An Angular Distribution Cookbook

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1 Introduction

This paper will compare and contrast two methods of parameterizing angular distributions — the helicity formalism and the partial wave formalism. It will then describe how to express the helicity amplitudes in terms of partial wave amplitudes and vice versa. Several cookbook formulae will then be derived for angular distributions which are frequently encountered in CLEO II analyses. Finally, a some examples will be worked out. One of these examples, section 17.8, provides a clear illustration of how CP violation can be observed in angular distributions.

The focus of the paper will be on two things: clear definitions of the minimum of ideas needed to get results and careful exposition of several subtleties, most of them elementary, which have confused the author in the past. The paper will also answer the perennial question: "What happened to L and to conservation of J_Z in the helicity formalism". The interested reader can find a much more complete and much more formal discussion in any of the standard references on the subject[1, 2, 3]. Of these, the paper by Richman [3] has the most pedagogical approach. As this paper neared completion another review of the subject has also become available [11]; that paper includes many details about how to deal with polarized particles.

2 Rotation of Systems with Angular Momentum

Consider the system shown in figure 1. A system with total angular momentum J is at rest and has a definite third component of J, m, along the z axis. Now perform an active rotatation of the system by the Euler angles α, β, γ so that the system now has the same value of m along the z' axis. The operator which performs this rotation is denoted $R(\alpha, \beta, \gamma)$. Following the rotation one can describe the system in either the rotated basis (quantization axis is z') or the original basis (quantization axis is z). The rotation operator changes a basis state in the original basis, $|jm\rangle$, into a linear combination of basis states in the new basis, $|jm'\rangle$,

$$R(\alpha, \beta, \gamma)|jm\rangle \quad = \quad \sum_{m=-j}^{j} |jm'\rangle \, D^j_{m'm}(\alpha, \beta, \gamma). \qquad (1)$$

An explicit representation of these D functions is,

$$D^j_{m'm}(\alpha,\beta,\gamma) \quad = \quad e^{-i\alpha m'} d^j_{m'm}(\beta) e^{-i\gamma m}, \tag{2}$$

where the functions $d^j_{m'm}(\beta)$ are real and are tabulated in many places. One such place is in the Particle Data Book on the page which also tabulates the Clebsch Gordon coefficients and the spherical harmonic functions. In order to use that table of the d functions one needs the identity,

$$d^j_{m'm}(\beta) = (-1)^{m-m'} d^j_{m\,m'}(\beta) = d^j_{-m\,-m'}(\beta). \tag{3}$$

The domain of the angles is $[0,\pi]$ for β but $[0,2\pi]$ for α and γ. Another important property of the D functions is the orthogonality relation,

$$\int D^{j\,*}_{mn}(\alpha\beta\gamma) D^{j'}_{m'n'}(\alpha\beta\gamma) d\alpha\, d\cos\beta\, d\gamma \quad = \quad \frac{8\pi^2}{2j+1}\delta_{jj'}\delta_{mm'}\delta_{nn'}. \tag{4}$$

Also, some of the D functions can be expressed in terms of the familiar spherical harmonic functions,

$$Y^m_\ell(\theta,\phi) \quad = \quad \sqrt{\frac{2\ell+1}{4\pi}} D^{\ell\,*}_{m\,0}(\phi,\theta,\gamma). \tag{5}$$

Finally, the rotation matrices are unitary, $R^{-1} = R^\dagger$.

3 The Parital Wave Basis

Consider the decay $1 \to 2\ 3$. The particles have spins (s_1, s_2, s_3) and spin components (m_1, m_2, m_3) along some quantization axis. For the purposes of this paper, the quantization axis may be chosen arbitrarily and it will be chosen as the direction of particle 1 in the lab frame. This direction is not changed by a boost from the lab frame to the rest frame of particle 1. So, if particle 1 has helicity λ_1 in the lab frame, it will have a spin component $m_1 = \lambda_1$ along this axis in its own rest frame. In order to discuss the angular distributions in the decay of particle 1 it is necessary to construct a coordinate system. This system is defined in the rest frame of particle 1 and has its z axis along the spin quantization axis of particle 1. For the time being it will not be necessary to carefully define the x and y axes, except to say that $\hat{x}, \hat{y}, \hat{z}$ form a righthanded orthonormal basis.

Consider an initial state in which particle 1 is at rest with some definite value of m_1. Following the decay $1 \to 2\ 3$, one can measure the direction of particle 2 and the spin components of particles 2 and 3 along the z axis, m_2 and m_3. When all three particles are long lived, known as the narrow resonance approximation, this fully specifies the final state. When the some of the particles are short lived then $p_2 = |\vec{p}_2|$, measured in the rest frame of 1, is also needed to completely specify the final state. As is usual for spin components, m_2 and m_3 are defined, respectively, in the rest frames of particles 2 and 3. The direction of the momentum of particle 2 is

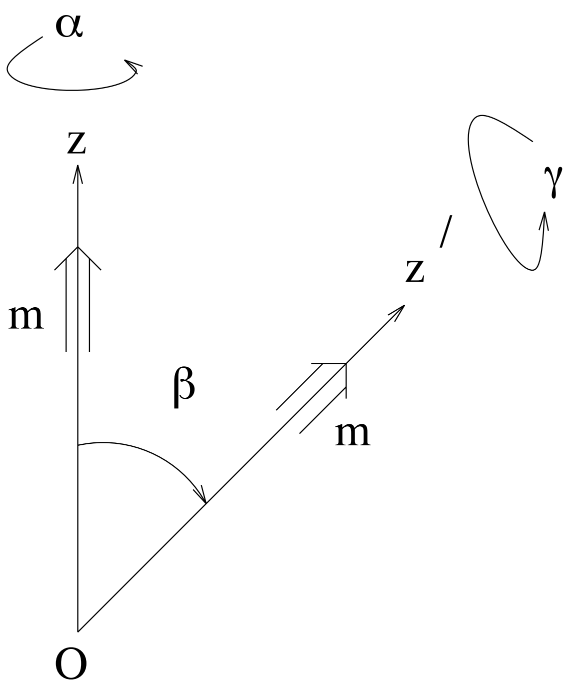

Figure 1: Rotation of a particle with Spin.

specified in the rest frame of particle 1 by the angles, $\Omega = (\theta, \phi)$. The orbital angular momentum of the final state particles, L, can be any of the values allowed by the selection rule,

$$\vec{s_1} \;=\; \vec{L} + \vec{s_2} + \vec{s_3}. \tag{6}$$

If parity is conserved in the decay there is a second selection rule,

$$\eta_1 \;=\; \eta_2\eta_3(-1)^L, \tag{7}$$

where η_i is the intrinsic parity of particle i. This decay is shown schematically in parts a) and b) of figure 2. Part a) of the figure shows the z axis and the initial state with definite m_1. Part b) shows the final state in the partial wave basis; particle 2 travels along the direction (θ, ϕ) and the three angular momenta, s_1, s_2, and L, are quantized along the z axis. Conservation of angular momentum tells us that $m_1 = m_2 + m_3 + m_L$.

In the general case, the decay may proceed through several different partial waves and the amplitude is given by any text on elementary quantum mechanics as,

$$A_{PW}(\Omega; m_1, m_2, m_3) = \sum_{L,\, s} C_{s_2 s_3}(s\ m_s; m_2\ m_3)\ C_{Ls}(s_1 m_1; m_L m_s)\ Y_L^{m_L}(\Omega)\ M(L, s). \tag{8}$$

In the above, the quantities $M(L, s)$ are known as the partial wave amplitudes and the quantities $C_{j_1 j_2}(jm; m_1 m_2)$ are Clebsch-Gordon coefficients. These Clebsch Gordon

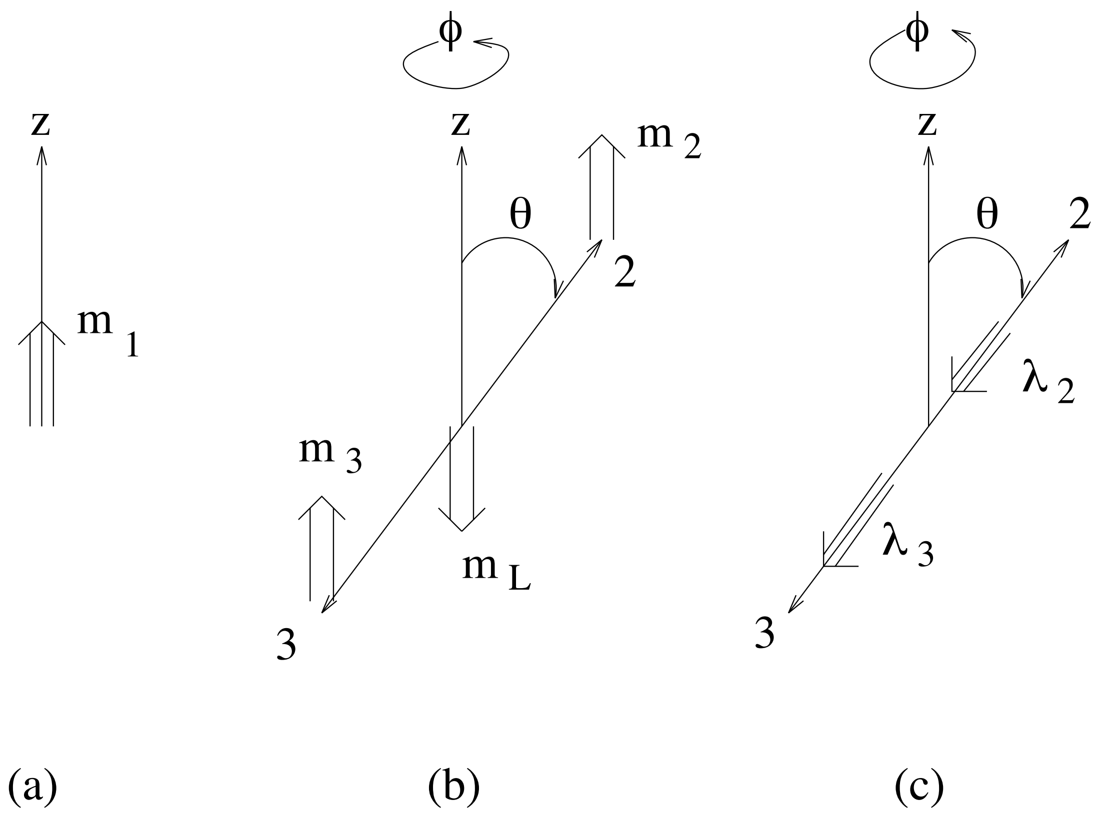

Figure 2: Schematic diagram of the decay $1 \to 2\ 3$. Unless otherwise stated all quantities are defined in the rest frame of 1; a) initial state; b) final state in the partial wave basis. Here m_2 and m_3 are defined in their respective rest frames; c) final state in the helicity basis.

coefficients enforce the conservation of the third component of angular momentum, $m_s = m_2 + m_3$ and $m_L = m_1 - m_2 - m_3$. When all of the particles are long lived the partial wave amplitudes are simply constants but, when some of the particles are broad resonances, they are functions of p_2. Often this momentum dependence is adequately approximated as being p_2^L. The angular distribution is given by,

$$\begin{aligned} \frac{d\Gamma}{d\Omega_{m_1,m_2,m_3}} &= |A_{PW}(\Omega; m_1, m_2, m_3)|^2 \\ &= \sum_{L,L',s,s'} \Big\{ C_{s_2 s_3}(s\ m_s; m_2\ m_3)\ C_{s_2 s_3}(s'\ m_{s'}; m_2\ m_3) \\ &\qquad C_{Ls}(s_1 m_1; m_L m_s)\ C_{L's'}(s_1 m_1; m_{L'} m_{s'}) \\ &\qquad Y_L^{m_L}(\Omega)\ M(L,s)\ Y_{L'}^{m_{L'}*}(\Omega)\ M^*(L',s') \Big\} \end{aligned} \tag{9}$$

Here $m_L = m_{L'} = m_1 - m_2 - m_3$, $m_s = m_{s'} = m_1 + m_2$ and the subscripts on the left hand side denote quantities which are fixed. Notice that the sum over different partial waves is coherent; that is, there is no measurement one can make on the final state particles which can be used to determine, on an event by event basis, through which partial wave the decay went. Also, because $m_L = m_{L'}$, this angular distribution has no ϕ dependence.

For most decays, the partial wave amplitudes are unknown quantities which can be measured. In a wavefunction model of decays, they are overlap integrals of momentum space radial wave functions. In some cases, such as leptonic τ decays, they can be calculated from the Electroweak lagrangian.

Finally, it is important to reiterate one important bit of physics which might have been lost in the above construction: the angle θ is defined with respect to the the spin quantization axis of particle 1, not with respect to some arbitrary "z axis". In up coming sections the physical meaning of ϕ will become clear.

4 The Helicity Basis

Now consider the same initial state as discussed above, particle 1 at rest with a definite value of m_1. However, in this case, the measurements made on the final state particles will be the direction of particle 2, again specified by Ω, and the two helicities, λ_2 and λ_3. These dynamical variables also fully specify the final state but they specify a final state which is distinct from that described in the previous section. All quantities in this basis are defined in the rest frame of 1, whereas, in the partial wave basis, the quantities m_2 and m_3 were defined, respectively, in the rest frames of particles 2 and 3. Figure 2 shows, schematically, the difference between the helicity basis and the partial wave basis. Parts a) and b) of the figure were discussed above. Part c) shows the final state in the helicity basis; particle 2 makes the same angles as before, (θ, ϕ), with the z axis but the quantization axis for the angular momenta has changed. It is now the direction of particle 2. (Actually particle 3 is quantized along its own direction of motion but this is not really a new quantization axis, just an additional sign convention.) The orbital angular momentum, $\vec{L} = \vec{r} \times \vec{p}$, must be perpendicular to $\vec{p_2}$ and, therefore, its component along the quantization axis is zero. Therefore the final state has a component of angular momentum along the direction of particle 2 of $\lambda_2 - \lambda_3$. Because the decay conserves angular momentum, the final state has a total angular momentum of s_1. Again the physics is that the angle θ is defined with respect to the spin quantization axis of particle 1.

Following reference [3], the amplitude, in the helicity basis, for the above process is,

$$A_H(\Omega; m_1 \lambda_2 \lambda_3) \;\; = \;\; \sqrt{\frac{2s_1+1}{4\pi}} D^{s_1\;*}_{m_1\, \lambda_2 - \lambda_3}(\phi, \theta, -\phi)\; A_{\lambda_2 \lambda_3}. \tag{10}$$

Here the quantity $A_{\lambda_2 \lambda_3}$, known as the helicity amplitude, describes the strength of this configuration compared to other configurations of helicities. The choice of $\gamma = -\phi$ is conventional and follows the use of [1] and [3]. In the following the argument of the D function will be written simply as Ω and should be understood as a shorthand for $(\phi, \theta, -\phi)$.

As the notation implies, there are $(2s_2+1)(2s_3+1)$ helicity amplitudes but not all are independent. Conservation of angular momentum requires,

$$|\lambda_2 - \lambda_3| \le s_1. \tag{11}$$

If parity is conserved in the decay,

$$A_{-\lambda_2 -\lambda_3} = \eta_1\eta_2\eta_3(-1)^{s_2+s_3-s_1} A_{\lambda_2\lambda_3}. \tag{12}$$

In the end, there are exactly as many independent helicity amplitudes as there are independent partial wave amplitudes. As will be shown in section 7, helicity amplitudes and partial wave amplitudes are simply linear combinations of each other. When some of the particles are broad resonances, the helicity amplitudes will depend on p_2. Because the partial wave amplitudes can often be approximated as being proportional to p_2^L, the helicity amplitudes can often be approximated as a power series in p_2.

As in the partial wave basis, the angular distribution is found by squaring the amplitude,

$$\begin{aligned}\frac{d\Gamma}{d\Omega}_{m_1\lambda_2\lambda_3} &= \frac{2s_1+1}{4\pi}\left|D^{s_1}_{m_1 \lambda_2-\lambda_3}(\phi,\theta,-\phi)\right|^2 \, |A_{\lambda_2\lambda_3}|^2 . \\ &= \frac{2s_1+1}{4\pi}|d^{s_1}_{m_1 \lambda_2-\lambda_3}(\theta)|^2 \, |A_{\lambda_2\lambda_3}|^2.\end{aligned} \tag{13}$$

This is a much simpler form than is equation 9; in particular there are no coherent sums. Also, there are no free parameters needed to describe the shape of this distribution. Equation 9, on the other hand, depends on many parameters, the relative magnitudes and phases of the partial wave amplitudes. An interesting consequence of this is shown in example 17.4. Finally, equation 13, like equation 9 has no ϕ dependence.

Now, integrating equation 13 over Ω gives,

$$\Gamma_{m_1\lambda_2\lambda_3} = |A_{\lambda_2\lambda_3}|^2. \tag{14}$$

This makes clear the meaning of the helicity amplitude: its magnitude squared is simply the decay rate into a final state with the specified helicities.

As mentioned earlier, the two final states shown in figure 2 are different and will not, in general, have the same angular distribution. However, when one sums over final state spins, the two final states are identical and they must, therefore, have the same angular distribution,

$$\begin{aligned}\frac{d\Gamma}{d\Omega}_{m_1} &= \sum_{\lambda_2\lambda_3} \frac{d\Gamma}{d\Omega}_{m_1\lambda_2\lambda_3} \\ &= \sum_{m_2 m_3} \frac{d\Gamma}{d\Omega}_{m_1 m_2 m_3}\end{aligned} \tag{15}$$

And, the total decay rate must also be the same in both bases,

$$\Gamma = \sum_{\lambda_2\lambda_3} |A_{\lambda_2\lambda_3}|^2 = \sum_{Ls} |M(L,s)|^2. \tag{16}$$

These expressions do not depend on m_1, which is expected because the decay rate of a free particle should not depend on which way its spin vector is pointing. To simplfy the notation, the angular distributions will be denoted,

$$I(\Omega; m_1) = \frac{1}{\Gamma}\frac{d\Gamma}{d\Omega_{m_1}}. \tag{17}$$

5 What happened to Conservation of J_z?

Now, how does conservation of angular momentum enter into this? Consider the specific example of a W^- boson, at rest, which has $m = +1$ along the z axis. Let it decay to $e^-\overline{\nu}_e$ and let the angles (θ, ϕ) be defined by the electron direction. In figure 2 c), this corresponds to $1 = W^-$, $2 = e^-$ and $3 = \overline{\nu}_e$. In the limit that the mass of the electron can be ignored, the final state will have a left handed electron and a right handed anti-neutrino. The component of orbital angular momentum along the electron direction is 0. This configuration is an eigenstate of $J = J_W = 1$ with a component $m = -1$ along the the electron direction. When the electron is emitted along the z axis, the quantization axes for the initial and final states are identical and one can easily check conservation of angular momentum. The initial state has $m = 1$ and the final state has $m = -1$ so the amplitude must vanish. When the electron is emitted along the $-z$ axis, the final state has $m = 1$ along the $+z$ axis so the amplitude may be non zero. Moreover the overlap between the initial and final state m is unity when the electron is emitted along the $-z$ axis. Since unity is the largest value for such an overlap, the most probable angle at which to emit the electron must be along the $-z$ direction.

When the electron is emitted at a general angle, (θ, ϕ), one must decompose the final state in a basis in which spins are quantized along the original z direction. This is done by using the inverse of equation 1. The amount of $m = 1$ basis state in this decomposition is just, $D^{J_W\,*}_{m\,\lambda_e-\lambda_{\overline{\nu}_e}}(\phi, \theta, -\phi) = D^{1\,*}_{1\,-1}(\phi, \theta, -\phi)$. For simplicity, consider the case that $\phi = 0$ for which the amount of $m = 1$ basis state is just $d^1_{1\,-1}(\theta) = (1 - \cos\theta)/2$. This function agrees with the limiting cases just discussed: it vanishes when $\theta = 0$ and is maximal when $\theta = \pi$.

When one includes the electron mass in the above example, there will be a second allowed configuration of final state helicities, a right handed electron and a right handed anti-neutrino. This configuration has a component of $m=0$ along the electron direction. Clearly this configuration must have zero amplitude when $\theta=0$ and when $\theta=\pi$. Again, one can decompose the final state in a basis in which J is quantized along the z direction. The amount of $m = 1$ basis state in this decomposition is, $D^{1*}_{1\,0}(\phi, \theta, -\phi)$. For $\phi = 0$, this function is $-\sin\theta/\sqrt{2}$, which agrees with the limiting cases.

6 Spin Density Matrix

In the above two examples the intial state has always been prepared with a definite value of m_1 and it has not been important to distinguish between m_1 and λ_1. For reasons which shortly will be come apparent, the following discussion will use λ_1. Consider the case when the particles 1 are produced via the process,

$$\begin{array}{rcl} e^+e^- & \to & 1\,X \\ & & \hookrightarrow 2\,3. \end{array} \tag{18}$$

Here X may be a single particle, many particles or even no particles. In cases that X is more than a single particle, there are no measurements which can be made on the initial and final state particles which will allow, on an event by event basis, a measurement of λ_1. In these cases the amplitude must contain a coherent sum over λ_1. In the general case the amplitude will also depend on the lab direction of particle 1, on the spins and momenta of the initial state particles and on the degrees of freedom needed to describe X. In order to discuss the structure of the amplitude for the above process, the dynamical variables of the problem will be broken into 4 groups.

1. The quantities involved in the decay of 1; that is Ω, λ_1, λ_2 and λ_3.
2. All of the other continuous variables (angles and momenta). These will be denoted by $\vec{\Theta}$.
3. Those of the remaining spins which need to be summed incoherently. This includes the spins of the inital state particles and the remaining final state particles. These will be denoted by M.
4. All of the remaining spins - these need to be summed coherently and will be denoted by N.

In treating the above process one other assumption is usually made: that the decay products of X do not interfere with the decay products of 1. Here "interfere" is used in its QM sense. The decay products of X may certainly affect how the detector responds to the decays of 1 but that is a separate issue. With this assumption one can write the amplitude for the above process as,

$$A = \sum_{\lambda_1} P(\vec{\Theta}; M\lambda_1) A(\Omega; \lambda_1\lambda_2\lambda_3), \tag{19}$$

where the symbol P stands for all of the messy dynamics of both the production of the system (1 X) and the decay of X. Among other things, it will contain a coherent sum over N. The differential decay rate is obtained by squaring the amplitude and summing over spins,

$$|A|^2 = \sum_{\lambda_1\lambda_1'\lambda_2\lambda_3 M} \left[P(\vec{\Theta}; M\lambda_1) A(\Omega; \lambda_1\lambda_2\lambda_3)\right]^* P(\vec{\Theta}; M\lambda_1') A(\Omega; \lambda_1'\lambda_2\lambda_3)$$

$$= \sum_{\lambda_1\lambda_1'\lambda_2\lambda_3 M} A^*(\Omega;\lambda_1\lambda_2\lambda_3)\left[P^*(\vec{\Theta};M\lambda_1)P(\vec{\Theta};M\lambda_1')\right]A(\Omega;\lambda_1'\lambda_2\lambda_3)\,d\Phi. \tag{20}$$

Here $d\Phi$ represents an element of Lorentz invariant phase space. Notice which sums are coherent and which are incoherent. In order to calculate the angular distribution, $I(\Omega)$, one must integrate out all of the other continuous dynamical variables and sum over all of the spins in the problem. In the above form, the dependence on all of the uninteresting variables $(\vec{\Theta}, M, N)$ is contained entirely within the square brackets. It is common to define the spin density matrix of particle 1 as,

$$\rho_{\lambda_1\lambda_1'} = \frac{1}{\sigma}\int\sum_M P^*(\vec{\Theta};M\lambda_1)P(\vec{\Theta};M\lambda_1')d\vec{\Theta}, \tag{21}$$

where the production cross-section of particle 1, σ, is given by,

$$\sigma = \int\sum_{M\lambda_1} P^*(\vec{\Theta};M\lambda_1)P(\vec{\Theta};M\lambda_1)d\vec{\Theta}. \tag{22}$$

By construction, $\rho_{\lambda\lambda'}$ is Hermitian and has a trace of 1. Also, if the production process is parity conserving [2],

$$\rho_{\lambda\lambda'} = (-1)^{\lambda-\lambda'}\rho_{-\lambda-\lambda'}. \tag{23}$$

With this definition, the angular distribution becomes,

$$\begin{aligned} I(\Omega) &= \frac{1}{\Gamma}\sum_{\lambda_1\lambda_1'\lambda_2\lambda_3} A^*(\Omega;\lambda_1\lambda_2\lambda_3)\,\rho_{\lambda_1\lambda_1'}\,A(\Omega;\lambda_1'\lambda_2\lambda_3) \\ &= \frac{1}{\Gamma}\frac{2s_1+1}{4\pi}\sum_{\lambda_1,\lambda_1',\lambda_2,\lambda_3} D^{s_1}_{\lambda_1\;\lambda_2-\lambda_3}(\Omega)\,A^*_{\lambda_2\lambda_3}\,\rho_{\lambda_1\lambda_1'}\,D^{s_1\;*}_{\lambda_1'\;\lambda_2-\lambda_3}(\Omega)\,A_{\lambda_2\lambda_3}. \end{aligned} \tag{24}$$

In most cases at CLEO II one should regard the spin density matrix as simply a matrix of unknown constants which are either interesting to measure, if one wants to study production dynamics, or a nuisance to measure, if one wants to study decay dynamics. Finally, if the off diagonal elements of ρ are non-zero, then $I(\Omega)$ will depend on ϕ. In this case one must ensure that the x and y axes are defined in a way that is consistent with the definitions used to define the spin density matrix.

need a proper discussion of the meaning of ϕ

Now that the spin density matrix has been defined, one can properly define the terms alignment and polarization. A state is said to be polarized if for at least one value of λ_1, $\rho_{\lambda_1\lambda_1} \neq \rho_{-\lambda_1\,-\lambda_1}$. This requires parity violation in the production process. An example of a polarized state is a spin 1 particle with $\rho_{11} \neq \rho_{-1\,-1}$. A state is said to be aligned if it is unpolarized but there is at least one pair of diagonal elements, $(\lambda_1, \lambda_1', \lambda_1 \neq \pm\lambda_1')$, for which $\rho_{\lambda_1\lambda_1} \neq \rho_{\lambda_1'\lambda_1'}$. An example of an aligned state is a spin 1 particle for which, $\rho_{11} = \rho_{-1\,-1} \neq \rho_{00}$. If a spin 1/2 particle is unpolarized it is also unaligned.

It is not necessary to quantize the spin of particle 1 along its direction in the lab frame. It is only required that the spin quantization axis be the same in the defintion of the decay amplitude as it is in the definition of $\rho_{\lambda_1\,\lambda_1'}$. Because, in this paper, the spin quantization axis of particle 1 was chosen as being along the direction of particle 1 in the lab, the indices on ρ are indeed λ_1, not some generic "m". Alternatively, one could, for example, quantize s_1 along the positron direction. In this case the above formalism can be repeated with three simple changes: the spin density matrix now refers to spin components along the new quantization axis, the angles (θ, ϕ) are now defined relative to the new quantization axis and relation 23 no longer holds. However parity conservation will then lead to some different relationship among the elements of ρ [2].

Finally, when the system X in equation 18 is either no particles or a single particle it is possible to make measurements on the initial and final state particles which determine, on an event by event basis, the helicity of particle 1. In these cases the sum over λ_1 will be incoherent. An example of such a case is the process $e^+e^- \to \gamma^* \to \mu^+\mu^-$.

7 Jacob Wick Transformation

In the preceeding sections, the decay $1 \to 2\ 3$ has been described in two different bases. In this section the transformation between the bases will be derived.

Consider the decay $1 \to 2\ 3$ when $\Omega = (0, 0)$ and m_1 is fixed. In this case the quantization axes for the final state particles are the same in both the partial wave basis and helicity basis. Therefore the final states are identical and the two approaches must give the same amplitude. Also $\lambda_2 = m_2$ and $\lambda_3 = -m_3$. Therefore, we can set equations 8 and 10 equal to each other and obtain,

$$\sqrt{\frac{2s_1+1}{4\pi}}\, D^{s_1\,*}_{m_1\lambda_2-\lambda_3}(0,0,0)\, A_{\lambda_2\lambda_3} = \sum_{L\,s} C_{s_2s_3}(s\,m_s; m_2m_3)\, C_{Ls}(s_1m_1; m_Lm_s)\, Y_L^{m_L}(0,0)\, M(L,s), \tag{25}$$

where $m_s = m_2 + m_3 = \lambda_2 - \lambda_3$ and $m_L = m_1 - m_s$. Now,

$$D^{s_1\,*}_{m_1\lambda_2-\lambda_3}(0,0,0) = \begin{cases} 0, & m_1 \neq \lambda_2 - \lambda_3 \\ 1, & m_1 = \lambda_2 - \lambda_3 \end{cases}$$
$$Y_L^{m_L}(0,0) = \begin{cases} 0, & m_L \neq 0 \\ \sqrt{\frac{2L+1}{4\pi}}, & m_L = 0. \end{cases} \tag{26}$$

Therefore,

$$A_{\lambda_2\lambda_3} = \sum_{L\,s} \sqrt{\frac{2L+1}{2s_1+1}}\, C_{s_2s_3}(s\,m_1; \lambda_2\ -\lambda_3)\, C_{Ls}(s_1m_1; 0\,m_1)\, M(L,s). \tag{27}$$

This can be inverted to give,

$$M(L,s) \;=\; \sqrt{\frac{2s_1+1}{2L+1}} \sum_{\lambda_2\lambda_3} C_{s_2s_3}(s\, m_1; \lambda_2 \; -\lambda_3)\, C_{Ls}(s_1 m_1; 0\, m_1)\, A_{\lambda_2\lambda_3}, \quad (28)$$

where the sum runs over all values of λ_2 and λ_3, not just over the independent combinations.

8 Sequential Two Body Decays

Now consider the decay,

$$\begin{aligned} 1 \;\rightarrow\; & 2\,3 \\ & \hookrightarrow 4\,5. \end{aligned} \quad (29)$$

In order to describe the second decay, one must first construct a new coordinate system in the rest frame of particle 2. This will be called the primed coordinate system. For the helicity basis, one first rotates the unprimed axes by $R(\phi, \theta, -\phi)$, which rotates the z axis onto the flight direction of particle 2. Then this system is boosted to the rest frame of 2 to define the primed coordinate system. In short, $\hat{z}'$ is along the direction of 2 in the rest frame of 1, while $\hat{x}'$ and $\hat{y}'$ are uniquely defined by the formal procedure just given. The axes, $(\hat{x}', \hat{y}', \hat{z}')$ form a righthanded orthonormal basis. This coordinate system was chosen so that, if particle 2 has helicity λ_2 in the rest frame of 1, it will have a spin component in its own rest frame of $m_2 = \lambda_2$ along the z' axis. The partial wave basis will be discussed later.

The angles $\Omega' = (\theta', \phi')$ are defined by the direction of particle 4 in the primed coordinate system. Again this means that the angle θ' is defined with respect to the spin quantization axis of particle 2. In the helicity formalism, the amplitude for the above process is,

$$A(\Omega\, \Omega'; \lambda_1, \lambda_3, \lambda_4, \lambda_5) = \sqrt{\frac{(2s_1+1)}{4\pi}\frac{(2s_2+1)}{4\pi}} \sum_{\lambda_2} D^{s_1 *}_{\lambda_1 \lambda_2 - \lambda_3}(\Omega)\, A_{\lambda_2\lambda_3} D^{s_2 *}_{\lambda_2 \lambda_4 - \lambda_5}(\Omega')\, B_{\lambda_4\lambda_5}. \quad (30)$$

Here $B_{\lambda_4\lambda_5}$ are the helicity amplitudes of the second decay. Notice that the sum over λ_2 is coherent. It must be emphasized that if one changes the definition of the primed coordinate system, then form of the amplitude must also change. In this sense there is no freedom in selecting the coordinate system; it is fully specified by the formalism. Again the angular distribution is obtained by squaring the amplitude and summing over spins,

$$I(\Omega, \Omega') = \frac{1}{\Gamma_1\Gamma_2}\frac{2s_1+1}{4\pi}\frac{2s_2+1}{4\pi} \sum_{\substack{\lambda_1\lambda_1'\lambda_2\lambda_2' \\ \lambda_3\lambda_4\lambda_5}} \rho_{\lambda_1\lambda_1'} \Big\{ \quad (31)$$

$$D^{s_1}_{\lambda_1\lambda_2-\lambda_3}(\Omega)\, D^{s_1 *}_{\lambda_1'\lambda_2'-\lambda_3}(\Omega)\, A^*_{\lambda_2\lambda_3} A_{\lambda_2'\lambda_3} D^{s_2}_{\lambda_2\lambda_4-\lambda_5}(\Omega')\, D^{s_2 *}_{\lambda_2'\lambda_4-\lambda_5}(\Omega')\, B^*_{\lambda_4\lambda_5} B_{\lambda_4\lambda_5}. \Big\}$$

While this expression may appear daunting, it is simple compared to the corresponding expression in the partial wave basis. In the partial wave basis, the decay $1 \to 2\ 3$ produces particle 2 in its rest frame with a definite value of m_2 along the quantization axis of 1. One must then express this state in terms of basis states with definite values of m_2 along the axis z' and then the calculation proceeds as above. This introduces a new coherent sum into the amplitude, which already contains coherent sums over m_2 and over the partial wave amplitudes of both decays. Constrast this with equation 30 which has only one coherent sum. As a result, if one wishes to obtain the angular distribution in the partial wave basis, it is often simpler to calculate it the helicity basis and then to use the Jacob Wick transformation, equations 27 and 28, to convert it to the partial wave basis. An example of this is in section 17.4.

There is a second way to look at the partial wave basis. One can define the primed axes as being at rest in the rest frame of 2 and parallel to the unprimed axes. Again the direction Ω' is defined by the direction of particle 4 in this frame. Again this means that θ' is defined with respect to the spin quantization axis of particle 2. This definition of Ω' is different from the one described above so the resulting angular distribution will be different, even after sums over spins. With this definition, the component of the spin of particle 2, m_2, is the same along both the z axis and the z' axis. Therefore the extra coherent sum described in the previous paragraph is no longer needed. Nevertheless the amplitude has three coherent sums whereas there is only one in the helicity formalism. There are probably cases for which the second definition of Ω' leads to simpler forms of the final equations. However the author is not aware of any examples.

Now consider the term in the braces in equation 31. The part of this term which depends on ϕ and ϕ' is,

$$e^{i(\lambda_1-\lambda_1')\phi}e^{i(\lambda_2-\lambda_2')(\phi'-\phi)}. \tag{32}$$

This is multiplied by a product of d functions which depend only on θ and θ'. It is common to define $\chi = \phi' - \phi$ and to make change of variables so that the amplitude depends on $(\theta, \theta', \phi, \chi)$. The Jacobian of this transformation is -1, an unmeasurable phase. The above definitions imply that χ is just the angle between the production and decay planes of particle 2, as measured in the rest frame of particle 2.

The measurement of all of the unknown parameters in equation 31 requires a tremendous amount of data, probably thousands of almost background free events for the case that $s_1 = s_2 = 1$, $s_3 = s_4 = s_5 = 0$. However one can reduce the number of observables to a manageable number by integrating out some of the angles. There are three useful tricks when doing this: an integral over ϕ brings in a factor of $\delta_{\lambda_1\lambda_1'}$, an integral over χ brings in a factor of $\delta_{\lambda_2\lambda_2'}$ and there is the orthogonality relation, equation 4.

Using the orthogonality relation on equation 31 gives the angular distribution in Ω',

$$I(\Omega') = \frac{1}{\Gamma_1\,\Gamma_2}\frac{(2s_2+1)}{4\pi}\sum_{\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5}\rho_{\lambda_1\lambda_1}|D^{s_2}_{\lambda_2\lambda_4-\lambda_5}(\Omega')|^2\,|A_{\lambda_2\lambda_3}|^2\,|B_{\lambda_4\lambda_5}|^2 \tag{33}$$

$$= \frac{1}{\Gamma_1 \, \Gamma_2} \frac{(2s_2+1)}{4\pi} \left[\sum_{\lambda_1} \rho_{\lambda_1 \lambda_1} \right] \sum_{\lambda_2 \lambda_3 \lambda_4 \lambda_5} |d^{s_2}_{\lambda_2 \lambda_4 - \lambda_5}(\theta')|^2 \, |A_{\lambda_2 \lambda_3}|^2 \, |B_{\lambda_4 \lambda_5}|^2$$

This function has no dependence on ϕ' and the trace of the spin density matrix is 1. So the angular distribution in θ' is,

$$I(\theta') = \frac{1}{\Gamma_1 \, \Gamma_2} \frac{2s_2+1}{2} \sum_{\lambda_2 \lambda_3 \lambda_4 \lambda_5} |d^{s_2}_{\lambda_2 \lambda_4 - \lambda_5}(\theta')|^2 \, |A_{\lambda_2 \lambda_3}|^2 \, |B_{\lambda_4 \lambda_5}|^2 \tag{34}$$

It is interesting that this distribution does not depend in any way on the alignment of the initial state. This occurs because, as remarked earlier, the helicity amplitudes are independent of λ_1.

Similarly, one can use the orthogonality relation to integrate out the dependence on $\Omega\prime$. This recovers equation 24.

Finally, the one can integrate out ϕ and χ from equation 31,

$$I(\theta, \theta') = \tag{35}$$
$$\frac{1}{\Gamma_1 \, \Gamma_2} \frac{2s_1+1}{2} \frac{2s_2+1}{2} \sum_{\lambda_1 \lambda_2 \lambda_3 \lambda_4 \lambda_5} \rho_{\lambda_1 \lambda_1} |d^{s_1}_{\lambda_1 \lambda_2 - \lambda_3}(\theta)|^2 \, |d^{s_2}_{\lambda_2 \lambda_4 - \lambda_5}(\theta')|^2 \, |A_{\lambda_2 \lambda_3}|^2 \, |B_{\lambda_4 \lambda_5}|^2.$$

9 Experimental Caveat

When the orthogonality relation was used to obtain equation 34, an unspoken assumption was made, that the detector acceptance is flat in Ω. If it is not, then $I(\theta')$ will depend on the alignment or polarization of the initial state in a complicated way. Moreover, one must be very careful with the common practice of cutting on $\cos\theta$ in order to reduce backgrounds. In general, if one cuts on $\cos\theta$, then the integral over Ω will leave $I(\Omega')$ dependent on ρ. However, in some cases an integral over $\cos\theta$ in either of the domains $[-1, 0]$ or $[0, 1]$ will also remove the dependence on ρ. In these cases one can cut at $\cos\theta = 0$ and extract the helicity amplitudes using half of the available data. This is a useful trick for the common situation that the detector acceptance is flat over most of $\cos\theta$ but dips near one of $\cos\theta = \pm 1$. This cut will work either when the decay $1 \to 2\ 3$ is parity conserving or when it has only one independent amplitude.

10 Real or Complex?

When a process is time reversal invariant, and when some other conditions hold, the paper by Chung [2] shows that the helicity amplitudes are relatively real. However for most cases which one might actually encounter at CLEO II there is no way to know if these other conditions hold. Therefore one must consider the helicity amplitudes as relatively complex. Because the partial wave amplitudes are just linear combinations of the helicity amplitudes, they too must be considered relatively complex.

Many models seem to explicitly predict that the amplitudes will be relatively real. One example is the Godfrey Kokoski [4] model of the decay of P-wave mesons

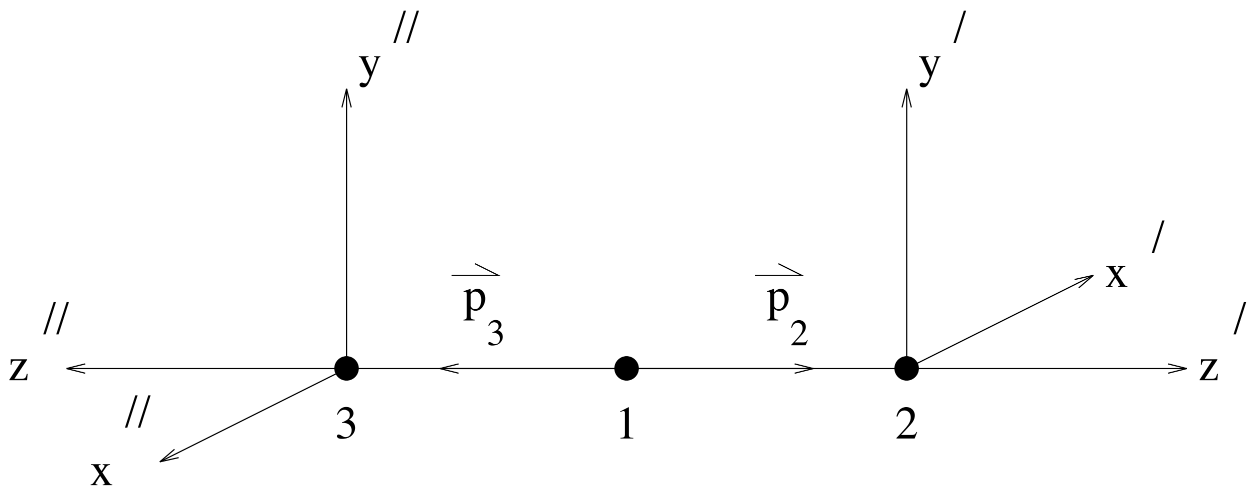

Figure 3: Relationship between the $'$ and $''$ coordinate systems. The figure is drawn in three different rest frames. The momentum vectors of particles 2 and 3 are shown in the rest frame of particle 1. The $'$ system is defined in the rest frame of particle 2 and the $''$ system is defined in the rest frame of particle 3.

containing one heavy quark. However, if one looks carefully at these models, most of them are only calculated in the Born approximation of some effective Hamiltonian. If a higher order of perturbation theory were used then these models would predict the helicity amplitudes to be relatively complex. An equivalent statement to the above is that final state interations may introduce phase shifts.

11 More Sequential Decays

In the helicity basis, consider the decay,

$$\begin{array}{lll} 1 & \to & 2\ 3 \\ & & \ \ \ \hookrightarrow 6\ 7 \\ & & \hookrightarrow 4\ 5. \end{array} \tag{36}$$

In section 8, the primed coordinate system, in the rest frame of 2, was constructed for the decay $2 \to 4\ 5$. In this section the double primed coordinate system, defined in the rest frame of 3, will be constructed for the decay $3 \to 6\ 7$. This system is obtained by rotating the unprimed basis by $R(\pi+\phi, \pi-\theta, -\pi-\phi)$ and then boosting to the rest frame of 3. The axes of the prime and double prime systems, each in their own rest frame, are shown in figure 3. The z'' axis points along the flight direction of 3 in the rest frame of 1. It is therefore antiparallel to the z' axis. The y'' axis is parallel to the y' axis and the x'' axis is antiparallel to the x' axis. The reason that it is y'', not x'', which is parallel to the corresponding singly primed axis is buried in the definition of the Euler angles, in which the β rotation is made around the y axis.

The angles (θ'', ϕ'') are defined by the direction of particle 6 in this coordinate system. Again this means that the angle θ'' is defined with respect to the spin quantization axis of particle 3.

The amplitude for the above process is then,

$$A(\Omega\,\Omega'\Omega''; \lambda_1, \lambda_4, \lambda_5, \lambda_6\lambda_7) = \sqrt{\frac{(2s_1+1)}{4\pi}\frac{(2s_2+1)}{4\pi}\frac{(2s_3+1)}{4\pi}}$$
$$\sum_{\lambda_2\lambda_3} D^{s_1*}_{\lambda_1\lambda_2-\lambda_3}(\Omega)\; A_{\lambda_2\lambda_3} D^{s_2*}_{\lambda_2\lambda_4-\lambda_5}(\Omega')\; B_{\lambda_4\lambda_5} D^{s_3*}_{\lambda_3\lambda_6-\lambda_7}(\Omega'')\; C_{\lambda_6\lambda_7}. \qquad (37)$$

The angular distribution is given by,

$$I(\Omega, \Omega', \Omega'') = |A(\Omega\,\Omega'\Omega''; \lambda_1, \lambda_4, \lambda_5, \lambda_6\lambda_7)|^2\, \frac{(2s_1+1)}{4\pi\,\Gamma_1}\frac{(2s_2+1)}{4\pi\,\Gamma_2}\frac{(2s_3+1)}{4\pi\,\Gamma_3} \qquad (38)$$
$$= \frac{(2s_1+1)}{4\pi\,\Gamma_1}\frac{(2s_2+1)}{4\pi\,\Gamma_2}\frac{(2s_3+1)}{4\pi\,\Gamma_3} \sum \Big\{ \rho_{\lambda_1\lambda_1'}$$
$$D^{s_1}_{\lambda_1\lambda_2-\lambda_3}(\Omega)\; D^{s_1*}_{\lambda_1'\lambda_2'-\lambda_3'}(\Omega)\; A^*_{\lambda_2\lambda_3} A_{\lambda_2'\lambda_3'} D^{s_2}_{\lambda_2\lambda_4-\lambda_5}(\Omega')\; D^{s_2*}_{\lambda_2'\lambda_4-\lambda_5}(\Omega')\; B^*_{\lambda_4\lambda_5} B_{\lambda_4\lambda_5}$$
$$D^{s_3}_{\lambda_3\lambda_6-\lambda_7}(\Omega'')\; D^{s_3*}_{\lambda_3'\lambda_6-\lambda_7}(\Omega'')\; C^*_{\lambda_6\lambda_7} C_{\lambda_6\lambda_7} \Big\},$$

where the sum runs over $(\lambda_1, \lambda_1', \lambda_2, \lambda_2', \lambda_3, \lambda_3', \lambda_4, \lambda_5, \lambda_6, \lambda_7)$. When the dependence on Ω and on Ω'' are integrated out, one can show that $I(\theta')$ is still given by equation 34. Similarly, $I(\theta'')$ is given by,

$$I(\theta'') \;=\; \frac{1}{\Gamma_1\,\Gamma_3}\frac{2s_3+1}{2} \sum_{\lambda_2\lambda_3\lambda_6\lambda_7} |d^{s_3}_{\lambda_3\lambda_6-\lambda_7}(\theta'')|^2\; |A_{\lambda_2\lambda_3}|^2\; |C_{\lambda_6\lambda_7}|^2. \qquad (39)$$

Again it should be emphasized that choosing a different set of coordinate systems will change the form of these expressions.

One observable related to the proper definition of the double primed system is the dihedral angle, $\chi = \phi' + \phi''$ between the decay planes of 2 and 3. This angle has the same value when measured in the rest frame of any of particles 1, 2 or 3. It is unfortunate that it is conventional to use the same notation, χ, both for the variable defined here and for the variable defined in section 8. In practice, one finds that the angular distributions do not depend directly on ϕ'' but rather that they depend on χ. See example 17.8. The physical interpretation of χ is shown in figure 4. That figure is drawn in the rest frame of 1 with the z' axis coming out of the page, the z'' axis going into the page, and the remaining axes as shown in the figure. Both of these coordinate systems are right handed. The momenta of particles 4, 5, 6 and 7, as measured in the rest frame of 1, are also shown on the figure. While, in this view particles 4 and 5 appear to be back to back, they are not back to back in three dimensions. Similarly for particles 6 and 7. In a typical case both of particles 4 and 5 will be coming out of the page and both of particles 6 and 7 will be going into the page. The angles ϕ', ϕ'' and $\chi = \phi'' + \phi'$, are also shown. In this view, ϕ' increases in

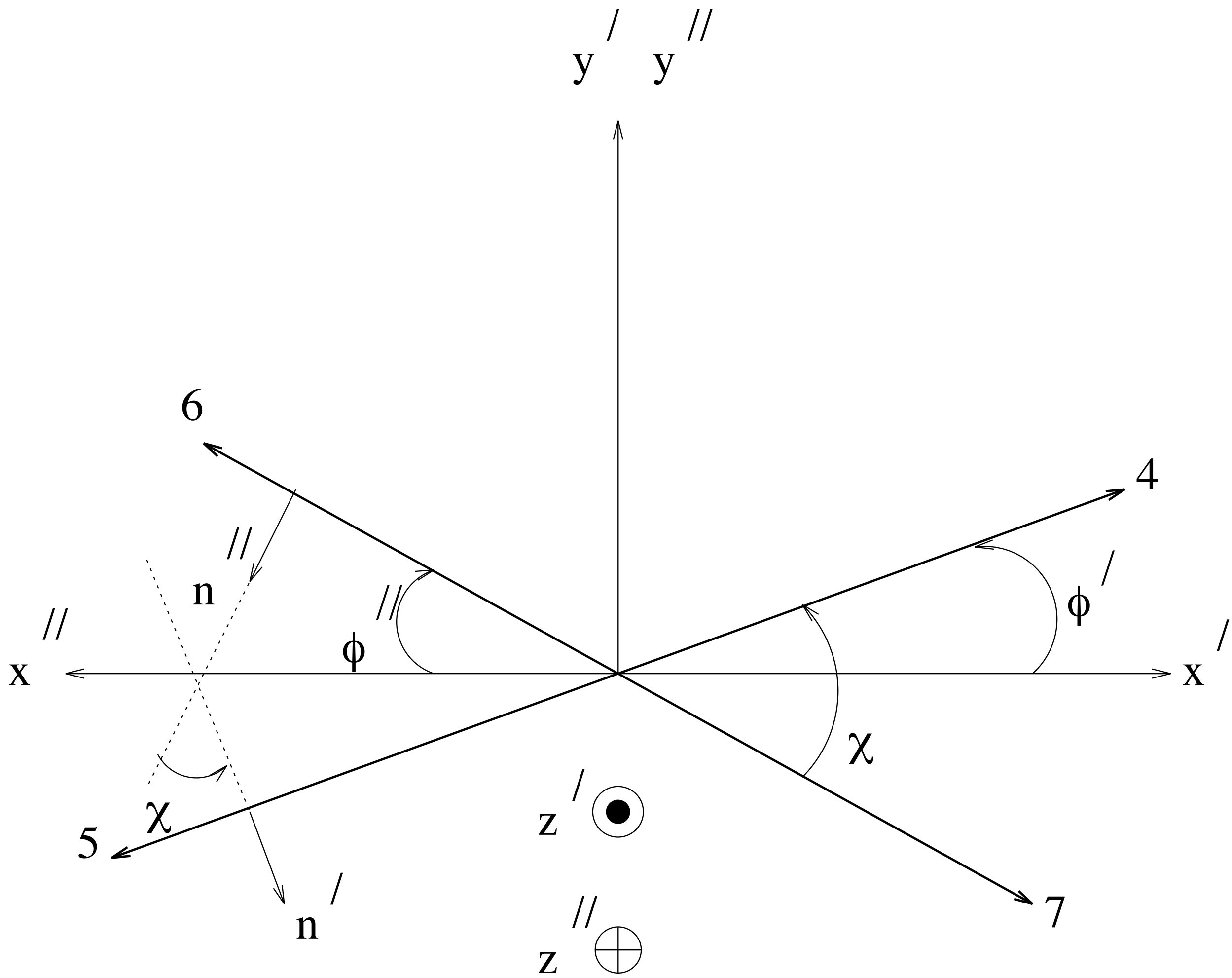

Figure 4: The definition of χ.

the usual counter-clockwise sense while ϕ'' increases in a clockwise sense. The angle χ increases in a counter-clockwise sense from particle 7 to particle 4, or, equivalently, from particle 6 to particle 5. Inspection of the figure shows that χ can be expressed in terms of the unit projections, $\hat{p}_4^\perp$ and $\hat{p}_7^\perp$, of $\vec{p}_4$ and $\vec{p}_7$ onto the plane perpendicular to $\hat{p}_2$,

$$\begin{aligned} \cos\chi &= \hat{p}_7^\perp \cdot \hat{p}_4^\perp \\ \sin\chi &= \hat{p}_7^\perp \times \hat{p}_4^\perp \cdot \hat{p}_2 \end{aligned} \tag{40}$$

Here $\hat{p}_2$ is the unit vector in the direction of particle 2 as measured in the rest frame of 1. In the figure, $\hat{p}_2$ comes out of the page. Finally the normals to the decay planes are also shown,

$$\hat{n}' = \frac{\vec{p}_4 \times \vec{p}_5}{|\vec{p}_4 \times \vec{p}_5|} \qquad \hat{n}'' = \frac{\vec{p}_6 \times \vec{p}_7}{|\vec{p}_6 \times \vec{p}_7|}. \tag{41}$$

The angle χ can also be expressed in terms of these,

$$\cos(\chi) = \hat{n}'' \cdot \hat{n}' \quad \sin(\chi) = \hat{n}'' \times \hat{n}' \cdot \hat{p}_2 \tag{42}$$

Now consider the case when $\theta' = 0$ or $\theta'' = 0$. In these cases one of the decay planes has shrunk to a line and one cannot define the dihedral angle. In practice one has to handle these cases specially but the correct procedure will usually be straightforward to work out. In all cases so far encountered by the author, the terms containing functions of χ also contain the product $\sin\theta' \sin\theta''$. Therefore they vanish when χ is undefined.

12 Theoretical Caveat

Other authors might choose to work in very similar but subtly different coordinate systems. For example one might define, $\chi = \pi - \phi' - \phi''$, which changes the sign of all terms in the amplitude which are proportional to $\cos\chi$. Another variation is to define the z'' axis as being in the same direction as the z' axis. Among other things, this leads to the replacement $\theta'' \to \pi - \theta''$ and changes the sign of all terms proportional to $\cos\theta''$. Such subtleties are rife in the literature on $B \to D^*\ell\nu$.

13 A Special Case in QED

Consider the decay of a virtual photon to a fermion anti-fermion pair. There are 4 possible helicity amplitudes for this process. Of these only two are independent and one can take them to be $A_{\frac{1}{2}\,\frac{1}{2}}$ and $A_{\frac{1}{2}\,-\frac{1}{2}}$. Because of the Lorentz structure of the QED vertex,

$$\left| \frac{A_{\frac{1}{2}\,\frac{1}{2}}}{A_{\frac{1}{2}\,-\frac{1}{2}}} \right| \propto \frac{m_f}{\sqrt{s}}, \tag{43}$$

where m_f is the mass of the fermion and $\sqrt{s}$ is the mass of the virtual photon. This property is often stated as, "at large s, $e^+e^- \to f\bar{f}$ proceeds through a virtual photon with $J_Z = \pm 1$". Similarly, one can usually treat $J/\psi \to e^+e^-$ in the limit that $A_{\frac{1}{2}\,\frac{1}{2}} = A_{-\frac{1}{2}\,-\frac{1}{2}} = 0$.

14 Decay of spin zero particles

There is a simple and well known argument for working out the angular distributions in decays of the form,

$$\begin{array}{lll} 1 & \to & 2\;3 \\ & & \hookrightarrow 4\;5, \end{array} \tag{44}$$

when particles 1, 3, 4 and 5 have spin 0 and particle 2 has spin s_2. There is only one partial wave allowed for the first decay, $M(s_2, s_2)$, and only one partial wave allowed for the second decay, $M(s_2, 0)$. Because the initial state has spin zero, the third component of the total angular momentum in the first decay must be zero along every direction. This allows one to choose the direction of particle 2 as the quantization axis. The component of the orbital angular momentum along this axis is 0 and, therefore, particle 2 is produced only with helicity 0. Therefore particle 2 will decay with the angular distribution $I(\theta') \propto |Y^0_{s_2}(\theta', \phi')|^2$. The decay chain $D_s \to \phi\pi, \phi \to K^+K^-$ is a familiar example. See example 17.3.

The above argument works because the only interesting quantization axis in the problem is the flight direction of particle 2. When particle 1 has a nonzero spin there is a second axis of interest, the quantization axis of the intial state, and one must resort to the full formalism to calculate the angular distributions.

15 Yet Another Method

There is another method of calculating decay angular distributions, sometimes known as the Zeemach formalism [5]. The advantages of this method are that it is manifestly covariant and that it is simple to include the exchange (anti-)symmetry between identical (fermions)bosons in the final state. This method will not be discussed further.

16 The Cookbook

Consider the process,

$$\begin{array}{lll} e^+e^- & \to & 1\ X \\ & & \hookrightarrow 2\ 3 \\ & & \quad \hookrightarrow 4\ 5. \end{array} \tag{45}$$

- $\vec{p}_i^{\,(j)}$ the momentum of particle i as measured in the rest frame of particle j.
- θ the angle between $\vec{p}_2^{\,(1)}$ and $\vec{p}_1^{\,e^+e^-}$, measured in the rest frame of 1.
- θ' the angle between $\vec{p}_4^{\,(2)}$ and $\vec{p}_2^{\,(1)}$, measured in the rest frame of 2.
- λ_i the helicity of i measured in the rest frame of its parent. The parent frame of particle 1 is the CM frame of the e^+e^-.
- $\rho_{\lambda_1\lambda_1'}$ spin density matrix of 1, measured in the CM frame of the e^+e^-.
- $A_{\lambda_2\lambda_3}$ Helicity amplitudes for $1 \to 2\ 3$.
- $B_{\lambda_4\lambda_5}$ Helicity amplitudes for $2 \to 4\ 5$.

The selection rules for the helicity amplitudes are,

$$\begin{array}{cc} |\lambda_2 - \lambda_3| \le s_1; & A_{-\lambda_2 -\lambda_3} = \eta_1\eta_2\eta_3(-1)^{s_2+s_3-s_1} A_{\lambda_2\lambda_3} \\ \text{(always)} & \text{(if parity is conserved in the decay)} \end{array} \tag{46}$$

$$I(\theta,\theta') = \tag{47}$$
$$\frac{1}{\Gamma_1\,\Gamma_2}\frac{2s_1+1}{2}\frac{2s_2+1}{2}\sum_{\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5} \rho_{\lambda_1\lambda_1}|d^{s_1}_{\lambda_1\lambda_2-\lambda_3}(\theta)|^2\ |d^{s_2}_{\lambda_2\lambda_4-\lambda_5}(\theta')|^2\ |A_{\lambda_2\lambda_3}|^2\ |B_{\lambda_4\lambda_5}|^2$$

$$I(\theta') = \frac{1}{\Gamma_1\,\Gamma_2}\frac{2s_2+1}{2}\sum_{\lambda_2\lambda_3\lambda_4\lambda_5} |d^{s_2}_{\lambda_2\lambda_4-\lambda_5}(\theta')|^2\ |A_{\lambda_2\lambda_3}|^2\ |B_{\lambda_4\lambda_5}|^2 \tag{48}$$

$$I(\theta) = \frac{1}{\Gamma_1}\frac{2s_1+1}{2}\sum_{\lambda_1\lambda_2\lambda_3} \rho_{\lambda_1\lambda_1}|d^{s_1}_{\lambda_1\lambda_2-\lambda_3}(\theta)|^2\ |A_{\lambda_2\lambda_3}|^2 \tag{49}$$

These can be expressed in terms of partial wave amplitudes, $M(L,s)$,

$$A_{\lambda_2\lambda_3} = \sum_{L\ s}\sqrt{\frac{2L+1}{2s_1+1}}C_{s_2s_3}(s\lambda_1;\lambda_2\ -\lambda_3)\ C_{Ls}(s_1\lambda_1;0\ \lambda_1)\ M(L,s) \tag{50}$$

$$M(L,s) = \sqrt{\frac{2s_1+1}{2L+1}}\sum_{\lambda_2\lambda_3} C_{s_2s_3}(s\lambda_1;\lambda_2\ -\lambda_3)\ C_{Ls}(s_1\lambda_1;0\ \lambda_1)\ A_{\lambda_2\lambda_3}, \tag{51}$$

where $\lambda_1 = \lambda_2 - \lambda_3$.

17 Examples

17.1 $A \to V\,\gamma$

The decay of an axial vector meson to a vector meson and a photon illustrates a complication which occurs in the partial wave basis. Because this is an electromagnetic decay it is parity conserving and the the selection rules 6 and 7 give three allowed partial wave amplitudes, $M(L,s) = (M(2,2), M(2,1), M(0,1))$. However, in the helicity basis, $A_{\lambda_V \lambda_\gamma}$, there are only two independent amplitudes: the three amplitudes $A_{\lambda_V 0}$ vanish because the photon does not have a helicity zero state; the amplitudes $A_{1\,-1} = A_{-1\,1} = 0$ because they violate the selection rule 11; finally, parity conservation gives, $A_{11} = -A_{-1\,-1}$ and $A_{01} = -A_{0\,-1}$.

This discrepancy arises because, in the partial wave basis, there is no simple way to account for the missing helicity state of the photon. Using the Jacob Wick relations, equations 27 and 28, one can show that only two of $(M(2,2), M(2,1), M(0,1))$ are independent. Similar discrepancies may arise in decays which contain a photon, a neutrino or a highly relativistic massive fermion. They do not necessarily arise because the $\lambda_\gamma = 0$ amplitude may already be forbidden by one of the selection rules. An example of this is the decay $D^* \to D\,\gamma$, in which the amplitude A_{00} is already forbidden by the selection rule 12.

17.2 $D^* \to D\pi$

These next two examples are included to give the reader some confidence by using the formalism to obtain well known results. In the partial wave basis there is only one amplitude allowed, $M(1,0)$, and in the helicity basis there is only one amplitude allowed, A_{00}. For this decay, equation 49 reads,

$$\begin{aligned} I(\theta) &= \frac{3}{2}\left[\rho_{11}\frac{\sin^2\theta}{2} + \rho_{00}\cos^2\theta + \rho_{-1-1}\frac{\sin^2\theta}{2}\right] \\ &= \frac{3}{2}\left[(1-\rho_{00})\frac{\sin^2\theta}{2} + \rho_{00}\cos^2\theta.\right] \end{aligned} \qquad (52)$$

The shape of this distribution does not depend on the helicity or partial wave amplitudes, which is a general feature of any decay with only one independent amplitude. The amplitudes do, however, do affect the decay rate. While the distribution does depend on the alignment of the initial state it is insensitive to any polarization. Therefore this form is correct even when the D^* is produced in a parity violating process such as B meson decay. When the state is unaligned $\rho_{00} = 1/3$ and $I(\theta)$ is flat.

17.3 $D_s \to \phi\pi, \phi \to K^+K^-$

In the partial wave basis the first decay has only one amplitude, $M(1,1)$, and the second decay has only one amplitude, $M(1,0)$. In the helicity basis the allowed helicty amplitudes are A_{00} for the first decay and B_{00} for the second decay. The amplitudes

$A_{\pm 10}$ vanish because of selection rule 11. As in the previous example these amplitudes do not affect the shape of the angular distributions. For this decay equation 48 reads,

$$I(\theta') = \frac{3}{2}\cos^2\theta. \tag{53}$$

17.4 $D_1(2420) \to D^*\pi, D^* \to D\pi$

Consider the production and decay of the $D_1(2420)$, a $J^P = 1^+$ meson,

$$\begin{aligned} e^+e^- \;\to\;& D_1\, X \\ & \hookrightarrow D^*\,\pi \\ & \quad \hookrightarrow D\,\pi. \end{aligned} \tag{54}$$

In the partial wave basis the two independent amplitudes for the decay of the D_1 are $S = M(0,1)$ and $D = M(2,1)$, while in the helicity basis they are $A_{10} = +A_{-10}$ and A_{00}. Recall that,

$$\Gamma = 2|A_{10}|^2 + |A_{00}|^2. \tag{55}$$

The one independent amplitude for the D^* decay does not affect the shape of angular distributions. For this decay equation 30 reads,

$$\begin{aligned} A(\Omega,\Omega';\lambda_1,\lambda_2) &= \frac{3}{4\pi}\sum_{\lambda_2} D^{1\,*}_{\lambda_1\,\lambda_2}(\Omega)\; D^{1\,*}_{\lambda_2\,0}(\Omega')\; A_{\lambda_2\,0} \\ &= \frac{3}{4\pi}\sum_{\lambda_2} e^{i\lambda_1\phi}\; e^{i\lambda_2(\phi'-\phi)}\; d^1_{\lambda_1\,\lambda_2}(\theta)\; d^1_{\lambda_2\,0}(\theta')\; A_{\lambda_2\,0} \end{aligned} \tag{56}$$

The angular distribution is then given by equation 24. It is common to define $\chi = \phi' - \phi$ and then to make a change of variables so that the amplitude is given in terms of $(\theta, \theta', \phi, \chi)$. The variable χ defined here is different from the variable χ defined in section 11. After integrating the angular distribution over ϕ, which removes any dependence on the off-diagonal elements of the spin-density matrix, one obtains,

$$\begin{aligned} I(\theta,\theta',\chi) \;\propto\; \frac{9}{8\pi\Gamma}\Bigg\{ & \;|A_{10}|^2\,\frac{\sin^2\theta'}{4}\left[\left(1+\cos^2\theta\right) + \rho_{00}\left(1-3\cos^2\theta\right)\right] \\ &+ |A_{00}|^2\,\frac{\cos^2\theta'}{2}\left[\left(1-\cos^2\theta\right) - \rho_{00}\left(1-3\cos^2\theta\right)\right] \\ &- \frac{1}{4}\,|A_{10}|^2\,(1-3\rho_{00})\;\sin^2\theta\sin^2\theta'\cos(2\chi) \\ &+ \Re(A^*_{10}A_{00})\;(1-3\rho_{00})\;\sin\theta\cos\theta\sin\theta'\cos\theta'\cos\chi\Bigg\}. \end{aligned} \tag{57}$$

This distribution is the simplest distribution which contains all of the information about the helicity amplitudes for the decay of the D_1. The shape of this angular distribution depends on three parameters which one can choose to be $|A_{10}|^2/\Gamma$, ρ_{00}

and $\cos\phi_{10}$, where $\phi_{10} = \arg(A^*_{10}A_{00})$. Instead of $|A_{10}|^2/\Gamma$, one could have chosen one of the free parameters to be $|A_{00}/A_{10}|$. However, when fitting distributions, it is often more useful to have a parameter which is bounded by $[0,1]$, than it is to have a parameter which is bounded by $[0,\infty]$.

In practice, it can be difficult to determine all of the free parameters in the above equation by fitting an experimental $I(\theta,\theta',\chi)$ distribution. The difficulty lies in knowing how the background behaves in θ, θ' and χ. In such a case one can determine the parameters by fitting the appropriate one dimensional distributions. For this decay chain equations 48 and 49 are,

$$I(\theta') \propto \frac{3}{2\Gamma}\left[|A_{10}|^2\sin^2\theta' + |A_{00}|^2\cos^2\theta'\right] \tag{58}$$

$$\begin{aligned} I(\theta) \propto \frac{3}{4\Gamma}\Big[&(1+\rho_{00})\,|A_{10}|^2 + (1-\rho_{00})\,|A_{00}|^2 \\ &+ (1-3\rho_{00})\left(|A_{10}|^2 - |A_{00}|^2\right)\cos^2\theta\Big] \end{aligned} \tag{59}$$

These also could have been obtained by integrating equation 57. The distributions 57 and 59 are are flat in θ for two cases, an unaligned initial state or when $|A_{00}| = |A_{10}|$. The $I(\theta')$ distribution is independent of the alignment of the initial state and it is flat when $|A_{00}| = |A_{10}|$.

Because $I(\theta')$ has the form $1 + C\cos^2\theta$, where C is a constant, it is possible to determine two parameters from a fit to an experimental $I(\theta')$ distribution. One of these is the overall normalization, leaving only one measurable shape parameter. Inspection of equation 59 shows that one can measure $|A_{10}|^2/\Gamma$ from a fit an experimental $I(\theta')$ distribution. Similarly, one can measure both $|A_{10}|^2/\Gamma$ and ρ_{00} from a simultaneous fit to experimental $I(\theta)$ and $I(\theta')$ distributions. However one cannot measure $\cos\phi_{10}$ from these distributions.

Determining $\cos\phi_{10}$ from a one dimensional distribution is a little more complicated. Firstly, one might think to integrate $\cos\theta$ and $\cos\theta'$ out of equation 57. This gives,

$$I(\chi) \propto \frac{1}{2\pi}\left[1 - (1-3\rho_{00})\frac{|A_{10}|^2}{\Gamma}\cos(2\chi)\right], \tag{60}$$

which cannot be used to determine $\cos\phi_{10}$. The right answer is to consider how the signs in equation 57 behave in the four quadrants of the $(\cos\theta, \cos\theta')$ plane. The sign of the first three terms does not depend on the sign of either $\cos\theta$ or $\cos\theta'$. But the sign of the fourth term does. The sign is positive in the first and third quadrants of the plane, while it is negative in the second and fourth quadrants. Now, let $I^{++}(\chi)$ denote the angular distribution in χ when $\cos\theta > 0$ and $\cos\theta' > 0$. Similarly, let $I^{+-}(\chi)$ denote the angular distribution in χ when $\cos\theta > 0$ and $\cos\theta' < 0$, and so on. Finally, consider the following combination of angular distributions,

$$I(\chi)^{++} + I(\chi)^{--} - I(\chi)^{+-} - I(\chi)^{-+} = \frac{4}{9}(1-3\rho_{00})\frac{|A_{10}||A_{00}|}{\Gamma}\cos\phi_{10}\cos\chi. \tag{61}$$

Inspection of the above equations show that one can measure all of $|A_{10}|^2/\Gamma$, ρ_{00} and $\cos\phi_{10}$ from a simultaneous fit to experimental distributions of $I(\theta)$, $I(\theta')$ and equation 61.

The relationship between the partial wave amplitudes and the helicity amplitudes is given by equations 27 and 28,

$$A_{10} = \frac{S}{\sqrt{3}} + \frac{D}{\sqrt{6}} \tag{62}$$

$$A_{00} = \frac{S}{\sqrt{3}} - \sqrt{\frac{2}{3}}D. \tag{63}$$

These relations differ from those given in Rosner [6] by an overall, unphysical, minus sign. In the partial wave basis, the equations 58 and 59 become,

$$I(\theta) \propto \frac{1}{4\Gamma}\Big[\ 2S^2 + \frac{D^2}{2}\left[(5-3\cos^2\theta) - 3\rho_{00}(1-3\cos^2\theta)\right] - \sqrt{2}|S|\,|D|\cos\phi_{SD}(1-3\rho_{00})(1-3\cos^2\theta)\Big] \tag{64}$$

$$I(\theta') \propto \frac{1}{2\Gamma}\left[|S|^2 + |D|^2\frac{1+3\cos^2\theta'}{2} + \sqrt{2}|S|\,|D|\cos\phi_{SD}\left(1-3\cos^2\theta'\right)\right], \tag{65}$$

where $\phi_{SD} = \arg(S^*D)$ and where $\Gamma = |S|^2 + |D|^2$.

This example illustrates one of the advantages of the helicity formalism. In the helicity basis it was possible to determine $|A_{10}|^2/\Gamma$ from a fit to an experimental $I(\theta')$ distribution. In the partial wave basis, however, the shape of this distribution depends on two parameters, $|S|^2/\Gamma$ and $\cos\phi_{SD}$, so neither can be determined by fitting an experimental $I(\theta')$ distribution. Instead, one can only obtain an allowed region in the plane of the two parameters, as has been done by CLEO II[16]. Similarly, the two distributions $I(\theta)$ and $I(\theta')$ depend on three parameters, $|S|^2/\Gamma$, $\cos\phi_{SD}$ and ρ_{00}. Therefore, the parameters are again underdetermined by a simultaneous fit to experimental $I(\theta)$ and $I(\theta')$ distributions.

Finally, equation 61 is given in the partial wave basis as,

$$I(\chi)^{++} + I(\chi)^{--} - I(\chi)^{+-} - I(\chi)^{-+} = \frac{4}{9}(1-3\rho_{00})\left(|S|^2 - |D|^2 - \frac{1}{\sqrt{2}}|S||D|\cos\phi_{SD}\right)\cos\chi. \tag{66}$$

One can determine all of $|S|^2/\Gamma$, $\cos\phi_{SD}$ and ρ_{00} from a simulateous fit to experimental distributions for $I(\theta)$, $I(\theta')$ and equation 66.

17.5 $B \to K^*\Psi, K^* \to K\,\pi, \Psi \to \mu^+\mu^-$

There are three independent partial waves for the first decay, $M(2,2)$, $M(1,1)$, $M(0,0)$ in the partial wave basis, or A_{11}, A_{00} and $A_{-1\,-1}$ in the helicity basis. As discussed in section 13, the amplitude $C_{\frac{1}{2}\,\frac{1}{2}}$ can be neglected when describing the decay of the Ψ. This leaves only one independent amplitude for each of the secondary decays, neither of which can affect the shape of the angular distributions. For the decay of the K^*, equation 48 reads,

$$\begin{aligned} I(\theta') &\propto |A_{11}|^2\frac{\sin^2\theta'}{2} + |A_{00}|^2\cos^2\theta' + |A_{-1\,-1}|^2\frac{\sin^2\theta'}{2} \\ &\propto 1 + \left(\frac{2\Gamma_L}{\Gamma_T} - 1\right)\cos^2\theta'. \end{aligned} \tag{67}$$

For the decay of the Ψ, equation 39 reads,

$$\begin{aligned} I(\theta'') &\propto |A_{11}|^2\frac{1+\cos^2\theta''}{2} + |A_{00}|^2\sin^2\theta'' + |A_{-1\,-1}|^2\frac{1+\cos^2\theta''}{2} \\ &\propto 1 + \frac{\Gamma - 3\Gamma_L}{\Gamma + \Gamma_L}\cos^2\theta''. \end{aligned} \tag{68}$$

Here, and in some of the subsequent examples, overall factors of $1/\Gamma$ and $(2s+1)/2$ have been left out, hence the $\propto$ instead of equal signs. In the above, the longitudinal width is given by $\Gamma_L = |A_{00}|^2$, the transverse width by $\Gamma_T = |A_{11}|^2 + |A_{-1\,-1}|^2$ and the total width by $\Gamma = \Gamma_L + \Gamma_T$. The nomenclature "longitudinal" and "transverse" refers to the direction of the polarization vector, which is perpendicular to the direction of the spin vector[7].

The reason that the above two angular distributions are different is simply that the Ψ is decaying into a pair of spin 1/2 particles while the K^* is decaying into a pair of spin 0 particles.

17.6 $B \to K^*\gamma, K^* \to K\pi$

In this decay there are two independent helicity amplitydes, A_{11} and $A_{-1\,-1}$. The amplitude A_{00} vanishes because photons have only transverse polarizations. The second decay has only one independent amplitude, which does not affect the shape of the angular distributions. For this decay, equation 48 reads,

$$I(\theta') = \frac{3}{4}\sin^2\theta'. \tag{69}$$

17.7 $\Lambda_c^* \to \Sigma_c\pi$

The lowest lying excitations of the Λ_c, ($J^P = \frac{1}{2}^+$), are believed to be a doublet of P-wave states with $J^P = (\frac{1}{2}^-, \frac{3}{2}^-)$ [8, 9]. Isospin invariance implies that, if kinematically allowed, the strong decays of these states should proceed via the decay chain,

$$\begin{aligned} \Lambda_c^* &\to \Sigma_c^{(*)}\pi \\ &\qquad \hookrightarrow \Lambda_c\pi. \end{aligned} \tag{70}$$

Here the notation $\Sigma_c^{(*)}$ means that the decay can proceed either via a Σ_c, ($J^P = \frac{1}{2}^+$), or via a Σ_c^*, ($J^P = \frac{3}{2}^+$). In this section the angular distributions for all of these decays will be derived. One further assumption will be made: since CLEO II analyses look for

Decay	Helicity Amplitudes	Partial Wave Amplitudes	$I(\theta)$	$I(\theta')$
	$\Lambda_c^* \to \Sigma_c^{(*)}\pi$			
$\frac{1}{2}^- \to \frac{1}{2}^+ 0^-$	$A_{\frac{1}{2}\,0} = +A_{-\frac{1}{2}\,0}$	$S/\sqrt{2}$	flat	flat
$\frac{1}{2}^- \to \frac{3}{2}^+ 0^-$	$A_{\frac{1}{2}\,0} = -A_{-\frac{1}{2}\,0}$	$D/\sqrt{2}$	flat	
$\frac{3}{2}^- \to \frac{1}{2}^+ 0^-$	$A_{\frac{1}{2}\,0} = -A_{-\frac{1}{2}\,0}$	$-D\sqrt{2}$		flat
$\frac{3}{2}^- \to \frac{3}{2}^+ 0^-$	$A_{\frac{1}{2}\,0} = +A_{-\frac{1}{2}\,0}$	$(S-D)/2$		
	$A_{\frac{3}{2}\,0} = +A_{-\frac{3}{2}\,0}$	$(S+D)/2$		
	$\Sigma_c^{(*)} \to \Lambda_c\pi$			
$\frac{1}{2}^+ \to \frac{1}{2}^+ 0^-$	$A_{\frac{1}{2}0} = -A_{-\frac{1}{2}\,0}$	$-P/\sqrt{2}$		
$\frac{3}{2}^+ \to \frac{1}{2}^+ 0^-$	$A_{\frac{1}{2}0} = +A_{-\frac{1}{2}\,0}$	$+P\sqrt{2}$		

Table 1: Partial wave and helicity amplitudes for various strong decays of P-wave charm baryons.

these charmed baryons in continuum production, one may assume that the Λ_c^* states are unpolarized. For the spin 1/2 states, this means that they are also unaligned.

Table 1 lists the helicity and partial wave amplitudes for the decays in process 70. Those entries above the line list quantities for the 4 possibilities for the first decay and those below the line list quantities for the 2 possibilities for the second decay. The first column of the table lists the specific decay by giving the J^P of the initial and final state particles. The second column lists the helicity amplitudes. Because these decays conserve parity the helicity amplitudes come in pairs which are equal to each other up to a sign; only one of each pair is an independent quantity. The third column lists the partial wave amplitudes for the decay, where the numerical factors were obtained using the Jacob Wick relation 27. The sign convention is that the partial wave amplitude is equal to the first amplitude of the corresponding pair in column 2.

For entries above the line there are an additional two columns in the table that indicate which angular distributions are trivially flat. Recall that θ is the polar angle in the first decay and that θ' is the polar angle in the second decay. Any state which is produced unaligned will have a flat distribution for $\cos\theta$ — therefore the first two entries in the fourth colunm show flat distributions. In the first and third decays, both of the helcity states of the Σ_c are equally populated, therefore the $\cos\theta'$ distribution is flat. The non-flat angular distributions are given below.
For $\frac{1}{2}^- \to \frac{3}{2}^+ 0^-$,

$$I(\theta') = \frac{1}{4}\left(1 + 3\cos^2\theta'\right). \tag{71}$$

For $\frac{3}{2}^- \to \frac{1}{2}^+ 0^-$,

$$I(\theta) = \frac{1}{4}(3 - 4\rho_{\frac{1}{2}\frac{1}{2}}) - \frac{3}{4}(1 - 4\rho_{\frac{1}{2}\frac{1}{2}})\cos^2\theta. \tag{72}$$

For $\frac{3}{2}^- \to \frac{3}{2}^+ 0^-$,

$$\begin{aligned} I(\theta) &= \frac{1}{2\Gamma}\Big[\left(3|A_{\frac{1}{2}0}|^2 + |A_{\frac{3}{2}0}|^2\right) - 4\rho_{\frac{1}{2}\frac{1}{2}}\left(|A_{\frac{1}{2}0}|^2 - |A_{\frac{3}{2}0}|^2\right) \\ &\quad -3\left(1 - 4\rho_{\frac{1}{2}\frac{1}{2}}\right)\left(|A_{\frac{1}{2}0}|^2 - |A_{\frac{3}{2}0}|^2\right)\cos^2\theta\Big] \qquad (73) \\ I(\theta') &= \frac{1}{2\Gamma}\Big[\left(3|A_{\frac{3}{2}0}|^2 + |A_{\frac{1}{2}0}|^2\right) + 3\left(|A_{\frac{1}{2}0}|^2 - |A_{\frac{3}{2}0}|^2\right)\cos^2\theta'\Big]. \qquad (74) \end{aligned}$$

Here $\Gamma = 2|A_{\frac{1}{2}0}|^2 + 2|A_{\frac{3}{2}0}|^2$.

Inspection of these last two equations shows that when $|A_{\frac{1}{2}0}|^2 = |A_{\frac{3}{2}0}|^2$ both distributions are flat. Not surprisingly, this happens when the decay occurs in a pure S wave but it also happens when the decay occurs in a pure D wave. In the limit that the charmed quark mass is very heavy all of the above decays are predicted to be either pure S wave or pure D wave [17]. Therefore any deviation from flatness indicates a violation of the heavy quark limit.

17.8 $D_s \to \phi\,\rho$

17.8.1 The angular distributions.

Consider the decay,

$$\begin{aligned} D_s^+ \;\to\;& \phi\,\rho^+ \\ & \;\;\hookrightarrow \pi^0\,\pi^+ \\ & \hookrightarrow K^+\,K^-. \end{aligned} \tag{75}$$

The first decay in this chain has three allowed amplitudes, $[M(0,0), M(1,1), M(2,2)]$ in the partial wave basis, or $[A_{11}, A_{00}, A_{-1\,-1}]$ in the helicity basis. In order to simplify the notation these will be written as $[S, P, D]$ in the partial wave basis and as $[H_+, H_0, H_-]$ in the helicity basis. The two secondary decays each have only one independent amplitude and neither of these can affect the shape of the angular distribution. For the full decay chain, the angular distribution is given by equation 38,

$$\begin{aligned} & I(\theta', \theta'', \chi) \\ \propto \quad & |H_+|^2\,|d^1_{10}(\theta')|^2|d^1_{10}(\theta'')|^2 \;+\; |H_-|^2\,|d^1_{-10}(\theta')|^2|d^1_{-10}(\theta'')|^2 \\ + \quad & |H_0|^2\,|d^1_{00}(\theta')|^2|d^1_{10}(\theta'')|^2 \\ & + 2\,\left[\Re\left(H_+H_0^*\right)\cos\chi - \Im\left(H_+H_0^*\right)\sin\chi\right]\; d^1_{10}(\theta')\,d^1_{00}(\theta')\,d^1_{10}(\theta'')\,d^1_{00}(\theta'') \\ & + 2\,\left[\Re\left(H_0H_-^*\right)\cos\chi - \Im\left(H_0H_-^*\right)\sin\chi\right]\; d^1_{00}(\theta')\,d^1_{-10}(\theta')\,d^1_{00}(\theta'')\,d^1_{-10}(\theta'') \\ & + 2\,\left[\Re\left(H_+H_-^*\right)\cos(2\chi) - \Im\left(H_+H_-^*\right)\sin(2\chi)\right]\; d^1_{10}(\theta')\,d^1_{-10}(\theta')\,d^1_{10}(\theta'')\,d^1_{-10}(\theta'') \end{aligned}$$

$$
\begin{aligned}
= & \frac{1}{4}\Big\{ \, |H_+|^2 \sin^2\theta' \sin^2\theta'' \; + \; |H_-|^2 \sin^2\theta' \sin^2\theta'' \\
& + 4\, |H_0|^2 \cos^2\theta' \cos^2\theta'' \\
& + 2\, \Big[\Re(H_+ H_-^*) \cos(2\chi) - \Im(H_+ H_-^*) \sin(2\chi)\Big] \sin^2\theta' \sin^2\theta'' \\
& + 4\, [\Re(H_+ H_0^*) \cos\chi - \Im(H_+ H_0^*) \sin\chi] \sin\theta' \cos\theta' \sin\theta'' \cos\theta'' \\
& + 4\, \Big[\Re(H_0 H_-^*) \cos\chi - \Im(H_0 H_-^*) \sin\chi\Big] \sin\theta' \cos\theta' \sin\theta'' \cos\theta'' \Big\}
\end{aligned}
\tag{76}
$$

Here $\chi = \phi + \phi'$, as discussed in section 11. To get a properly normalized distribution the right hand side of the above equations must be multiplied by $9/8\pi\Gamma$.

Now consider the decay,

$$
\begin{aligned}
D_s^- \;\; \rightarrow \;\; & \phi \; \rho^- \\
& \;\; \hookrightarrow \pi^0 \, \pi^- \\
& \hookrightarrow K^- \, K^+ .
\end{aligned}
\tag{77}
$$

This decay is the charge conjugate of decay 75, and its decay angular distribution will be denoted by $\overline{I}(\theta', \theta'', \chi)$. Careful reading of section 11 shows that the angles θ', θ'' and χ are defined in a way which is symmetric under charge conjugation. The following statement of those definitions shows the symmetry,

- θ': the angle, in the domain $[0, \pi]$, between the momentum of the kaon with the same charge as the D_s, measured in the rest frame of the ϕ, and the momentum of the ϕ, measured in the rest frame of the D_s.
- θ'': the angle, in the domain $[0, \pi]$, between the momentum of the π^0, measured in the rest frame of the ρ, and the momentum of the ρ, measured in the rest frame of the D_s.
- χ: Consider a righthanded orthonormal coordinate system. Look at the decay from the viewpoint that the momentum of the ϕ is along the z axis and the momentum of the charged pion is along the x axis. Because the D_s has spin zero this is not a special configuration. The angle χ is the azimuth of the kaon with the same charge as the D_s.

While the above definition of the angles is symmetric under charge conjugation the weak interactions are not. Therefore one should not expect to find $I(\theta', \theta'', \chi) = \overline{I}(\theta', \theta'', \chi)$. In the standard model, however, this decay does conserve CP: every Feynman diagram which one can draw is proportional to $V_{cs} V_{ud}^*$ but, to observe CP violation, one requires that at least two diagrams have different CKM phases. This will be further explained in section 17.8.3.

The consequences of CP conservation for this decay are illustrated in figure 5. Part a) of that figure shows the decay 75 with some particular values of θ', θ'', χ. The momentum vectors of the ϕ and of the ρ are drawn in the rest frame of the D_s^+, while those of the two pions are drawn in the rest frame of the ρ and those of the two kaons are drawn in the rest frame of the ϕ. The inset, which defines χ is to be viewed with the momentum of the ϕ coming out of the page. Therefore the main part of this figure shows the decay of the ρ taking place in the plane of the page while the K^+ comes out of the page, by an angle χ. Because the D_s has spin zero this is a completely general configuration. Part b) shows the CP conjugate of this configuration: particles have been changed into their anti-particles and the directions of all momenta have been reversed. There are no observable spin components in the final state; if there were, they would not have been reversed. Again the inset shows the sense of χ, which is that the K^- is now going into the page. Finally, part c) shows the CP conjugated state rotated by 180 degrees about an axis perpendicular to the plane of the page. Because the D_s has spin zero, the configuration shown in part c) is indistinguishable from the configuration shown in b); that is, part c) is also the CP conjugate of part a). If one compares parts a) and c) it can be seen that momentum of the π^- from the D_s^- decay is coincident with that of the π^+ from the D_s^+ decay. Similarly for the π^0. However this is not the case for the decay products of the ϕ: in part a), the kaon with the same charge as the D_s comes out of the page whereas, in part c), it goes into the page. Refering to the above definition of the angles, one sees that the CP conjugate of a D_s^+ decay to the configuration $(\theta', \theta'', \chi)$ is simply the decay of a D_s^- to the configuration $(\theta', \theta'', -\chi)$. If CP is a good symmetry, then the decay rates to these configurations must be equal,

$$I(\theta', \theta'', \chi) \;=\; \overline{I}(\theta', \theta'', -\chi). \tag{78}$$

That is, $\overline{I}(\theta', \theta'', \chi)$ is given by equation 76 but with the sign flipped for the terms proportional to $\sin\chi$ and $\sin(2\chi)$. The significance of this sign flip is simple: while the process under consideration is symmetric under CP but not C, it has been described by angles whose definition is symmetric under C. One could redefine χ for the D_s^- to be the CP conjugate of χ for the D_s^+ and the angular distributions would then have the same form, regardless of the sign of the charge of the initial state.

Another view of this sign flip is as follows: instead of thinking of the dynamic variables as θ', θ'' and χ, one can choose $\cos\theta'$, $\cos\theta''$, $\cos\chi$ and $\sin\chi$. Recall that the two polar angles are defined in the domain $[0, \pi]$ but that χ is defined in the domain $[0, 2\pi]$. Therefore $\sin\theta'$ and $\sin\theta''$ are not independent dynamic variables. Of these four dynamic variables, the three cosines are defined by dot products of momenta and are, therefore, invariant under P. However $\sin\chi$ is defined as a triple scalar product of momenta, equation 40, and does change sign under P. Further, the angles are defined in a way which does not change sign under C. Therefore, in a CP conserving angular distribution, all terms which are proportional to $\sin\chi$ must change sign. The jargon is that the polar angles and $\cos\chi$ are "CP-even" variables while $\sin\chi$ is a "CP-odd" variable. Similarly the coefficient of a term containing a CP-even combination of variables is a CP-even observable while the coefficient of a term containing a CP-odd combination of variables is a CP-odd observable.

17.8.2 Another new basis

Using the Jacob Wick relations 27 and 28, the helicity amplitudes and the partial wave amplitudes for decay 75 are related by,

$$\begin{aligned} H_+ &= \frac{S}{\sqrt{3}} + \frac{D}{\sqrt{6}} - \frac{P}{\sqrt{2}} \\ H_0 &= -\frac{S}{\sqrt{3}} + \sqrt{\frac{2}{3}}D \\ H_- &= \frac{S}{\sqrt{3}} + \frac{D}{\sqrt{6}} + \frac{P}{\sqrt{2}} \end{aligned} \tag{79}$$

Each of the partial wave amplitudes is either parity conserving or parity violating: P is parity conserving while S and D are parity violating. An helicity amplitude, on the other hand, can contain both parity conserving and parity violating parts,

$$H_+^e = -\frac{P}{\sqrt{2}} \qquad H_+^o = \frac{S}{\sqrt{3}} + \frac{D}{\sqrt{3}} \qquad H_0^o = -\frac{S}{\sqrt{3}} + \sqrt{\frac{2}{3}}D. \tag{80}$$

Here the superscript e denotes a parity conserving amplitude and the superscript o denotes a parity violating amplitude. As required by equation 12, the amplitude H_0 has no parity conserving part. Therefore one can rewrite the helicity amplitudes as,

$$H_+ = H_+^e + H_+^o, \qquad H_0 = H_0^o, \qquad \begin{aligned} H_- &= H_-^e + H_-^o \\ &= -H_+^e + H_+^o, \end{aligned} \tag{81}$$

where H_+^e, H_+^o, H_0^o are taken to be the independent amplitudes. The angular distribution 76 is then,

$$\begin{aligned} &I^+(\cos\theta', \cos\theta'', \chi) \\ &= \left[|H_+^e|^2 \sin^2\chi + |H_+^o|^2 \cos^2\chi - \Im(H_+^e H_+^{o*}) \sin(2\chi)\right] \sin^2\theta' \sin^2\theta'' \\ &+ 2\left[\Re(H_+^o H_0^{o*}) \cos\chi - \Im(H_+^e H_0^{o*}) \sin\chi\right] \sin\theta' \cos\theta' \sin\theta'' \cos\theta'' \\ &+ |H_0^o|^2 \cos^2\theta' \cos^2\theta''. \end{aligned} \tag{82}$$

Again the missing normalization factor is $9/8\pi\Gamma$. As discussed in the previous subsection the angular distribution for D_s^- decay will be equation 82 but with a sign change for the terms which contain $\sin\chi$ or $\sin(2\chi)$. When the angular distribution is written in this way, one sees that the CP-odd observables are the imaginary parts of interference terms between parity even and parity odd amplitudes.

At first glance, equation 82 is identical to equation 1.3 in reference [14]. However the definitions of the angles used in that note, $\theta_K, \theta_\pi, \tilde{\chi}$, are different from the ones used in this note, θ', θ'', χ. They are related by, $\theta_K = \pi - \theta'$, $\theta_\pi = \pi - \theta''$ and $\tilde{\chi} = -\chi$[15]. If one writes equation 82 in terms of $\theta_K, \theta_\pi, \tilde{\chi}$ then the form changes: the two CP odd observables change sign. Therefore the helicity amplitudes used in that paper do not mean the same thing as the helicity amplitudes used in this paper: they differ by a sign change in H_+^e. (Equivalently, one can say that they differ by a sign change in both H_+^o and H_0^o). Provided one fully understands the sign conventions both of these equations are correct.

17.8.3 Aside on CP violation

While the decay just discussed does conserve CP, it suggests a framework in which to discuss CP violation. Consider a decay of the form,

$$\begin{array}{lll} 0^- & \rightarrow & 1^-\ 1^- \\ & & \quad \hookrightarrow 0^-\ 0^- \\ & & \hookrightarrow 0^-\ 0^-, \end{array} \tag{83}$$

in which the first decay is weak and the subsequent decays are strong. Here each particle is labeled by its spin-parity, J^P. The allowed amplitudes for the first decay are as dicussed above, H_+, H_- and H_0. There is only one independent helicity amplitude for each of the subsequent decays and neither of these can affect the angular distributions. To further simplify the problem, consider that each of the 4 final state particles is distinguishable from the others. Now consider that there are several different Feynman diagrams which contribute to the decay. Following [12] and [13], the helicity amplitudes for this decay have the form,

$$H_\lambda = \sum_k h_\lambda^k \, e^{i(\phi_k + \delta_\lambda^k)}, \tag{84}$$

where the sum runs over all of the diagrams, ϕ_k is the CKM phase of the k^{th} diagram and where $h_\lambda^k e^{i\delta_\lambda^k}$ is the rest of the amplitude for that diagram. The quantities h_λ^k, ϕ_k and δ_λ^k are all real. The jargon is that ϕ_k are the weak phases and δ_λ^k are the strong phases. The angular distribution for this process is simply equation 76.

Now consider the charge conjugate decay, for which the helicity amplitudes will be denoted $\overline{H}_\lambda$. The angular distribution for the charge conjugate decay is simply equation 76, without any sign changes, but with all of the H_λ replaced by $\overline{H}_\lambda$. This is true whether or not CP is conserved in the decay. Following [12] and [13], the amplitudes for the charge conjugate decay are,

$$\overline{H}_\lambda = \sum_k h_{-\lambda}^k \, e^{i(-\phi_k + \delta_{-\lambda}^k)}. \tag{85}$$

Notice that the weak phases change sign while the strong phases remain unchanged. Also notice the sign change on the helicity index of $h_\lambda^k e^{i\delta_\lambda^k}$.

If CP is conserved in the first decay, substitution of equations 84 and 85 into equation 76 must recover the relation, $I(\theta', \theta'', \chi) = \overline{I}(\theta', \theta'', -\chi)$. In the example considered above, $D_s \rightarrow \phi\rho$, there is only one CKM phase which is common to all diagrams and the helicity amplitudes can be written as,

$$\begin{aligned} H_\lambda &= e^{i\phi} \sum_k h_\lambda^k e^{i\delta_\lambda^k} \\ \overline{H}_\lambda &= e^{-i\phi} \sum_k h_{-\lambda}^k e^{i\delta_{-\lambda}^k}. \end{aligned} \tag{86}$$

If one substitutes these expressions into equation 76, it is easily shown that the angular distribution conserves CP, that is that $I(\theta', \theta'', \chi) = \overline{I}(\theta', \theta'', -\chi)$.

Finally, the partial widths of the particle and anti-particle decays are given by,

$$\Gamma = \sum_\lambda |H_\lambda|^2 \tag{87}$$

$$\overline{\Gamma} = \sum_\lambda |\overline{H}_\lambda|^2. \tag{88}$$

Now consider what is necessary to observe CP violation. There are three sorts of observables in which one might observe CP violation:

- the partial rates, Γ and $\overline{\Gamma}$. If CP is conserved these are equal.
- the coefficients of the CP-even interference terms. If CP is conserved then these coefficients are the same for a given decay and for its charge conjugate decay.
- the coefficients of the CP-odd interference terms. If CP is conserved these change sign in the charge conjugate decay.

Any deviation from the above pattern is a manifestation of CP violation. By substitution of equations 84 and 85 into equation 76, one can verify,

- If there is only one term in the sum over k then the angular distribution will conserve CP, as will the partial rates.
- If there are several terms in the sum but all have the same CKM phase then, again, both the angular distribution and the partial rates will conserve CP.
- If there are two or more diagrams with different CKM phases, and if all of the h_λ^k are relatively real, then one can see a CP violating asymmetry in the coefficients of the CP-odd interference terms. However there will be no observable CP violation in either the partial widths or in the coefficients of the CP-even interference terms.
- If there are two or more diagrams with different CKM phases and if some of the h_λ^k are relatively complex, then one can observe CP violation in all of the observables.

One can also show that the CP violating parts of the CP odd interference terms contain factors of, $\sin(\phi_k - \phi_{k'})\cos(\delta_\lambda^k - \delta_{\lambda'}^{k'})$. On the other hand, the CP violating parts of the decay rate and of the CP even interference terms contain factors of $\sin(\phi_k - \phi_{k'})\sin(\delta_\lambda^k - \delta_{\lambda'}^{k'})$.

Also, if there are two terms in the sum, and if one of them is due to particle-anti-particle mixing in the initial state, then the CP violating observables will depend on the decay time of the initial state. Depending on details of the production mechanism, the time integral of these asymmetries may vanish. Such is the case for asymmetries arising from $B^0\overline{B^0}$ mixing in decays of the $\Upsilon(4S)$.

Another bit of jargon found in the literature is “direct” CP violation. This simply refers to a CP violating observable which comes from the interference of two diagrams, neither of which involves particle anti-particle mixing. These observables do not depend on the decay time of the initial state and the CP asymmetries will remain when the measurements are integrated over all decay times.

Finally, when the amplitude is given simply by the product two weak currents there are no strong phases, δ_λ^k. This also goes by the names "factorization" and "no final state interactions".

17.9 $B \to D^* \ell \nu$

Consider the decay chain,

$$\begin{array}{lll} \overline{B^0} & \to & D^{*+}\, W^{-*} \\ & & \quad \hookrightarrow \ell^- \, \overline{\nu} \\ & & \hookrightarrow D^0 \, \pi^+ . \end{array} \tag{89}$$

Here W^{-*} denotes a virtual W. The three independent helicity amplitudes for the first decay are $[A_{11}, A_{00}, A_{-1\,-1}]$ and these depend on q^2, the invariant mass squared of the virtual W. A more conventional notation for these amplitudes is, $H_+(q^2)$, $H_0(q^2)$, $H_-(q^2)$. There is only one independent helicity amplitude for the D^* decay and, in the limit of massless leptons, there is only one independent helicity amplitude for the W^{-*} decay. The angular distribution is independent of the helicity amplitudes of these last two decays. The dynamical variables for this decay are defined as follows:

θ' the angle between the D direction, measured in the rest frame of the D^*, and the D^* direction, measured in the B rest frame.

θ'' the angle between the ℓ direction, measured in the W^* rest frame, and the W^* direction measured in the B rest frame.

χ: Consider a righthanded orthonormal coordinate system. Look at the decay from the viewpoint that the momentum of the D^* is along the z axis and the momentum of the charged lepton is along the x axis. The angle χ is the azimuth of the π.

These definitions are appropriate both for the decay of a B and for the decay of a $\overline{B}$.

Now consider the allowed form of these amplitudes. There are only two Feynman diagrams which contribute to this decay, a direct diagram and a mixing diagram. These do have different CKM phases. At CLEO II, however, all of the B mesons are produced by the process $\Upsilon(4S) \to B\overline{B}$ and the angular distributions are integrated over all decay times. Therefore the interference between the direct and mixing terms vanishes. Therefore there are two contributions to the decay rate, the time integrated square of the direct term and the time integrated square of the mixing term. These are identical. Also, because the W materializes into a lepton and an anti-lepton, there are no strong final state interactions. Therefore the strong phases vanish and one can write, $H_\lambda(q^2) = V_{cb} h_\lambda(q^2)$, where $h_\lambda(q^2)$ is a real function of q^2.

Using the above notation, the angular distribution for decay 89 is given by equation 38,

$$
\begin{aligned}
&I(\theta', \theta'', \chi) \\
&\propto \frac{1}{8}\Big\{ |h_+(q^2)|^2 \sin^2\theta' \,(1-\cos\theta'')^2 + |h_-(q^2)|^2 \sin^2\theta' \,(1+\cos\theta'')^2 \\
&\quad + 4\,|h_0(q^2)|^2 \cos^2\theta' \sin^2\theta'' \\
&\quad - 2\, h_+(q^2) h_-(q^2) \cos(2\chi) \sin^2\theta' \sin^2\theta'' \\
&\quad + 4\, h_+(q^2) h_0(q^2) \cos\chi \sin\theta' \cos\theta' \sin\theta'' \,(1-\cos\theta'') \\
&\quad - 4\, h_0(q^2) h_-(q^2) \cos\chi \sin\theta' \cos\theta' \sin\theta'' \,(1+\cos\theta'')\Big\} \\
&= \frac{1}{8}\Big\{ \Big(|h_+(q^2)|^2 + |h_-(q^2)|^2\Big) \sin^2\theta' \Big(1+\cos^2\theta''\Big) \\
&\quad - 2\,\Big(|h_+(q^2)|^2 - |h_-(q^2)|^2\Big) \sin^2\theta' \cos\theta'' \\
&\quad + 4\,|h_0(q^2)|^2 \cos^2\theta' \sin^2\theta'' \\
&\quad - 2\, h_+(q^2) h_-(q^2) \sin^2\theta' \sin^2\theta'' \cos 2\chi \\
&\quad + 4\Big(h_+(q^2) h_0(q^2) - h_0(q^2) h_-(q^2)\Big) \sin\theta' \cos\theta' \sin\theta'' \cos\chi \\
&\quad - 4\Big(h_+(q^2) h_0(q^2) + h_0(q^2) h_-(q^2)\Big) \sin\theta' \cos\theta' \sin\theta'' \cos\theta'' \cos\chi\Big\}
\end{aligned}
\tag{90}
$$

The normalization factor is $9/8\pi\Gamma$.

Now consider the charge conjugate decay chain,

$$
\begin{array}{l}
B^0 \;\rightarrow\; D^{*-}\; W^{+*} \\
\qquad\qquad\quad \hookrightarrow \ell^+ \,\nu \\
\qquad \hookrightarrow \overline{D^0}\,\pi^- .
\end{array}
\tag{91}
$$

For this decay the helicity amplitudes will be denoted by, $\overline{H}_+(q^2)$, $\overline{H}_-(q^2)$, $\overline{H}_0(q^2)$. In terms of these amplitudes, the angular distribution $\overline{I}(\theta', \theta'', \chi)$ is given by equation 90 but with sign changes in front of the terms proportional to $(|h_+(q^2)|^2 - |h_-(q^2)|^2)$ and $(h_+(q^2)h_0(q^2) - h_0(q^2)h_-(q^2))$. These sign changes arise from the change in the handedness of the leptons, $\lambda_{\ell^-} - \lambda_{\overline{\nu}} = -1$ while $\lambda_{\ell^+} - \lambda_\nu = +1$. These factors enter into the second index of the D function for the decay of the W^*. However, from equations 84 and 85 one has $\overline{h}_\lambda(q^2) = h_{-\lambda}(q^2)$. This cancels the above sign changes and one finds that equation 90 is correct both for the decay of a $\overline{B^0}$ and for the decay of a B^0.

When comparing different calculations of this angular distribution one must check three things: the signs in front of each of the terms, the definition of the angles and the definition of the helicity amplitudes. That is, one must know whether $h_+(q^2)$ is the amplitude for a $\overline{B^0}$ to decay into a D^{*+} with helicity $+1$ or whether it is the amplitude for a B^0 to decay into a D^{*-} with helicity $+1$.

For example, equation 90 can be compared with the same angular distribution as derived by Korner and Schuler [10]. Their angular variables, $(\theta_{KS}, \theta^*_{KS}, \chi_{KS})$ are related to the ones defined in this section by $\theta_{KS} = \theta'$, $\theta^*_{KS} = \pi - \theta''$ and $\chi_{KS} = \pi + \chi$. Both equation 90 and their equation 22 are for the decay of a $\overline{B}^0$; therefore their helicity amplitudes are the same as the ones used in equation 90. In order to compare their results with the above equation one must also drop the terms proportional to the lepton mass from their equation. If one expresses their equation using the variables from this paper, equation 90 from this paper is reproduced.

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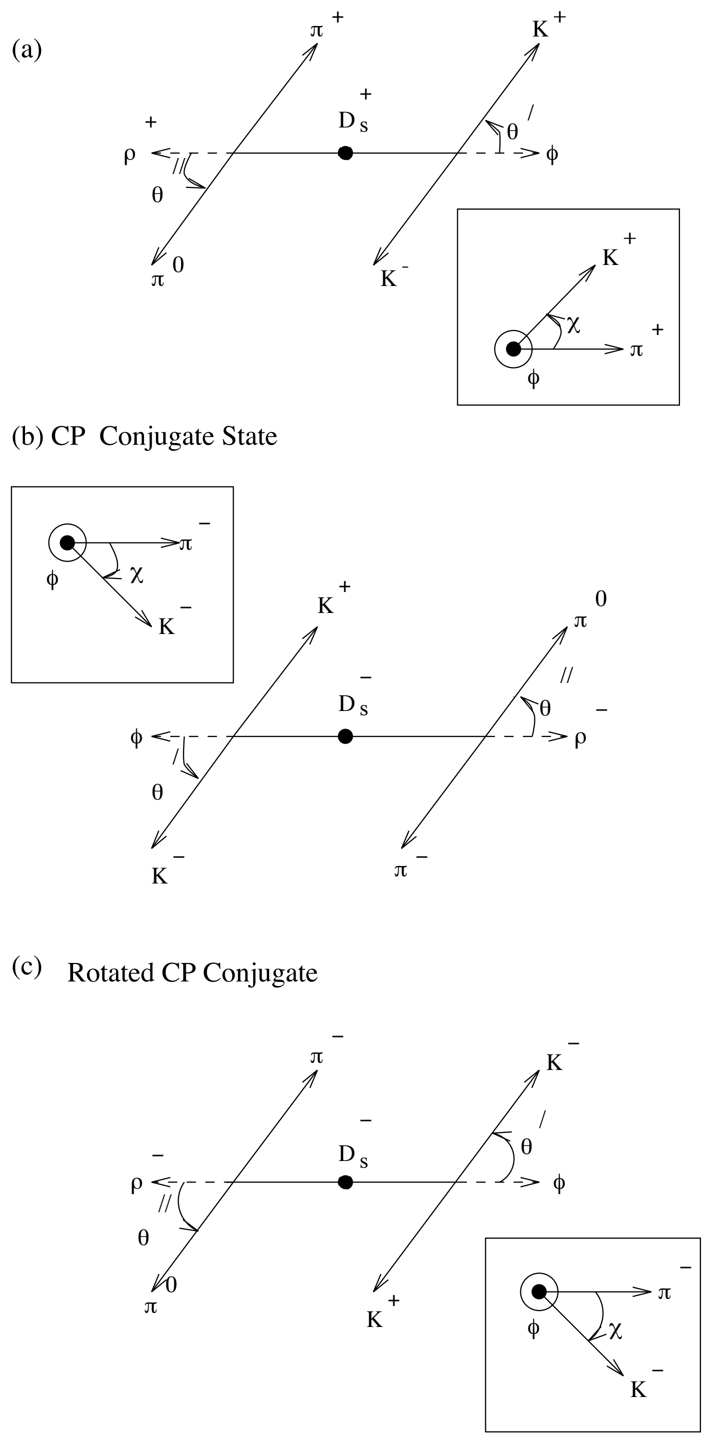

Figure 5: The decay a) $D_s^- \to \phi\rho^+$ and b) its CP conjugate. Part c) also shows the CP conjugate of a). The angles θ' and θ'' are defined, respectively, in the rest frames of the ϕ and the ρ. The angle χ is the same when measured in the rest frame of any of the ϕ, the ρ or the D_s. The sign change in χ is discussed in the text.