NNLO Computation in Heavy Quarkonium Process

冯 锋
中国矿业大学（北京）

2018.09 北京工业大学

Collaborators:

- Yu JIA （贾宇）GROUP
  高能物理研究所（IHEP）

- Wen-Long SANG （桑文龙）
  西南大学 （SWU）

Outline:

- A Brief Introduction.
- Our Recent Works @ NNLO corrections to Heavy Quarkonium Processes.
- Summary

Based on:

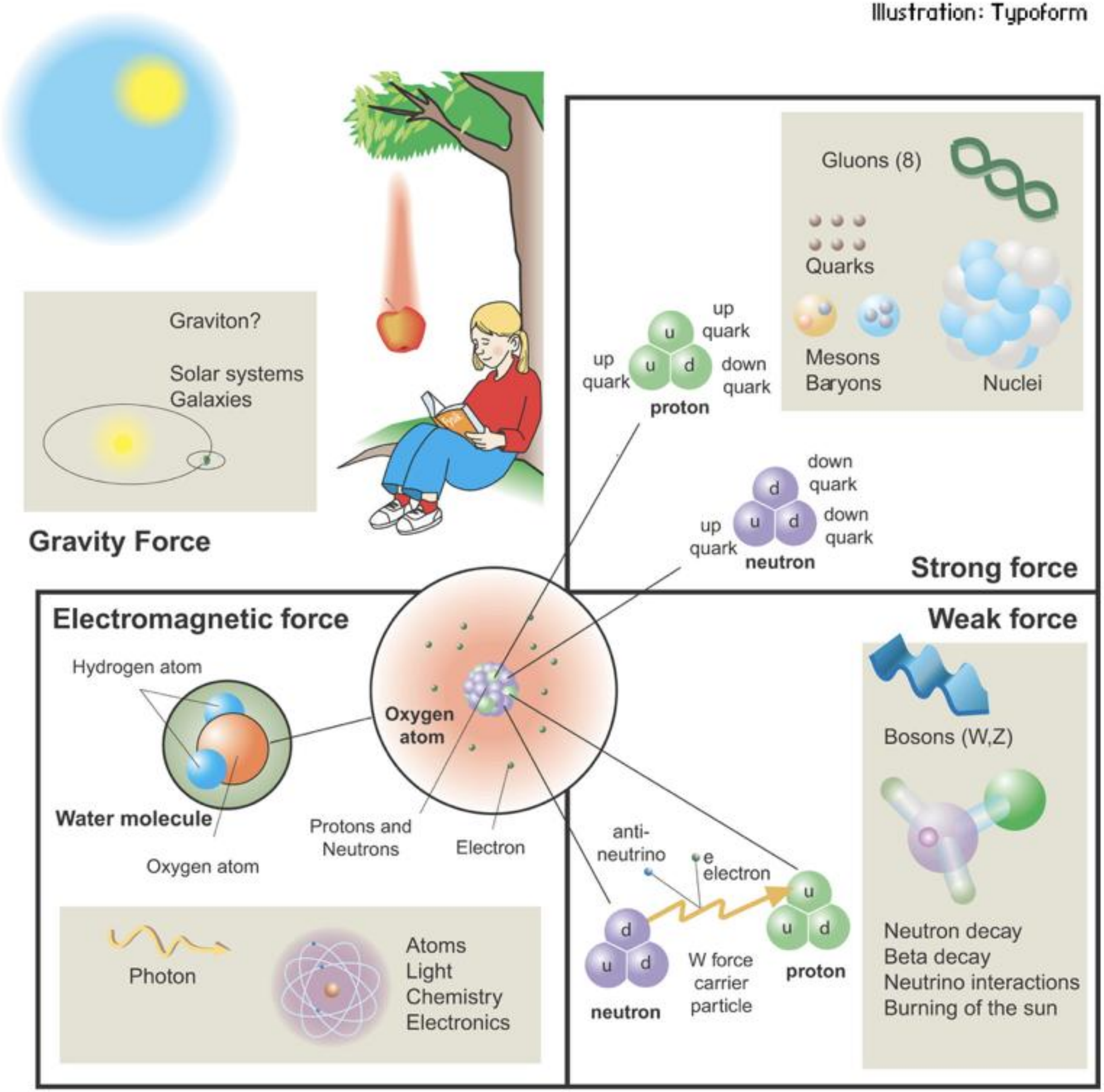
- arXiv:1505.02665（PRL 115, 222001）
  Can Nonrelativistic QCD Explain the $\gamma\gamma^* \to \eta_c$ Transition Form Factor Data?

- arXiv:1511.06288（PRD 94, 111501）
  Next-to-next-to-leading-order QCD corrections to $\chi_{c0,2} \to \gamma\gamma$

- arXiv:1707.05758（PRL 119, 252001）
  Next-to-next-to-leading-order QCD corrections to hadronic width of pseudoscalar quarkonium

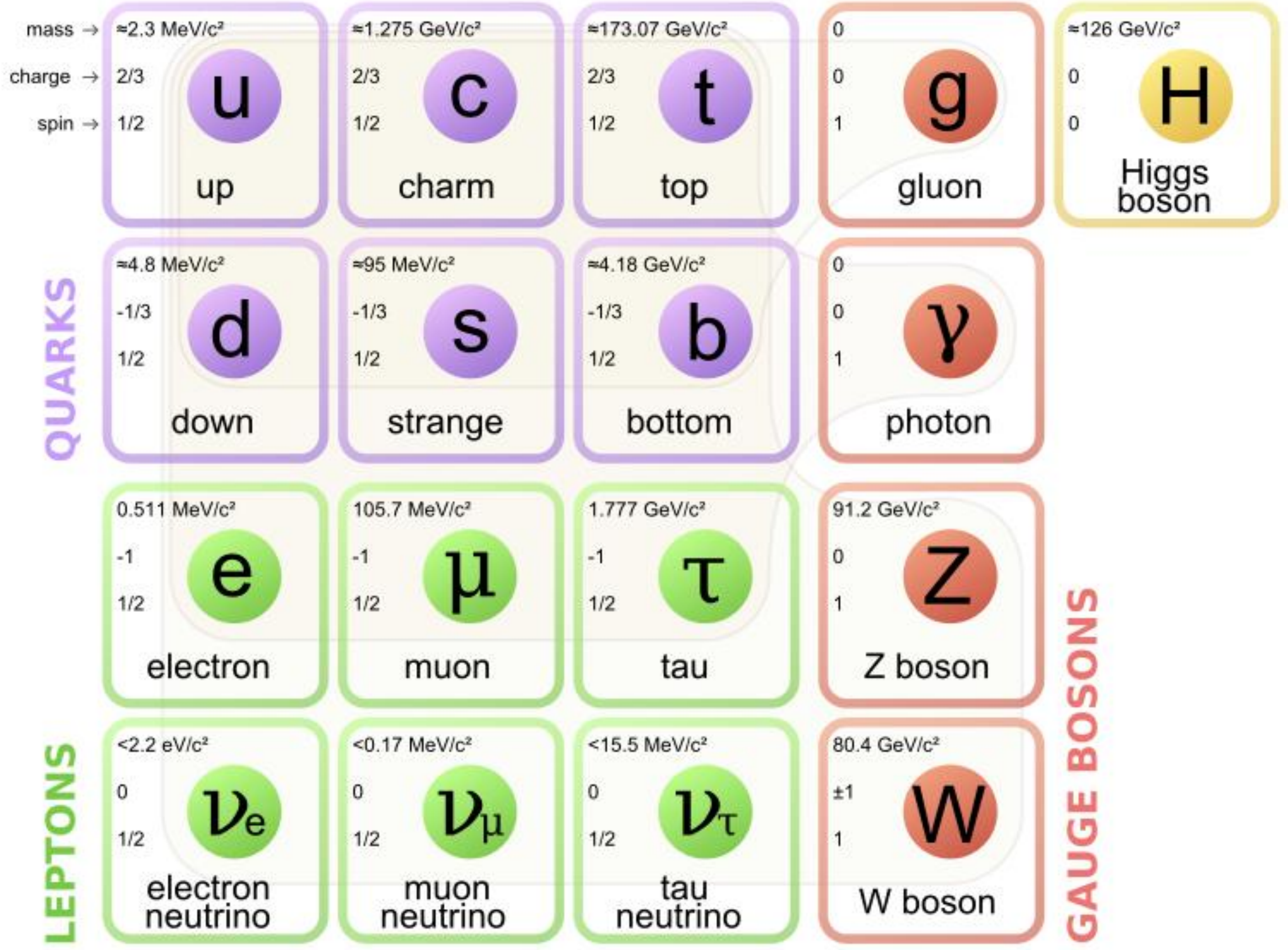
A Brief Introduction

Illustration: Typoform



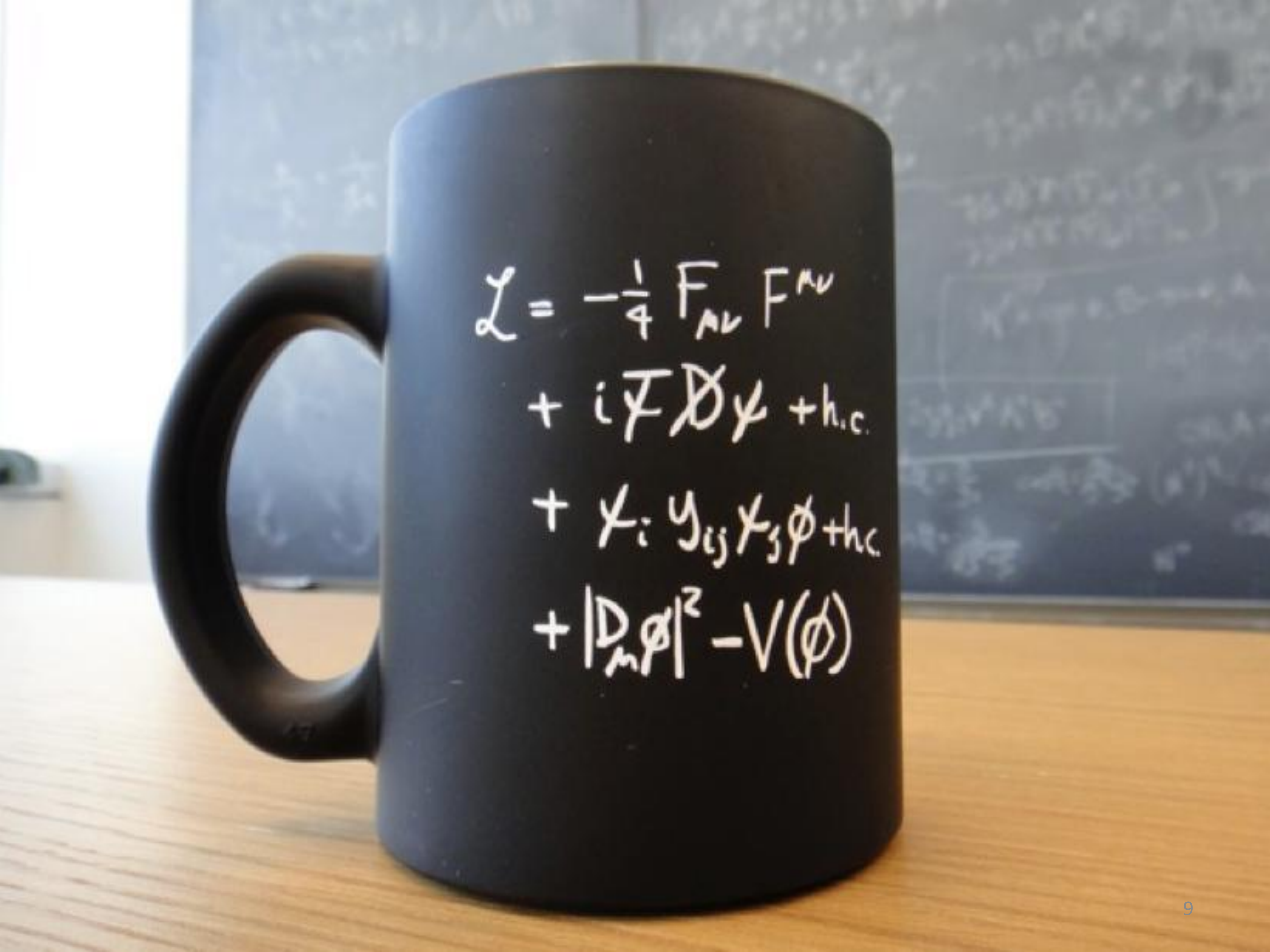
Forces in Nature :

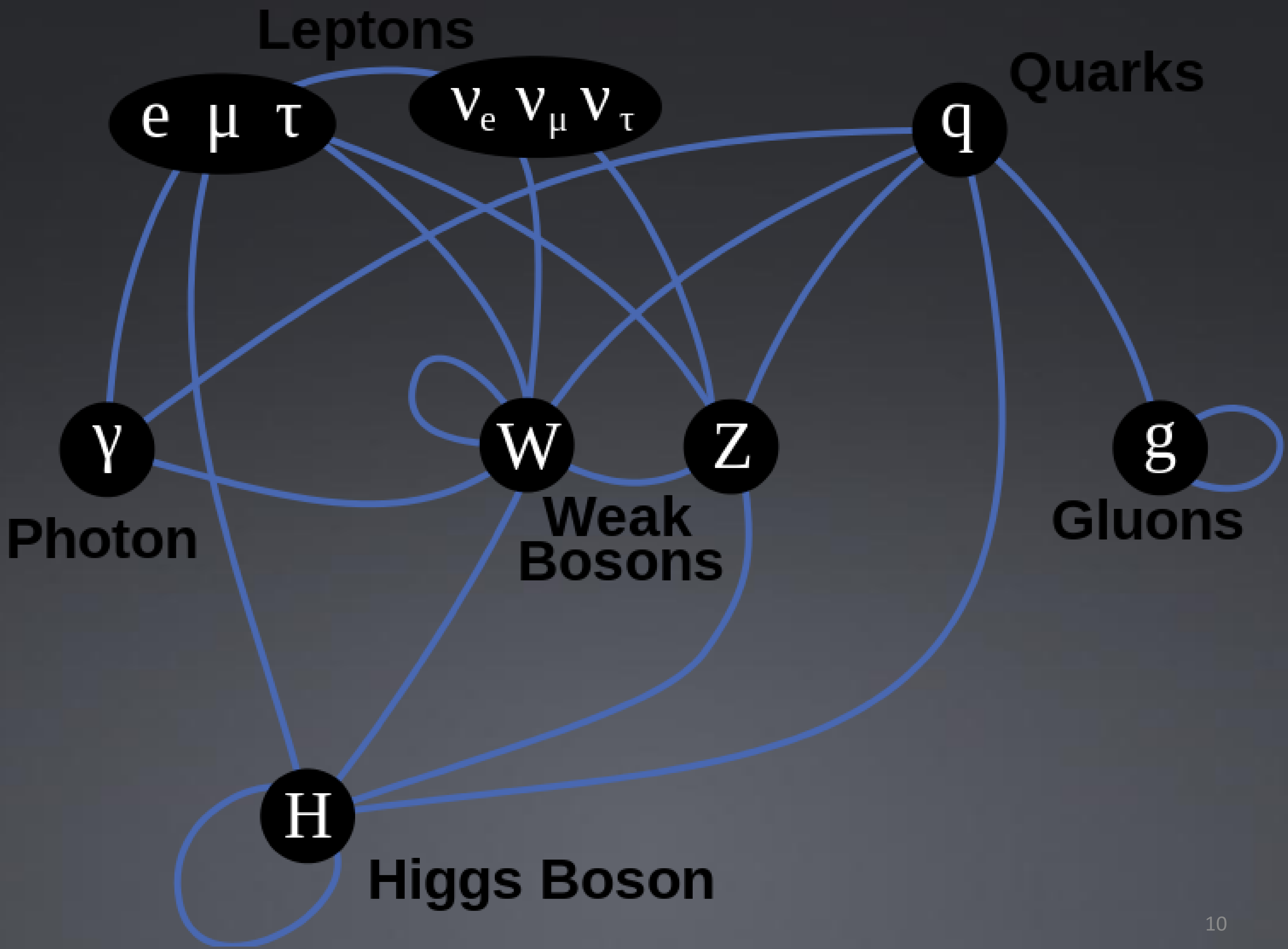
- Strong Force: Quantum Chromodynamics
  - QCD
- Electroweak Force: Electro-Weak theory
  - EW
- Standard Model (SM) = QCD + EW



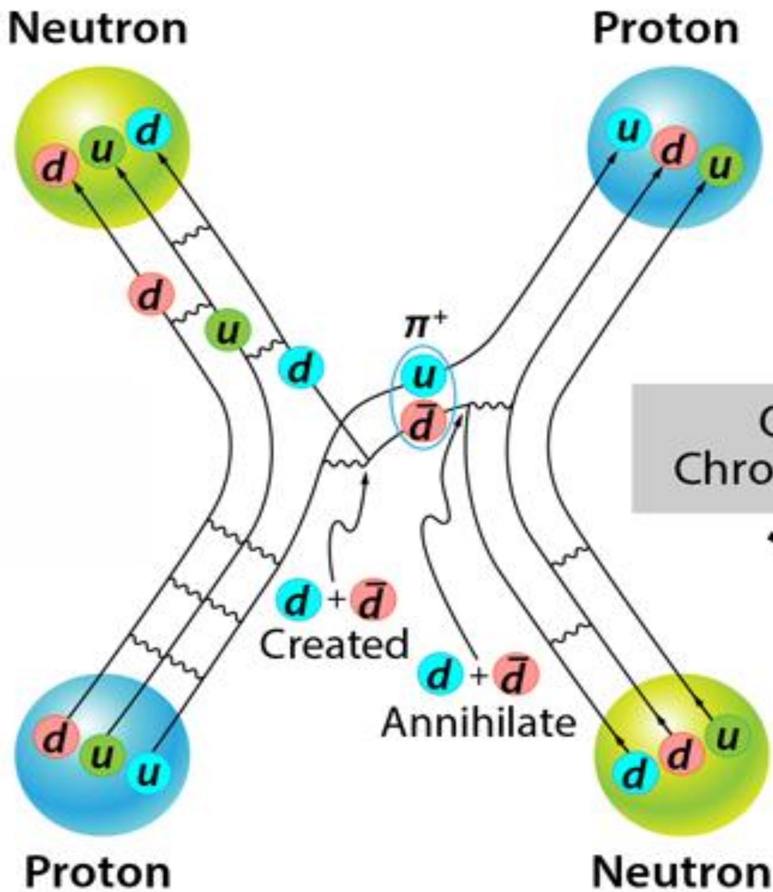
Particles in Standard Model

X

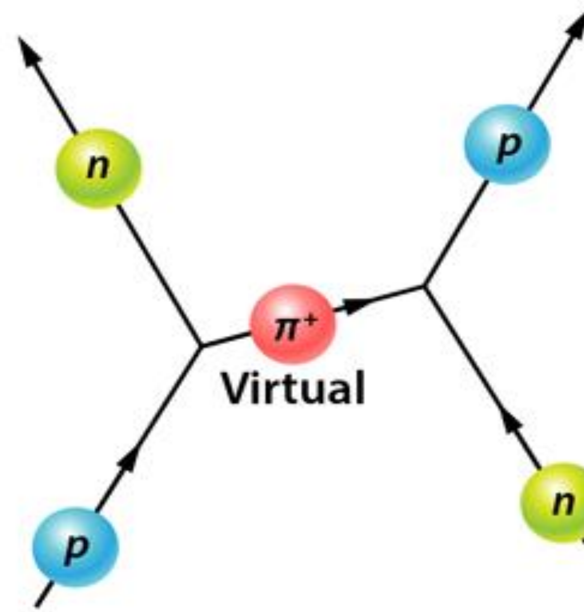

$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$
$+ i \bar{\psi} \not{D} \psi + h.c.$
$+ \psi_i y_{ij} \psi_j \phi + h.c.$
$+ |D_\mu \phi|^2 - V(\phi)$


Leptons
e μ τ
νe νμ ντ
Quarks
q
γ
W
Z
g
Photon
Weak Bosons
Gluons
H
Higgs Boson

QCD & Factorization



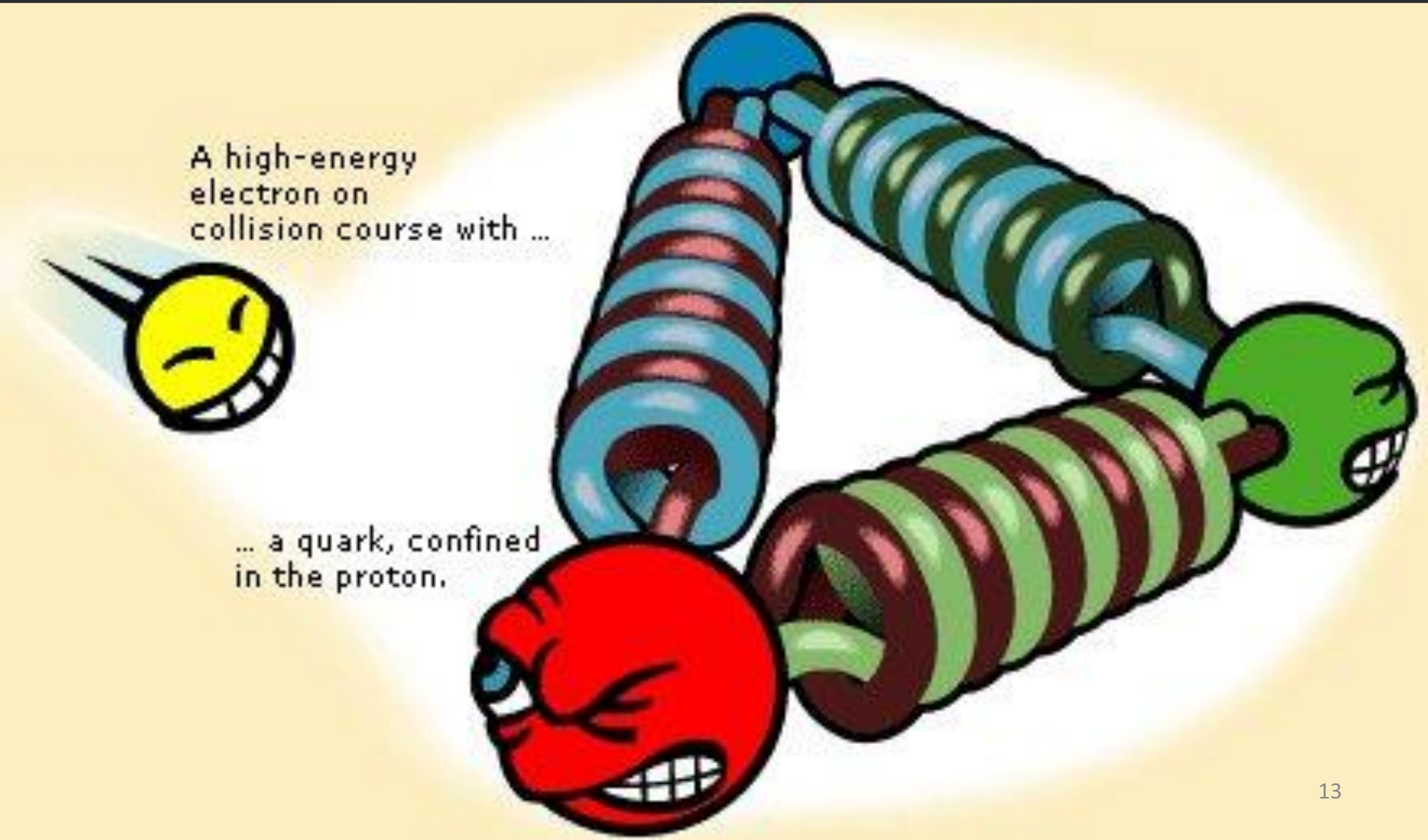
Quantum Chromodynamics



Chiral Effective-Field Theory

Nuclear Few- and Many-Body Problems

Confinement



Confinement

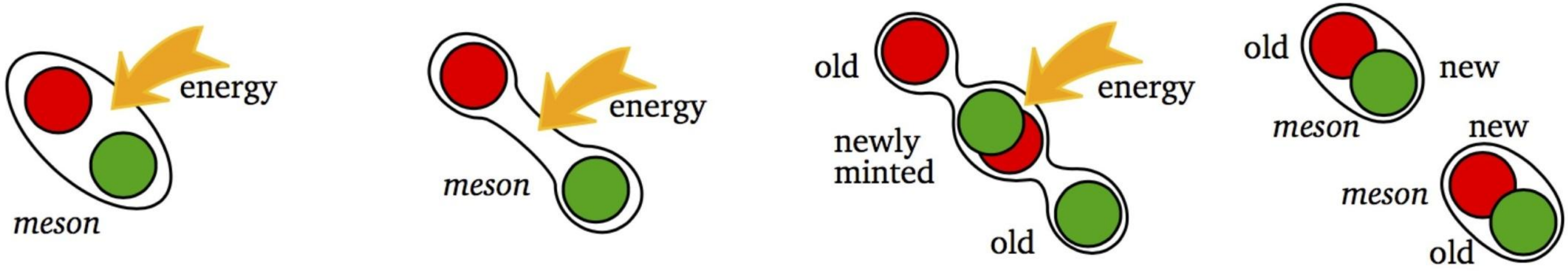
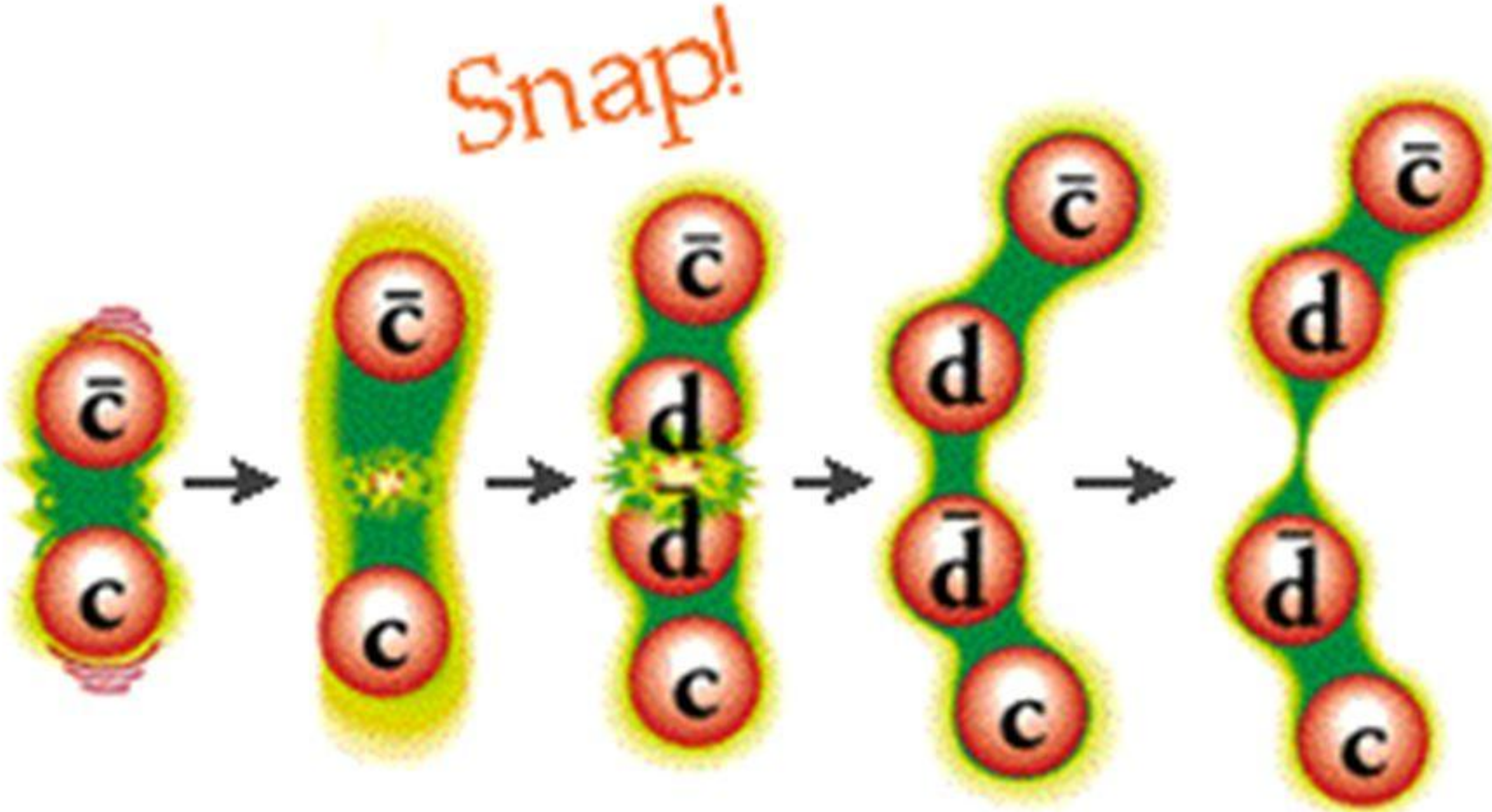
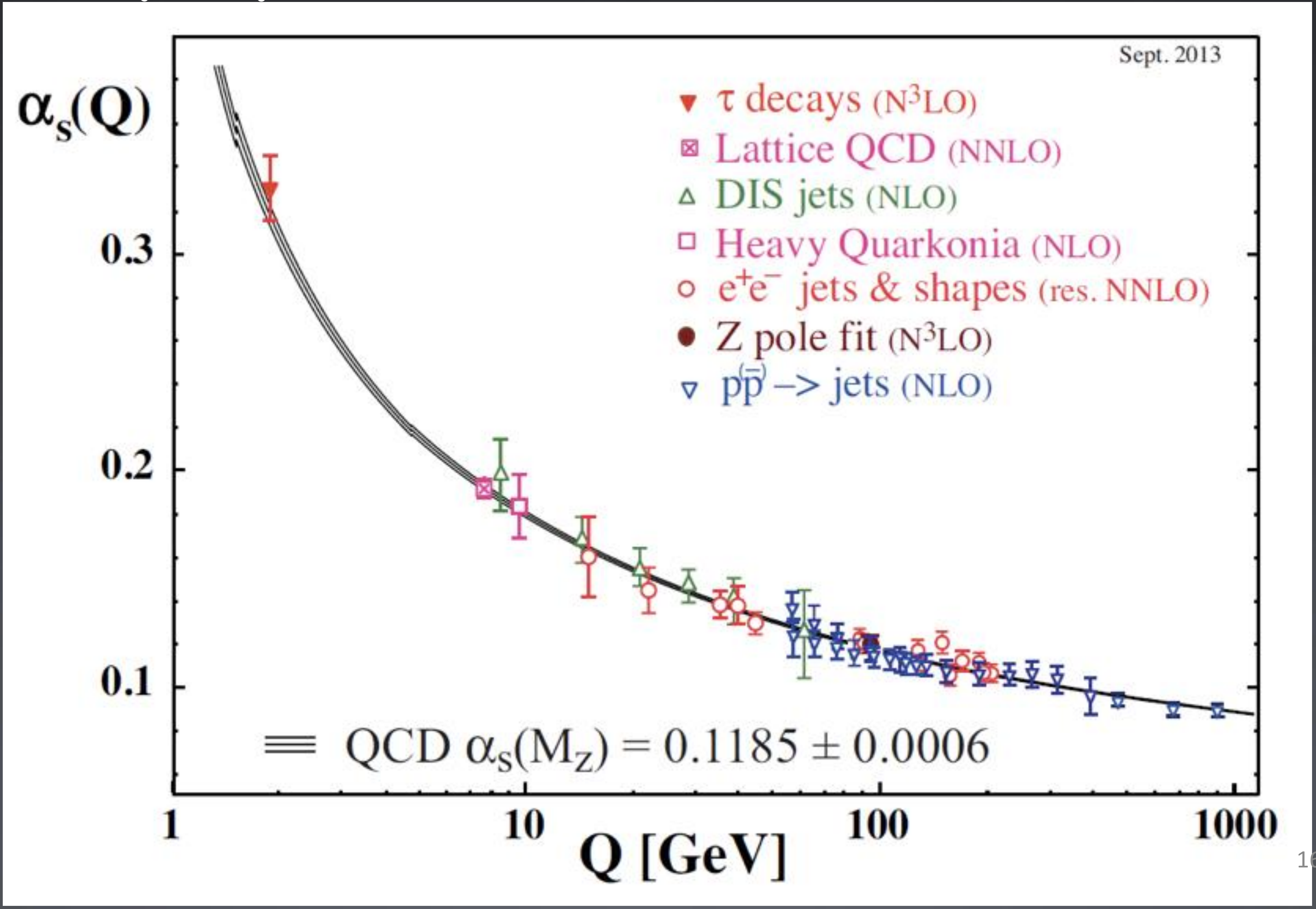


Figure –1.5: Inseparability of quarks and antiquarks in spite of investing ever more energy

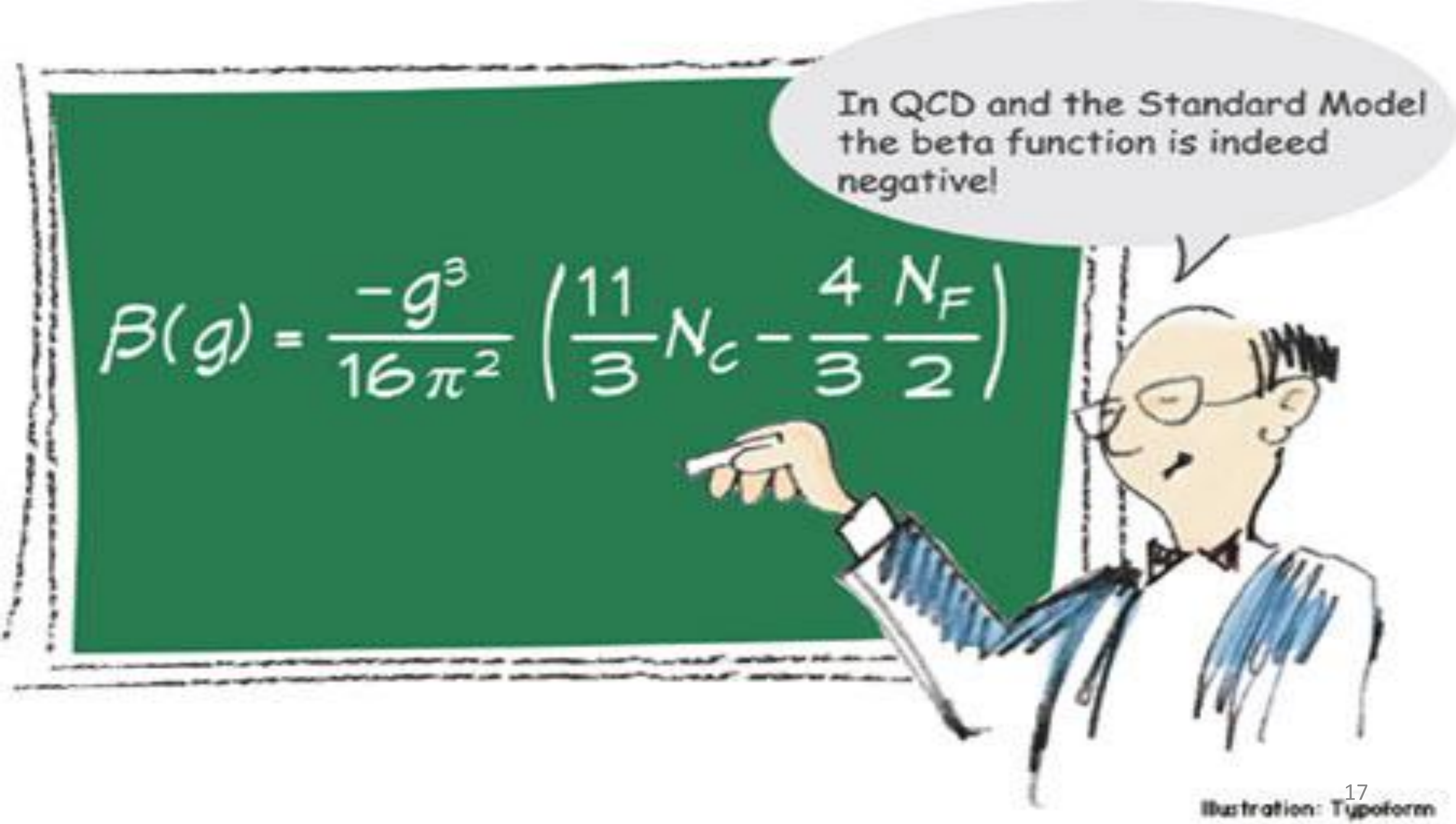
Quark confinement



Asymptotic Freedom



Asymptotic Freedom



Factorization:

- Factorization: to separate "short-distance" and "long-distance"、"high-energy" and "low-energy" effects.

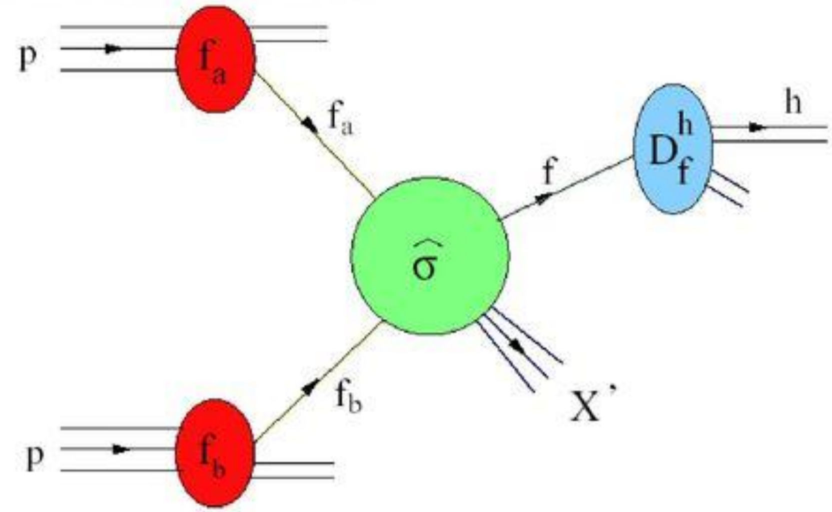
$$\mathcal{F} = \psi \otimes \mathcal{H} + \cdots$$

- Non-perturbative Part: "long-distance"、"low-energy" effects, due to confinement.

- Perturbative Part: "short-distance"、"high-energy" effects, due to asymptotic freedom.

QCD factorization

– ex. hadron production in proton collisions
  • pp→hX

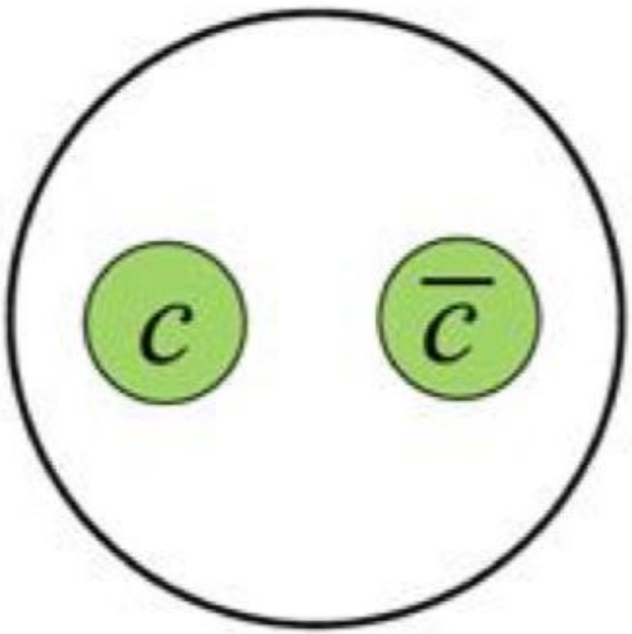


$$d\sigma = \sum_{a,b,c} \int dx_a \int dx_b \int dz_c \; f_a(x_a,\mu) \; f_b(x_b,\mu) \; D_c^h(z_c,\mu) \; d\hat{\sigma}_{ab}^c(x_a P_A, x_b P_B, P_h / z_c, \mu)$$

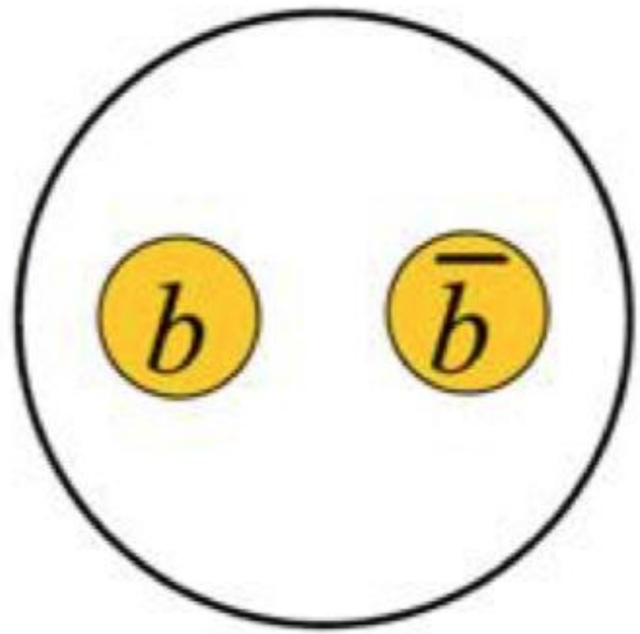
$f_a(x_a,\mu)\,,\, f_b(x_b,\mu)$	parton distribution function (PDF)	long distance term
$D_c^h(z_c,\mu)$	fragmentation function (FF)	
$d\hat{\sigma}_{ab}^c(x_a P_A, x_b P_B, P_h / z_c, \mu)$	partonic cross section	short distance term
μ	factorization scale – boundary between short and long distance	

Quarkonium & NRQCD

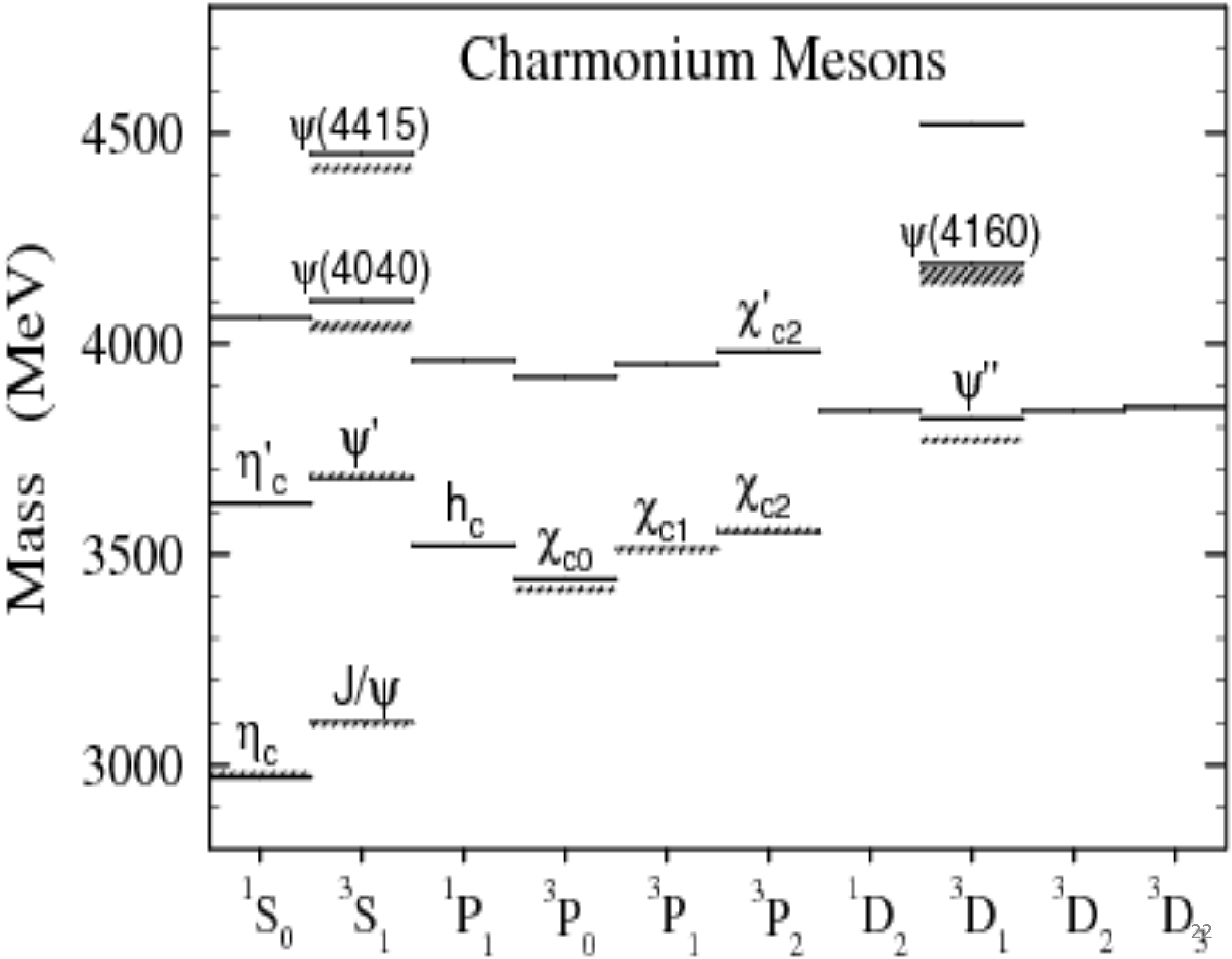
Heavy Quarkonium



"Charmonium" meson

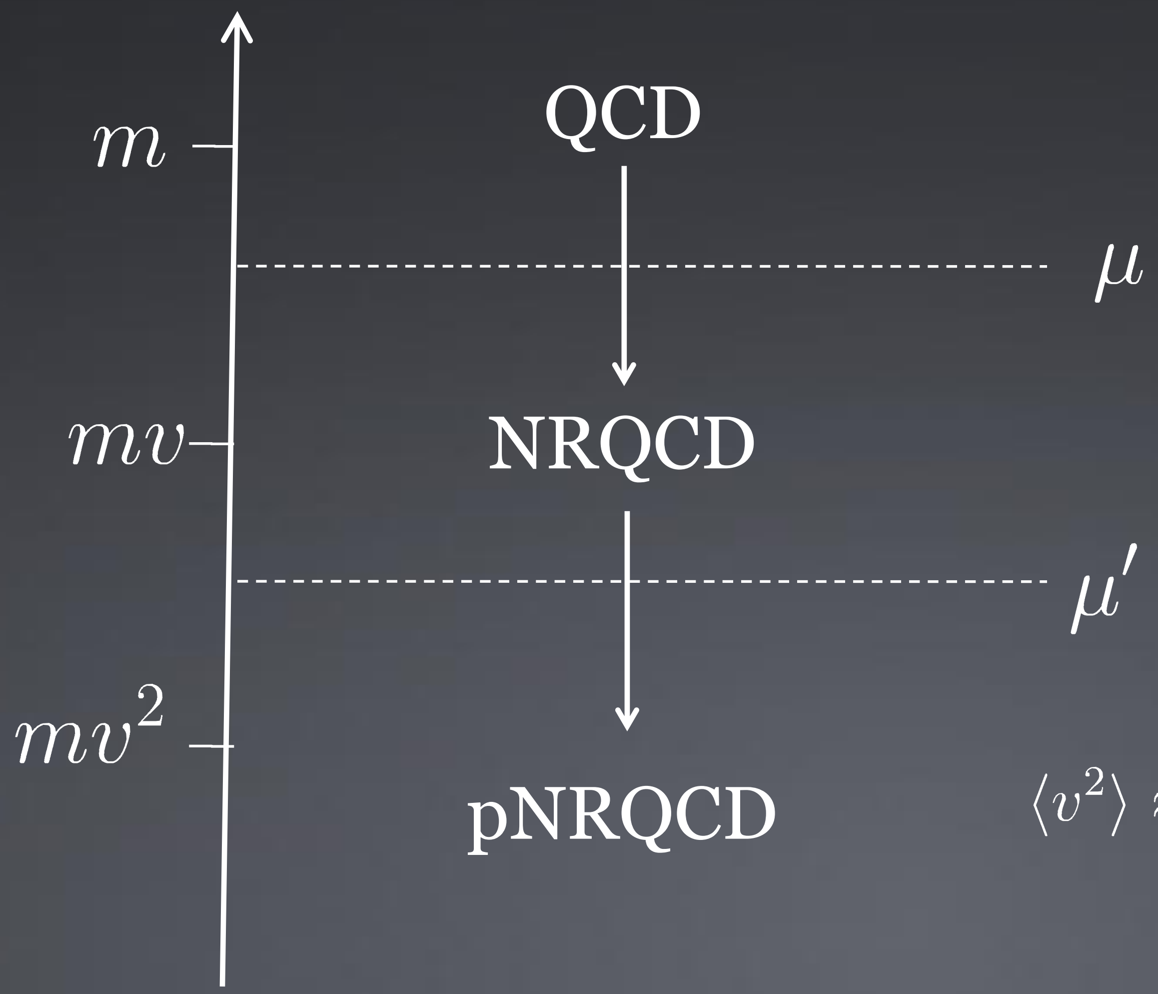


"Bottomonium" meson



Charmonium Mesons
Mass (MeV)
4500
4000
3500
3000
ψ(4415)
ψ(4040)
ψ(4160)
χ'c2
ψ''
η'c
ψ'
hc
χc0
χc1
χc2
J/ψ
ηc
1S0
3S1
1P1
3P0
3P1
3P2
1D2
3D1
3D2
3D3

NRQCD Factorization:



This scale separation is usually referred to as **NRQCD Factorization**

$$\langle v^2 \rangle \approx \begin{cases} 0.3 & \text{Charmonium} \\ 0.1 & \text{Bottomonium} \end{cases}$$

NRQCD Factrozaition for $\Gamma(^3S_1 \to \mathrm{LH})$

$$
\begin{aligned}
& \frac{F_1(^3S_1)}{m^2}\langle ^3S_1|\mathcal{O}_1(^3S_1)|^3S_1\rangle + \frac{G_1(^3S_1)}{m^4}\langle ^3S_1|\mathcal{P}_1(^3S_1)|^3S_1\rangle \\
+ & \frac{F_8(^1S_0)}{m^2}\langle ^3S_1|\mathcal{O}_8(^1S_0)|^3S_1\rangle + \frac{F_8(^3S_1)}{m^2}\langle ^3S_1|\mathcal{O}_8(^3S_1)|^3S_1\rangle \\
+ & \sum_{J=0,1,2} \frac{F_8(^3P_J)}{m^4}\langle ^3S_1|\mathcal{O}_8(^3P_J)|^3S_1\rangle + \frac{H_1^1(^3S_1)}{m^6}\langle ^3S_1|\mathcal{Q}_1^1(^3S_1)|^3S_1\rangle \\
+ & \frac{H_1^2(^3S_1)}{m^6}\langle ^3S_1|\mathcal{Q}_1^2(^3S_1)|^3S_1\rangle + \cdots
\end{aligned}
$$

Recent Works

NRQCD Status:

- Nowadays, NRQCD becomes an important approach to tackle various quarkonium production and decay processes.

- Large Corrections @NLO.

$e^+ + e^- \to J/\psi + \eta_c$ $\quad \mathcal{K} \approx 1.8 - 2.1$ Zhang *et.al.*

$p + p \to J/\psi + X$ $\quad \mathcal{K} \approx 2$ Campbell *et.al.*

$\cdots \quad \cdots$

Motivation

- Status @NNLO

$$\Gamma(J/\psi \to \ell\ell) = \Gamma^{(0)}\left\{1 - \frac{8}{3}\frac{\alpha_s}{\pi} - (44.55 - 0.41 n_f)\frac{\alpha_s^2}{\pi^2} + (-2091 + 120.66 n_f - 0.82\, n_f^2)\frac{\alpha_s^3}{\pi^3}\right\}$$

$$\Gamma(B_c \to \ell\nu) = \Gamma^{(0)}\left\{1 - 1.39\frac{\alpha_s}{\pi} - 23.7\frac{\alpha_s^2}{\pi^2} + \mathcal{O}(\alpha_s^3)\right\}$$

$$\Gamma(\eta_c \to \gamma\gamma) = \Gamma^{(0)}\left\{1 - 1.69\frac{\alpha_s}{\pi} - 56.52\frac{\alpha_s^2}{\pi^2} + \mathcal{O}(\alpha_s^3)\right\}$$

Motivation

- Status @NNLO

Decay

S-Wave

Exclusive

$$\Gamma(J/\psi \to \ell\ell) = \Gamma^{(0)}\left\{1 - \frac{8}{3}\frac{\alpha_s}{\pi} - (44.55 - 0.41 n_f)\frac{\alpha_s^2}{\pi^2} + (-2091 + 120.66 n_f - 0.82\, n_f^2)\frac{\alpha_s^3}{\pi^3}\right\}$$

$$\Gamma(B_c \to \ell\nu) = \Gamma^{(0)}\left\{1 - 1.39\frac{\alpha_s}{\pi} - 23.7\frac{\alpha_s^2}{\pi^2} + \mathcal{O}(\alpha_s^3)\right\}$$

$$\Gamma(\eta_c \to \gamma\gamma) = \Gamma^{(0)}\left\{1 - 1.69\frac{\alpha_s}{\pi} - 56.52\frac{\alpha_s^2}{\pi^2} + \mathcal{O}(\alpha_s^3)\right\}$$

Motivation:

- 1^{st} S-wave Production: $\gamma\gamma^* \to \eta_c$

TABLE III. The Q^2 interval and the weighted average Q^2 value ($\overline{Q^2}$), the $e^+e^- \to e^+e^-\eta_c$ cross section multiplied by $\mathcal{B}(\eta_c \to K\bar{K}\pi)$ $[d\sigma/dQ^2(\overline{Q^2})]$, and the normalized $\gamma\gamma^* \to \eta_c$ transition form factor ($|F(\overline{Q^2})/F(0)|$). The statistical and systematic errors are quoted separately for the cross section, but are combined in quadrature for the form factor. Only Q^2-dependent systematic errors are quoted; the Q^2-independent error is 6.6% for the cross section and 4.3% for the form factor.

Q^2 interval (GeV^2)	$\overline{Q^2}$ (GeV^2)	$d\sigma/dQ^2(\overline{Q^2})$ (fb/GeV^2)	$\lvert F(\overline{Q^2})/F(0)\rvert$
2–3	2.49	$18.7 \pm 4.2 \pm 0.8$	0.740 ± 0.085
3–4	3.49	$10.6 \pm 2.1 \pm 0.8$	0.680 ± 0.073
4–5	4.49	$6.62 \pm 1.18 \pm 0.19$	0.629 ± 0.057
5–6	5.49	$4.00 \pm 0.80 \pm 0.10$	0.555 ± 0.056
6–8	6.96	$3.00 \pm 0.43 \pm 0.17$	0.563 ± 0.043
8–10	8.97	$1.58 \pm 0.30 \pm 0.08$	0.490 ± 0.049
10–12	10.97	$0.72 \pm 0.17 \pm 0.05$	0.385 ± 0.048
12–15	13.44	$0.55 \pm 0.13 \pm 0.03$	0.395 ± 0.047
15–20	17.35	$0.34 \pm 0.07 \pm 0.01$	0.385 ± 0.038
20–30	24.53	$0.084 \pm 0.026 \pm 0.004$	0.261 ± 0.041
30–50	38.68	$0.019 \pm 0.009 \pm 0.001$	0.204 ± 0.049

(BaBar 2010)

Motivation:

- 1st P-wave Decay: $\chi_{c0,2} \to \gamma\gamma$

$$\Gamma_{\gamma\gamma}(\chi_{c0}) = (2.33 \pm 0.20 \pm 0.13 \pm 0.17)\,\mathrm{keV}$$

$$\mathcal{R} = \frac{\Gamma_{\gamma\gamma}(\chi_{c2})}{\Gamma_{\gamma\gamma}(\chi_{c0})} = 0.271 \pm 0.029 \pm 0.013 \pm 0.027$$

$$f_{0/2} = \frac{\Gamma^{\lambda=0}_{\gamma\gamma}(\chi_{c2})}{\Gamma^{\lambda=2}_{\gamma\gamma}(\chi_{c2})} = 0.00 \pm 0.02 \pm 0.02$$

(BESIII 2012)

Motivation:

- 1^{st} Inclusive Decay: $\eta_c \longrightarrow$ Light Hadrons

NLO Correction

1. R. Barbieri, E. d'Emilio, G. Curci and E. Remiddi, Nucl. Phys. B **154**, 535 (1979).

2. K. Hagiwara, C. B. Kim and T. Yoshino, Nucl. Phys. B **177**, 461 (1981).

(PDG 2017)

Motivation:

- 1st Inclusive Decay: $\eta_c \to$ Light Hadrons

1. R. Barbieri, E. d'Emilio, G. Curci and E. Remiddi, Nucl. Phys. B **154**, 535 (1979).
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NLO Correction

(PDG 2017)

40 Years Elapsed from **NLO** to **NNLO**

Another **???** Years to **NNNLO** ?

Basic Procedures:

1. Feynman Diagrams & Amplitudes
   - **FeynArts** - Mathematica Package
     http://www.feynarts.de
   - **QGRAF** - Fortran Program
     http://cfif.ist.utl.pt/~paulo/qgraf.html

2. Color- & Spin-Traces
   - **FeynCalc** - Mathematica Package
     https://github.com/FeynCalc
   - **FeynCalc/FormLink** - Mathematica Package
     https://github.com/FormLink

Basic Procedures:

3. Partial Fragmentation & IBP Reduction
   - **APart** - Mathematica Package
     https://github.com/F-Feng
   - **FIRE** - Mathematica Program & C++ version
     http://science.sander.su

4. Master Integrals - Numerical
   - **FIESTA** - Mathematica Package
     http://science.sander.su
   - **CubPack** - Fortran Code
     http://nines.cs.kuleuven.be/software/cubpack/

1st Production Process

$\gamma\gamma^* \to \eta_c$:

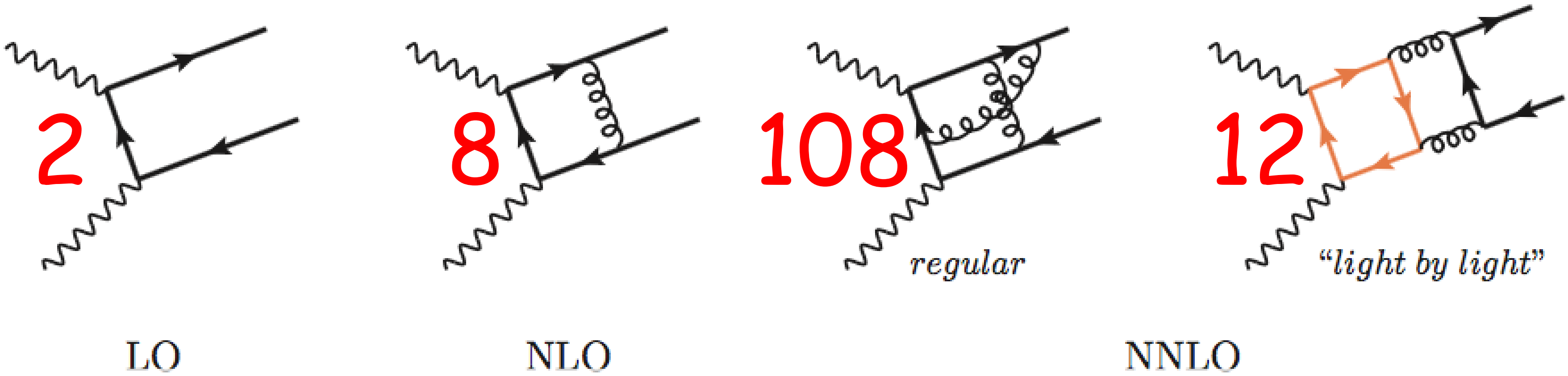


FIG. 1: Sample Feynman diagrams for $\gamma^*\gamma \to c\bar{c}({}^1S_0^{(1)})$.

$\gamma\gamma^* \to \eta_c$:

- Form Factor

$$\langle \eta_c(p) | J^\mu | \gamma(k,\varepsilon) \rangle = i e^2 \epsilon^{\mu\nu\rho\sigma} \varepsilon_\nu q_\rho k_\sigma F(Q^2)$$

- NRQCD Factorization

$$F(Q^2) = C(Q, m, \mu_R, \mu_\Lambda) \, \frac{\langle \eta_c | \psi^\dagger \chi(\mu_\Lambda) | 0 \rangle}{\sqrt{m}} + \mathcal{O}(v^2)$$

$\gamma\gamma^* \to \eta_c$:

- Form Factor

$$\langle \eta_c(p) | J^\mu | \gamma(k,\varepsilon) \rangle = i e^2 \epsilon^{\mu\nu\rho\sigma} \varepsilon_\nu q_\rho k_\sigma F(Q^2)$$

- NRQCD Factorization

$$F(Q^2) = C(Q, m, \mu_R, \mu_\Lambda) \frac{\langle \eta_c | \psi^\dagger \chi(\mu_\Lambda) | 0 \rangle}{\sqrt{m}} + \mathcal{O}(v^2)$$

Long Distance Matrix Element

$\gamma\gamma^* \to \eta_c$:

- Form Factor

$$\langle \eta_c(p)|J^\mu|\gamma(k,\varepsilon)\rangle = ie^2\epsilon^{\mu\nu\rho\sigma}\varepsilon_\nu q_\rho k_\sigma F(Q^2)$$

- NRQCD Factorization

$$F(Q^2) = C(Q, m, \mu_R, \mu_\Lambda)\frac{\langle \eta_c|\psi^\dagger\chi(\mu_\Lambda)|0\rangle}{\sqrt{m}} + \mathcal{O}(v^2)$$

Short Distance Coefficient

$\gamma\gamma^* \to \eta_c$:

- Form Factor

$$\langle \eta_c(p)|J^\mu|\gamma(k,\varepsilon)\rangle = ie^2\epsilon^{\mu\nu\rho\sigma}\varepsilon_\nu q_\rho k_\sigma F(Q^2)$$

- NRQCD Factorization

$$F(Q^2) = C(Q,m,\mu_R,\mu_\Lambda)\,\frac{\langle \eta_c|\psi^\dagger\chi(\mu_\Lambda)|0\rangle}{\sqrt{m}} + \mathcal{O}(v^2)$$

- Form Factor Ratio

$$\left|\frac{F(Q^2)}{F(0)}\right| = \left|\frac{C(Q,m,\mu_R,\mu_\Lambda)}{C(0,m,\mu'_R,\mu_\Lambda)}\right| + \mathcal{O}(v^2)$$

$\gamma\gamma^* \to \eta_c$:

- Perturbative Expansion

$$C(Q, m, \mu_R, \mu_\Lambda) = C^{(0)}(Q, m)\Bigg\{1 + C_F \frac{\alpha_s(\mu_R)}{\pi} f^{(1)}(\tau)$$

$$+\frac{\alpha_s^2}{\pi^2}\left[\frac{\beta_0}{4}\ln\frac{\mu_R^2}{Q^2+m^2}C_F f^{(1)}(\tau) - \pi^2 C_F\left(C_F + \frac{C_A}{2}\right)\right.$$

$$\left.\times \ln\frac{\mu_\Lambda}{m} + f^{(2)}(\tau)\right] + \mathcal{O}(\alpha_s^3)\Bigg\},$$

where $\tau = \frac{Q^2}{m^2}$

$\gamma\gamma^* \to \eta_c$:

- Result @LO

$$C^{(0)}(Q,m) = \frac{4e_c^2}{Q^2 + 4m^2}$$

- Result @NLO

$$f^{(1)}(\tau) = \frac{\pi^2(3-\tau)}{6(4+\tau)} - \frac{20+9\tau}{4(2+\tau)} - \frac{\tau(8+3\tau)}{4(2+\tau)^2}\ln\frac{4+\tau}{2}$$
$$+3\sqrt{\frac{\tau}{4+\tau}}\tanh^{-1}\sqrt{\frac{\tau}{4+\tau}} + \frac{2-\tau}{4+\tau}\left(\tanh^{-1}\sqrt{\frac{\tau}{4+\tau}}\right)^2$$
$$-\frac{\tau}{2(4+\tau)}\mathrm{Li}_2\left(-\frac{2+\tau}{2}\right)$$

$\gamma\gamma^* \to \eta_c$:

- Numerical Result @NNLO

τ	1	5	10	25	50
$f_{\text{reg}}^{(2)}$	-59.420(6)	-61.242(6)	-61.721(7)	-61.843(8)	-61.553(8)
$f_{\text{lbl}}^{(2)}$	$0.49(4)$ $-0.65(3)i$	$-0.48(5)$ $-0.72(4)i$	$-1.09(5)$ $-0.71(4)i$	$-2.12(6)$ $-0.69(4)i$	$-3.10(6)$ $-0.68(4)i$
$f_{\text{reg}}^{(2)}$	-59.636(6)	-61.278(6)	-61.716(7)	-61.864(8)	-61.668(8)
$f_{\text{lbl}}^{(2)}$	$0.8(1)$ $-12.44(8)i$	$-5.6(2)$ $-13.5(2)i$	$-9.4(2)$ $-13.8(2)i$	$-15.3(2)$ $-14.0(2)i$	$-20.3(2)$ $-14.1(2)i$

$f_{\text{reg}}^{(2)}(\tau)$ and $f_{\text{lbl}}^{(2)}(\tau)$ at some typical values of τ.
The first two rows for η_c and the last two for η_b

$\gamma\gamma^* \to \eta_c$:

- Confront the *BaBar* Data

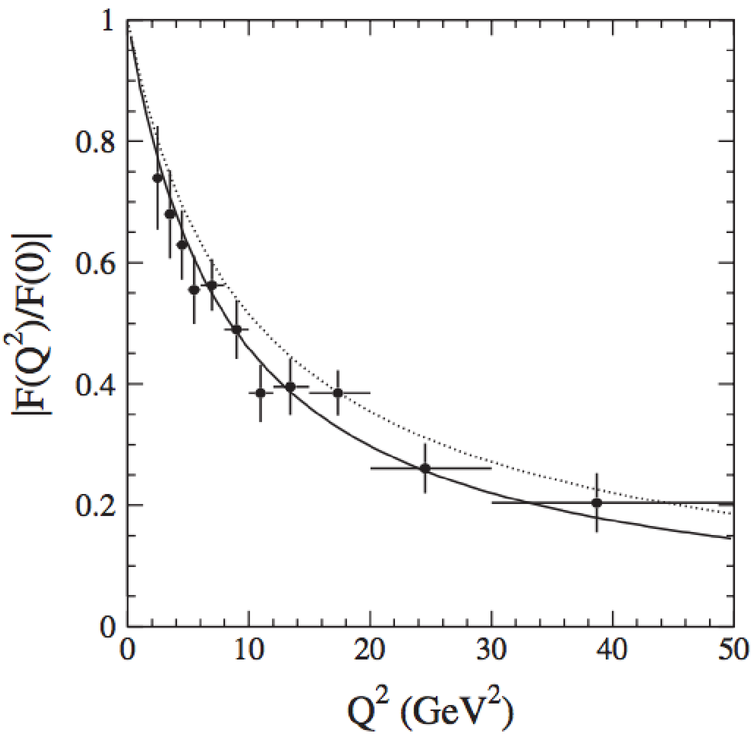


FIG. 26. The $\gamma\gamma^* \to \eta_c$ transition form factor normalized to $F(0)$ (points with error bars). The solid curve shows the fit to Eq. (22). The dotted curve shows the leading order pQCD prediction from Ref. [3].

$$\left|\frac{F(Q^2)}{F(0)}\right| = \frac{1}{1+Q^2/\Lambda}$$

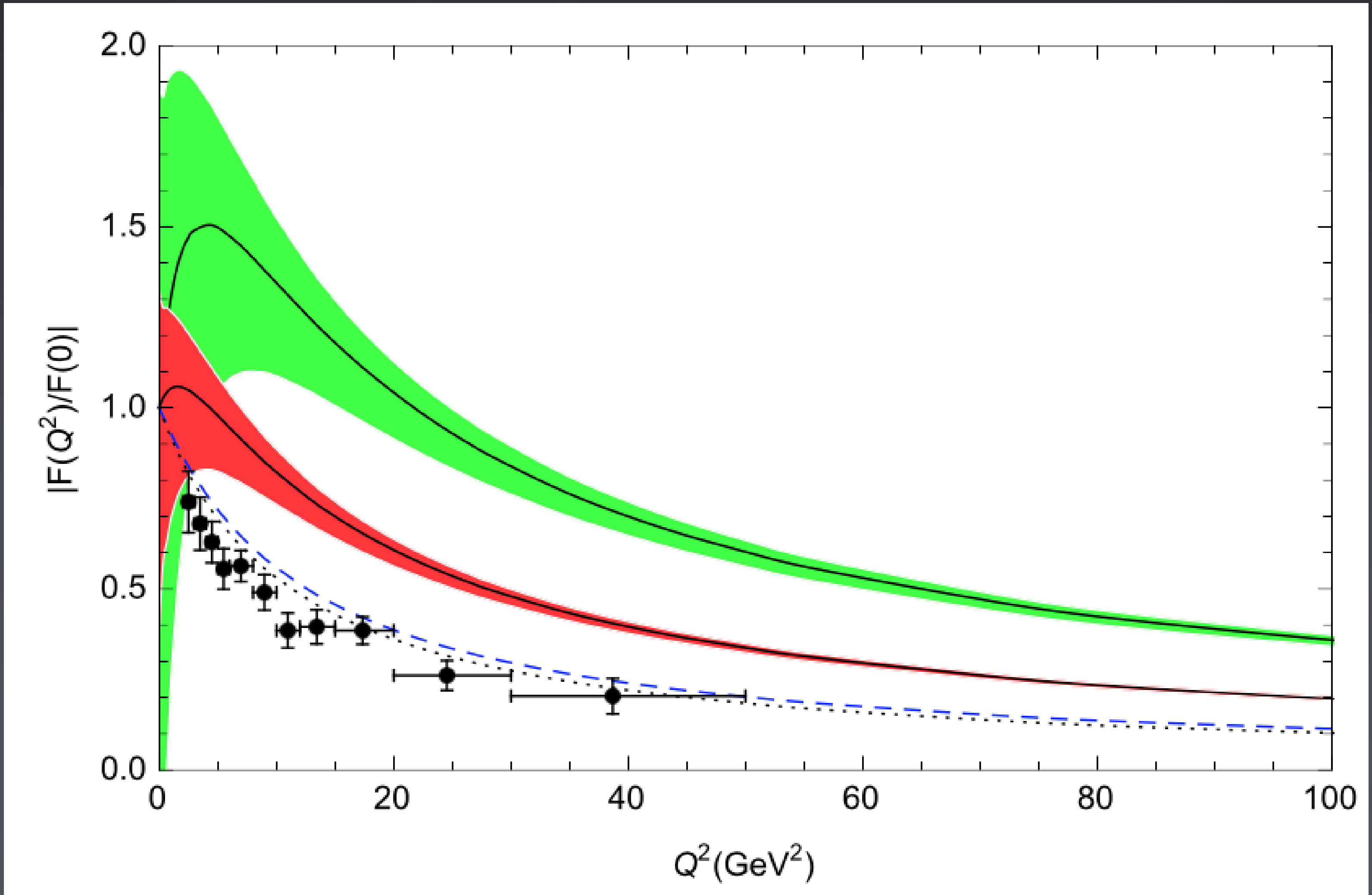
$$\Lambda = 8.5 \pm 0.6 \pm 0.7\ \mathrm{GeV}^2$$

$$\left|\frac{F(Q^2)}{F(0)}\right| = \frac{4m_c^2}{Q^2+4m_c^2}$$

$$4m_c^2 \approx 4 \times 1.5^2 = 9\mathrm{GeV}^2$$

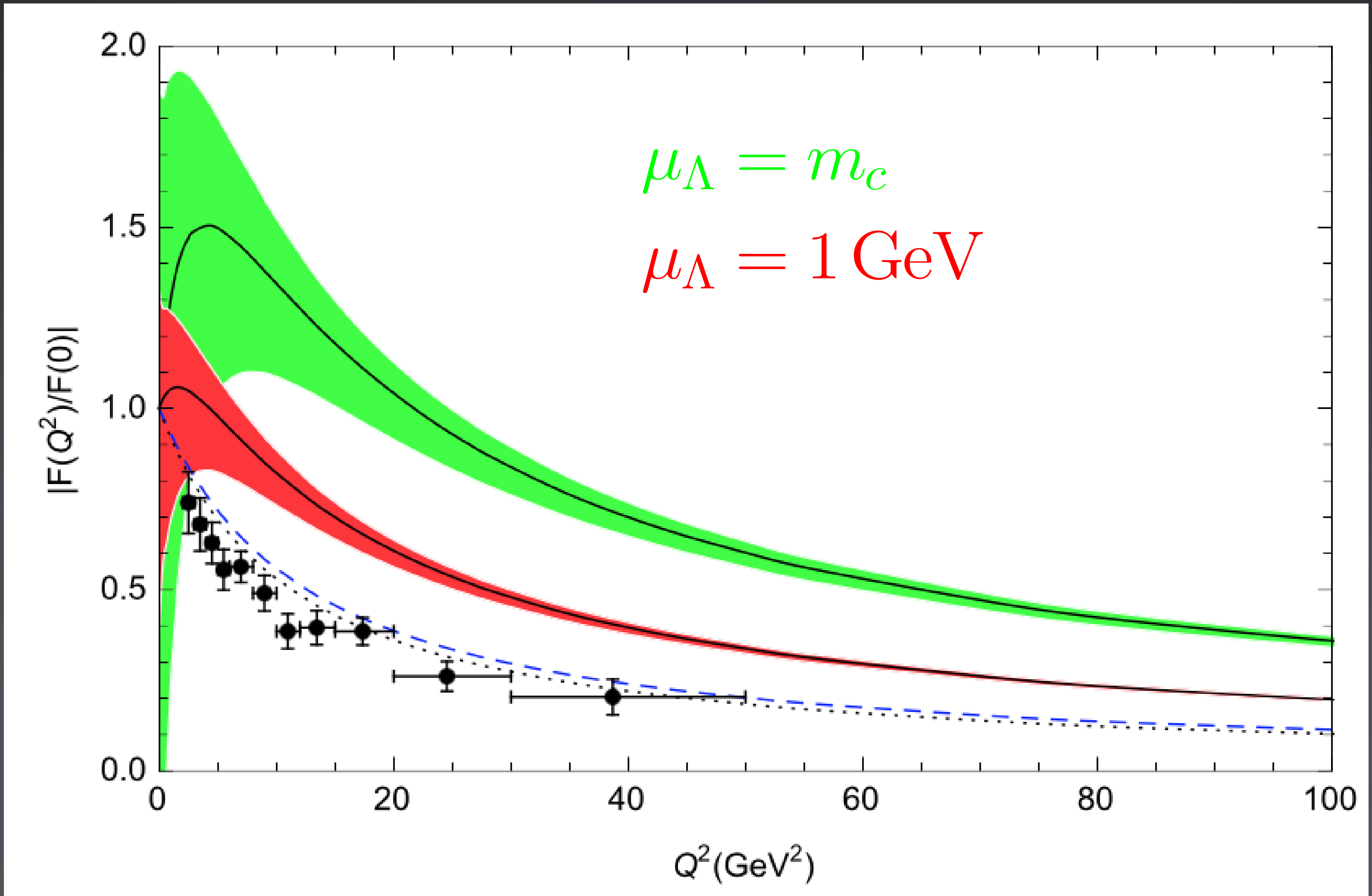
$\gamma\gamma^* \to \eta_c$:

- Confront the *BaBar* Data



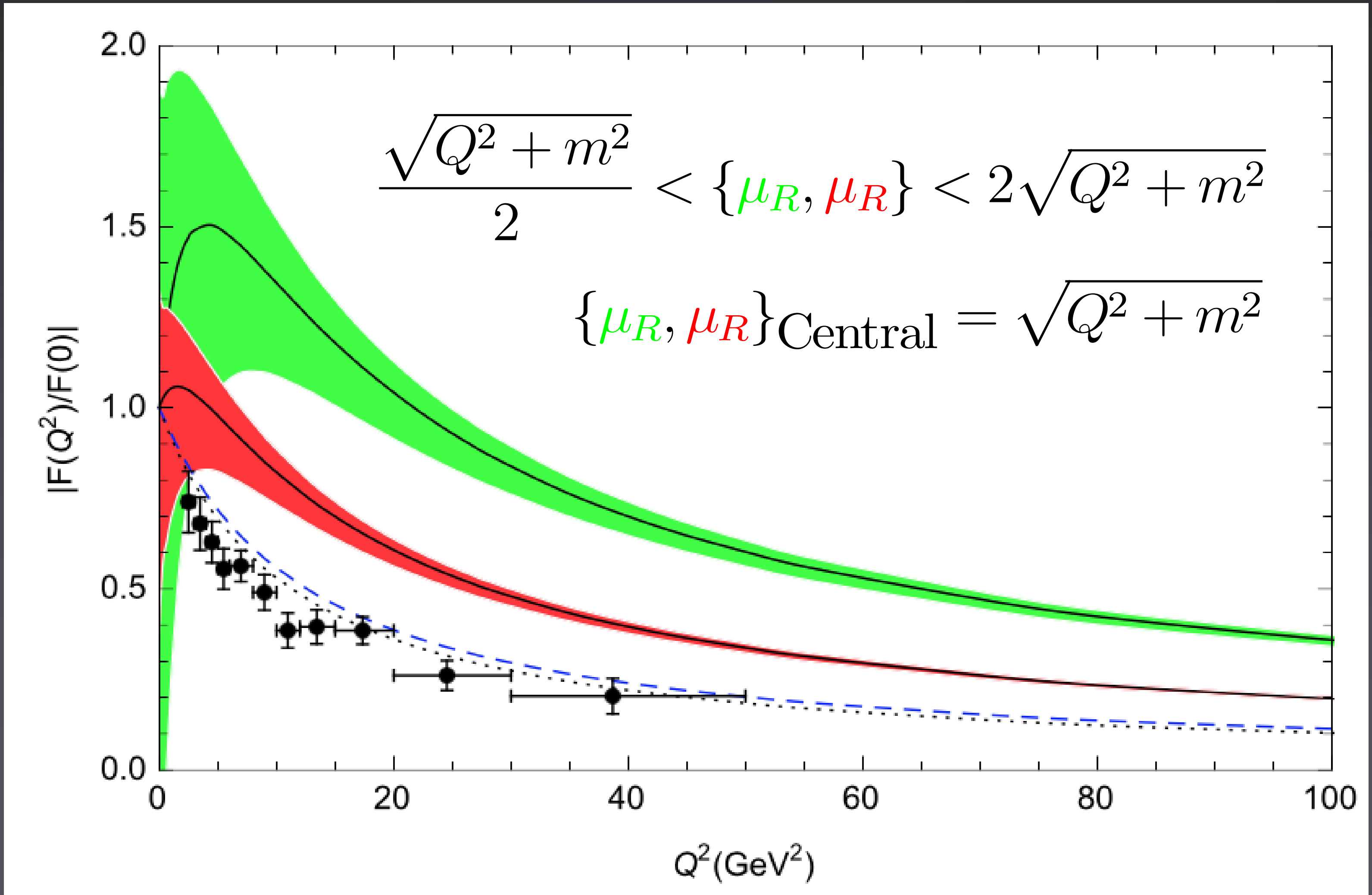
$\gamma\gamma^* \to \eta_c$:

- Confront the *BaBar* Data



$\gamma\gamma^* \to \eta_c$:

- Confront the *BaBar* Data



1st Inclusive Process

$\eta_c \to \text{Light Hadrons}$:

$$\Gamma(\eta_c \to \mathrm{LH}) = \frac{F_1({}^1S_0)}{m^2}\langle \eta_c|\mathcal{O}_1({}^1S_0)|\eta_c\rangle$$

$$+\frac{G_1({}^1S_0)}{m^4}\langle \eta_c|\mathcal{P}_1({}^1S_0)|\eta_c\rangle + \mathcal{O}(v^3\Gamma)$$

$\eta_c \to$ Light Hadrons :

$$\Gamma(\eta_c \to \mathrm{LH}) = \frac{F_1(^1S_0)}{m^2}\langle\eta_c|\mathcal{O}_1(^1S_0)|\eta_c\rangle + \frac{G_1(^1S_0)}{m^4}\langle\eta_c|\mathcal{P}_1(^1S_0)|\eta_c\rangle + \mathcal{O}(v^3\Gamma),$$

$$F_1(^1S_0) = \frac{\pi C_F \alpha_s^2}{N_c}\left\{1 + \frac{\alpha_s}{\pi} f_1 + \frac{\alpha_s^2}{\pi^2} f_2 + \cdots\right\}$$

$$G_1(^1S_0) = -\frac{4\pi C_F \alpha_s^2}{3N_c}\left\{1 + \frac{\alpha_s}{\pi} g_1 + \cdots\right\}$$

$\eta_c \to \text{Light Hadrons}$:

$$\Gamma(\eta_c \to \text{LH}) = \frac{F_1(^1S_0)}{m^2}\langle \eta_c|\mathcal{O}_1(^1S_0)|\eta_c\rangle + \frac{G_1(^1S_0)}{m^4}\langle \eta_c|\mathcal{P}_1(^1S_0)|\eta_c\rangle + \mathcal{O}(v^3\Gamma),$$

$$F_1(^1S_0) = \frac{\pi C_F \alpha_s^2}{N_c}\left\{1 + \frac{\alpha_s}{\pi}f_1 + \frac{\alpha_s^2}{\pi^2}f_2 + \cdots\right\},$$

$$G_1(^1S_0) = -\frac{4\pi C_F \alpha_s^2}{3N_c}\left\{1 + \frac{\alpha_s}{\pi}g_1 + \cdots\right\}.$$

$$f_1 = \frac{\beta_0}{2}\ln\frac{\mu_R^2}{4m^2} + \left(\frac{\pi^2}{4} - 5\right)C_F + \left(\frac{199}{18} - \frac{13\pi^2}{24}\right)C_A - \frac{8}{9}n_L - \frac{2n_H}{3}\ln 2,$$

Barbieri et al.,1979, Hagiwara et al.,1980

$$g_1 = \frac{\beta_0}{2}\ln\frac{\mu_R^2}{4m^2} - C_F\ln\frac{\mu_\Lambda^2}{m^2} - \left(\frac{49}{12} - \frac{5\pi^2}{16} - 2\ln 2\right)C_F + \left(\frac{479}{36} - \frac{11\pi^2}{16}\right)C_A - \frac{41}{36}n_L - \frac{2n_H}{3}\ln 2.$$

Guo,Ma,Chao, 2011

$\eta_c \to$ Light Hadrons :

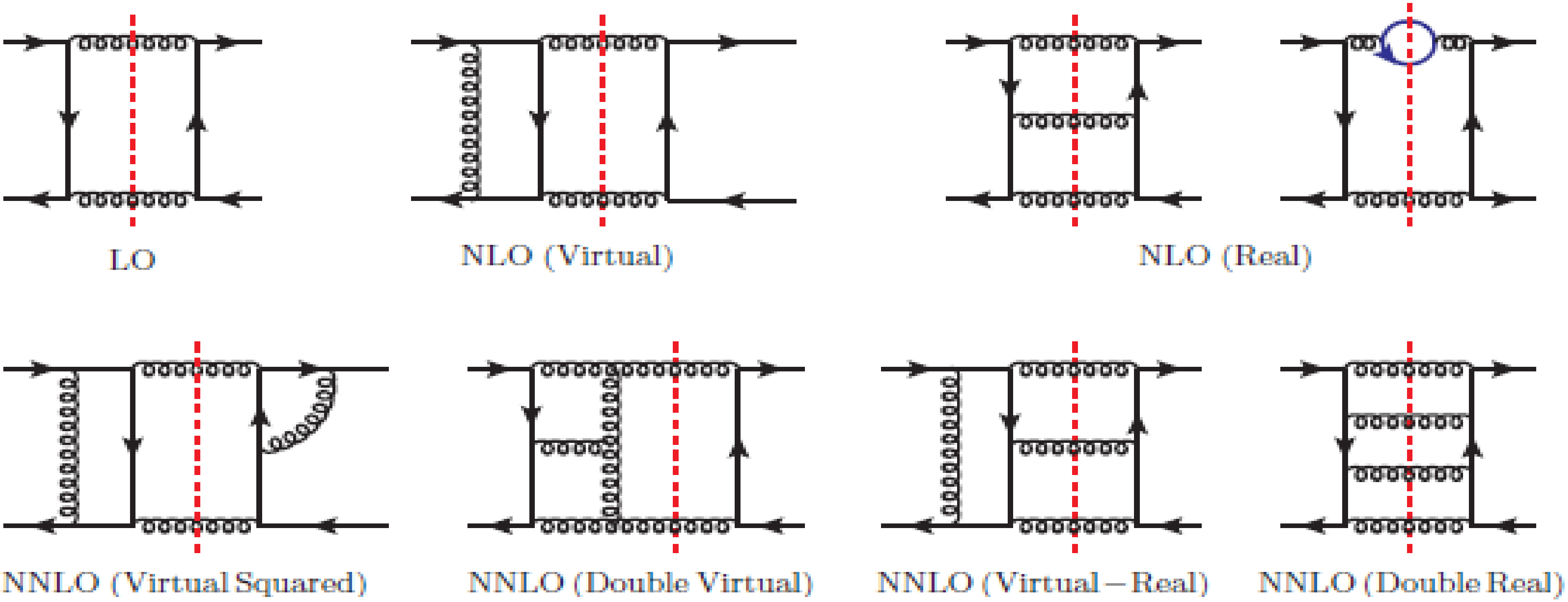


FIG. 1: Representative cut Feynman diagrams responsible for the quark reaction $c\bar{c}(^1S_0^{(1)}) \to c\bar{c}(^1S_0^{(1)})$ through NNLO in α_s. The vertical dashed line denotes the Cutkosky cut.

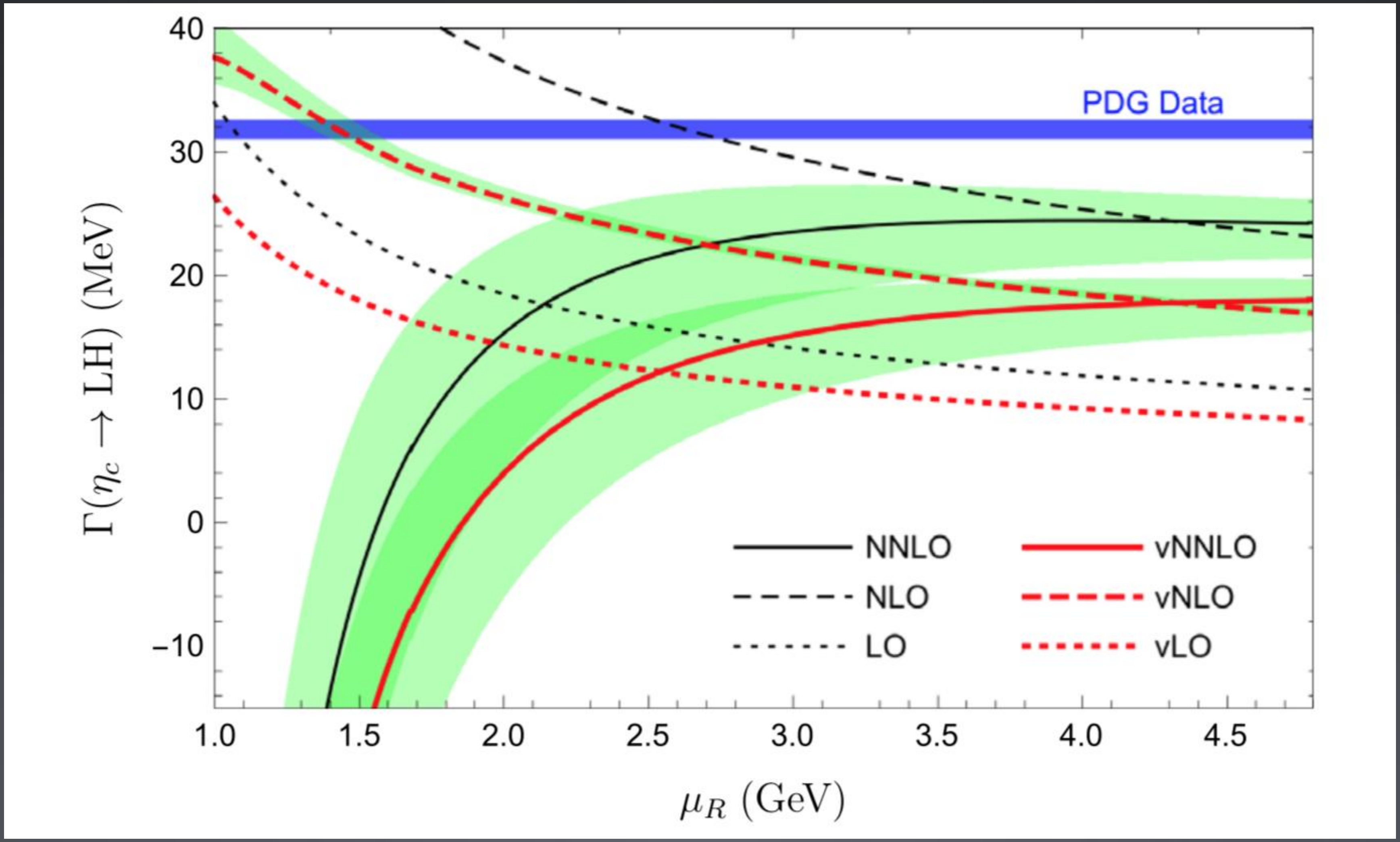
Roughly 1700 3-loop forward-scattering diagrams
Cutkosky rule is imposed

$\eta_c \to$ Light Hadrons :

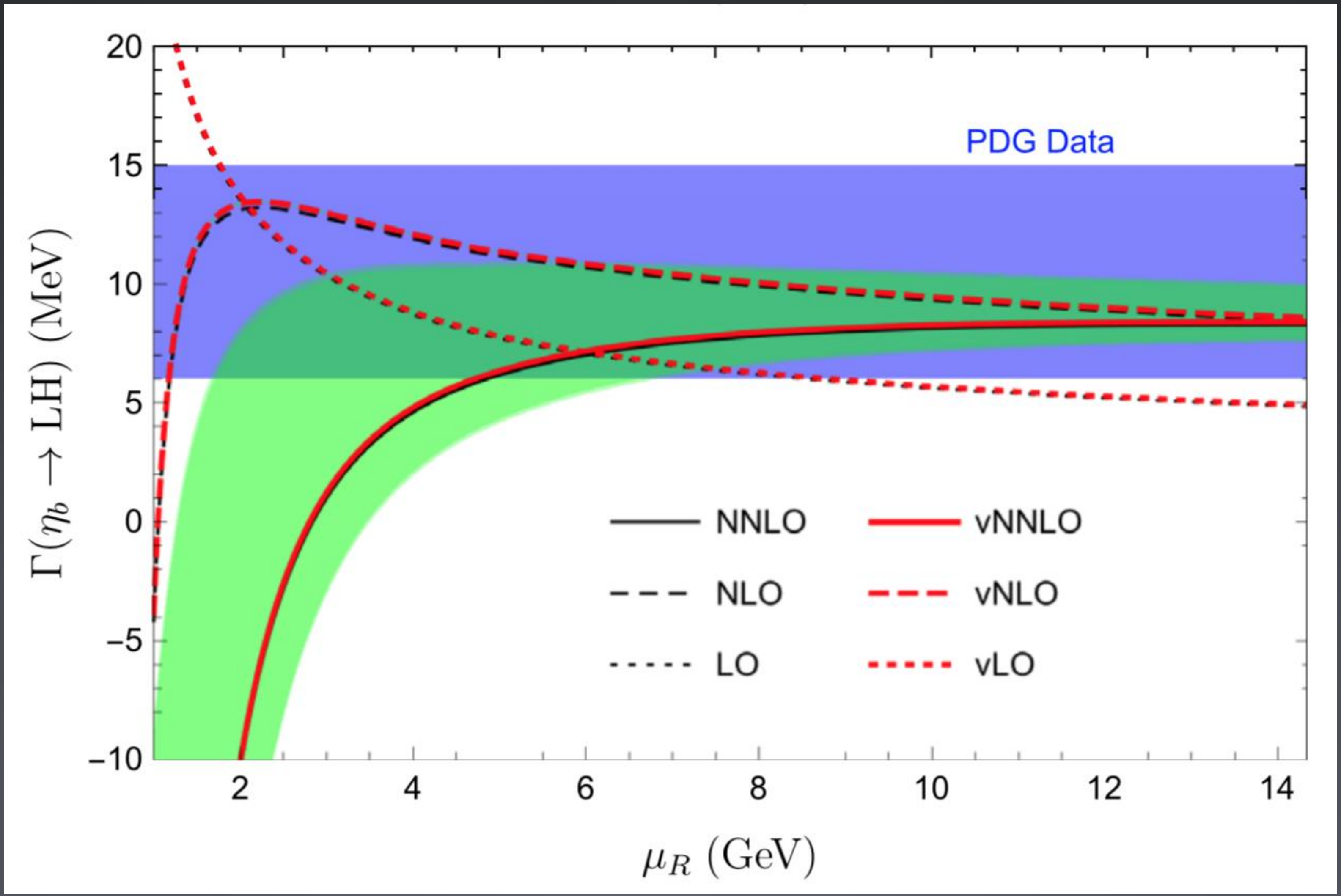
$$f_2 = \hat{f}_2 + \frac{3\beta_0^2}{16}\ln^2\frac{\mu_R^2}{4m^2} + \left(\frac{\beta_1}{8} + \frac{3}{4}\beta_0\hat{f}_1\right)\ln\frac{\mu_R^2}{4m^2}$$
$$-\pi^2\left(C_F^2 + \frac{C_A C_F}{2}\right)\ln\frac{\mu_\Lambda^2}{m^2}$$

$$\begin{aligned}\hat{f}_2 \;=\; & -0.799(13)N_c^2 - 7.4412(5)n_L N_c - 3.6482(2)N_c \\ & +0.37581(3)n_L^2 + 0.56165(5)n_L + 32.131(5) \\ & -0.8248(3)\frac{n_L}{N_c} - \frac{0.67105(3)}{N_c} - \frac{9.9475(2)}{N_c^2}\end{aligned}$$

$\eta_c \to$ Light Hadrons :



$\eta_c \to$ Light Hadrons :

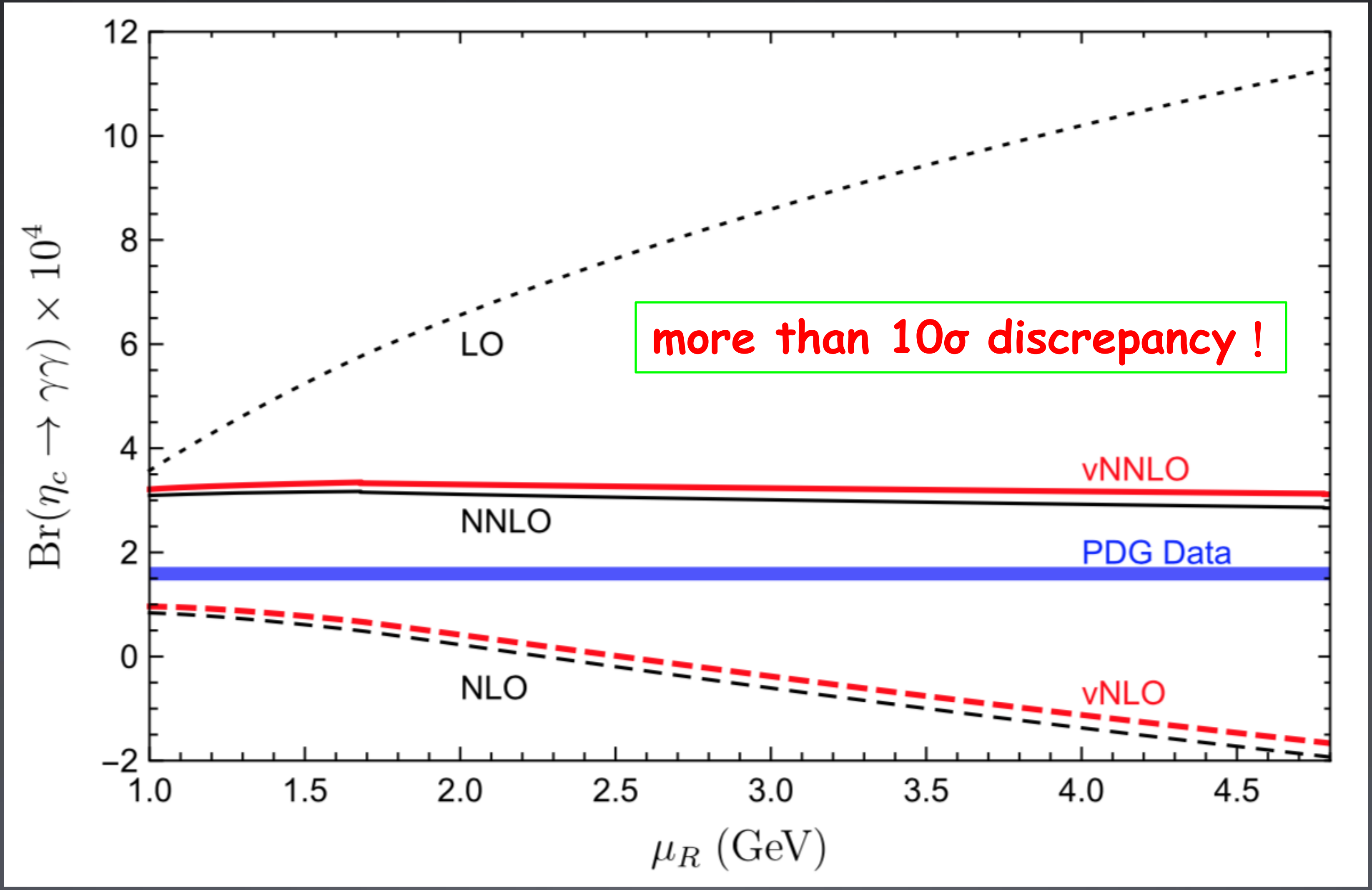


$\eta_c \to$ Light Hadrons :

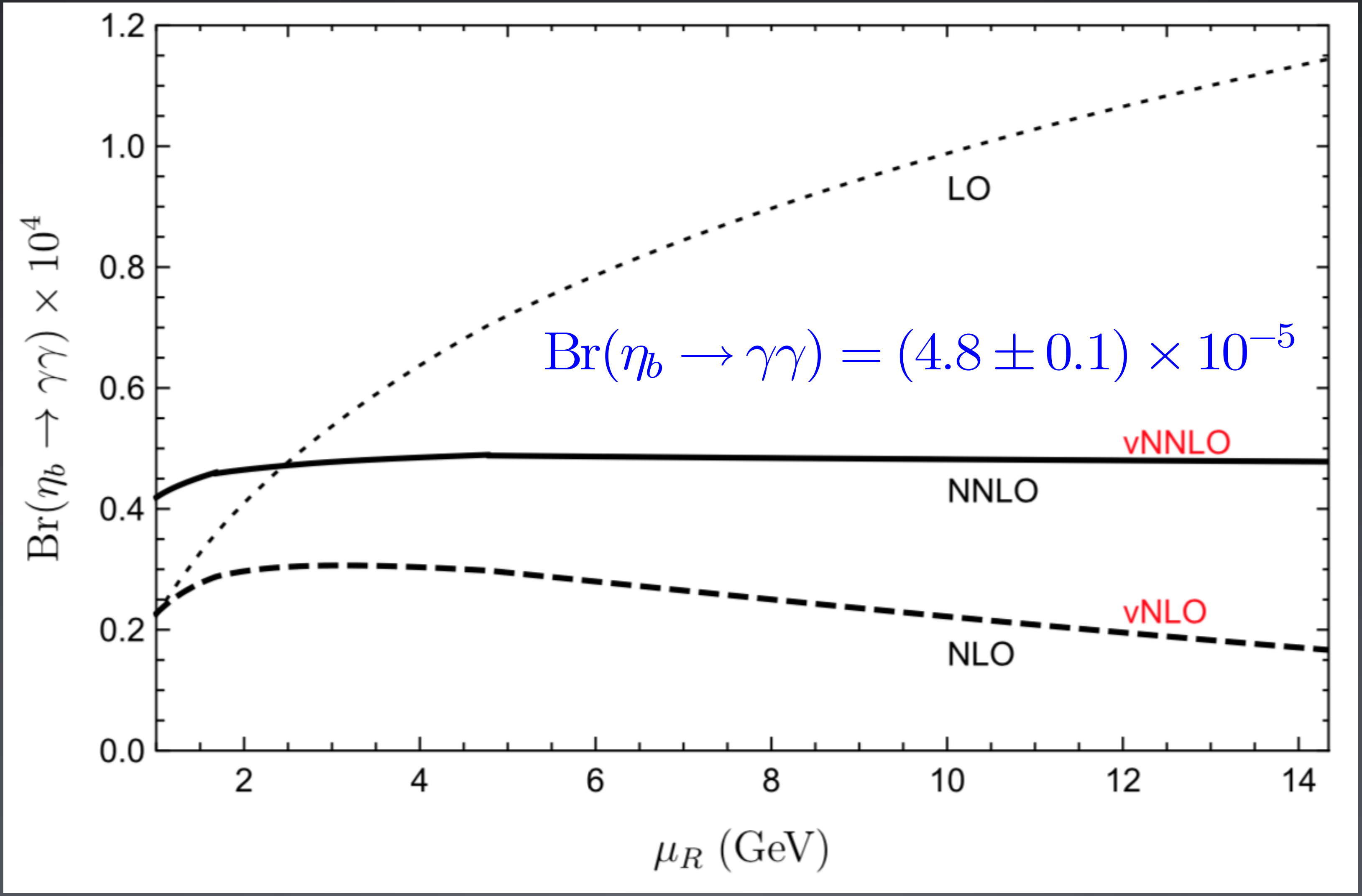
$$\mathrm{Br}(\eta_c \to \gamma\gamma) = \frac{8\alpha^2}{9\alpha_s^2}\left\{1 - \frac{\alpha_s}{\pi}\left[4.17\ln\frac{\mu_R^2}{4m_c^2} + 14.00\right] + \frac{\alpha_s^2}{\pi^2}\left[4.34\ln^2\frac{\mu_R^2}{4m_c^2} + 22.75\ln\frac{\mu_R^2}{4m_c^2} + 78.8\right] + 2.24\langle v^2\rangle_{\eta_c}\frac{\alpha_s}{\pi}\right\}$$

$$\mathrm{Br}(\eta_b \to \gamma\gamma) = \frac{\alpha^2}{18\alpha_s^2}\left\{1 - \frac{\alpha_s}{\pi}\left[3.83\ln\frac{\mu_R^2}{4m_b^2} + 13.11\right] + \frac{\alpha_s^2}{\pi^2}\left[3.67\ln^2\frac{\mu_R^2}{4m_b^2} + 20.30\ln\frac{\mu_R^2}{4m_b^2} + 85.5\right] + 1.91\langle v^2\rangle_{\eta_b}\frac{\alpha_s}{\pi}\right\}$$

$\eta_c \to$ Light Hadrons :



$\eta_c \rightarrow$ Light Hadrons :



Summary

- Investigated NNLO QCD corrections to $\gamma\gamma^* \to \eta_c, \chi_{c0,2} \to \gamma\gamma, \eta_c \to \text{Light Hadrons}$.
  Observe significant NNLO corrections.
  Alarming discrepancy with the existing measurements.
- Perturbative expansion seems to have poor convergence behavior for Charmonium.
- Perturbative expansion bears much better behavior for Bottomonium.

Thanks for your attention!

Backup Slides

Basic Procedures:

1. Feynman Diagrams & Amplitudes
2. Trace & Contraction
3. Partial Fragmentation & IBP Reduction
4. Master Integrals – Numerical
5. Other Processing – Plots etc.

Basic Procedures:

1. Feynman Diagrams & Amplitudes

- FeynArts - Mathematica Package
  http://www.feynarts.de
- QGRAF - Fortran Program
  http://cfif.ist.utl.pt/~paulo/qgraf.html
- ...

Basic Procedures:

1. Feynman Diagrams & Amplitudes

Load Package

```
<< Qgraf`

LoadFeynRules["sm"];
```

QgRun

```
HAmp = QgRun["
output='out';
style='math.sty';
model='sm.mod';
in=C[p1],Cbar[p2];
out=g[k1],g[k2];
loops=2;
loop_momentum=q;
options=notadpole,onshell;
true=vsum[QCD,6,6];
true=vsum[QED,0,0];
"];
```

```
----------------------------------------------------------------
            qgraf-3.1.4
----------------------------------------------------------------

output='out';
style='math.sty';
model='sm.mod';
in=C[p1],Cbar[p2];
out=g[k1],g[k2];
loops=2;
loop_momentum=q;
options=notadpole,onshell;
true=vsum[QCD,6,6];
true=vsum[QED,0,0];

----------------------------------------------------------------

 24P --- 7+ 17- --- 5N+ 2C+ 17C-

 169V --- 3^142 4^27

----------------------------------------------------------------

 -  4^3 --- 0 diagrams
 3^2 4^2 --- 23 diagrams
 3^4 4^1 --- 244 diagrams
 3^6  -  --- 1122 diagrams

 total = 1389 diagrams
```

Basic Procedures:

1. Feynman Diagrams & Amplitudes

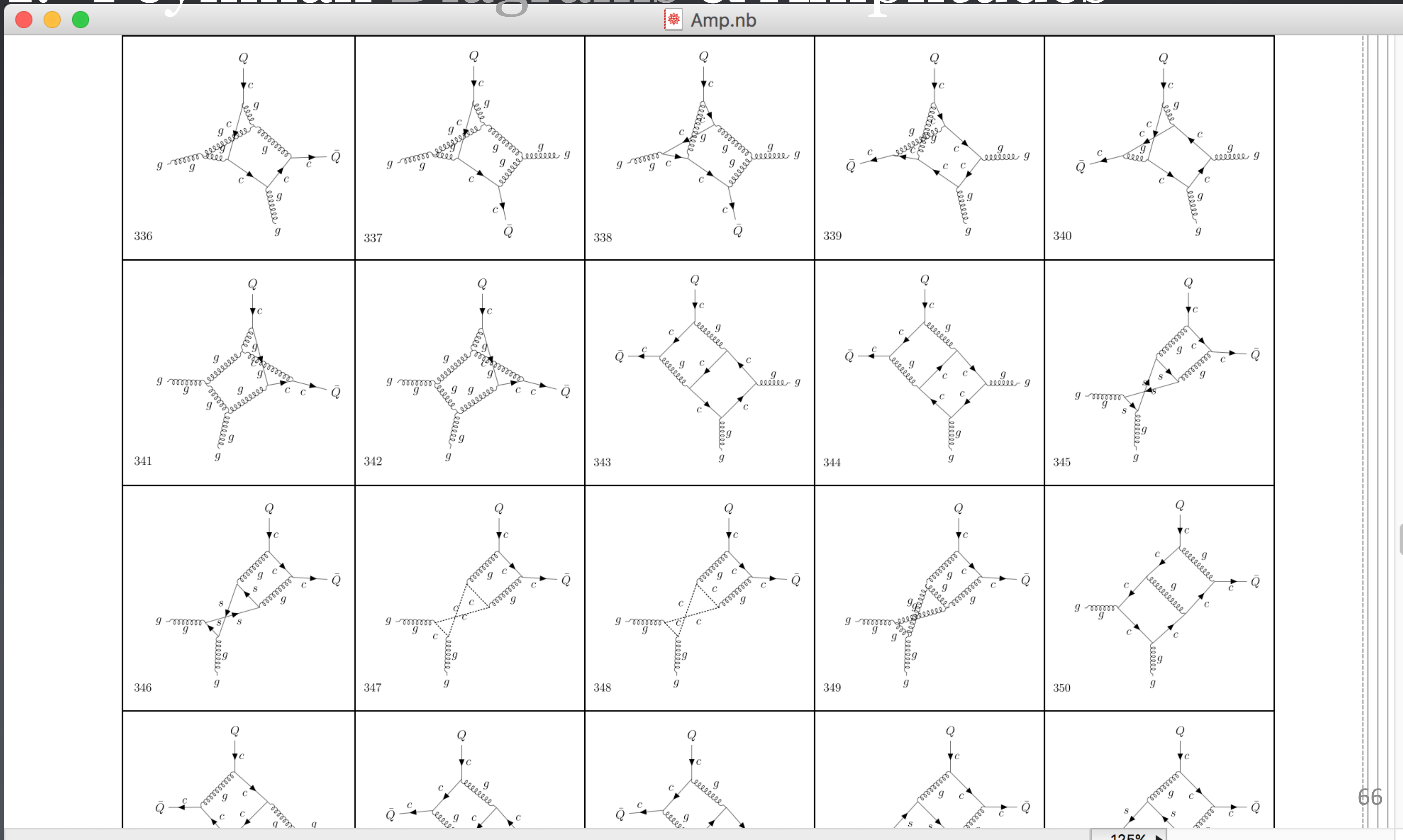
math.sty

math.sty 〉 No Selection

```
<prologue>
(* QGraf Output in Mathematica Style By F.Feng *)

<diagram>

Amp[[<diagram_index>]]:= Hold[[(<sign><symmetry_factor>)*
<in_loop>InField[[<field>,<field_index>,<momentum>]]*
<end>
<back><out_loop>OutField[[<field>,<field_index>,<momentum>]]*
<end>
<back><propagator_loop>Propagator[[
<back>Field[[<field>,<field_index>]],
<back>Field[[<dual-field>,<dual-field_index>]],
<back><momentum>]]*
<end>
<back><vertex_loop>Vertex[[
<back><ray_loop>Field[[<field>,<field_index>,<momentum>]],
<back><end><back>]]*
<end><back><back>]];

<epilogue>

<exit>
```

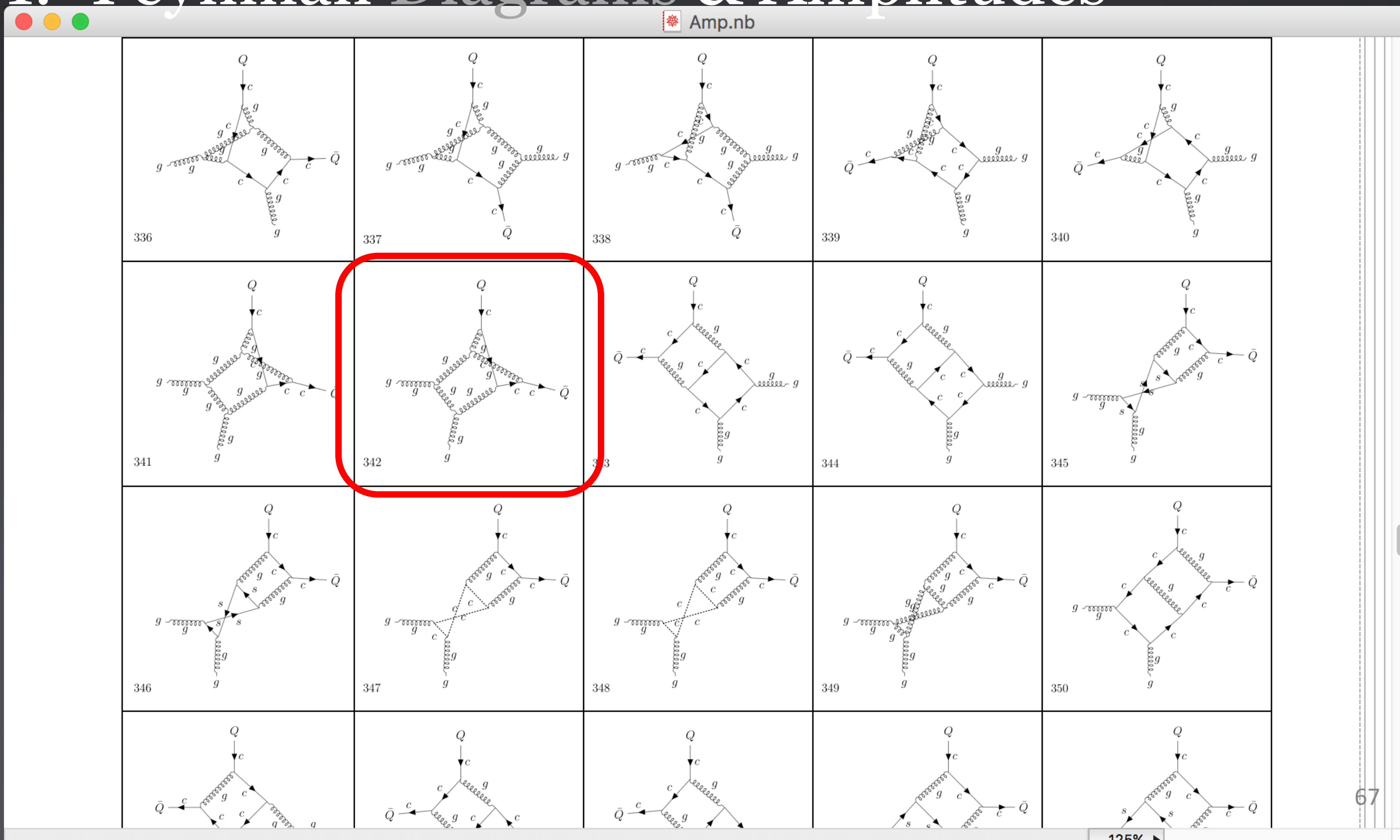
Basic Procedures:

1. Feynman Diagrams & Amplitudes



Basic Procedures:

1. Feynman Diagrams & Amplitudes



Basic Procedures:

1. Feynman Diagrams & Amplitudes

Example.nb

HAmp[[342]]

$$\Big(g_s^6\, \delta_{\text{ci1 ci2}}\, \delta_{\text{ci11 ci12}}\, \delta_{\text{ci3 ci4}}\, \delta_{\text{ci5 ci6}}\, \delta_{\text{ci7 ci8}}\, g^{\text{li1 li2}}\, g^{\text{li11 li12}}\, g^{\text{li3 li4}}\, g^{\text{li5 li6}}\, g^{\text{li7 li8}}\, f_{\text{cim2 ci1 ci3}}\, f_{\text{ci4 ci8 ci12}}\, f_{\text{cim4 ci2 ci5}}$$
$$\left(g^{\text{li1 li3}}\left(-\text{k1}-2\,\text{q1}^{\text{lim2}}\right)+g^{\text{li3 lim2}}\left(2\,\text{k1}+\text{q1}^{\text{li1}}\right)+g^{\text{lim2 li1}}\left(\text{q1}-\text{k1}^{\text{li3}}\right)\right)$$
$$\left(g^{\text{li2 li5}}\left(2\,\text{q1}-\text{k2}^{\text{lim4}}\right)+g^{\text{li5 lim4}}\left(2\,\text{k2}-\text{q1}^{\text{li2}}\right)+g^{\text{lim4 li2}}\left(-\text{k2}-\text{q1}^{\text{li5}}\right)\right)$$
$$\left(g^{\text{li12 li4}}\left(2\,\text{k1}+2\,\text{q1}-\text{q2}^{\text{li8}}\right)+g^{\text{li4 li8}}\left(-\text{k1}-\text{q1}-\text{q2}^{\text{li12}}\right)+g^{\text{li8 li12}}\left(-\text{k1}-\text{q1}+2\,\text{q2}^{\text{li4}}\right)\right)$$
$$\text{ColorLine}(T_{\text{ci11}}.T_{\text{ci6}}.T_{\text{ci7}},\{\text{p2},\text{p1}\})\,\text{SpinLine}(\gamma^{\text{li11}}.(\gamma\cdot(\text{k1}-\text{p2}+\text{q1}-\text{q2})+\text{mc}).\gamma^{\text{li6}}.(\text{mc}+\gamma\cdot(\text{p1}-\text{q2})).\gamma^{\text{li7}},\{\text{p2},\text{p1}\})\Big)\Big/$$
$$\Big(\text{q1}^2\,\text{q2}^2\,(\text{k1}+\text{q1})^2\,(\text{k2}-\text{q1})^2\,(-\text{k1}-\text{q1}+\text{q2})^2\left((\text{p1}-\text{q2})^2-\text{mc}^2\right)\left((\text{k1}-\text{p2}+\text{q1}-\text{q2})^2-\text{mc}^2\right)\Big)$$

200%

Basic Procedures:

1. Feynman Diagrams & Amplitudes

Example.nb

HAmp[[342]]

$$\Big(g_s^6\, \delta_{\text{ci1 ci2}}\, \delta_{\text{ci11 ci12}}\, \delta_{\text{ci3 ci4}}\, \delta_{\text{ci5 ci6}}\, \delta_{\text{ci7 ci8}}\, g^{\text{li1 li2}}\, g^{\text{li11 li12}}\, g^{\text{li3 li4}}\, g^{\text{li5 li6}}\, g^{\text{li7 li8}}\, f_{\text{cim2 ci1 ci3}}\, f_{\text{ci4 ci8 ci12}}\, f_{\text{cim4 ci2 ci5}}$$
$$\left(g^{\text{li1 li3}}\left(-\text{k1}-2\,\text{q1}^{\text{lim2}}\right)+g^{\text{li3 lim2}}\left(2\,\text{k1}+\text{q1}^{\text{li1}}\right)+g^{\text{lim2 li1}}\left(\text{q1}-\text{k1}^{\text{li3}}\right)\right)$$
$$\left(g^{\text{li2 li5}}\left(2\,\text{q1}-\text{k2}^{\text{lim4}}\right)+g^{\text{li5 lim4}}\left(2\,\text{k2}-\text{q1}^{\text{li2}}\right)+g^{\text{lim4 li2}}\left(-\text{k2}-\text{q1}^{\text{li5}}\right)\right)$$
$$\Big(g^{\text{li12 li4}}\left(2\,\text{k1}+2\,\text{q1}-\text{q2}^{\text{li8}}\right)+ g^{\text{li4 li8}}\left(-\text{k1}-\text{q1}-\text{q2}^{\text{li12}}\right)+g^{\text{li8 li12}}\left(-\text{k1}-\text{q1}+2\,\text{q2}^{\text{li4}}\right)\Big)$$
$$\text{ColorLine}(T_{\text{ci11}}.T_{\text{ci6}}.T_{\text{ci7}},\{\text{p2},\text{p1}\})\ \text{SpinLine}(\gamma^{\text{li11}}.(\gamma\cdot(\text{k1}-\text{p2}+\text{q1}-\text{q2})+\text{mc}).\gamma^{\text{li6}}.(\text{mc}+\gamma\cdot(\text{p1}-\text{q2})).\gamma^{\text{li7}},\{\text{p2},\text{p1}\})\Big)\Big/$$
$$\left(\text{q1}^2\,\text{q2}^2\,(\text{k1}+\text{q1})^2\,(\text{k2}-\text{q1})^2\,(-\text{k1}-\text{q1}+\text{q2})^2\left((\text{p1}-\text{q2})^2-\text{mc}^2\right)\left((\text{k1}-\text{p2}+\text{q1}-\text{q2})^2-\text{mc}^2\right)\right)$$

200%

Basic Procedures:

2. Trace & Contraction

- FeynCalc - Mathematica Package
  https://github.com/FeynCalc
- FORM - C Program (tFROM, parFORM)
  https://github.com/vermaseren/form
- FeynCalc/FormLink - Combined
  https://github.com/FormLink
- ...

Basic Procedures:

2. Trace & Contraction

Untitled-1

In[1]:=
```
<< FeynCalcFormLink`
```

In[8]:=
```
exp = DiracTrace[GAD[μ].(GSD[p1 + p2] + m).GAD[ν].(GSD[k1 + k2] + m) GSD[p1, p2, k1, k2]]
  (FVD[p1 + k1, μ] FVD[p2 + k2, ν] + MTD[μ, ν])
```

Out[8]= $(g^{\mu\nu} + (\text{k1} + \text{p1}^{\mu})(\text{k2} + \text{p2}^{\nu}))\,\text{tr}((\gamma\cdot\text{p1}).(\gamma\cdot\text{p2}).(\gamma\cdot\text{k1}).(\gamma\cdot\text{k2})\,\gamma^{\mu}.(m + \gamma\cdot(\text{p1}+\text{p2})).\gamma^{\nu}.(\gamma\cdot(\text{k1}+\text{k2}) + m))$

In[9]:=
```
FeynCalcFormLink[exp] // Factor
```

```
Symbols D,m;
Dimension D;
Vectors k1,k2,p1,p2;
AutoDeclare Index lor;
Format Mathematica;
AutoDeclare Symbol str;
L ltr1 =
(g_(1,lor1)*(m*gi_(1)+g_(1,p1)+g_(1,p2))*g_(1,lor2)*(m*gi_(1)+g_(1,k1)+g_(1,k2))*g_(1,p1)*g_(1,p2)*g_(1,k1)*g
_(1,k2));
tracen,1;
.sort;
L resFL = (str1*(d_(lor1,lor2)+(k1(lor1)+p1(lor1))*(k2(lor2)+p2(lor2))));
id str1=ltr1;
.sort;
contract 0;
.sort;
#call put("%E", resFL)
.end
```

125%

Basic Procedures:

2. Trace & Contraction

Untitled-1

Piping the script to FORM and running FORM

Time needed by FORM : 0.006 seconds. FORM finished. Got the result back to Mathematica as a string.

Start translation to Mathematica / FeynCalc syntax

Total wall clock time used: 0.15 seconds. Translation to Mathematica and FeynCalc finished.

Out[9]=

$$
\begin{aligned}
&4\big(2\,k1\cdot p2\,k2\cdot p1^2\,m^2 + 2\,k1\cdot k2\,p1\cdot p2^2\,m^2 + D\,k1\cdot p2\,k2\cdot p1\,m^2 + 2\,k1\cdot k2\,k1\cdot p2\,k2\cdot p1\,m^2 - D\,k1\cdot p1\,k2\cdot p2\,m^2 - 2\,k1\cdot k2\,k1\cdot p1\,k2\cdot p2\,m^2 - \\
&2\,k1\cdot p1\,k2\cdot p1\,k2\cdot p2\,m^2 - k1\cdot p2\,k2^2\,p1^2\,m^2 + 2\,k1\cdot k2^2\,p1\cdot p2\,m^2 + D\,k1\cdot k2\,p1\cdot p2\,m^2 + 4\,k1\cdot k2\,k1\cdot p2\,p1\cdot p2\,m^2 - k1^2\,k2^2\,p1\cdot p2\,m^2 + \\
&2\,k1\cdot k2\,k2\cdot p1\,p1\cdot p2\,m^2 + 2\,k1\cdot p2\,k2\cdot p1\,p1\cdot p2\,m^2 - 2\,k1^2\,k2\cdot p2\,p1\cdot p2\,m^2 - 2\,k1\cdot p1\,k2\cdot p2\,p1\cdot p2\,m^2 - 2\,k1\cdot k2\,k1\cdot p1\,p2^2\,m^2 + \\
&k1^2\,k2\cdot p1\,p2^2\,m^2 - k1\cdot k2\,p1^2\,p2^2\,m^2 - 4\,k1\cdot p2^2\,k2\cdot p1^2 - 2\,D\,k1\cdot p2\,k2\cdot p1^2 + 2\,k1^2\,k1\cdot p2\,k2\cdot p1^2 + 4\,k1\cdot p2\,k2\cdot p1^2 - 4\,k1\cdot p1^2\,k2\cdot p2^2 + \\
&2\,D\,k1\cdot p1\,k2\cdot p2^2 - 2\,k1^2\,k1\cdot p1\,k2\cdot p2^2 - 4\,k1\cdot p1\,k2\cdot p2^2 + 4\,k1\cdot k2^2\,p1\cdot p2^2 + 2\,k1^2\,k1\cdot k2\,p1\cdot p2^2 + 2\,k1\cdot k2\,k2^2\,p1\cdot p2^2 - \\
&2\,D\,k1\cdot p2^2\,k2\cdot p1 - 4\,k1\cdot k2\,k1\cdot p2^2\,k2\cdot p1 + 4\,k1\cdot p2^2\,k2\cdot p1 - 2\,D\,k1\cdot p1\,k1\cdot p2\,k2\cdot p1 - 4\,k1\cdot k2\,k1\cdot p1\,k1\cdot p2\,k2\cdot p1 + 4\,k1\cdot p1\,k1\cdot p2\,k2\cdot p1 + \\
&2\,D\,k1\cdot p1^2\,k2\cdot p2 + 4\,k1\cdot k2\,k1\cdot p1^2\,k2\cdot p2 - 4\,k1\cdot p1^2\,k2\cdot p2 + 2\,D\,k1\cdot p1\,k1\cdot p2\,k2\cdot p2 + 4\,k1\cdot k2\,k1\cdot p1\,k1\cdot p2\,k2\cdot p2 - \\
&4\,k1\cdot p1\,k1\cdot p2\,k2\cdot p2 + 2\,D\,k1\cdot p1\,k2\cdot p1\,k2\cdot p2 - 2\,k1^2\,k1\cdot p1\,k2\cdot p1\,k2\cdot p2 - 4\,k1\cdot p1\,k2\cdot p1\,k2\cdot p2 - 2\,D\,k1\cdot p2\,k2\cdot p1\,k2\cdot p2 + \\
&2\,k1^2\,k1\cdot p2\,k2\cdot p1\,k2\cdot p2 + 8\,k1\cdot p1\,k1\cdot p2\,k2\cdot p1\,k2\cdot p2 + 4\,k1\cdot p2\,k2\cdot p1\,k2\cdot p2 + 4\,k1\cdot k2\,k1\cdot p2^2\,p1^2 - 2\,k1\cdot p1\,k2\cdot p2^2\,p1^2 + \\
&4\,k1\cdot k2^2\,k1\cdot p2\,p1^2 + 2\,D\,k1\cdot k2\,k1\cdot p2\,p1^2 - 4\,k1\cdot k2\,k1\cdot p2\,p1^2 + 2\,k1\cdot p2^2\,k2^2\,p1^2 + D\,k1\cdot p2\,k2^2\,p1^2 - k1^2\,k1\cdot p2\,k2^2\,p1^2 + \\
&2\,k1\cdot k2\,k1\cdot p2\,k2^2\,p1^2 - 2\,k1\cdot p2\,k2^2\,p1^2 + 2\,k1\cdot p2^2\,k2\cdot p1\,p1^2 + k1\cdot p2\,k2^2\,k2\cdot p1\,p1^2 - D\,k1^2\,k2\cdot p2\,p1^2 + 2\,k1^2\,k2\cdot p2\,p1^2 - \\
&2\,k1^2\,k1\cdot k2\,k2\cdot p2\,p1^2 - 2\,k1^2\,k1\cdot p2\,k2\cdot p2\,p1^2 - 2\,k1\cdot p1\,k1\cdot p2\,k2\cdot p2\,p1^2 - k1^2\,k2^2\,k2\cdot p2\,p1^2 - k1\cdot p1\,k2^2\,k2\cdot p2\,p1^2 + \\
&2\,k1\cdot p2\,k2\cdot p1\,k2\cdot p2\,p1^2 - 4\,k1\cdot k2^2\,k1\cdot p1\,p1\cdot p2 - 2\,D\,k1\cdot k2\,k1\cdot p1\,p1\cdot p2 + 4\,k1\cdot k2\,k1\cdot p1\,p1\cdot p2 + 4\,k1\cdot k2^2\,k1\cdot p2\,p1\cdot p2 + \\
&2\,D\,k1\cdot k2\,k1\cdot p2\,p1\cdot p2 - 4\,k1\cdot k2\,k1\cdot p2\,p1\cdot p2 - 8\,k1\cdot k2\,k1\cdot p1\,k1\cdot p2\,p1\cdot p2 + 2\,k1^2\,k1\cdot p1\,k2^2\,p1\cdot p2 + 2\,D\,k1\cdot p2\,k2^2\,p1\cdot p2 + \\
&4\,k1\cdot k2\,k1\cdot p2\,k2^2\,p1\cdot p2 - 4\,k1\cdot p1\,k1\cdot p2\,k2^2\,p1\cdot p2 - 4\,k1\cdot p2\,k2^2\,p1\cdot p2 - 2\,D\,k1\cdot k2\,k2\cdot p1\,p1\cdot p2 + 2\,k1^2\,k1\cdot k2\,k2\cdot p1\,p1\cdot p2 + \\
&4\,k1\cdot k2\,k2\cdot p1\,p1\cdot p2 + 2\,k1^2\,k1\cdot p2\,k2\cdot p1\,p1\cdot p2 + 2\,k1\cdot p2\,k2^2\,k2\cdot p1\,p1\cdot p2 - 2\,D\,k1^2\,k2\cdot p2\,p1\cdot p2 + 4\,k1^2\,k2\cdot p2\,p1\cdot p2 - \\
&2\,D\,k1\cdot k2\,k2\cdot p2\,p1\cdot p2 - 2\,k1^2\,k1\cdot k2\,k2\cdot p2\,p1\cdot p2 + 4\,k1\cdot k2\,k2\cdot p2\,p1\cdot p2 + 2\,k1^2\,k1\cdot p1\,k2\cdot p2\,p1\cdot p2 - 2\,k1^2\,k2^2\,k2\cdot p2\,p1\cdot p2 -
\end{aligned}
$$

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Basic Procedures:

3. Partial Fragmentation & IBP Reduction

- APart - Mathematica Package
  https://github.com/F-Feng
- FIRE - Mathematica Program & C++ version
  http://science.sander.su
- AIR - Maple Program
  https://www.phys.ethz.ch/~pheno/air/
- Reduze - C Program (MPI)
  https://reduze.hepforge.org
- …

Basic Procedures:

3. Partial Fragmentation & IBP Reduction

- Integrate By Part (IBP) [JHEP **0810**, 107 (2008)]

$$F(a_1, \cdots, a_n) = \int \cdots \int \frac{d^d k_1 \cdots d^d k_h}{E_1^{a_1} \cdots E_n^{a_n}}$$

where k_i, $i = 1, \cdots, h$, are loop momenta and the denominators E_r are either quadratic or linear with respect to the loop momenta k_i of the graph. Irreducible polynomials in the numerator can be represented as denominators raised to negative powers.

- Basic idea:

$$\int \cdots \int d^d k_1 d^d k_2 \cdots \frac{\partial}{\partial k_i} \left[\frac{p_j}{E_1^{a_1} \cdots E_n^{a_n}} \right] = 0$$

- List of equations:

$$\sum \alpha_i F(a_1 + b_{i,1}, \cdots, a_n + b_{i,n}) = 0$$

Basic Procedures:

3. Partial Fragmentation & IBP Reduction

Apart function in MATHEMAITCA

In[1]:= $$\text{Apart}\left[\frac{1}{(x-a)\,(x-b)\,(x-c)}\right]$$

Out[1]= $$-\frac{1}{(a-b)(b-c)(x-b)}-\frac{1}{(a-c)(c-b)(x-c)}+\frac{1}{(a-b)(a-c)(x-a)}$$

In[2]:= $$\text{Apart}\left[\frac{1}{(x+a)\,(y+2\,b)\,(3\,x+4\,y)}\right]$$

Out[2]= $$\frac{1}{(a+x)(3\,x-8\,b)(2\,b+y)}-\frac{4}{(a+x)(3\,x-8\,b)(3\,x+4\,y)}$$

Basic Procedures:

3. Partial Fragmentation & IBP Reduction

APART Package:

```
In[1]:= << CalcExt`Apart`
```

In[2]:= $\text{\$Apart}\left[\frac{1}{(x-a)\ (x-b)\ (x-c)},\ \{x\}\right]$

Out[2]= $-\frac{\left\|\frac{1}{a-x}\right\|}{(a-b)(a-c)}+\frac{\left\|\frac{1}{b-x}\right\|}{(a-b)(b-c)}+\frac{\left\|\frac{1}{c-x}\right\|}{(a-c)(c-b)}$

In[3]:= $\text{\$Apart}\left[\frac{1}{(x+a)\ (y+2\,b)\ (3\,x+4\,y)},\ \{x,\ y\}\right]$

Out[3]= $-\frac{\left\|\frac{1}{(a+x)(2\,b+y)}\right\|}{3\,a+8\,b}+\frac{4\left\|\frac{1}{(a+x)(3\,x+4\,y)}\right\|}{3\,a+8\,b}+\frac{3\left\|\frac{1}{(2\,b+y)(3\,x+4\,y)}\right\|}{3\,a+8\,b}$

Basic Procedures:

3. Partial Fragmentation & IBP Reduction

$$F[l,m,n] = \int \frac{d^4k}{(2\pi)^4} \frac{(k\cdot p_2)^{-l}}{(m^2-k^2-2k\cdot p_1-p_1^2)^m(-m^2+k^2+2k\cdot p_2+p_2^2)^n}$$

```
In[1]:= << HighEnergyPhysics`fc`

In[2]:= << FIRE`

In[3]:= Replacement = {p1^2 → m^2, p2^2 → m^2, p1 p2 → SP[p1, p2]};
        Internal = {k};
        External = {p1, p2};
        Propagators = {k p2, -2 k p1 - k^2 + m^2 - p1^2, 2 k p2 + k^2 - m^2 + p2^2};
        PrepareIBP[];
        startinglist = {IBP[k, k], IBP[k, p1], IBP[k, p2]} /. Replacement;
        Prepare[];

In[10]:= Burn[];

In[11]:= F[{-1, 1, 2}]
```

Out[11]= $\frac{(d-2)\,G(\{0,0,1\})}{8\,(m^2-\mathrm{p1}\cdot\mathrm{p2})}+\frac{(d-2)\,G(\{0,1,0\})}{8\,(m^2-\mathrm{p1}\cdot\mathrm{p2})}+\frac{1}{4}(4-d)\,G(\{0,1,1\})$

Basic Procedures:

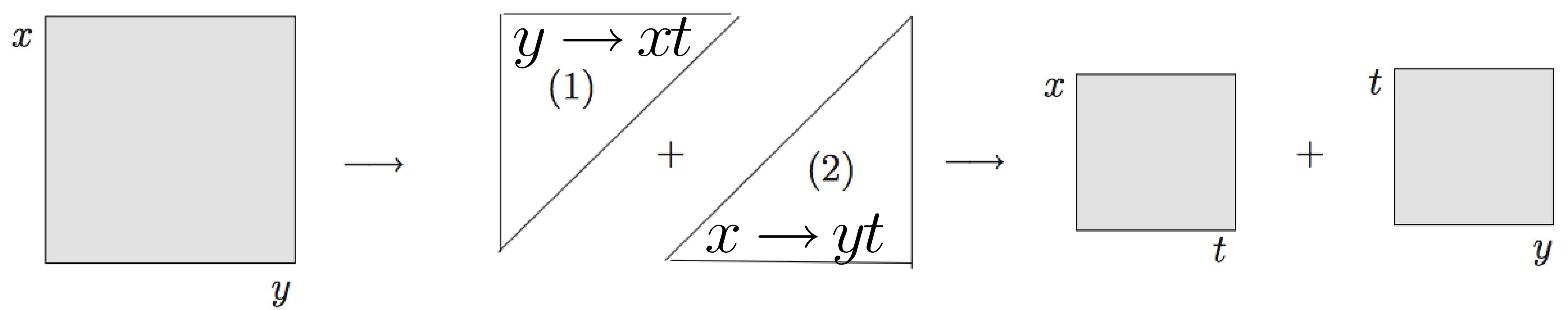
4. Master Integrals - Numerical
   - FIESTA - MATHEMATICA Package
     http://science.sander.su
   - SECDEC - MATHEMATICA/PYTHON Package
     http://secdec.hepforge.org
   - CUBPACK - FORTRAN Code
     http://nines.cs.kuleuven.be/software/cubpack/
   - CUBA - FORTRAN/C Code
     http://www.feynarts.de/cuba/
   - MPI-supported $\cdots$

Basic Procedures:

4. Master Integrals - Numerical

Sector Decomposition

$$
\begin{aligned}
I &= \int_0^1 dx \int_0^1 dy\, x^{-1-a\varepsilon} y^{-b\varepsilon} (x+(1-x)y)^{-1} \\
&= \int_0^1 dx \int_0^1 dy\, x^{-1-a\varepsilon} y^{-b\varepsilon} (x+(1-x)y)^{-1} \Big[\theta(x-y)+\theta(y-x)\Big]
\end{aligned}
$$



Basic Procedures:

4. Master Integrals - Numerical

Sector Decomposition

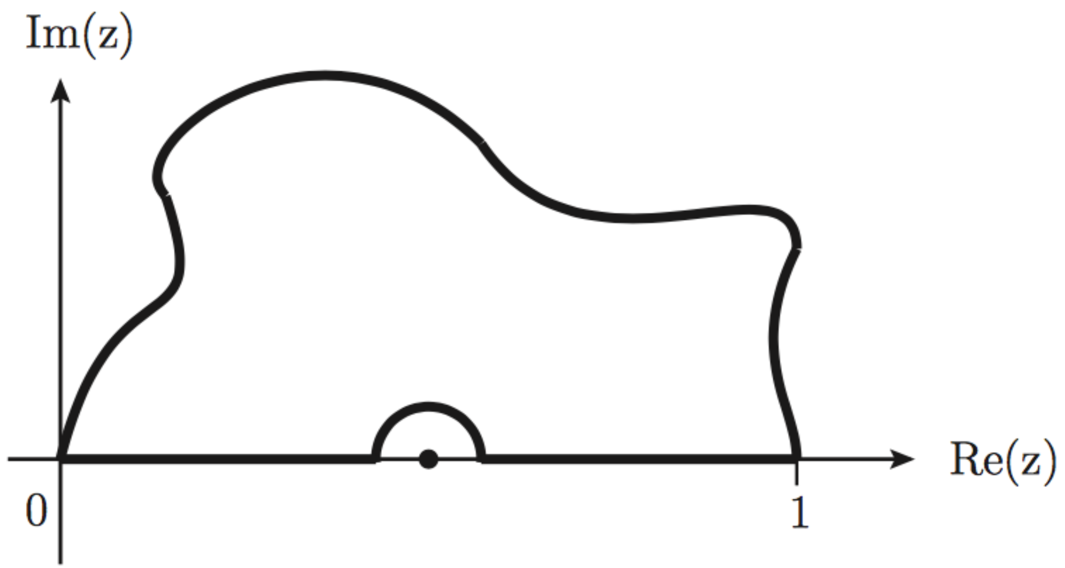
$$
\begin{aligned}
I &= \int_0^1 dx \int_0^1 dy\, x^{-1-a\varepsilon} y^{-b\varepsilon} (x + (1-x)y)^{-1} \\
&= \int_0^1 dx \int_0^1 dy\, x^{-1-a\varepsilon} y^{-b\varepsilon} (x + (1-x)y)^{-1} \Big[\theta(x-y) + \theta(y-x)\Big] \\
&= \int_0^1 dx x^{-1-(a+b)\varepsilon} \int_0^1 t^{-b\varepsilon} \Big(1 + (1-x)t\Big)^{-1} \\
&\quad + \int_0^1 dy y^{-1-(a+b)\varepsilon} \int_0^1 t^{-1-a\varepsilon} \Big(1 + (1-y)t\Big)^{-1}
\end{aligned}
$$

Basic Procedures:

4. Master Integrals - Numerical

Sector Decomposition

Deformation of the integration contour



Basic Procedures:

4. Master Integrals - Numerical

Sector Decomposition

$$\vec{z}(\vec{x}) = \vec{x} - i\,\vec{\tau}(\vec{x})$$

$$\tau_k = \lambda\, x_k(1 - x_k)\, \frac{\partial \mathcal{F}(\vec{x})}{\partial x_k}\,.$$

SecDec 2.0

$$\mathcal{F}(\vec{z}(\vec{x})) = \mathcal{F}(\vec{x}) - i\,\lambda \sum_j x_j(1 - x_j)\left(\frac{\partial \mathcal{F}}{\partial x_j}\right)^2 + \mathcal{O}(\lambda^2)$$

Basic Procedures:

4. Master Integrals - Numerical

- Prepare Database
  Sector decomposion done with FIESTA

- Extract Sectors
  Expression in each sector exported to FORTRAN **format**

- Numerical Integration
  Run FORTRAN **code using different numerical integration routines**

- Combine Sector Results

Basic Procedures:

4. Master Integrals - Numerical

Float Precision

- Long-Double Precision $(18 \sim 19$ digits)
  Fortran native: REAL(KIND=10)

- Quadruple Precision $(33 \sim 36$ digits)
  Fortran native: REAL(KIND=16)

- Double-Quadruple Precision $(\sim 66$ digits)
  http://www.davidhbailey.com/dhbsoftware/

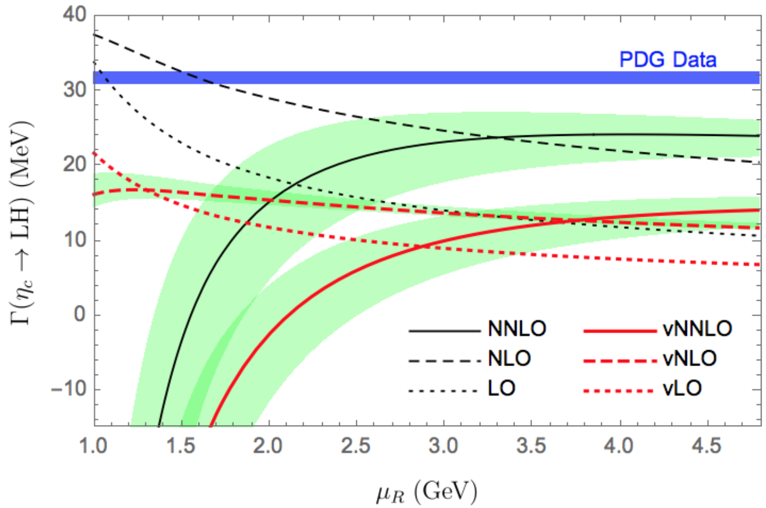
- Arbitrarily High Numeric Precision
  http://www.davidhbailey.com/dhbsoftware/

Basic Procedures:

5. Other Processing – Plots etc.

Plots-v2.nb

```
    Text[Style["vLO"], {1 + r1 + 0.05, -10}, {-1, 0}]
   }]]
Export[FileNameJoin[{NotebookDirectory[], "TeX", "draft", "Gmc.eps"}], %];
```



Bottom-Gamma

```
Show[bGammaPlot, Graphics[{
   Text[Style["PDG Data", Blue], {10.4, 17}, {-1, 0}],
   , {Thickness[0.003], Line[{{6.3, -5}, {7.3, -5}}]}, Text[Style["NNLO"], {7.5, -5}, {-1, 0}]
   , {Dashed, Thickness[0.003], Line[{{6.3, -10}, {7.3, -10}}]},
```

Basic Procedures:

1. Feynman Diagrams & Amplitudes
2. Trace & Contraction
3. Partial Fragmentation & IBP Reduction
4. Master Integrals – Numerical
5. Other Processing – Plots etc.