Third Asian School on Superconductivity & Cryogenics for Accelerators (ASSCA2018), IHEP, Beijing, China, December 10-16, 2018

Solutions of the Final Exam

1. (10 points) Introduction

a) (5 points) What should be the circumference of a proton circular accelerator of 1 TeV energy by using a SC magnet with a central field of 10 Tesla?

b) (5 points) Which SC material (NbTi or Nb_3Sn) and at what temperature we should use for the 10 Tesla magnet and why? Based on their value of T_c (10 K & 18 K) & H_{c2} at 0 K (15 T & 24 T).

Solution:

a) Use the equation $E_{beam} = 0.3B_{dipole}R$. We get R = 333.3 m. Circumference will be approximately 2.1 km. This is the length of dipole Magnet. Total length including focusing magnet, beam instrument can be double, i.e 4 km.

b) For NbTi at 4.2 K, H_{c2} (4.2 K) value will be 12 T and at 2 K it will be 14.4 T. To have an operation safety, it is wise to have a margin between design field and calculated value of H_{c2} (T), 2 K will be better choice for NbTi.

For Nb_3Sn, at 4.2 K, H_{c2} (4.2K) will be much higher than operation field. Hence Nb_3Sn can be used at 4.2 K.

2. (10 points) Superconducting Magnet

a) (3 points) What's the basic superconducting wire and cable applied for accelerator magnets?

b) (4 points) Why do we need cables? What are the problems in a cable and the characteristic of the magnet cable?

c) (3 points) What is the specialty of the cables for detector magnets?

Solution:

a) Accelerator magnets rely on standard NbTi and Nb3Sn multifilamentary conductor in a copper matrix. All superconducting accelerators to date have used magnets wound from Rutherford cable.

b) Clearly need conductors able to carry several kA and the best way of achieving this is to combine many wires in a cable.

Interfilament coupling is the main problem in a cable. When a cable is subjected to a time varying magnetic field, current loops are generated between filaments. If filaments are straight, large loops are generated with large currents. It will cause the big energy loss. If the strands are magnetically coupled, the effective filament size is larger. It will cause flux jumps.

To reduce these effects, filaments are twisted with a twist pitch of the order of 20-30 times of the wire diameter. But twisting this cable doesn't help if the inner wires are always inside and the outer outside. Wires must be fully transposed, i.e. every wire must change places with every other wire along the length of the cable so that, on the average, no flux is enclosed.

c) Detector magnets rely on the cryo-stabilized NbTi cable with a pure aluminum stabilizer. The pure aluminum stabilizing material is low electrical resistivity and high residual resistivity ratio (RRR).

3. (15 points) Criterion for flux jumping.

This problem deals with the derivation of the critical conductor size above which flux jumping will occur. Flux jumping was once a major source of instabilities in the first superconducting magnets of engineering significance in the early 1960s. Flux jumping is a thermal instability peculiar to a Type II superconductor that permits the magnetic field to penetrate its interior. A time-varying magnetic field, $\dot{H}_e$, at the conductor surface induces an electric field $\vec{E}$ in the conductor, which interacts with the supercurrent (density J_c). This $\vec{E}\cdot\vec{J}_c$ interaction heats the conductor. Since J_c decreases with temperature, the field (flux) penetrates further into the conductor, generating more heat, which further decreases J_c. The field penetration and temperature rise can cascade until the conductor loses its superconductivity. This thermal runaway event is called a flux jump.

a) (5 points) Using the Bean model and computing the $\vec{E}\cdot\vec{J_c}$ interaction over the positive half ($0 \le x \le a$) of the slab, show that an expression for the dissipative energy density, $e_\varphi [J/m^3]$, generated within the slab when the critical current density J_c is suddenly decreased by $|\Delta J_c|$ is given by:

$$e_\phi = \frac{\mu_\circ J_c |\Delta J_c| a^2}{3} \tag{1}$$

Note that the entire slab is in the critical state with its surface ($\pm a$) exposed to an external field of $H_e\vec{\iota}_y$.

b) (5 points) Derive the last equation by computing the Poynting energy flow into the slab at $x = a$ and equating it with the change in magnetic energy storage and dissipation energy ε_ϕ in the positive half of the slab.

c) (5 points) To relate ΔJ_c to an equivalent temperature rise in the conductor, we may assume a linear temperature dependence for $J_c(T)$:

$$J_c(T) = J_{c_\circ}\left(\frac{T_c - T}{T_c - T_{op}}\right) \tag{2}$$

where J_{c_0} is the critical current density at the operating temperature T_{op}. T_c is the critical temperature at a given magnetic induction B_o. From last equation (2), ΔJ_c in Eq.(1) may be related to an equivalent temperature rise

ΔT:

$$\Delta J_c = -J_{c_\circ}\left(\frac{\Delta T}{T_c - T_{op}}\right) \tag{3}$$

Now, by requiring that $\Delta T_s = e_\phi/\tilde{C}_s \leq \Delta T$, where $\tilde{C}_s$ is the superconductor's average heat capacity [J/m^{3}K] in the range from T_{op} to T_c, show thermal stability implies a critical slab half width a_c of:

$$a_c = \sqrt{\frac{3\tilde{C}_s(T_c - T_{op})}{\mu_\circ J_{c_\circ}^2}} \tag{4}$$

Solution:

a) Because of symmetry about $x = 0$, we shall consider only one half of the slab, between $x = 0$ and $x = a$. As illustrated in Fig. 5.18, the solid line corresponds to $H_{s1}(x)$, which gives the initial field distribution within the slab, with $J = J_c$. The dotted line corresponds to $H_{s2}(x)$ for the slab carrying $J_c - |\Delta J_c|$. Note that the field at the surface is H_e in both cases. We thus have:

$$H_{s1}(x) = H_e + J_c(x - a) \tag{S2.1a}$$

$$H_{s2}(x) = H_e + (J_c - |\Delta J_c|)(x - a) \tag{S2.1b}$$

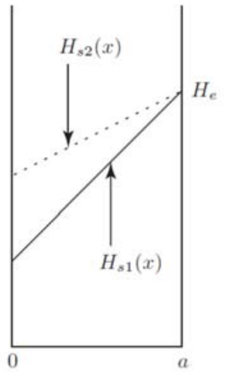


Fig. 5.18 Field profiles.

Because there is a change in magnetic field within the slab, an electric field $\vec{E}$ is generated, which from Faraday's law of induction is given by:

$$\oint_C \vec{E} \cdot d\vec{s} = -\mu_\circ \int_S \frac{\Delta H_s(x)\,\vec{\imath}_y \cdot d\vec{A}}{\Delta t} \tag{S2.2}$$

From symmetry we have $\vec{E}(x = 0) = 0$ and $\vec{E}$ points in the z-direction. $\Delta H_s(x)$ is given by:

$$\begin{aligned}\Delta H_s(x) &= H_{s2}(x) - H_{s1}(x)\\ &= |\Delta J_c|(a - x)\end{aligned} \tag{S2.3}$$

Combining Eqs. $S2.2$ and $S2.3$, we obtain:

$$\begin{aligned}E_z(x) &= \mu_\circ \frac{|\Delta J_c|}{\Delta t}\int_0^x (a - x)\,dx\\ &= \mu_\circ \frac{|\Delta J_c|}{\Delta t}\left(ax - \frac{x^2}{2}\right)\end{aligned} \tag{S2.4}$$

Dissipation power density, $p(x)$, is given by $E_z(x)J_c$; the total energy density per unit length dissipated in the slab or per unit slab surface area in the y-z plane, $\mathcal{E}_\phi$ [J/m^2], is given by:

$$\begin{aligned}\mathcal{E}_\phi &= \int_0^a p(x)\Delta t\,dx\\ &= \mu_\circ J_c|\Delta J_c|\int_0^a \left(ax - \frac{x^2}{2}\right)dx = \frac{\mu_\circ J_c|\Delta J_c|a^3}{3}\end{aligned} \tag{S2.5}$$

The average dissipation energy density, e_ϕ, is given by $\mathcal{E}_\phi/a$:

$$e_\phi = \frac{\mu_\circ J_c|\Delta J_c|a^2}{3} \tag{5.37}$$

b) The Poynting energy flux [J/m^2] in the y-z plane into the slab (in the $-x$-direction) at $x = a$ is equal to the change in magnetic energy storage flux ΔE_m [J/m^2] and dissipation energy flux $\mathcal{E}_\phi$ in the slab. Thus:

$$\int S_x(a)\,dt = \Delta E_m + \mathcal{E}_\phi \tag{S2.6}$$

We can verify the direction of $\vec{S}$ by computing $\vec{S} = \vec{E} \times \vec{H}$ at $x = a$. At $x = a$, $\vec{H} = H_e\,\vec{\imath}_y$; from $E_z(x)$ derived in Eq. $S2.4$:

$$E_z(a) = \mu_\circ \frac{|\Delta J_c| a^2}{2\Delta t} \tag{S2.7}$$

Thus:

$$\vec{S}(a) = \mu_\circ \frac{|\Delta J_c| a^2}{2\Delta t}\,\vec{\imath}_z \times H_e\,\vec{\imath}_y = -\mu_\circ \frac{H_e |\Delta J_c| a^2}{2\Delta t}\,\vec{\imath}_x \tag{S2.8}$$

As expected, $\vec{S}(a)$ points in the $-x$-direction; energy indeed flows into the slab. Thus:

$$\int S_x(a)\,dt = \mu_\circ \frac{H_e |\Delta J_c| a^2}{2} \tag{S2.9}$$

The difference in magnetic energy flux ΔE_m in the slab is given by:

$$\begin{aligned}
\Delta E_m &= \frac{\mu_\circ}{2} \int_0^a [H_{s2}^2(x) - H_{s1}^2(x)]\,dx \qquad (S2.10)\\
&= \frac{\mu_\circ}{2} \int_0^a \{[H_e + (J_c - |\Delta J_c|)(x-a)]^2 - [H_e + J_c(x-a)]^2\}\,dx \\
&= \frac{\mu_\circ}{2} \int_0^a [-2H_e|\Delta J_c|(x-a) - 2J_c|\Delta J_c|(x-a)^2 + |\Delta J_c|^2(x-a)^2]\,dx
\end{aligned}$$

Neglecting the $|\Delta J_c|^2$ term in the above integral, we obtain:

$$\Delta E_m = \mu_\circ \left(\frac{H_e |\Delta J_c| a^2}{2} - \frac{J_c |\Delta J_c| a^3}{3} \right) \tag{S2.11}$$

From Eq. $S2.6$, we have:

$$\mathcal{E}_\phi = \int S_x(a)\,dt - \Delta E_m \tag{S2.12}$$

Combining Eqs. $S2.9$, $S2.11$, and $S2.12$, we obtain:

$$\begin{aligned}
\mathcal{E}_\phi &= \mu_\circ \frac{H_e |\Delta J_c| a^2}{2} - \mu_\circ \left(\frac{H_e |\Delta J_c| a^2}{2} - \frac{J_c |\Delta J_c| a^3}{3} \right) \\
&= \mu_\circ \frac{J_c |\Delta J_c| a^3}{3} \qquad (S2.13)
\end{aligned}$$

Equation $S2.13$ leads directly to Eq. 5.37:

$$e_\phi = \frac{\mathcal{E}_\phi}{a} = \frac{\mu_\circ J_c |\Delta J_c| a^2}{3} \tag{5.37}$$

c) As given by Eq. 5.38, $J_c(T)$ is a decreasing function of temperature. We thus have:

$$\Delta J_c = -J_{c_\circ}\left(\frac{\Delta T}{T_c - T_{op}}\right) \tag{5.39}$$

From Eq. 5.39, we have:

$$|\Delta J_c| = \frac{J_{c_\circ}\Delta T}{T_c - T_{op}} \tag{S2.14}$$

Replacing J_c with $J_{c_\circ}$ in Eq. 5.37 and combining it with Eq. $S2.14$, we obtain:

$$e_\phi = \frac{\mu_\circ J_{c_\circ}^2 \Delta T a^2}{3(T_c - T_{op})} \tag{S2.15}$$

Note that e_ϕ is proportional not only to ΔT but also, more importantly, to a^2. Under adiabatic conditions, the dissipation energy density e_ϕ increases the superconductor's temperature by ΔT_s, given by:

$$\Delta T_s = \frac{e_\phi}{\tilde{C}_s} > 0 \tag{S2.16}$$

$\tilde{C}_s$ is the superconductor's average heat capacity [J/m^3 K] in the temperature range from T_{op} to T_c. Combining Eqs. $S2.15$ and $S2.16$, and requiring $\Delta T_s < \Delta T$ for thermal stability, we have:

$$\frac{\Delta T_s}{\Delta T} < \frac{\mu_\circ J_{c_\circ}^2 a^2}{3\tilde{C}_s(T_c - T_{op})} \tag{S2.17}$$

For a given superconducting material and operating temperature, a is the only parameter that can be varied by the magnet designer to satisfy Eq. $S2.17$. That is, thermal stability can be satisfied only if the slab half-width a is less than the critical size a_c, given by:

$$a_c = \sqrt{\frac{3\tilde{C}_s(T_c - T_{op})}{\mu_\circ J_{c_\circ}^2}} \tag{5.40}$$

Equation 5.40 is applied to compute approximate values of a_c for NbTi (LTS) operating at 4.2 K and YBCO (HTS) operating at 77.3 K. Table 5.3 lists approximate values of parameters appearing in Eq. 5.40 for both superconductors.

We may conclude that for a circular filament of NbTi, $a_c = 140\,\mu$m means a critical diameter of ~300 μm (Eq. 5.29) and a coated YBCO tape of width 8 mm.

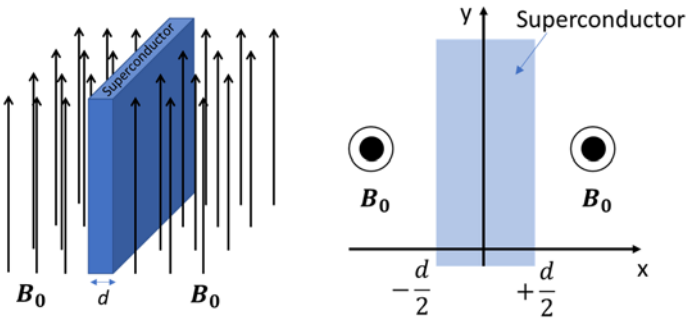
Table 5.3: Application of Eq. 5.40 to NbTi and YBCO

Superconductor	T_{op} [K]	T_c [K]	$J_{c_\circ}$ [A/m^2]	$\tilde{C}_s$ [J/m^3 K]	a_c [mm]
NbTi	4.2	9.8	2×10^9	6×10^3	0.14
YBCO	77.3	93	2×10^9	2×10^6	4

4. (10 points) RF Superconductivity.

Consider the system shown in the figure. A superconducting slab with thickness d is placed in the uniform magnetic field parallel to the slab. The field distribution can be written as

$$\boldsymbol{B} = \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ B(x) \end{pmatrix} \qquad \begin{matrix} B(-d/2) = B_0 \\ B(+d/2) = B_0 \end{matrix}$$



$B(x)$ at - $d/2 \le x \le d/2$ is obtained by solving the London equation. Fill the following blanc boxes [1]-[4]:

When the London penetration depth is given by λ, London equation is $\mathrm{d}^2 B/dx^2 = B/\lambda^2$. The solution can be written as

$$B(x) = C_1 \cosh\frac{x}{\lambda} + C_2 \sinh\frac{x}{\lambda}$$

where C_1 and C_2 are constants. By using the boundary conditions, we find

$$C_1 = [1] \quad \frac{B_0}{\cosh\frac{d}{2\lambda}}$$

and

$$C_2 = [2] \;\; 0$$

Thus $B(x)$ is given by

$$B(x) = B_0 \frac{[3] \;\; \cosh\frac{x}{\lambda}}{\cosh\frac{d}{2\lambda}}$$

The current distribution in the slab is obtained by using the Maxwell equation,

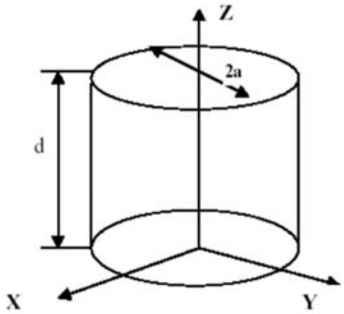
$\mathrm{rot}\,\boldsymbol{B} = \mu_0 \boldsymbol{j}$ or $\boldsymbol{j}(x) = \begin{pmatrix} 0 \\ j_y(x) \\ 0 \end{pmatrix}$. Thus

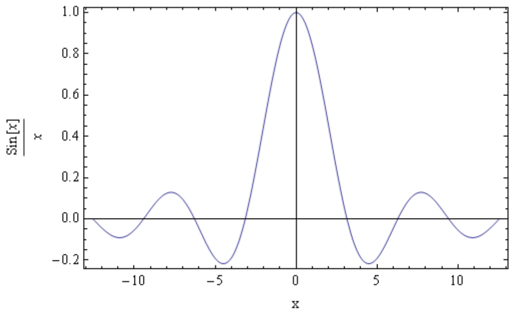
$$j_y(x) = -\frac{1}{\mu_0}\frac{dB}{dx} = [4] \quad -\frac{B_0}{\mu_0 \lambda}\frac{\sinh\frac{x}{\lambda}}{\cosh\frac{d}{2\lambda}}$$

Now consider the case the thickness d is much smaller than the penetration depth λ, namely, $\frac{d}{\lambda} \to 0$. In this case, we have $B(x) \to B_0$ and $j_y \to 0$. This is an example that engineering material geometry allows us to control the current that breaks superconductivity.

5. (15 points) Pillbox cavity at 2 K.

Beam is on axis (assuming field is not distorted by the beam ports). 10 mA proton of β=0.75 is accelerated by TM010 mode of this cavity, and the beam gains energy of 1.0 MeV with synchronized phase 0 deg. Given: a = 382.77 mm, d = 500.00 mm.





Calculate:

a) (2 points) Frequency of the TM010 mode.

b) (2 points) Transient time factor as function of β, i.e. TTF(β).

c) (2 points) Optimal beta β_0.

d) (3 points) Given R/Q = 196 Ω for particle at β=1, what is the R/Q for particle at β=0.75?

e) (3 points) Matched Q_{in}.

f) (3 points) If the microphonics introduces a frequency deviation of 125 Hz, what is the minimum power that the power source has to provide?

Solution:

a) $f_{TM010}=\frac{2.405\times c}{2\pi\times a}=\frac{2.405\times 2.9979e8[\frac{m}{s}]}{2\pi\times 0.38277[m]}$=299.8 MHz

b) $$TTF(\beta)=\frac{\int_{-d/2}^{d/2} E0\cdot\cos\left(\frac{\omega z}{\beta c}\right)\cdot dz}{\int_{-d/2}^{d/2} E0\cdot dz}=\frac{\sin\left(\frac{\omega d}{2\beta c}\right)}{\frac{\omega d}{2\beta c}}=\frac{\sin\left(\frac{\pi f d}{\beta c}\right)}{\frac{\pi f d}{\beta c}}=\frac{\sin\left(\frac{\pi\times 2.998e8[Hz]\times 0.5[m]}{\beta\times 2.998e8[\frac{m}{s}]}\right)}{\frac{\pi\times 2.998e8[Hz]\times 0.5[m]}{\beta\times 2.998e8[\frac{m}{s}]}}=$$

$$\frac{\sin\left(\frac{\pi}{2\beta}\right)}{\frac{\pi}{2\beta}}$$

c) Noticing sin(x)/x has maximum at x=0, and it has first zero at x=π, so as long as β>0.5, the TTF(β) will increase as β increases. So β_0=1.

d) $$R/_{Q}(\beta)=\frac{V_{acc}^2(\beta)}{\omega U}=\frac{[\int_{-d/2}^{d/2} E0\cdot\cos\left(\frac{\omega z}{\beta c}\right)\cdot dz]^2}{\omega U}=\frac{\left[\int_{-d/2}^{d/2} E0\cdot dz\right]^2}{\omega U}TTF^2(\beta)$$

$$R/_{Q}(0.75)=R/_{Q}(1)\times\left(\frac{TTF(0.75)}{TTF(1)}\right)^2=196[\Omega]\times\left(\frac{\sin\left(\frac{2\pi}{3}\right)}{\frac{2\pi}{3}}\cdot\frac{\frac{\pi}{2}}{\sin\left(\frac{\pi}{2}\right)}\right)^2$$

$$=82.69\ \ \Omega$$

g) $$Q_{in}=\frac{V_{acc}}{R/Q\times I_{beam}}=\frac{1e6\ V}{82.69\ [\Omega]\ \times\ 1e{-2}\ A}=1.209e6$$

h) $P_f = \frac{V_{acc}^2}{4\times\frac{R}{Q}\times Q_{in}}\left[\left(1+\frac{Q_{in}}{Q_0}+\frac{Q_{in}\times I_{beam}\times\frac{R}{Q}}{V_{acc}}\right)^2+\left(2\frac{df}{f_0}Q_{in}\right)^2\right] =$

$\frac{(1e6[V])^2}{4\times 82.69[\Omega]\times 1.209e6}\left[\left(1+\frac{1.209e6}{1e10}+\frac{1.209e6\times 0.01[A]\times 82.69[\Omega]}{1e6[V]}\right)^2+\right.$

$\left.\left(2\frac{125[Hz]}{2.998e8[Hz]}1.209e6\right)^2\right] = 12.5kW$

6. (10 points) Show the method to measure the accelerating gradient of superconducting cavity at 2 K in a cryomodule. Prove the relation $P_t Q_t = 4 P_g Q_L$.

Solution:

1) Connect NA Port 1 to the Input Coupler, Port 2 to the Pickup, HOM1 or HOM2; terminate the other ports with 50 Ω matched loads
2) Measure Q_L (i.e. external Q of the input coupler at 2K)
3) Measure s_{21}(dB)
4) Calculate Q_t (external Q of the Pickup, HOM1, HOM2):

Q_0 is much higher than Q_e, so Q_L~Q_e. Because of strong over coupling, the cavity emitted voltage is twice of the generator forward voltage, thus the cavity emitted power P_e is four times of the generator forward P_g.

$$P_t \cdot Q_t = P_e \cdot Q_e = 4P_g \cdot Q_e = 4P_g \cdot Q_L$$

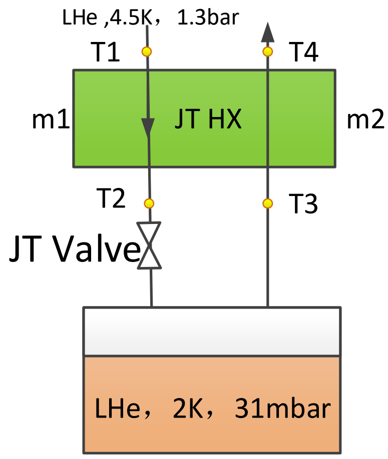
$$P_t \cdot Q_t = 4P_g \cdot Q_L$$

$$Q_t = 4Q_L / \frac{10^{\frac{S_{21}}{10}}}{Cable\ \ Loss}$$

$$E_{acc} = \frac{\sqrt{\frac{R}{Q}Q_t P_t}}{L}$$

7. (15 points) In ADS 2 K cryogenic system, the heat exchanger and JT valve are used to achieve 2 K saturated helium. The 4.5 K saturated liquid helium is subcooled by the 2 K return gas helium with JT heat exchanger. Then the subcooled liquid helium is Isenthalpic Expand by the JT valve. T1=4.5 K, T2=2.37 K, T3=2 K. The enthalpy for different temperature is as follows: H1=11.64 J/g, H2=4.31 J/g, H3= 25.04 J/g. The latent heat of 4.5 K saturated helium is 18.75 J/g. The latent heat of 2K saturated helium is 23.05 J/g. The mass flow of 2 K gas helium is equal to the mass flow of 4.5K. m1=m2=5 g/s.

a) (5 points) Calculate the enthalpy value of T4.
b) (5 points) Calculate the mass flow of 2 K liquid helium after the JT valve. (The enthalpy for 2 K liquid helium is 1.64 J/g, the enthalpy for 2 K gas helium is 25.04 J/g)
c) (5 points) The gas helium doesn't return to the refrigerator. In ADS cryogenic system, 1 g/s 4.5 K liquid helium corresponds to the refrigeration capacity of 100 W. Calculate the refrigeration capacity at 2 K.



Solution:

(1) Q1=Q2
m1*(H4-H3)=m2*(H1-H2)
m1=m2
H4=(H1-H2)+H3=(11.64-4.31)+25.04=32.37 J/g
(2) H2=H2'
The percentage of 2K liquid helium x:
x*H3'+(1-x)*H3=H2
x*1.64+(1-x)*25.04=4.31
x=0.886
m=0.886*5=4.43 g/s.
(3) We can only use the latent heat. 1 g/s 4.5 K liquid helium can get 0.886 g/s 2 K liquid helium.
The refrigeration capacity at 2 K: Q = 0.886*23.04 = 20.4 W.

8. (15 points) In BEPCII cryogenic system, the superconducting cavity was immersed in 4.4 K, 1.2 bar saturated helium. The latent heat of helium is used to cool the cryomodule. (ρ = 121.1 kg/m^3, latent heat = 19.49 kJ/kg)
a) (5 points) The liquid helium supply valve was closed. With zero cavity voltage, the helium level was decreased from 97 % to 86 % with 27 minutes. Please calculate the static heat load.
b) (10 points) The liquid helium supply valve was closed. With cavity voltage 1.5 MV, the liquid helium volume was decreased from 97 % to 86 % with 21 minutes. Calculate cavity Q_0. (R/Q = 95.3 Ω)

Helium Level (%)	Volume (L)
97	264.69
94	256.91
91	248.68
87	237.25
86	234.35

Solution:

(1) $Ps = \frac{\rho \Delta V}{t} \times H = \frac{121.1 \times (264.7 - 234.3)}{27 \times 60} \times 19.49 = 44.3$ W

(2) $Pt = \frac{\rho \Delta V}{t} \times H = \frac{121.1 \times (264.7 - 234.3)}{21 \times 60} \times 19.49 = 56.9$ W

Pt = Ps + Pc

Pc = 56.9 - 44.3 = 12.6 W

$$Q0 = \frac{Vc^2}{(R/Q) * Pc} = \frac{(1.5 * 10E6)^2}{95.3 * 12.6} = 1.2E9$$