A perturbative analytical solution of viscous hydrodynamics with longitudinal accelerating flow

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Xin-Nian Wang, T. S. Biró, M. I. Nagy.

Motivation

The next phase … will focus on detailed investigations of the QGP, "both to quantify its properties and to understand precisely how they emerge from the fundamental properties of QCD"

-- The frontiers of nuclear science, a long range plan

- What is the initial temperature and thermal evolution of the produced matter?

- What is the viscosity of the produced matter? … http://www.bnl.gov/physics/rhiciiscience/

Results of perfect fluid

- The exact solutions and results of the perfect fluid with longitudinal accelerating flow. (CNC, CKCJ solutions)

Csörgő, Nagy, Csanád (CNC) arXiv: 0605070, 0710.0327, 0805.1562,
Csanád, et. arXiv:1609.07176.
Z. F. Jiang, et. arXiv: 1711.10740, 1806.05750.
Csörgő, et. arXiv: 1805.01427, 1806.11309, 1810.00154.
(See Z. F. Jiang, C. B. Yang, M. Csanád, T. Csörgő, Phys. Rev. C 97,(2018) 064906,
T. Csörgő, G. Kasza, M. Csanád, Z. F. Jiang, Universe 4, (2018), 69.)

Outline

The next phase … will focus on detailed investigations of the QGP, "both to quantify its properties and to understand precisely how they emerge from the fundamental properties of QCD"

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Outline

1. A perturbative solution of viscous hydrodynamics.
2. Final state spectrum (a toy model) compared with RHIC and the LHC data.
3. Summary and outlook.

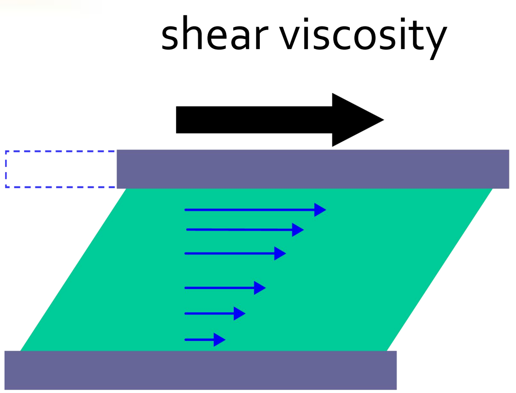
Z. F. Jiang, et. arXiv:1808.10287 (Accepted by Chin. Phys. C).

Relativistic accelerated viscous hydrodynamic

Longitudinal acceleration effect makes the fluid cooling faster. (CNC, CKCJ solutions)
The viscosity will creating heat and makes the fluid cooling slower.

$$T^{\mu\nu} = eu^{\mu}u^{\nu} - (p + \boxed{\Pi})\Delta^{\mu\nu} + \boxed{\pi^{\mu\nu}}$$

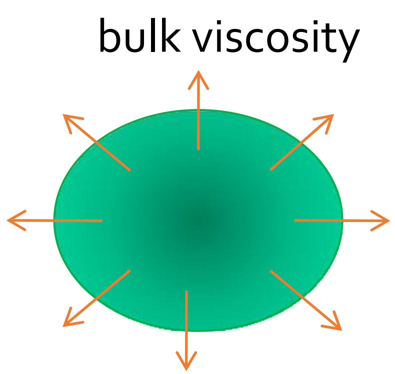
$$u^{\mu} = (\cosh\Omega, 0, 0, \sinh\Omega) \quad \Delta^{\mu\nu} = g^{\mu\nu} - u^{\mu}u^{\nu}$$



Shear viscosity tensor: $\pi^{\mu\nu}$ **Bulk viscosity:** Π.

Shear tensor:

$$\sigma^{\mu\nu} \equiv \partial^{\langle\mu}u^{\nu\rangle} \equiv (\frac{1}{2}(\Delta^{\mu}_{\alpha}\Delta^{\nu}_{\beta} + \Delta^{\mu}_{\beta}\Delta^{\nu}_{\alpha}) - \frac{1}{d}\Delta^{\mu\nu}\Delta_{\alpha\beta})\partial^{\alpha}u^{\beta}.$$



The fundamental equations of the viscous fluid:

$$e = \kappa p\,,\ \partial_{\mu}T^{\mu\nu} = 0.$$

Equations of viscous hydrodynamic

The second law of thermodynamics: $\partial_\mu S^\mu \geq 0$

$$\tau_\pi \Delta^{\alpha\mu} \Delta^{\beta\nu} \dot{\pi}_{\alpha\beta} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} - \frac{1}{2}\pi^{\mu\nu} \frac{\eta T}{\tau_\pi} \partial_\lambda \left(\frac{\tau_\pi}{\eta T} u^\lambda \right)$$

$$\tau_\Pi \dot{\Pi} + \Pi = -\zeta(\partial \cdot u) - \frac{1}{2}\Pi \frac{\zeta T}{\tau_\Pi} \partial_\lambda \left(\frac{\tau_\Pi}{\zeta T} u^\lambda \right)$$

Israel-Stewart equations.

viscous hydro: near-equilibrium system

The Navier-Stokes approximation,

$$\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} \qquad \Pi = -\zeta\left(\partial_\rho u^\rho\right)$$

The shear viscosity and bulk viscosity,

Strongly coupled AdS/CFT prediction: $\eta/s \geq 1/4\pi \approx 0.08$ D.T. Son, et,al. 05

Via lattice calculation: $\zeta/s \leq 0.015\,(\text{for } 3T_c)$ H.B. Meyer, et,al. 07 10.3717

Accelerating viscous hydrodynamic equation

Based on the conservation law, the energy equation and Euler equations are :

$$\begin{cases} D = u^{\mu}\partial_{\mu}, \quad \theta = \nabla_{\mu}u^{\mu} \\ \nabla^{\alpha} = \Delta^{\mu\alpha}\partial_{\mu}, \quad \Delta^{\mu\nu} = g^{\mu\nu} - u^{\mu}u^{\nu} \end{cases}$$

$$\begin{cases} De = -(e+P+\Pi)\theta + \sigma_{\mu\nu}\pi^{\mu\nu} \\ (e+P+\Pi)Du^{\alpha} = \nabla^{\alpha}(P+\Pi) - \Delta^{\alpha}_{\nu}u_{\mu}D\pi^{\mu\nu} - \Delta^{\alpha}_{\nu}\nabla_{\mu}\pi^{\mu\nu} \end{cases}$$

Accelerating viscous hydrodynamic equation

In Rindler coordinates (accelerate coordinates), the energy equation and Euler equation reduce to:

$$\begin{cases} \tau \frac{\partial T}{\partial \tau} + \tanh(\Omega - \eta_s) \frac{\partial T}{\partial \eta_s} + \frac{\Omega'}{\kappa} T = \frac{\Pi_d}{\kappa} \frac{\Omega'^2}{\kappa} \cosh(\Omega - \eta_s), \\ \tanh(\Omega - \eta_s)\left[\tau \frac{\partial T}{\partial \tau} + T\Omega'\right] + \frac{\partial T}{\partial \eta_s} = \frac{\Pi_d}{\kappa}[2\Omega'(\Omega'-1) + \Omega''\coth(\Omega - \eta_s)] \sinh(\Omega - \eta_s) \end{cases}$$

$$\begin{cases} \Omega' = \frac{\partial \Omega}{\partial \eta_s} \\ \Pi_d = \left(\frac{\zeta}{s} + \frac{2\eta}{s}(1 - \frac{1}{d})\right) \end{cases}$$

Bjorken approximation:

$$\Pi_d = 0,\ \Omega(\eta_s) = \eta_s$$

J.D.Bjorken, Phys.Rev. D27 (1983) 140-151.

Without accelerating:

$$\Pi_d \neq 0\ \ \Omega(\eta_s) = \eta_s$$

A. Muronga, Phys. Rev. C69 (2004), 034904.

Without Viscosity:

$$\Pi_d = 0,\ \Omega(\eta_s) \neq \eta_s$$

CKCJ solutions, Universe 4 (2018), 69.

Both are non-zero, a perturbative case.

$$\Omega = \lambda \eta_s = (1 + \varepsilon)\eta_s, \qquad |\varepsilon| << 1.$$

$$\Pi_d \neq 0$$

λ: the constant proper acceleration.
ε: the acceleration parameter.

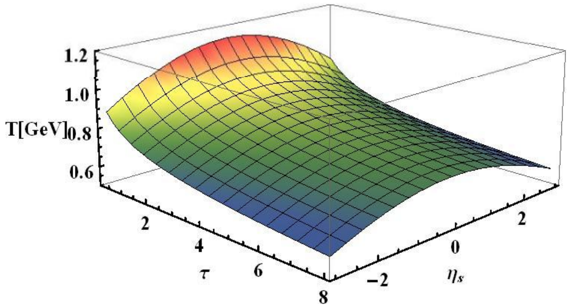
M. Csanád, 2017. Universe. 3. 1-9.
Z. F. Jiang, et. arXiv: 1711.10740

Up to $\mathcal{O}(\epsilon)$,

$$\begin{cases} \tau \frac{\partial T}{\partial \tau} + \frac{\epsilon + 1}{\kappa} T = \frac{\Pi_d}{\kappa} \frac{2\epsilon + 1}{\tau} + \mathcal{O}(\epsilon^2). \\ T_1(\eta_s)\left(1 - \frac{1}{\kappa}\right)\epsilon\eta_s + \frac{\epsilon \Pi_d}{(\kappa - 1)\tau_0}\left(1 - \frac{1}{\kappa}\right)\eta_s + \frac{\partial T_1(\eta_s)}{\partial \eta_s} + \mathcal{O}(\epsilon^2) = 0 \end{cases}$$

Solutions form hydrodynamic equations

The temperature profile:



$$T(\tau,\eta_s) = T_0\left(\frac{\tau_0}{\tau}\right)^{\frac{1+\epsilon}{\kappa}} \times \left[\exp[-\frac{1}{2}\epsilon(1-\frac{1}{\kappa})\eta_s^2] + \frac{R_0^{-1}}{\kappa-1}\left(2\epsilon + \exp[-\frac{1}{2}\epsilon(1-\frac{1}{\kappa})\eta_s^2] - (2\epsilon+1)\left(\frac{\tau_0}{\tau}\right)^{\frac{\kappa-\epsilon-1}{\kappa}}\right)\right]$$

Contribution from ideal terms.

Contribution from viscous effect

$$R_0^{-1} = \frac{\Pi_d}{T_0\tau_0}$$

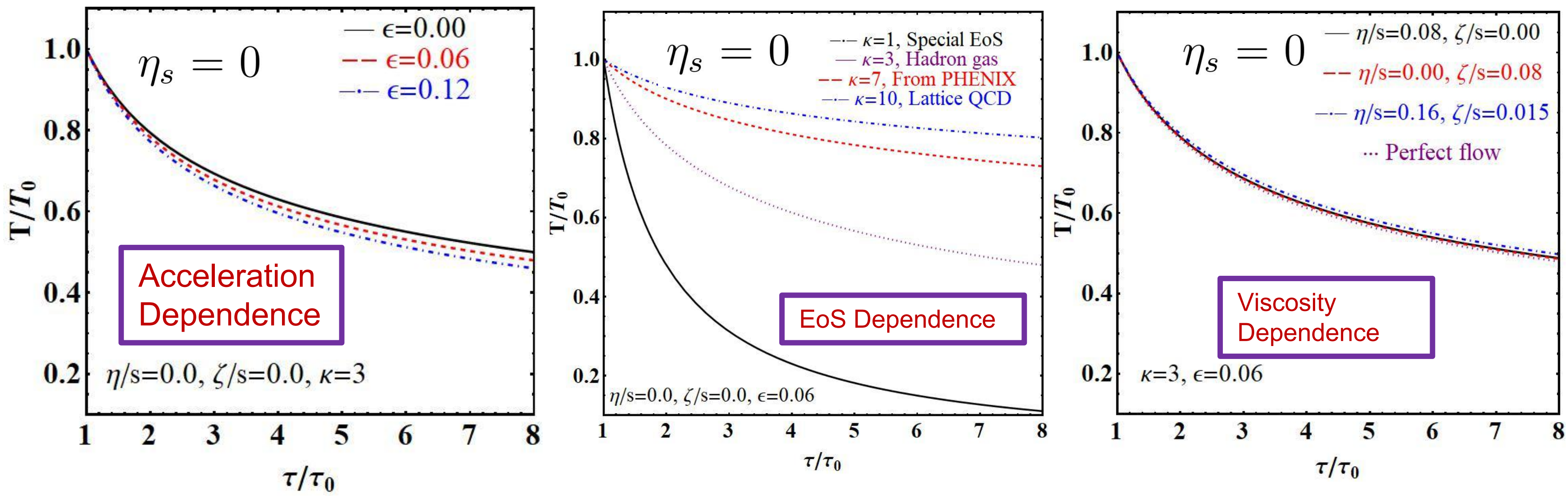
Reynolds number
[A. Muronga, arXiv: 0309055]

- A non-zero Reynolds numbers R_0^{-1} makes cooling rate smaller,
- A non-vanishing acceleration ϵ makes the cooling rate is larger.

Open question: setting viscosity as the perturbative term.

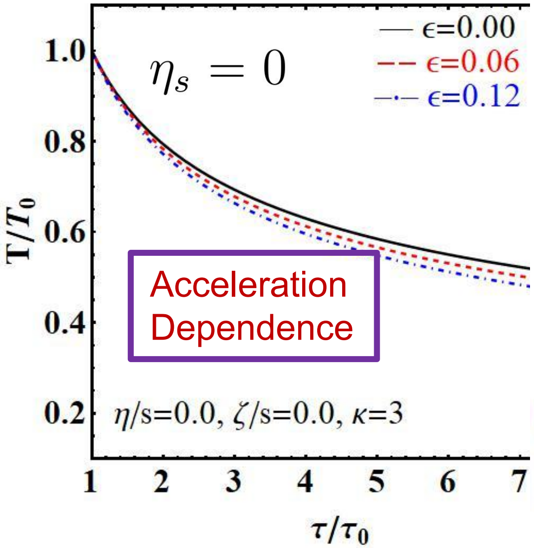
(Z.F. Jiang, C.B. Yang, T. Csörgő, M. Csanád, M. Nagy. In preparing.)

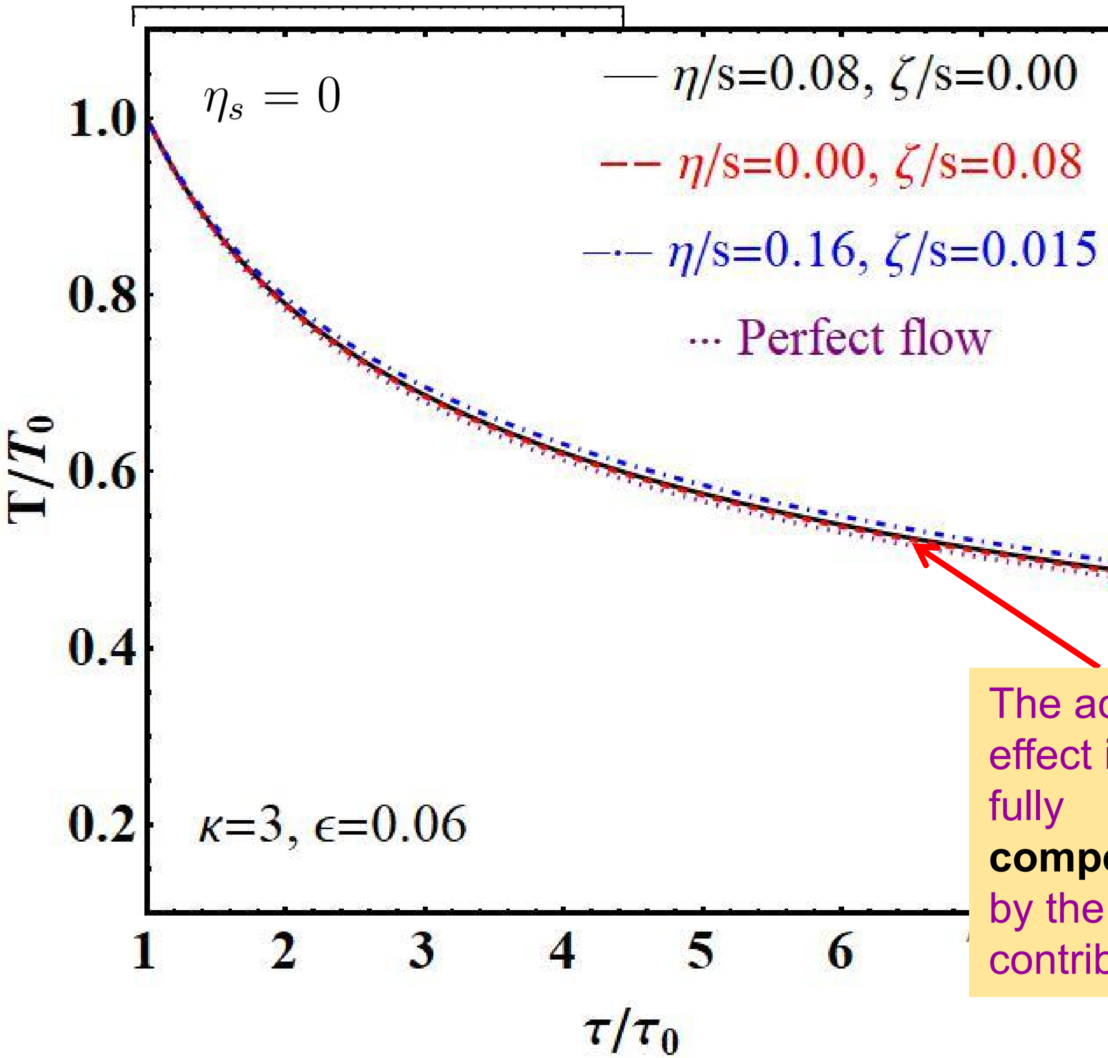
Temperature evolution



- Acceleration effect comes from the pressure gradient, makes the cooling ratio **larger** than non-acceleration flow.
  [M. Nagy, T. Csörgő, M. Csanád: arXiv:0709.3677v1]
- EoS is an important modified factor.
  κ=1 a very special case, CNC solution.
  κ=7 comes from [PHENIX, arXiv:nucl-ex/0608033v1] .
- Viscosity effect make the cooling rate samller. [H. Song, S. Bass, U. Heinz. et, PRL2011]

Temperature evolution





- Acceleration effect co
cooling ratio **larger** thar
[M. Nagy, T. Csörgő,
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Heinz. et, PRL2011]

The acceleration effect is almost fully **compensated** by the viscous contribution.

The final state spectrum

Freeze-out hypersurface:

$$p_\mu d\Sigma^\mu = m_T \tau_f \cosh^{\frac{2-\Omega'}{\Omega'-1}}((\Omega'-1)\eta_s)\cosh(\Omega - y) r dr d\phi d\eta_s$$

[M. I. Nagy, T. Csörgő, M. Csanád: arXiv:0709.3677v1]

The transverse momentum distribution (a toy model):

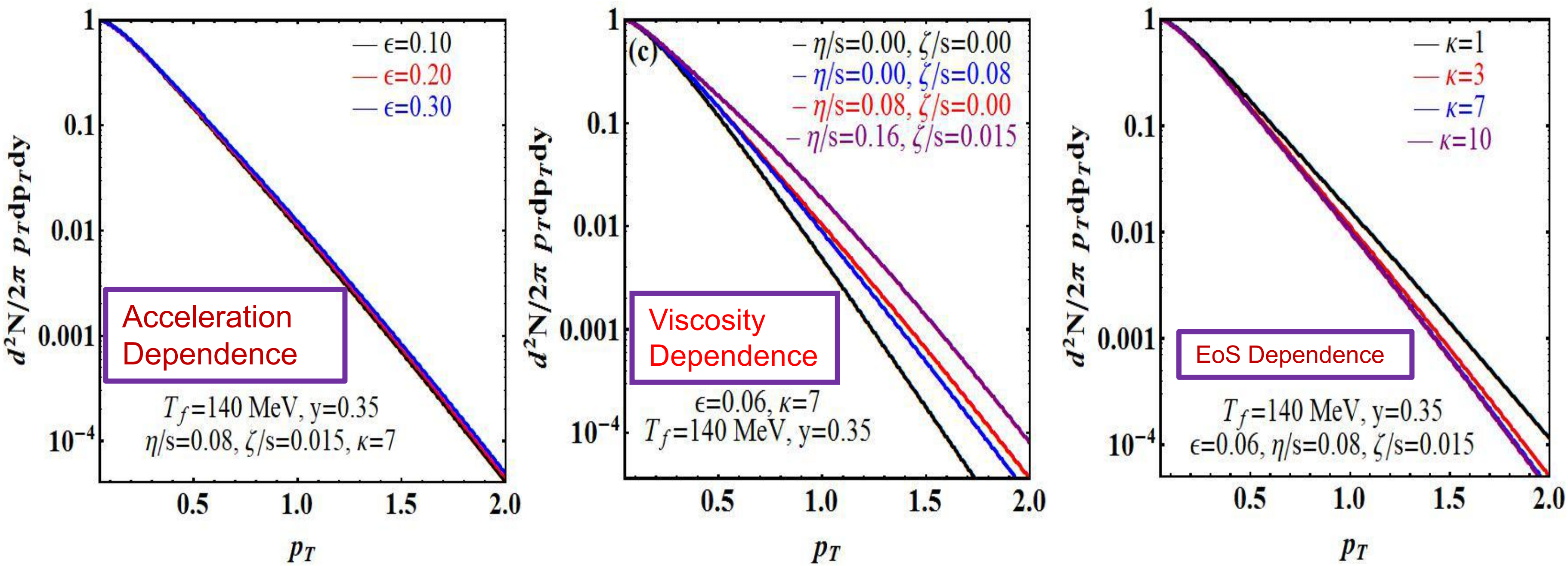
$$\frac{d^2N}{2\pi p_T dp_T dy} = \frac{\pi R_0^2}{(2\pi)^3}\int_{-\infty}^{+\infty} m_T \cosh((\epsilon+1)\eta_s - y)\exp\left[-\frac{m_T}{T(\tau,\eta_s)}\cosh((\epsilon+1)\eta_s - y)\right]$$
$$\times\left(\tau_f \cosh^{\frac{1-\epsilon}{\epsilon}}(\epsilon\eta_s) + \frac{1+\epsilon}{T^3(\tau,\eta_s)}\left[\frac{1}{3}\frac{\eta}{s}(p_T^2 - 2m_T^2\sinh^2((\epsilon+1)\eta_s - y))\right.\right.$$
$$\left.\left. - \frac{1}{5}\frac{\zeta}{s}(p_T^2 + m_T^2\sinh^2((\epsilon+1)\eta_s - y))\right]\right) d\eta_s$$

- $f_0 + \delta f$, Boltzmann approximation, from K. Dusling and D. Teaney (2010).
- Temperature solution, viscosity, acceleration parameter, mass, ...

D. Teaney, 2003. P. R. C 68, 034913, a special case when there is no acceleration effect (ε=0).

Transverse momentum distribution

Numerical results:



Transverse momentum distribution:
- the longitudinal acceleration effect is little,
- the viscous effect play a important role for distribution,
- the distribution is sensitive to the EoS.

(Pseudo-) Rapidity distribution

Rapidity distribution

Contribution from perfect fluid

$$\frac{dN}{dy} = \frac{\pi R_0^2}{(2\pi)^3} \int_0^{+\infty} \left\{ \cosh^{\frac{1-\epsilon}{\epsilon}}(\epsilon\eta_s) \frac{4\tau_f T^3(\tau,\eta_s)}{\cosh^2((\epsilon+1)\eta_s - y)} + \frac{48(1+\epsilon)T^2(\tau,\eta_s)}{\cosh^4((\epsilon+1)\eta_s - y)} \right.$$
$$\left. \times \left[\frac{1}{3}\frac{\eta}{s}(1 - 2\sinh^2((\epsilon+1)\eta_s - y)) - \frac{1}{5}\frac{\zeta}{s}\cosh^2((\epsilon+1)\eta_s - y) \right] \right\} d\eta_s$$

Rapidity distribution,
- the integral value $error \propto m^3$, this is a good approximation for the particle that mass m is little.

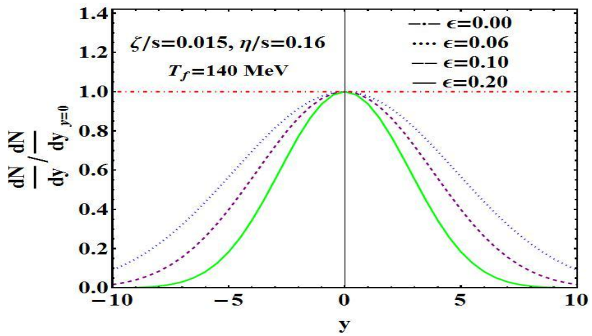
Pseudo-rapidity distribution

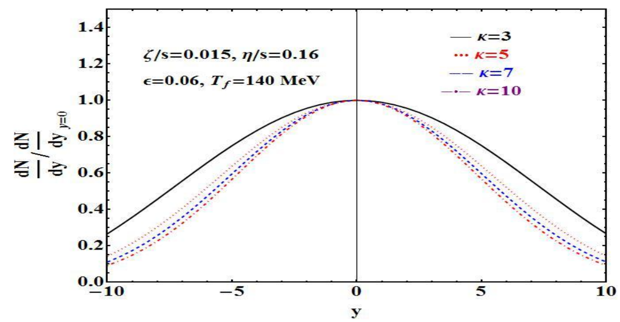
$$\frac{dN}{d\eta} = \frac{\pi R_0^2}{(2\pi)^3} \int_{-\infty}^{+\infty} d\eta_s \int_0^{+\infty} dp_T \sqrt{1 - \frac{m^2}{m_T^2 \cosh^2 y}} m_T p_T \cosh((\epsilon+1)\eta_s - y) \exp\left[-\frac{m_T}{T(\tau,\eta_s)} \cosh((\epsilon+1)\eta_s - y) \right]$$
$$\times \left(\tau_f \cosh^{\frac{1-\epsilon}{\epsilon}}(\epsilon\eta_s) + \frac{1+\epsilon}{T^3(\tau,\eta_s)} \left[\frac{1}{3}\frac{\eta}{s}(p_T^2 - 2m_T^2\sinh^2((\epsilon+1)\eta_s - y)) - \frac{1}{5}\frac{\zeta}{s}(p_T^2 + m_T^2\sinh^2((\epsilon+1)\eta_s - y)) \right] \right)$$

Contribution from perfect fluid | **Contribution from viscous effect**

Particle's distribution (toy model)

Numerical results (Rapidity distribution):





Acceleration parameter extracted from RHIC and the LHC data:

$\sqrt{s_{NN}}$ /[GeV]		$\frac{dN}{d\eta}\big\vert_{\eta=\eta_0}$	ϵ	χ^2/NDF
130	Au+Au	563.9±59.5	0.076±0.003	9.41/53
200	Au+Au	642.6±61.0	0.062±0.002	12.23/53
200	Cu+Cu	179.5±17.5	0.060±0.003	2.41/53
2760	Pb+Pb	1615±39.0	0.035±0.003	5.50/41
5020	Pb+Pb	1929±47.0	0.032±0.002	33.0/27
5440	Xe+Xe	1167±26.0[41]	0.030±0.003	−/−

$$\epsilon = A\left(\frac{\sqrt{s_{NN}}}{\sqrt{s_0}}\right)^{-B}$$

$$\sqrt{s_0} = 1\ \text{GeV}$$

$$A = 0.045 \text{ and } B = 0.23$$

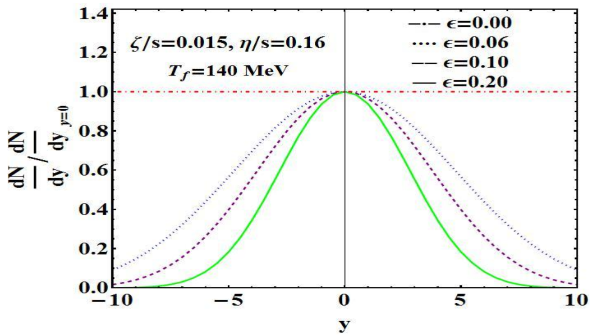
Z.F. Jiang, C.B. Yang, Chi Ding, Xiang-Yu Wu. arXiv: 1808.10287.

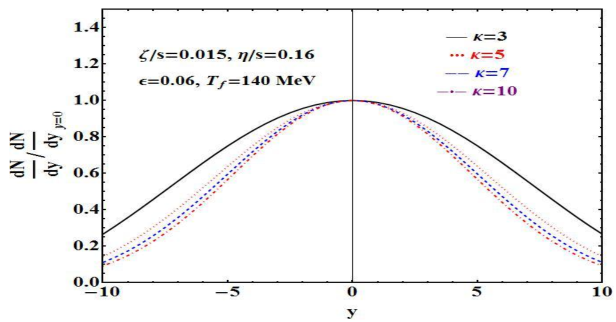
Particle's distribution

- at finial state, the dn/dy and $dn/d\eta$ are effected sensitively by the acceleration parameter.
- This toy model's prediction for XeXe@5440 GeV works well!
- A simple description of acceleration parameters is obtained.

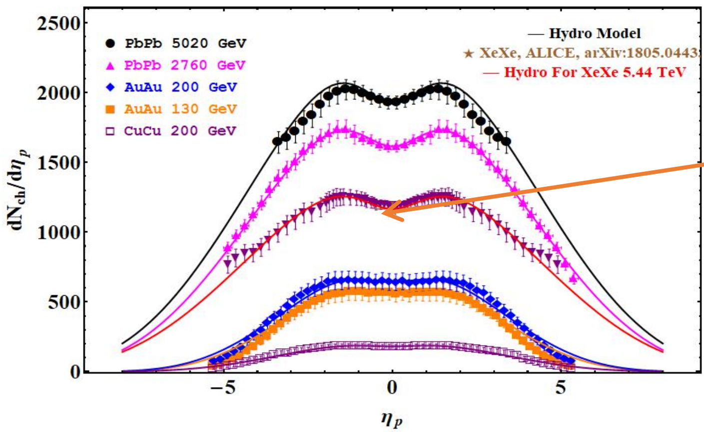
Particle's distribution (toy model)

Numerical Results (Rapidity distribution):





Numerical results (pseudo-rapidity distribution):



Red line: prediction for XeXe @ 5440 GeV.
Data: ALICE@QM2018, 1805.04432.

Z.F. Jiang, et al. arXiv: 1711.10740
Z.F. Jiang, C.B. Yang, Chi Ding, Xiang-Yu Wu. arXiv: 1808.10287.

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A brief summary and outlook

Summary:

1. A perturbative solution with both longitudinal accelerating effect and viscous correction are obtained.
2. The final state spectrum are obtained, this toy model described well the RHIC and the LHC data.

Outlook:

1. 2rd I-S problem, magnetohydrodynamics (MHD) and CLVisc 3+1D numerical code;
2. Fast parton disturbance evolution on medium background...

Thank you for your attention

arXiv: 1609.07176, 1711.10740, 1805.01427, 1806.05750, 1808.10287…

Backup

CNC solutions of relativistic hydrodynamics

Hydrodynamics can be a universal tool to study the QGP. In Rindler coordinate, 4 different parameters of the parameters λ, d, κ and ϕ for 5 possible cases as follows (λ is the accelerate parameter.):

Csörgő, Nagy, Csanád(CNC)
arXiv: 0605070,
0710.0327, 0805.1562,....

Case	λ	d	κ	ϕ	
a.)	2	**R**	d	0	⬅ accelerating, d dimension
b.)	1/2	**R**	*1*	$(\kappa+1)/\kappa$	⬅ d dimensional (T. S. Biró)
c.)	3/2	**R**	$(4d-1)/3$	$(\kappa+1)/\kappa$	⬅
d.)	1	**R**	**R**	0	⬅ **Hwa-Bjorken, Buda-Lund type**
e.)	**R**	**R**	1	0	⬅ Special EoS, but general velocity

In all ideal cases, the velocity field and the pressure is expressed as :

$$v = \tanh \lambda\eta_s, \quad p = p_0 \left(\frac{\tau_0}{\tau}\right)^{\lambda d \frac{(\kappa+1)}{\kappa}} \left(\cosh \frac{\eta_s}{2}\right)^{-(d-1)\phi}$$

The initial energy density estimation

Rapidity distribution:

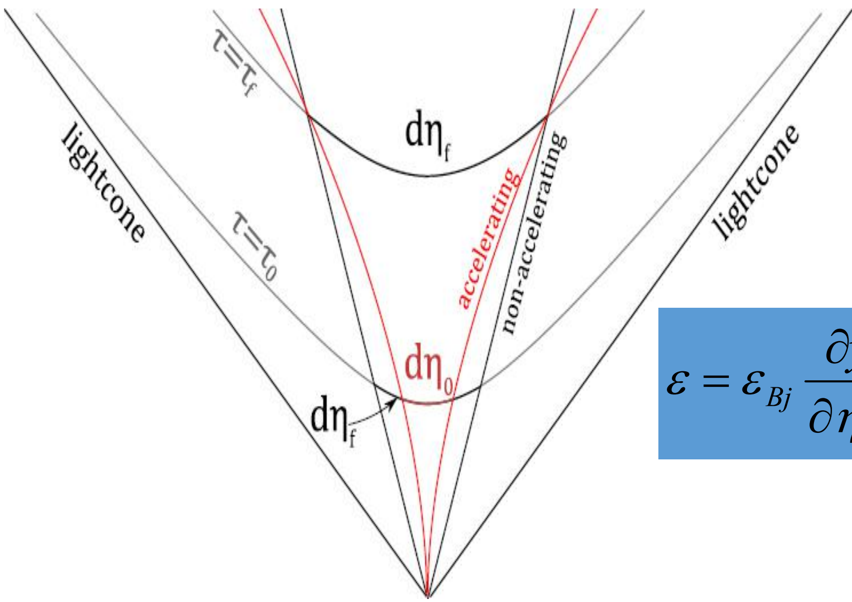
$$\frac{dN}{dy} \approx \left.\frac{dN}{dy}\right|_{y=0} \cosh^{\pm\frac{\alpha}{2}-1}\left(\frac{y}{\alpha}\right) e^{-\frac{m}{T_f}\left[\cosh^{\alpha}\left(\frac{y}{\alpha}\right)-1\right]}$$

⇔

Pseudorapidity distribution:

$$\frac{dN}{d\eta} \approx \frac{\overline{p}}{\overline{E}}\frac{dn}{dy} = \frac{\overline{p_T}\cosh\eta}{\sqrt{m^2+\overline{p_T^2}}}\frac{dN}{dy}$$

$$\alpha = \frac{2\lambda - 1}{\lambda - 1}.$$



Flow element's motion path

$$\varepsilon_{Bj} = \frac{1}{R^2\pi\tau_0}\frac{dE}{d\eta} = \frac{\langle E\rangle}{R^2\pi\tau_0}\frac{dn}{d\eta}$$

J. D. Bjroken, Phys. Rev. D 27, 140 (1983)

$$\varepsilon = \varepsilon_{Bj}\frac{\partial y}{\partial \eta_f}\frac{\partial \eta_f}{\partial \eta_i}$$

For an accelerating flow,
Two main modifications for **Initial conditions:** $y \neq \eta$; $\eta_{final} \neq \eta_{initial}$.

$$\lambda = 1.18 \pm 0.01, \quad \tau_f/\tau_0 = 8 \pm 2,$$

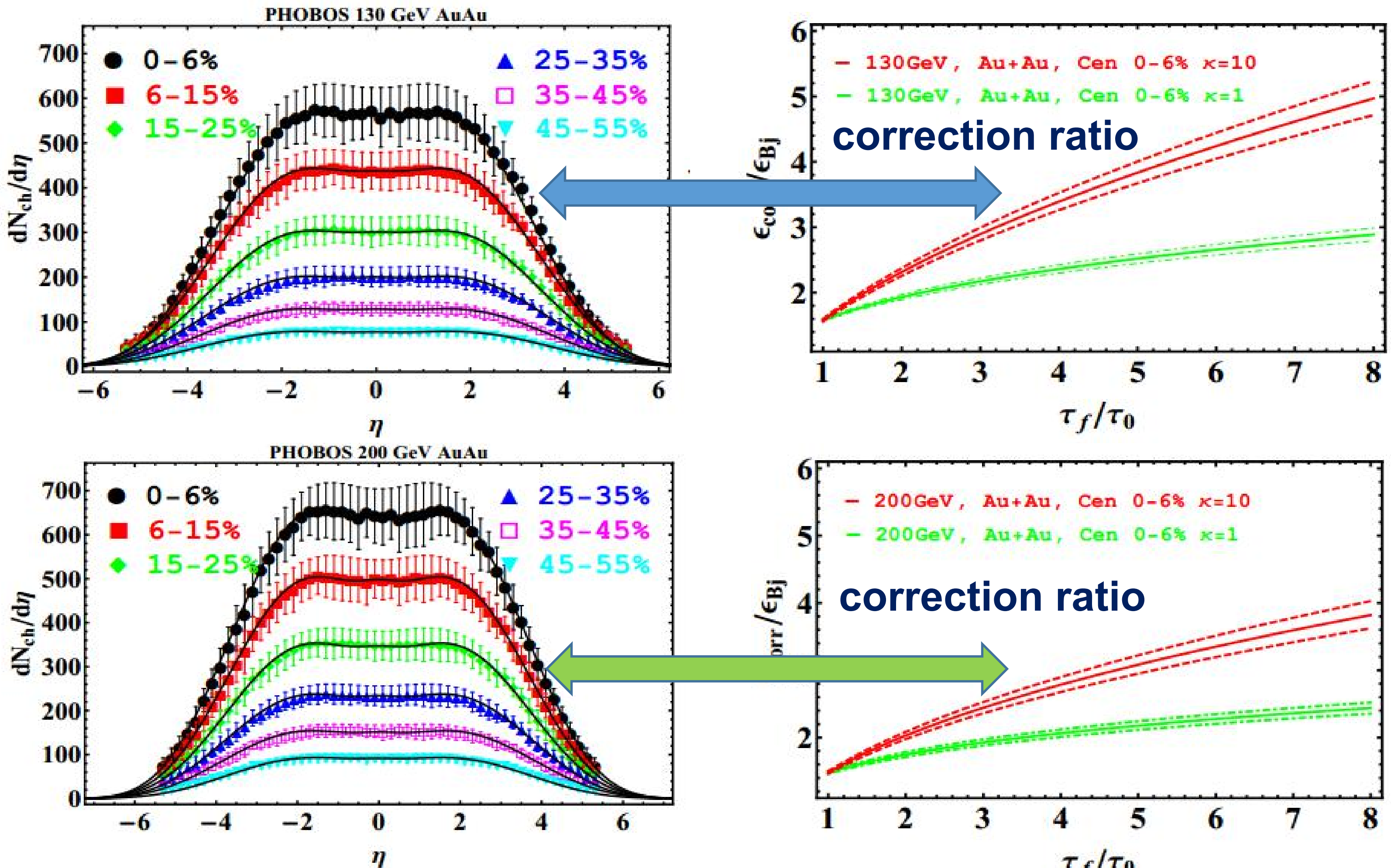
$$\varepsilon_{corr} = (2.0 \pm 0.1)\varepsilon_{Bj} = 10.0 \pm 0.5 \mathrm{GeV/fm^3}.$$

BRAHMS 200 GeV Au+Au data
arXiv: 0805.1562

For more recent, exact results that include the work done by pressure, see G.Kasza's talk.

The pseudorapidity distributions

Results for 130 GeV Au+Au and 200 GeV Au+Au collisions



(See Z.F. Jiang, C.B. Yang, M. Csanád, T. Csörgő, Phys. Rev. C 97, 064906)

The initial energy density estimation

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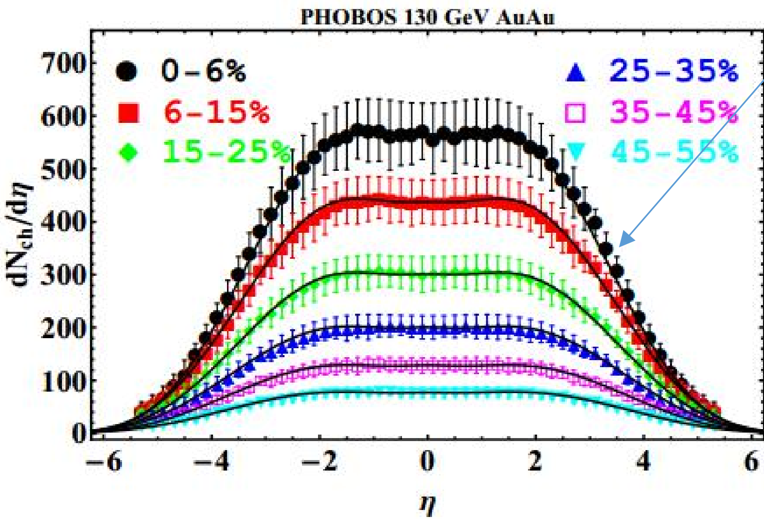
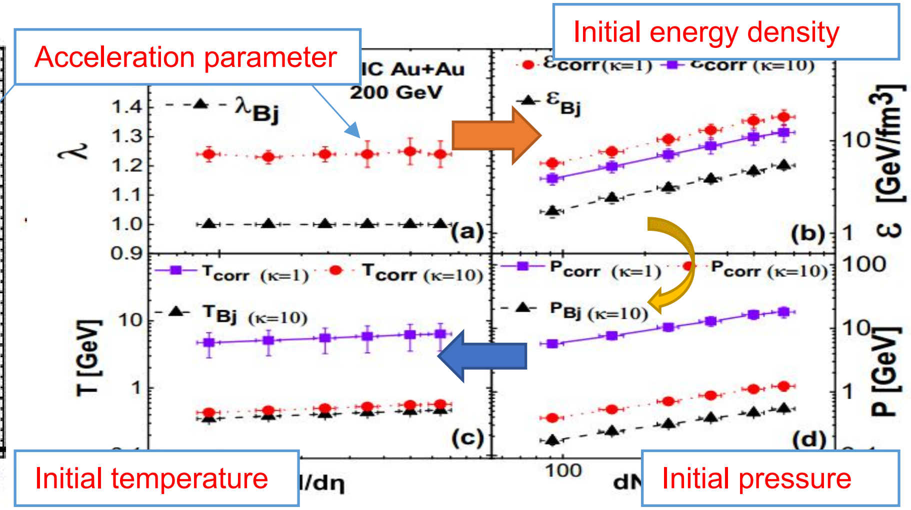
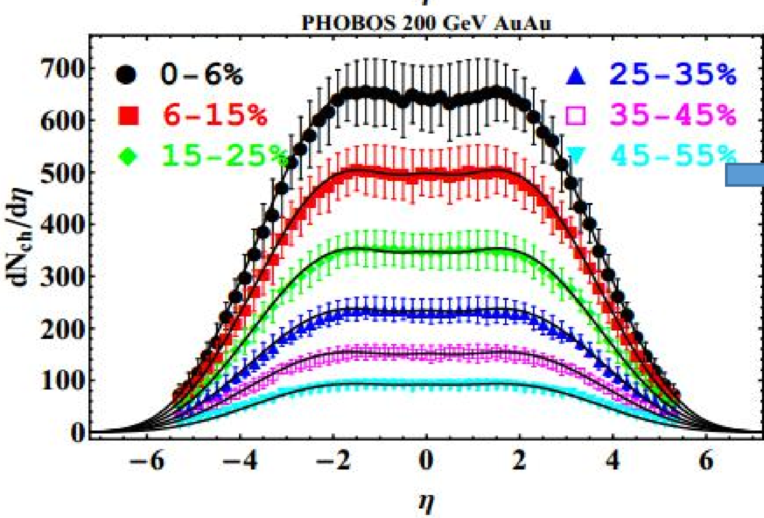
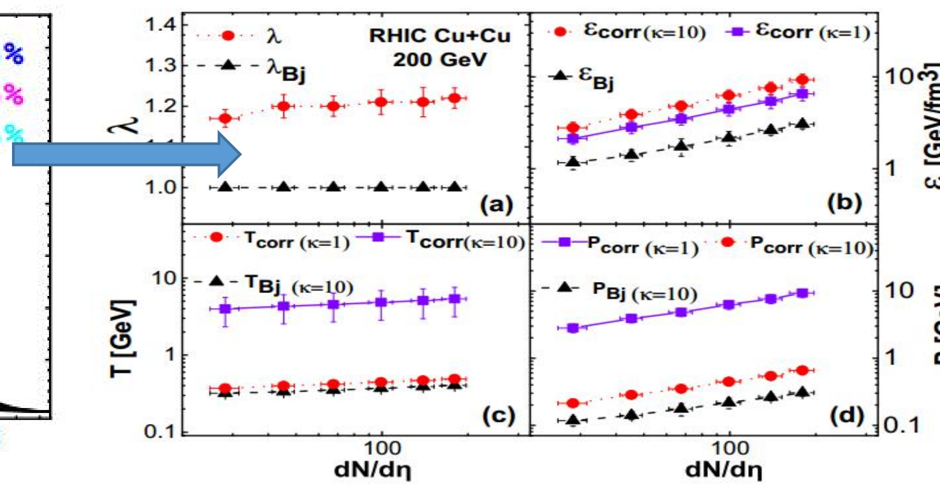




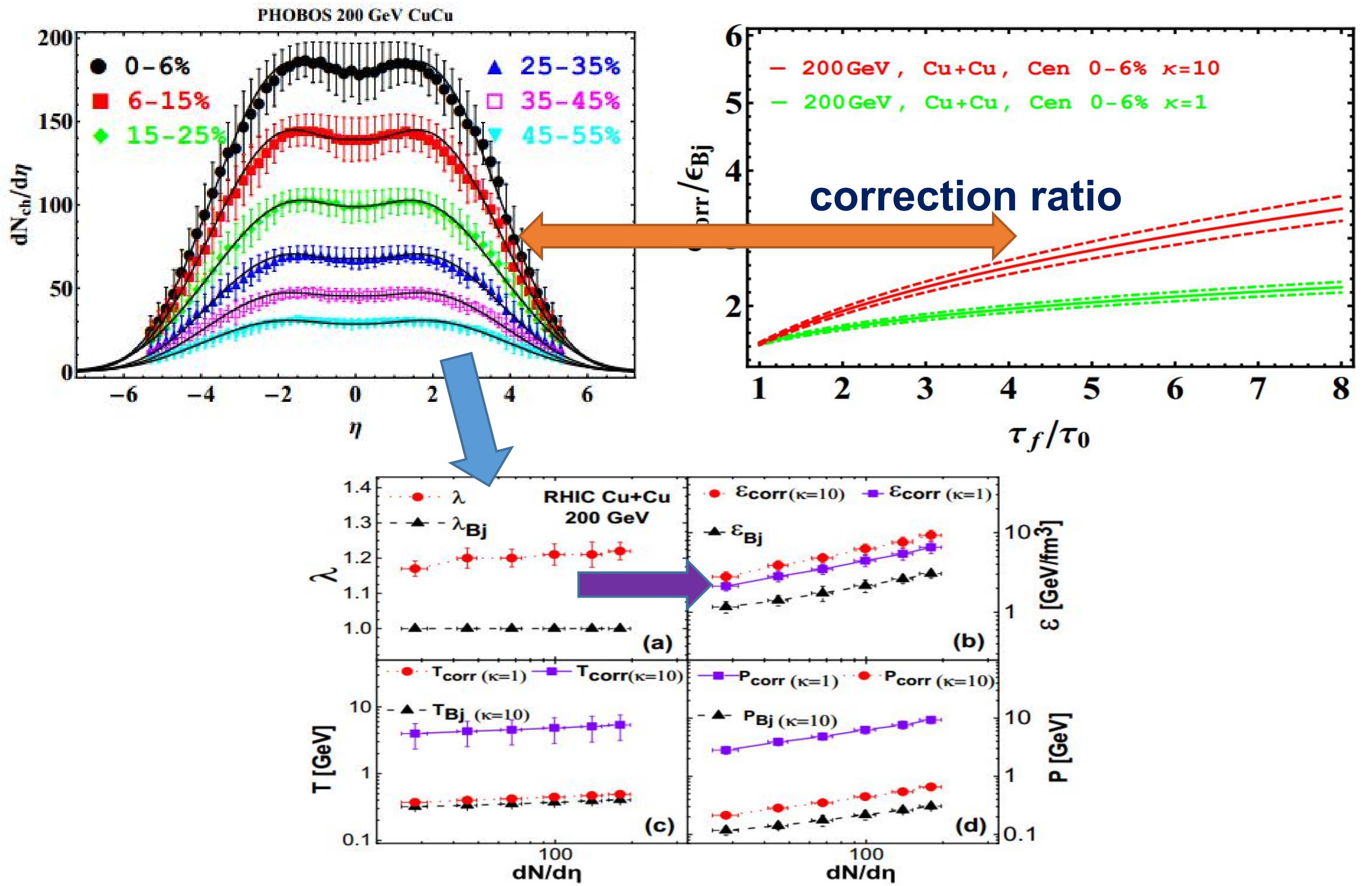




(See Z.F. Jiang, C.B. Yang, M. Csanád, T. Csörgő, Phys. Rev. C 97, 064906)

The initial energy density estimation

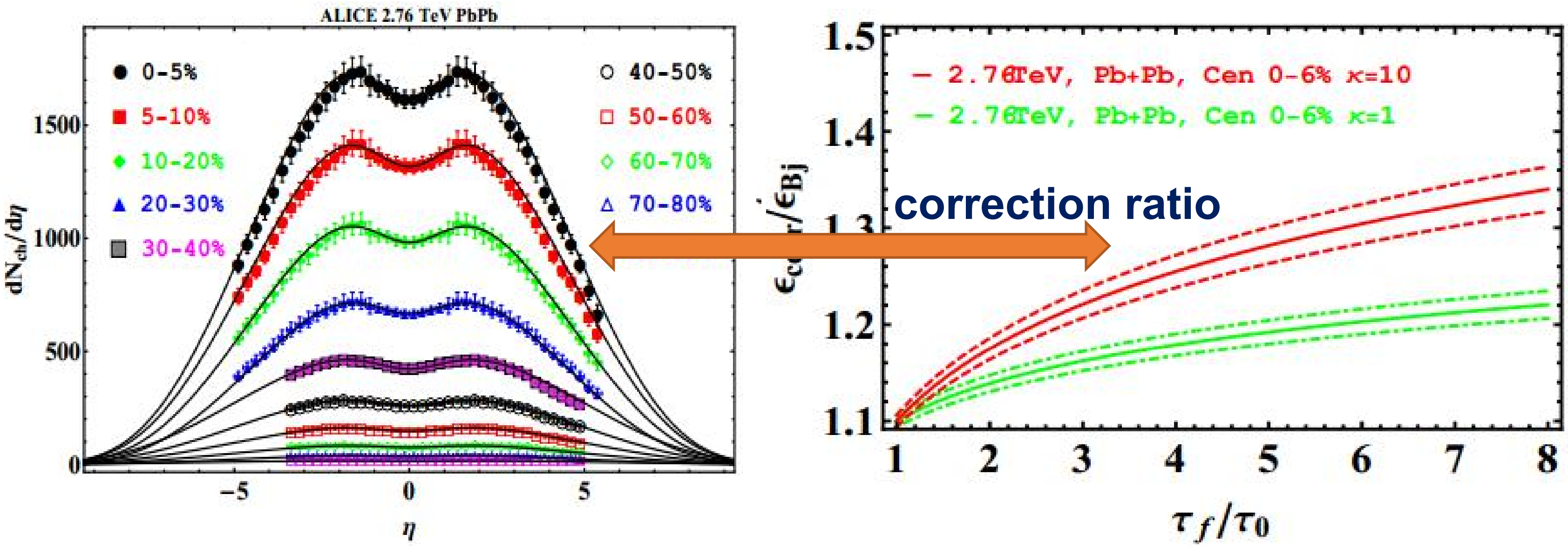
Results for 200 GeV Cu+Cu collisions



(See Z.F. Jiang, C.B. Yang, M. Csanád, T. Csörgő, Phys. Rev. C 97, 064906)

The initial energy density estimation

Results for 2.76 TeV Pb+Pb collisions



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The initial energy density estimation

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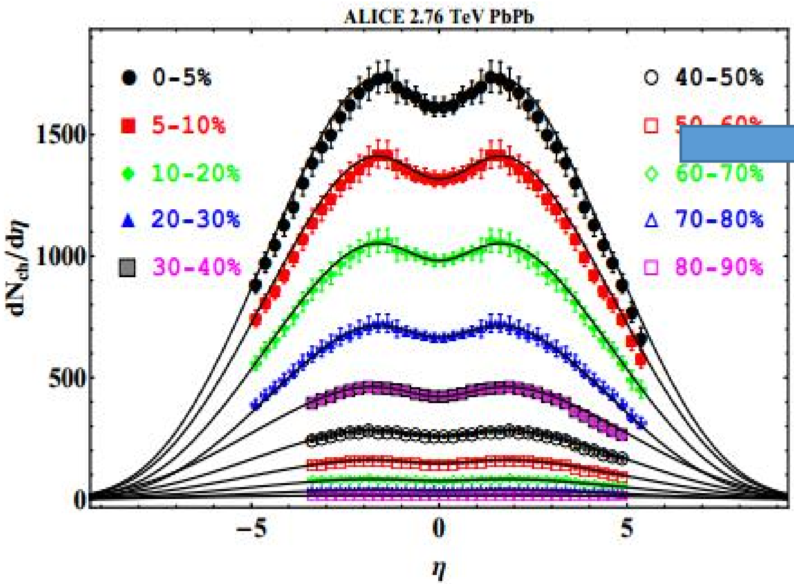
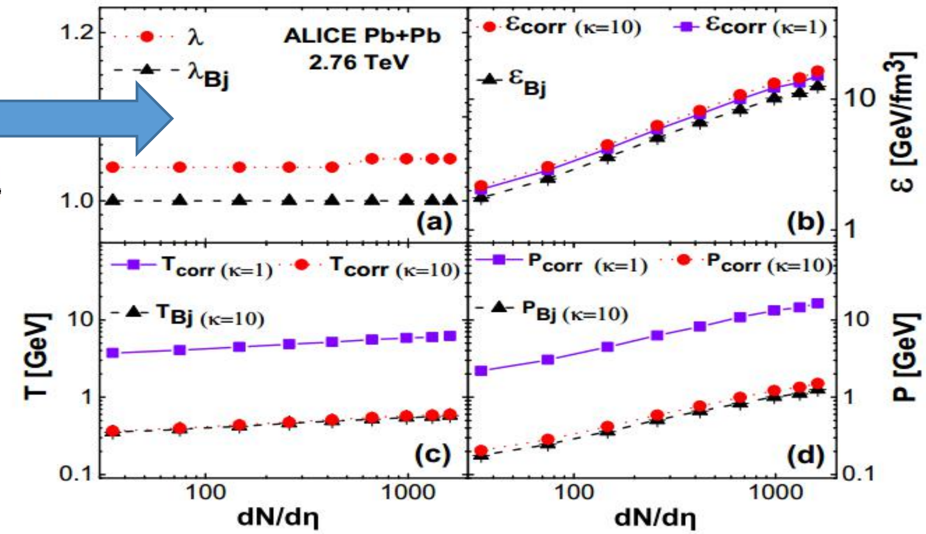




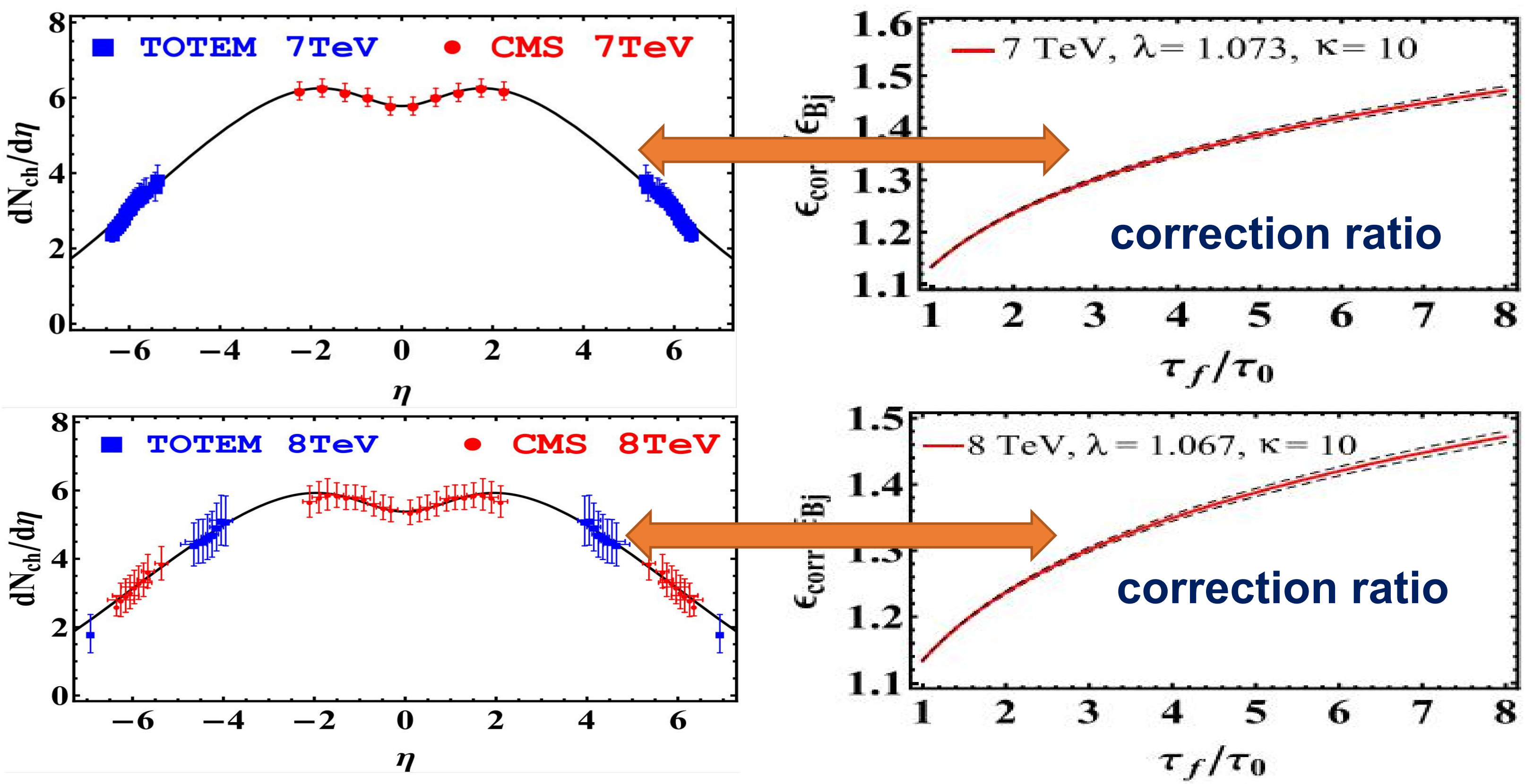
(See Z.F. Jiang, C.B. Yang, M. Csanád, T. Csörgő, Phys. Rev. C 97, 064906)

- Hydro can be used to estimate the initial ε_0 . (J. D. Bjorken, Phys. Rev. D27 (1983) 140-151. Gyulassy M, Matsui T. 1984. Phys. Rev. D29: 419)
- The larger the center of mass energy, the smaller the acceleration parameter λ. The bigger $\langle N_{part} \rangle$, the bigger λ.
- From the fits, the initial energy density, initial temperature and initial pressure are reconstructed.

New results for 13 TeV pp collision and 5.02 TeV PbPb collision, see arXiv:1806.05750.

The pseudorapidity distributions

Results for 7 TeV & 8 TeV p+p collisios:



(M. Csanád, T. Csörgő, Z.F. Jiang. C.B. Yang, Universe. vol 3(2017). pp 1-9.)

$\sqrt{S}$	ε_{Bj}	f_c	ε_{corr}	λ	c_s^2	$dN/d\eta\|_{\eta=0}$
7 TeV	0.507	1.262	0.640	1.073	0.10	5.895(NSD)
8 TeV	0.500	1.240	0.644	1.067	0.10	5.38(Inelastic)

The initial energy density estimation

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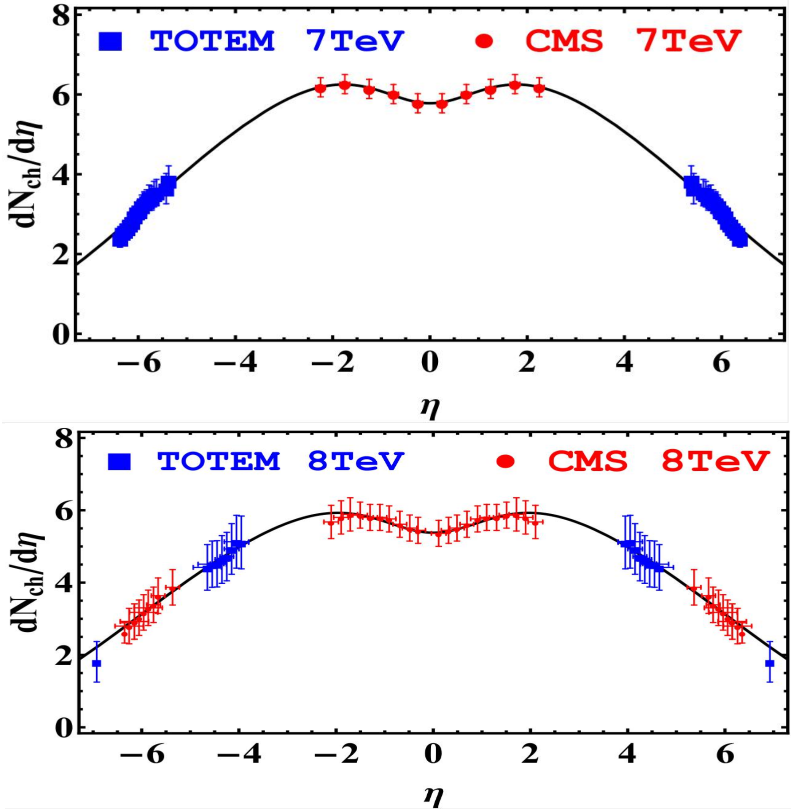

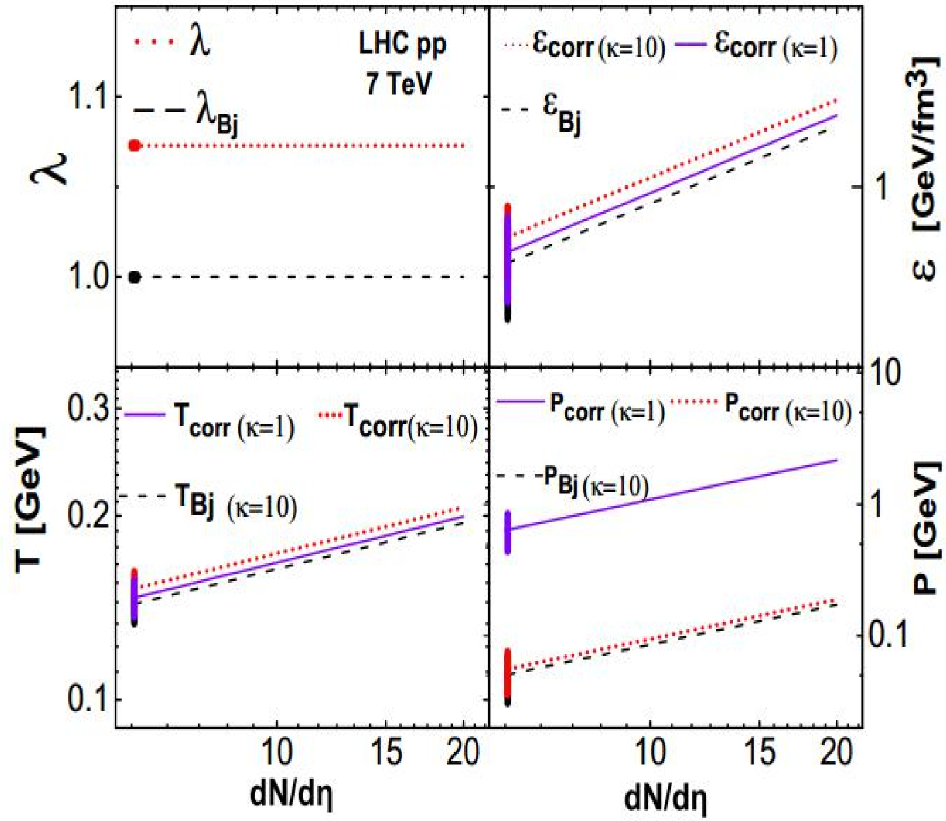

(M. Csanád, T. Csörgő, Z.F. Jiang. C.B. Yang, Universe. vol 3(2017). pp 1-9.)

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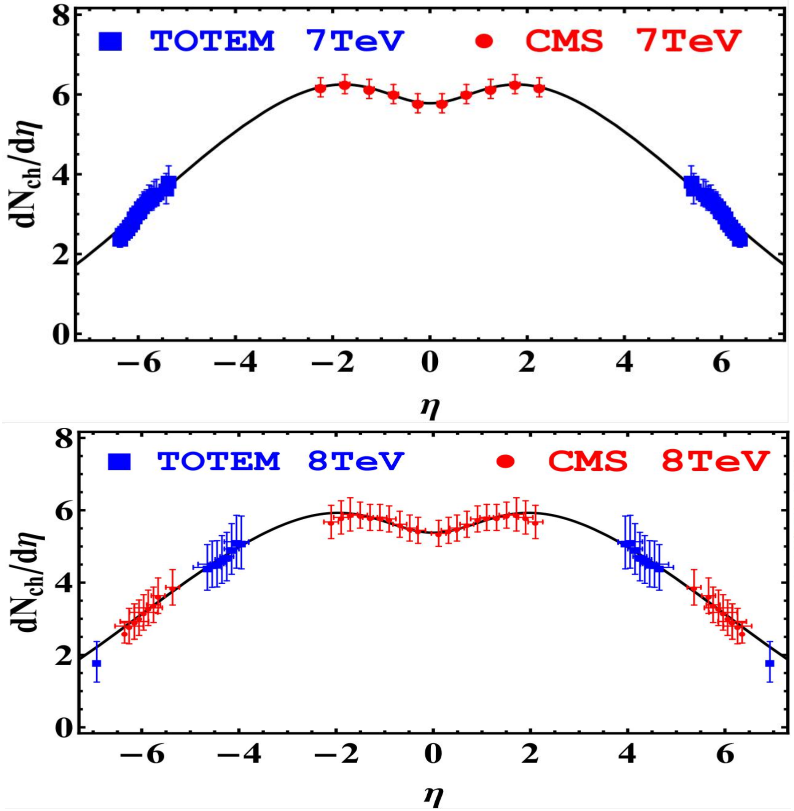

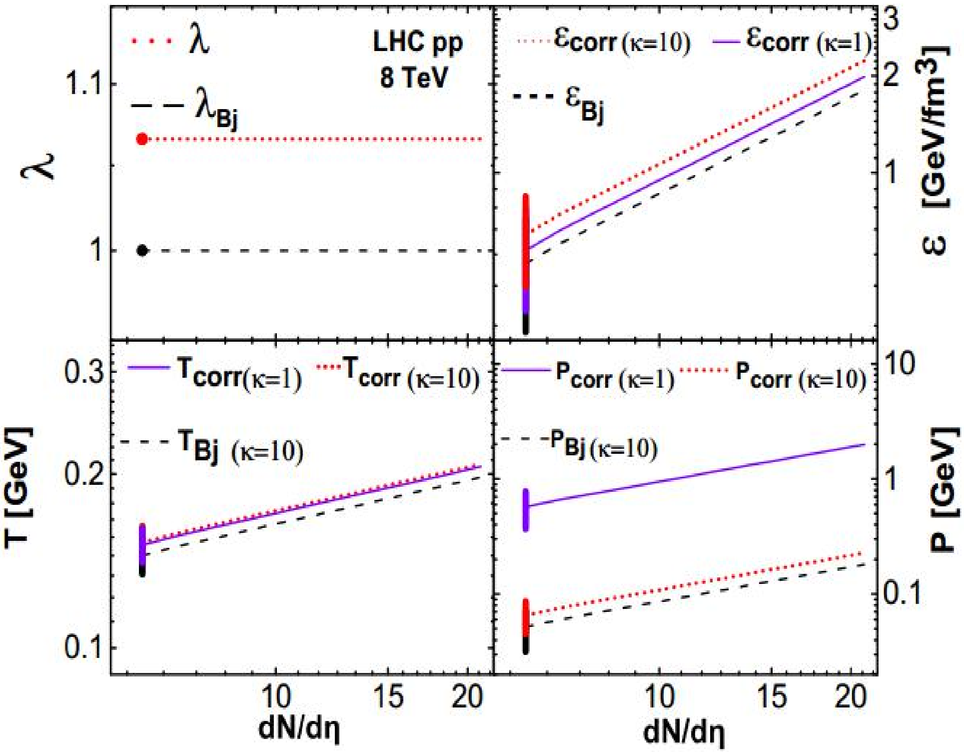

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The CKCJ solutions

The new CKCJ family of exact hydro solutions

$$
\begin{aligned}
\eta_x(H) &= \Omega(H) - H, \\
\Omega(H) &= \frac{\lambda}{\sqrt{\lambda-1}\sqrt{\kappa-\lambda}} \arctan\left(\sqrt{\frac{\kappa-\lambda}{\lambda-1}} \tanh(H)\right), \\
\sigma(\tau, H) &= \sigma_0 \left(\frac{\tau_0}{\tau}\right)^{\lambda} \mathcal{V}_\sigma(s) \left[1 + \frac{\kappa-1}{\lambda-1} \sinh^2(H)\right]^{-\frac{\lambda}{2}}, \\
T(\tau, H) &= T_0 \left(\frac{\tau_0}{\tau}\right)^{\frac{\lambda}{\kappa}} \mathcal{T}(s) \left[1 + \frac{\kappa-1}{\lambda-1} \sinh^2(H)\right]^{-\frac{\lambda}{2\kappa}}, \\
\mathcal{T}(s) &= \frac{1}{\mathcal{V}_\sigma(s)}, \\
s(\tau, H) &= \left(\frac{\tau_0}{\tau}\right)^{\lambda-1} \sinh(H) \left[1 + \frac{\kappa-1}{\lambda-1} \sinh^2(H)\right]^{-\lambda/2}.
\end{aligned}
$$

T. Csörgő, G. Kasza, M. Csanád, Z.F. Jiang, CKCJ solutions:
arXiv: 1805.01427 Universe. vol 4(2018). 69.

For more details, see:

HBT from the CKCJ solution - theoretical results for the lifetime determination.

The CKCJ solutions

Final state observables:

$$\frac{dn}{dy} \approx \left.\frac{dn}{dy}\right|_{y=0} \cosh^{-\frac{1}{2}\alpha(\kappa)-1}\left(\frac{y}{\alpha(1)}\right) \exp\left(-\frac{m}{T_f}\left[\cosh^{\alpha(\kappa)}\left(\frac{y}{\alpha(1)}\right)-1\right]\right),$$

$$\frac{dn}{d\eta_p} \approx \left.\frac{dn}{dy}\right|_{y=0} \frac{\langle p_T(y)\rangle \cosh(\eta_p)}{\sqrt{m^2+\langle p_T(y)\rangle^2 \cosh(\eta_p)}} \cosh^{-\frac{1}{2}\alpha(\kappa)-1}\left(\frac{y}{\alpha(1)}\right) \exp\left(-\frac{m}{T_f}\left[\cosh^{\alpha(\kappa)}\left(\frac{y}{\alpha(1)}\right)-1\right]\right),$$

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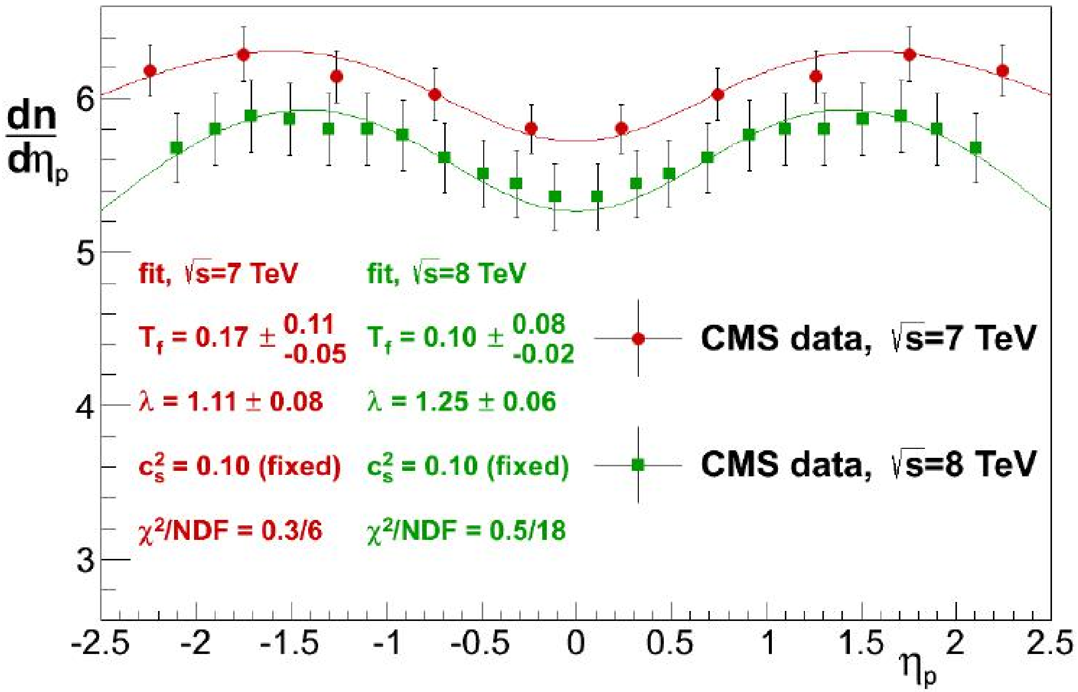
The CKCJ solutions

Results for 7 TeV & 8 TeV p+p collisios at CMS:



For a new method to estimate the initial energy density from the CKCJ solution.

See Fits of the CKCJ solutions to RHIC and LHC data (dn/deta, Rlong, initial energy density).