frac{dx d^2k_\perp}{(2\pi)^3}\frac{\chi_V}{\bar{x}}\frac{2}{M_V}\left[xM_0^2 - m_1(m_1-m_2) - \left(1-\frac{m_1+m_2}{D_{V,{\rm con}}}\right)k_\perp^2\right],$$

$[f_V]^{\lambda=\pm}_{\rm SLF}$ is usually ignored in previous works due to the traditional bias.

In order to clearly show their self-consistence we define:

$$\Delta^M_{\rm full}(x) \equiv \frac{d[f_M]^{\lambda=0}_{\rm full}}{dx} - \frac{d[f_M]^{\lambda=\pm}_{\rm full}}{dx}, \qquad \Delta^M_{\rm SLF}(x) \equiv \frac{d[f_M]^{\lambda=0}_{\rm SLF}}{dx} - \frac{d[f_M]^{\lambda=\pm}_{\rm SLF}}{dx}.$$

The valence contributions:

$$[f_V]^{\lambda=0}_{\rm val.} = N_c\int \frac{dx d^2k_\perp}{(2\pi)^3}\frac{\chi_V}{\bar{x}}\frac{2}{M_V}\left[k_\perp^2 + x\bar{x}M_V^2 + m_1m_2 + \frac{\bar{x}^2M_V^2 - m_2^2 - k_\perp^2}{\bar{x}D_{V,{\rm con}}}(\bar{x}m_1 - xm_2)\right],$$

$$[f_V]^{\lambda=\pm}_{\rm val.} = N_c\int \frac{dx d^2k_\perp}{(2\pi)^3}\frac{\chi_V}{\bar{x}}\frac{2}{M_V}\left[\frac{\bar{x}M_V^2 + xM_0^2 - (m_1-m_2)^2}{2} - \left(1-\frac{m_1+m_2}{D_{V,{\rm con}}}\right)k_\perp^2\right].$$

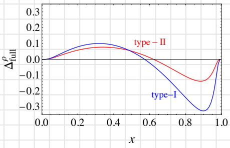
We do not find any relation in type-I scheme except for $[f_V]^{\lambda=0}_{\rm full} = [f_V]^{\lambda=\pm}_{\rm full} + \ldots\ldots B_1^{(2)}$

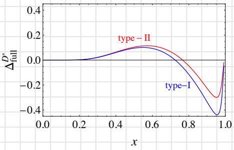
Numerical results: taking ρ and D^* as examples

	$[f_\rho]^{\lambda=0}_{\rm SLF}$	$[f_\rho]^{\lambda=\pm}_{\rm SLF}$	$[f_\rho]^{\lambda=0}_{\rm full}$	$[f_\rho]^{\lambda=\pm}_{\rm full}$	$[f_\rho]^{\lambda=0}_{\rm val.}$	$[f_\rho]^{\lambda=\pm}_{\rm val.}$
type-I	211.1	226.9	248.7	288.9	229.1	212.1
type-II	211.1	211.1	211.1	211.1	211.1	211.1
	$[f_{D^*}]^{\lambda=0}_{\rm SLF}$	$[f_{D^*}]^{\lambda=\pm}_{\rm SLF}$	$[f_{D^*}]^{\lambda=0}_{\rm full}$	$[f_{D^*}]^{\lambda=\pm}_{\rm full}$	$[f_{D^*}]^{\lambda=0}_{\rm val.}$	$[f_{D^*}]^{\lambda=\pm}_{\rm val.}$
type-I	252.6	273.5	275.3	305.6	244.6	258.9
type-II	252.6	252.6	252.6	252.6	252.6	252.6

Findings:

- Self-consistence of CLF QM: $\Delta^V_{\rm full}(x) = N_c \int \frac{{\rm d}^2 k_\perp}{(2\pi)^3} \frac{\chi_V}{\bar x} \frac{2}{M_V} \frac{D_{V,{\rm con}} - m_1 - m_2}{D_{V,{\rm con}}} 2B_1^{(2)}$



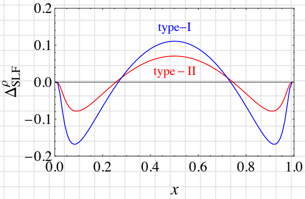


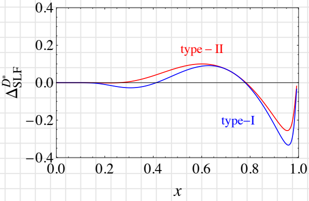
(i) $[f_V]^{\lambda=0}_{\rm full} \neq [f_V]^{\lambda=\pm}_{\rm full}$ (type-I) $\longrightarrow$ self-consistency problem of the CLF QM

(ii) Interestingly, we find: $[f_V]^{\lambda=0}_{\rm full} \doteq [f_V]^{\lambda=\pm}_{\rm full}$ (type-II) due to $\int {\rm d}x \Delta^V_{\rm full} = 0$

Type-II scheme provides a self-consistent correspondence between manifest covariant and LF approaches for f_V.

- Self-consistence of SLF QM: $\Delta^{M}_{\rm SLF}(x)$





(i) $[f_V]^{\lambda=0}_{\rm SLF} < [f_V]^{\lambda=\pm}_{\rm SLF}$ (type-I) $\longrightarrow$ self-consistency problem exists also in the traditional SLF QM

(ii) $[f_V]^{\lambda=0}_{\rm SLF} \doteq [f_V]^{\lambda=\pm}_{\rm SLF}$ (type-II) due to $\int {\rm d}x \Delta^{V}_{\rm SLF} = 0$

Type-II scheme is also favored by the self-consistency of the SLF QM.

- Relation between $[f_V]^{\lambda=0,\pm}_{\rm SLF}$ and $[f_V]^{\lambda=0,\pm}_{\rm val.}$:

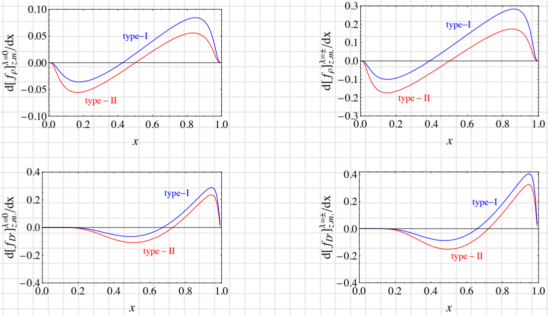
(i) No relation can be found (traditional type-I scheme).

(ii) Taking type-II scheme and making some simplifications, we find surprisingly

$[f_V]^{\lambda=0}_{\rm SLF} = [f_V]^{\lambda=0}_{\rm val.}$ and $[f_V]^{\lambda=\pm}_{\rm SLF} = [f_V]^{\lambda=\pm}_{\rm val.}$ (type-II)

which are exactly ones expected.

- Zero-mode effects: $[f_V]_{\rm z.m.}$



(i) $0 < [f_V]^{\lambda=0}_{\rm z.m.} < [f_V]^{\lambda=\pm}_{\rm z.m.}$ **(type-I)** $\longrightarrow$ $[f_V]_{\rm z.m.}$ are non-zero and dependent on λ.

(ii) $[f_V]^{\lambda=0,\pm}_{\rm z.m.} \doteq 0$ **(type-II)** $\longrightarrow$ $[f_V]^{\lambda=0}_{\rm full} \doteq [f_V]^{\lambda=0}_{\rm val.}$ and $[f_V]^{\lambda=\pm}_{\rm full} \doteq [f_V]^{\lambda=\pm}_{\rm val.}$

Summarizing the findings above:

$$[f_V]^{\lambda=0}_{\rm SLF} = [f_V]^{\lambda=0}_{\rm val.} \doteq [f_V]^{\lambda=0}_{\rm full} \doteq [f_V]^{\lambda=\pm}_{\rm full} \doteq [f_V]^{\lambda=\pm}_{\rm val.} = [f_V]^{\lambda=\pm}_{\rm SLF} \quad \text{(type-II)}$$

Our Updated predictions for f_V (in unit of MeV):

	data	LQCD	QCD SR	this work
f_ρ	210 ± 4	199 ± 4	206 ± 7	211 ± 1
f_{K^*}	204 ± 7	–	222 ± 8	223 ± 1
f_ϕ	228.5 ± 3.6	238 ± 3	215 ± 5	236 ± 1
f_{D^*}	–	223.5 ± 8.4	250 ± 8	253 ± 7
$f_{D_s^*}$	301 ± 13	268.8 ± 6.6	290 ± 11	314 ± 6
$f_{J/\psi}$	411 ± 5	418 ± 9	401 ± 46	382 ± 96
f_{B^*}	–	185.9 ± 7.2	210 ± 6	205 ± 5
$f_{B_s^*}$	–	223.1 ± 5.4	221 ± 7	246 ± 4
$f_{B_c^*}$	–	422 ± 13	453 ± 20	465 ± 7
$f_{\Upsilon(1S)}$	708 ± 8	–	–	713 ± 34

LQCD: Nucl. Phys. B 883 (2014) 306; Phys. Rev. D 96 (2017) no. 7, 074502; JHEP 1704 (2017) 082; PoS LATTICE 2016 (2017) 291; Phys. Rev. D 91 (2015) no.11, 114509.

QCD SR: Nucl. Phys. B 883 (2014) 306; Phys. Rev. D 75 (2007) 054004; Part. Phys. Proc. 270-272 (2016) 143.

4. Results and discussion: f_A

Definition:

$$\langle 0|\bar{q}_2\gamma^\mu\gamma_5 q_1|A(p,\lambda)\rangle = f_A\, M_A \epsilon^\mu_\lambda$$

^3A: $^{2S+1}L_J = {}^3P_1$; ^1A: $^{2S+1}L_J = {}^1P_1$.

Theoretical results for ^1A:

$$[f_{1_A}]^{\lambda=0}_{\rm SLF} = -\sqrt{N_c}\int \frac{{\rm d}x{\rm d}^2k_\perp}{(2\pi)^3}\frac{\psi_p(x,k_\perp)}{\sqrt{x\,\bar{x}}}\frac{1}{\sqrt{2}\hat{M}_0}\frac{2}{M_0}\frac{(\bar{x}m_1+xm_2)[(\bar{x}-x)k_\perp^2+\bar{x}^2m_1^2-x^2m_2^2]}{x\bar{x}D_{1_{A,{\rm LF}}}},$$

$$[f_{1_A}]^{\lambda=\pm}_{\rm SLF} = -\sqrt{N_c}\int \frac{{\rm d}x{\rm d}^2k_\perp}{(2\pi)^3}\frac{\psi_p(x,k_\perp)}{\sqrt{x\,\bar{x}}}\frac{1}{\sqrt{2}\hat{M}_0}\frac{2}{M_{1_A}}\frac{m_1-m_2}{D_{1_{A,{\rm LF}}}}k_\perp^2;$$

$$[f_{1_A}]^{\lambda=0}_{\rm full} = -N_c\int \frac{{\rm d}x{\rm d}^2k_\perp}{(2\pi)^3}\frac{\chi_{1_A}}{\bar{x}}\frac{2}{M_{1_A}}\frac{m_1-m_2}{D_{1_{A,{\rm con}}}}\left(k_\perp^2-\mathbf{2B_1^{(2)}}\right),$$

$$[f_{1_A}]^{\lambda=\pm}_{\rm full} = -N_c\int \frac{{\rm d}x{\rm d}^2k_\perp}{(2\pi)^3}\frac{\chi_{1_A}}{\bar{x}}\frac{2}{M_{1_A}}\frac{m_1-m_2}{D_{1_{A,{\rm con}}}}k_\perp^2;$$

$$[f_{1_A}]^{\lambda=0}_{\rm val.} = -N_c\int \frac{{\rm d}x{\rm d}^2k_\perp}{(2\pi)^3}\frac{\chi_{1_A}}{\bar{x}}\frac{2}{M_{1_A}}\frac{M_{1_A}^2\bar{x}^2-m_2^2-k_\perp^2}{\bar{x}D_{1_{A,{\rm con}}}}(\bar{x}m_1+xm_2),$$

$$[f_{1_A}]^{\lambda=\pm}_{\rm val.} = -N_c\int \frac{{\rm d}x{\rm d}^2k_\perp}{(2\pi)^3}\frac{\chi_{1_A}}{\bar{x}}\frac{2}{M_{1_A}}\frac{m_1-m_2}{D_{1_{A,{\rm con}}}}k_\perp^2.$$

Theoretical results for ^3A:

$$[f_{3_A}]^{\lambda=0}_{\rm SLF} = \sqrt{N_c}\int \frac{{\rm d}x{\rm d}^2k_\perp}{(2\pi)^3}\frac{\psi_p(x,k_\perp)}{\sqrt{x\,\bar{x}}}\frac{1}{\sqrt{2}\hat{M}_0}\frac{\hat{M}_0^2}{2\sqrt{2}M_0}\frac{2}{M_0}\Big\{2k_\perp^2+(m_1-m_2)(\bar{x}m_1-xm_2) - \frac{(\bar{x}m_1+xm_2)[(\bar{x}-x)k_\perp^2+\bar{x}^2m_1^2-x^2m_2^2]}{x\bar{x}\,D_{3_A,{\rm LF}}}\Big\},$$

$$[f_{3_A}]^{\lambda=\pm}_{\rm SLF} = \sqrt{N_c}\int \frac{{\rm d}x{\rm d}^2k_\perp}{(2\pi)^3}\frac{\psi_p(x,k_\perp)}{\sqrt{x\,\bar{x}}}\frac{1}{\sqrt{2}\hat{M}_0}\frac{\hat{M}_0^2}{2\sqrt{2}M_0}\frac{2}{M_{3_A}}\Big[\frac{k_\perp^2-2\bar{x}xk_\perp^2+(\bar{x}m_1-xm_2)^2}{2\bar{x}x} - \frac{k_\perp^2(m_1-m_2)}{D_{3_A,{\rm LF}}}\Big];$$

$$[f_{3_A}]^{\lambda=0}_{\rm full} = N_c\int \frac{{\rm d}x{\rm d}^2k_\perp}{(2\pi)^3}\frac{\chi_{3_A}}{\bar{x}}\frac{2}{M_{3_A}}\Big\{xM_0^2-m_1(m_1+m_2)-\Big(1+\frac{m_1-m_2}{D_{3_A,{\rm con}}}\Big)(k_\perp^2-\mathbf{2B_1^{(2)}})\Big\},$$

$$[f_{3_A}]^{\lambda=\pm}_{\rm full} = N_c\int \frac{{\rm d}x{\rm d}^2k_\perp}{(2\pi)^3}\frac{\chi_{3_A}}{\bar{x}}\frac{2}{M_{3_A}}\Big[xM_0^2-m_1(m_1+m_2)-\Big(1+\frac{m_1-m_2}{D_{3_A,{\rm con}}}\Big)k_\perp^2\Big];$$

$$[f_{3_A}]^{\lambda=0}_{\rm val.} = N_c\int \frac{{\rm d}x{\rm d}^2k_\perp}{(2\pi)^3}\frac{\chi_{3_A}}{\bar{x}}\frac{2}{M_{3_A}}\Big[k_\perp^2+x\bar{x}M_{3_A}^2-m_1m_2-\frac{M_{3_A}^2\bar{x}^2-m_2^2-k_\perp^2}{\bar{x}D_{3_A,{\rm con}}}(\bar{x}m_1+xm_2)\Big],$$

$$[f_{3_A}]^{\lambda=\pm}_{\rm val.} = N_c\int \frac{{\rm d}x{\rm d}^2k_\perp}{(2\pi)^3}\frac{\chi_{3_A}}{\bar{x}}\frac{2}{M_{3_A}}\Big[\frac{\bar{x}M_{3_A}^2+xM_0^2-(m_1+m_2)^2}{2}-\Big(1+\frac{m_1-m_2}{D_{3_A,{\rm con}}}\Big)k_\perp^2\Big].$$

Numerical results: taking $^1A_{(q\bar{q})}$, $^3A_{(q\bar{q})}$, $^1A_{(c\bar{q})}$ and $^3A_{(c\bar{q})}$ ($b_1(1235)$, $a_1(1260)$, $D_1(2420)$ and $D_1(2430)$) as examples

	$[f_{^1A_{(q\bar{q})}}]^{\lambda=0}_{\rm SLF}$	$[f_{^1A_{(q\bar{q})}}]^{\lambda=\pm}_{\rm SLF}$	$[f_{^1A_{(q\bar{q})}}]^{\lambda=0}_{\rm full}$	$[f_{^1A_{(q\bar{q})}}]^{\lambda=\pm}_{\rm full}$	$[f_{^1A_{(q\bar{q})}}]^{\lambda=0}_{\rm val.}$	$[f_{^1A_{(q\bar{q})}}]^{\lambda=\pm}_{\rm val.}$
type-I	0	0	0	0	-47.4	0
type-II	0	0	0	0	0	0
	$[f_{^1A_{(c\bar{q})}}]^{\lambda=0}_{\rm SLF}$	$[f_{^1A_{(c\bar{q})}}]^{\lambda=\pm}_{\rm SLF}$	$[f_{^1A_{(c\bar{q})}}]^{\lambda=0}_{\rm full}$	$[f_{^1A_{(c\bar{q})}}]^{\lambda=\pm}_{\rm full}$	$[f_{^1A_{(c\bar{q})}}]^{\lambda=0}_{\rm val.}$	$[f_{^1A_{(c\bar{q})}}]^{\lambda=\pm}_{\rm val.}$
type-I	-78.5	-84.6	-78.4	-84.6	-65.2	-84.6
type-II	-78.5	-78.5	-78.5	-78.5	-78.5	-78.5

	$[f_{^3A_{(q\bar{q})}}]^{\lambda=0}_{\rm SLF}$	$[f_{^3A_{(q\bar{q})}}]^{\lambda=\pm}_{\rm SLF}$	$[f_{^3A_{(q\bar{q})}}]^{\lambda=0}_{\rm full}$	$[f_{^3A_{(q\bar{q})}}]^{\lambda=\pm}_{\rm full}$	$[f_{^3A_{(q\bar{q})}}]^{\lambda=0}_{\rm val.}$	$[f_{^3A_{(q\bar{q})}}]^{\lambda=\pm}_{\rm val.}$
type-I	218.7	223.6	260.6	223.6	263.1	263.1
type-II	218.7	218.7	218.7	218.7	218.7	218.7
	$[f_{^3A_{(c\bar{q})}}]^{\lambda=0}_{\rm SLF}$	$[f_{^3A_{(c\bar{q})}}]^{\lambda=\pm}_{\rm SLF}$	$[f_{^3A_{(c\bar{q})}}]^{\lambda=0}_{\rm full}$	$[f_{^3A_{(c\bar{q})}}]^{\lambda=\pm}_{\rm full}$	$[f_{^3A_{(c\bar{q})}}]^{\lambda=0}_{\rm val.}$	$[f_{^3A_{(c\bar{q})}}]^{\lambda=\pm}_{\rm val.}$
type-I	231.7	256.7	244.7	256.7	228.5	228.5
type-II	231.7	231.7	231.7	231.7	231.7	231.7

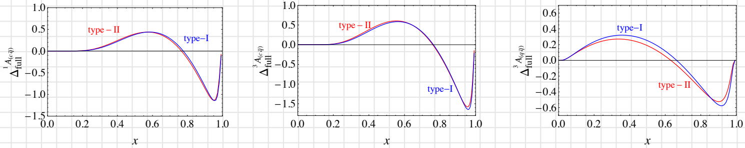
(i). Self-consistency problem also exists in ^1A and ^3A systems

(ii). $^1A_{(q\bar{q})}$ meson is not ideal for testing the self-consistency due to $m_1 = m_2$.

$[f_{^1A_{(q\bar{q})}}]^{\lambda=\pm}_{\rm val.,SLF,full}$, $[f_{^1A_{(q\bar{q})}}]^{\lambda=0}_{\rm full}$ $:\propto m_1 - m_2$

$[f_{^1A_{(q\bar{q})}}]^{\lambda=0}_{\rm SLF}$: anti-symmetry under $x \leftrightarrow \bar{x}$.

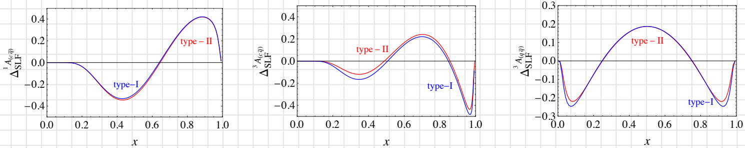
- Self-consistence of CLF QM:



(i) The violation of self-consistence is very small but non-zero in traditional type-I scheme.

(ii) $[f_A]^{\lambda=0}_{\rm full} \doteq [f_A]^{\lambda=\pm}_{\rm full}$ **(type-II)** due to $\int {\rm d}x \Delta^{1(3)A}_{\rm full} = 0$

- Self-consistence of SLF QM:



Self-consistency holds only in type-II scheme: $[f_A]^{\lambda=0}_{\rm SLF} \doteq [f_A]^{\lambda=\pm}_{\rm SLF}$ **(type-II)**

Above findings are similar to the case of V meson.

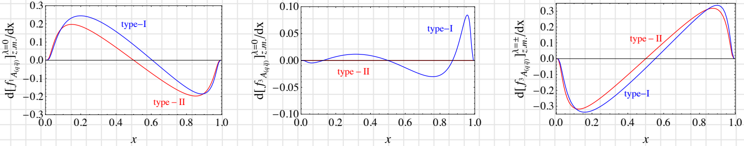
- Relation between $[f_A]^{\lambda=0,\pm}_{\rm SLF}$ and $[f_A]^{\lambda=0,\pm}_{\rm val.}$:

  Taking type-II scheme and making some simplifications, we find that:

  $[f_{1(3)A}]^{\lambda=0}_{\rm SLF}=[f_{1(3)A}]^{\lambda=0}_{\rm val.}$, $[f_{1(3)A}]^{\lambda=\pm}_{\rm SLF}=[f_{1(3)A}]^{\lambda=\pm}_{\rm val.}$, (type-II)

  in which, only $[f_{1A}]^{\lambda=\pm}_{\rm SLF}=[f_{1A}]^{\lambda=\pm}_{\rm val.}$ holds in the type-I scheme.

- Zero-mode effects: $[f_A]_{\rm z.m.}$



(i) $[f_{1A}]^{\lambda=\pm}_{\rm z.m.}=0$ (type-I and -II) , $[f_{1A}]^{\lambda=0}_{\rm z.m.}\neq 0$ (type-I) $\longrightarrow$ The existence or absence of $[f_{1A}]_{\rm z.m.}$ depends on the choice of λ in type-I scheme.

(ii) $[f_{3A}]^{\lambda=0,\pm}_{\rm z.m.}\neq 0$ (type-I) $\longrightarrow$ Its contribution depends on the choice of λ .

(iii) $[f_A]^{\lambda=0,\pm}_{\rm z.m.}\doteq 0$ (type-II) $\longrightarrow$ $[f_A]^{\lambda=0}_{\rm full}\doteq[f_A]^{\lambda=0}_{\rm val.}$ and $[f_A]^{\lambda=\pm}_{\rm full}\doteq[f_A]^{\lambda=\pm}_{\rm val.}$

Combining the findings for the V and A mesons,

$$[\mathcal{Q}]^{\lambda=0}_{\rm SLF} = [\mathcal{Q}]^{\lambda=0}_{\rm val.} \doteq [\mathcal{Q}]^{\lambda=0}_{\rm full} \doteq [\mathcal{Q}]^{\lambda=\pm}_{\rm full} \doteq [\mathcal{Q}]^{\lambda=\pm}_{\rm val.} = [\mathcal{Q}]^{\lambda=\pm}_{\rm SLF}, \qquad \text{(type-II)}$$

where $\mathcal{Q} = f_V$, $f_{^1A}$ and $f_{^3A}$, and the first and the last "$\doteq$" should be replaced by "$=$" for the $^3A_{(q\bar{q})}$ and ^1A mesons, respectively.

Updated predictions for $f_{^1A}$ and $f_{^3A}$ (in unit of MeV)

	$f_{q\bar{q}}$	$f_{s\bar{q}}$	$f_{s\bar{s}}$	$f_{c\bar{q}}$	$f_{c\bar{s}}$
^1A	0	-27 ± 1	0	-78 ± 2	-62 ± 2
^3A	220 ± 1	219 ± 2	203 ± 2	231 ± 8	257 ± 8
	$f_{c\bar{c}}$	$f_{b\bar{q}}$	$f_{b\bar{s}}$	$f_{b\bar{c}}$	$f_{b\bar{b}}$
^1A	0	-95 ± 3	-88 ± 2	-86 ± 3	0
^3A	250 ± 90	176 ± 6	180 ± 5	281 ± 7	353 ± 25

5. Covariance of CLF QM: $f_{V,A}$

Taking $\mathcal{A}_V \equiv \langle 0|\bar{q}_2\gamma^\mu q_1|V(p)\rangle$ as an example,

$$\hat{\mathcal{A}}_V^\mu = M_V(\epsilon^\mu f_V + \omega^\mu g_V)\,,$$

Note: covariance holds only when $g_V = 0$.

Origin of violation:
After integrating out the k^- component and taking into account the zero-mode contributions (most of ω dependences are eliminated), we can decompose $\hat{S}_{\mathcal{A}}$ (integrand) as

$$\hat{S}_V^\mu = 4\left\{2\left(1-\frac{m_1+m_2}{D_{V,\mathrm{con}}}\right)\frac{\omega\cdot\epsilon}{\omega\cdot p}p^\mu B_1^{(2)} + \epsilon^\mu\,[\cdots]\right\}\,,$$

- Second term: the physical contribution to f_V
  First term: the ω-dependent part
- Case of $\lambda = \pm$: the ω dependence vanishes due to $\omega\cdot\epsilon_\pm = 0 \longrightarrow$ Covariant

  Note that: $\lambda = \pm$ is not alway a “good choice” to avoid the covariance problem

- Case of $\lambda = 0$:
  In order to separate the physical and unphysical contributions, we have to use the identity

$$p^\mu \frac{\epsilon\cdot\omega}{\omega\cdot p} = \epsilon^\mu - \frac{\omega^\mu}{\omega\cdot p}\left(\epsilon\cdot p - \epsilon\cdot\omega\frac{p^2}{\omega\cdot p}\right) - \frac{i\lambda}{\omega\cdot p}\varepsilon^{\mu\nu\alpha\beta}\omega_\nu\epsilon_\alpha p_\beta\,.$$

  The third term: $= 0$

  The first term: gives an additional contribution to f_V that results in the self-consistency problem;

  The second term: the residual ω-dependent part that contributes to g_V and may violate the Lorentz covariance.

  The problems of self-consistency and covariance of the CLF quark model within the type-I scheme have the same origin!

Theoretical results

$$[g_V]^{\lambda=0} = \frac{N_c}{2}\int\frac{\mathrm{d}x\mathrm{d}^2k_\perp}{(2\pi)^3}\frac{\chi_V(x,k_\perp^2)}{\bar{x}}4\left(1-\frac{m_1+m_2}{D_{V,\mathrm{con}}}\right)\frac{2}{\omega\cdot p}B_1^{(2)}\,,$$

$$[g_{3A}]^{\lambda=0} = \frac{N_c}{2}\int\frac{\mathrm{d}x\mathrm{d}^2k_\perp}{(2\pi)^3}\frac{\chi_{3A}(x,k_\perp^2)}{\bar{x}}4\left(1+\frac{m_1-m_2}{D_{3A,\mathrm{con}}}\right)\frac{2}{\omega\cdot p}B_1^{(2)}\,,$$

$$[g_{1A}]^{\lambda=0} = \frac{N_c}{2}\int\frac{\mathrm{d}x\mathrm{d}^2k_\perp}{(2\pi)^3}\frac{\chi_{1A}(x,k_\perp^2)}{\bar{x}}4\frac{m_1-m_2}{D_{1A,\mathrm{con}}}\frac{2}{\omega\cdot p}B_1^{(2)}\,,$$

- Conditions for the covariance: $[g_V] = [g_A] = 0 \Longrightarrow$

$$\int \mathrm{d}x\mathrm{d}^2k_\perp \frac{\chi_M(x,k_\perp^2)}{\bar{x}} B_1^{(2)} = 0\,, \qquad \int \mathrm{d}x\mathrm{d}^2k_\perp \frac{\chi_M(x,k_\perp^2)}{\bar{x}} \frac{B_1^{(2)}}{D_{M,\mathrm{con}}} = 0\,,$$

  which is much stricter than the one given by Jaus.

- $[g_{V,A}] \propto 1/p^+$: the size of covariance violation within the type-I scheme is in fact out of control because p^+ is reference-frame dependent.
- Covariance is violated in the type-I scheme but can be recovered in the type-II scheme.
  In the rest frame ($p^+ = M$),

$$[g_{V,A}]^{\lambda=0} = [f_{V,A}]^{\lambda=0}_{\mathrm{full}} - [f_{V,A}]^{\lambda=\pm}_{\mathrm{full}} = \int \mathrm{d}x\, \Delta^{V,A}_{\mathrm{full}}(x)\,,$$

So,

$$[g_{\rho,\,D^*,\,{}^1A_{(c\bar{q})},\,{}^3A_{(q\bar{q})},\,{}^3A_{(c\bar{q})}}]^{\lambda=0} = (-40.2,\,-30.3,\,6.2,\,37.0,\,-12.0)\,\mathrm{MeV} \neq 0\,, \quad \text{(type-I)}$$

$$[g_{\rho,\,D^*,\,{}^1A_{(c\bar{q})},\,{}^3A_{(q\bar{q})},\,{}^3A_{(c\bar{q})}}]^{\lambda=0} = 0\,, \quad \text{(type-II)}$$

The problems of self-consistency and covariance of the CLF quark model can be “resolved” simultaneously within the type-II scheme.

- The type-I and -II schemes are consistent with each other in the heavy quark limit.
  $M \sim m_Q \gg m_{\bar{q}}$
  $x \sim m_Q/M$ and $\bar{x} \sim m_q/M$ $\Longrightarrow$ $M_0 \to M$
  $f(g)_{V,A}$ are dominated by $|k_\perp| \lesssim 1\ \mathrm{GeV}$

Some comments and conclusions for the form factors:

- $F_{P\to V}, {}_{V\to V,\dots}$ $(B_1^{(2)}, B_3^{(3)}, \dots)$ in CLF QM also suffer from the self-consistency and covariance problems, which can be "resolved" within type-II scheme.
- For all of the form factors of $P\to P$, $P\to V$, $V\to V\dots$ transitions,

$$[\mathcal{Q}]_{\rm SLF} = [\mathcal{Q}]_{\rm val.} \doteq [\mathcal{Q}]_{\rm full}\,, \qquad \text{(type-II)}$$

  Zero-mode contributions vanish numerically! (Two viewpoints for the SLF QM)
- $\lambda = \pm$ is not alway a good choice to avoid the covariance problem. An example is $P\to V$ transition.
- All of the form factors, for instance $a_-(q^2)$, are in fact calculable in the SLF QM after taking $M\to M_0$, and also satisfy the relation given above.

PHYSICAL REVIEW D, VOLUME 60, 054026

Covariant analysis of the light-front quark model

Wolfgang Jaus
Institut für Theoretische Physik der Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland
(Received 21 December 1998; published 9 August 1999)

the standard approach are reproduced, except for those that de covariant approach permits also the calculation of the scalar form factor for transitions between pseudoscalar mesons, and the form factor $a_-(q^2)$ for transitions between pseudoscalar and vector mesons, which is not possible in the standard light-front formalism. The practical application of the covariant extension of the

6. Brief summary

- In the traditional SLF and CLF QMs (type-I scheme), $f_{V,{}^1A,{}^3A}$ suffer from the self-consistency and covariance problems.
- In the CLF QMs, the self-consistency and covariance problems can be resolved simultaneously by taking type-II correspondence.
- The zero-mode contributions exist only formally but vanish numerically (type-II).
- For the decay constants of spin-1 systems,

$$[\mathcal{Q}]_{\rm SLF}^{\lambda=0} = [\mathcal{Q}]_{\rm val.}^{\lambda=0} \doteq [\mathcal{Q}]_{\rm full}^{\lambda=0} \doteq [\mathcal{Q}]_{\rm full}^{\lambda=\pm} \doteq [\mathcal{Q}]_{\rm val.}^{\lambda=\pm} = [\mathcal{Q}]_{\rm SLF}^{\lambda=\pm}\,; \qquad \text{(type-II)}$$

For the form factors,

$$[\mathcal{Q}]_{\rm SLF} = [\mathcal{Q}]_{\rm val.} \doteq [\mathcal{Q}]_{\rm full}\,, \qquad \text{(type-II)}$$

- The two schemes are consistent with each other in the heavy-quark limit.

Thank you !