AN INTRODUCTION TO TOPOLOGICAL FLUID DYNAMICS

Lecture notes of the Graduate Course delivered by

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Preface

These lecture notes were produced to assist the participants of a short course held at the Beijing University of Technology. They cover some fundamental aspects of topological fluid dynamics, starting from the basic equations. Focus is put on the construction of conservation laws from a lagrangian viewpoint, and on the role of geometric and topological features in the dynamics of vortex and magnetic fields in ideal conditions. On more advanced topics, particular emphasis is put on the concept of magnetic relaxation of knots and links and their groundstate energy spectrum, and on the topological interpretation of helicity in terms of linking numbers, writhe and twist.

My visit has been possible thanks to the generous support of the Institute of Theoretical Physics (ITP) of Beijing University of Technology (BJUT). For this I am particularly grateful to the endless organizational efforts of my host, Professor Xin Liu, and to the warm hospitality of Professor Wenyu Wang, Head of ITP. I would also like to express my gratitude to the students and the researchers, with whom I shared many inspiring discussions.

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R.L.R.

Contents

Chapter 1

Fundamentals of fluid flows

1.1 150 years of topological fluid mechanics

A topological approach to fluid mechanics originates from the very fundamental work of Cauchy on lagrangian conservation of vorticity (1815) and is rooted in two remarkable papers of 1858, one due to Helmholtz on the conservation laws of vortex dynamics, and the other due to Riemann on the implications of multiply connectedness of the ambient space on potentials. These works prompted Lord Kelvin to propose a fundamental theory of matter — the vortex atom theory (1868–1882) — that stimulated Tait's first tabulations of knots and links (1872–1880)and motivated Maxwell (1870) to investigate possible applications of Gauss' linking number formula (1833) for electric currents and magnetic fields in multiply connected domains. This led to a period of intensive research (mostly in Britain) that culminated in 1882 with the Adams Prize essay of J.J. Thomson on vortex links. With the abandonment of the vortex atom theory (and the subsequent discovery of the electron by the same J.J. Thomson in 1905) topological techniques received new emphasis with the works of Poincaré and the formal developments due to de Rham. In the period 1950-1960 a number of applications started to appear in quantum field theory, but it is in the 70s that we see a resurgence of interest, mainly due to the topological interpretation of helicity — a fundamental invariant of ideal fluid mechanics and magnetohydrodynamics — by Moffatt (1969), and novel applications of limiting forms of the Gauss' linking number in the study of DNA biology. With the continuous developments and refinements of computational investigations and technological applications topological field theory is now a very rich and diversified subject, under continuous developments (see Figure 1.1).

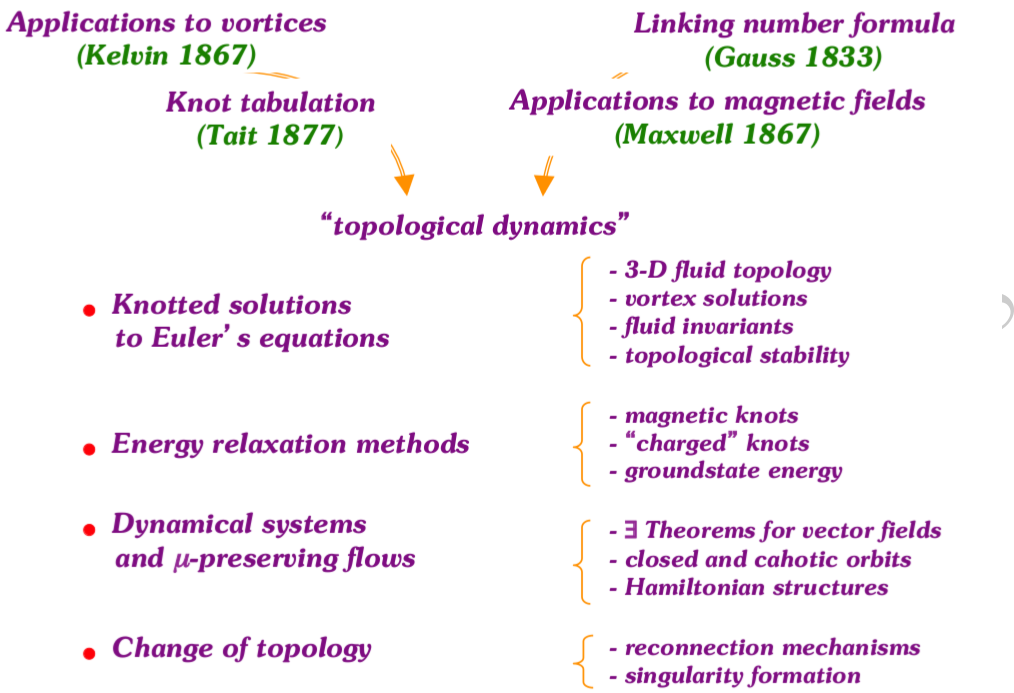


Figure 1.1: 150 years of developments in topological classical field theory.

1.2 Flow map and topological equivalence

We consider a continuum, fluid medium in $\mathbb{R}^3$ occupying an infinite connected region without boundaries. Kinematic and dynamical properties are derived from elementary considerations of the fluid medium considered from a macroscopic viewpoint, by assuming everything smooth and analytic. By $\mathbf{x}_0 = (x_0, y_0, z_0)$ we denote the vector position of a fluid particle at the initial position at point P and by $\mathbf{x} = (x, y, z)$ the vector position of the same particle at the new position Q. The trajectory followed by the particle to go from P to Q is described by the *flow map* φ that sends $\mathbf{x}_0$ to $\mathbf{x}$. In general we don't have an equation for φ, but we assume that such a flow exists.

Example 1.2.1. Within the fluid domain consider for example a sub-region of material (coloured) particles occupying a volume W_0 at $t = 0$ and the evolution of such region under the action of the flow map φ (see Figure 1.2); at each time t we have

$$\varphi_t : W_0 \;\to\; W \;, \tag{1.1}$$

that maps W_0 to the final state W. Note that the material volume W is a sub-region of $\mathbb{R}^3$ made always of the same material particles throughout the motion:

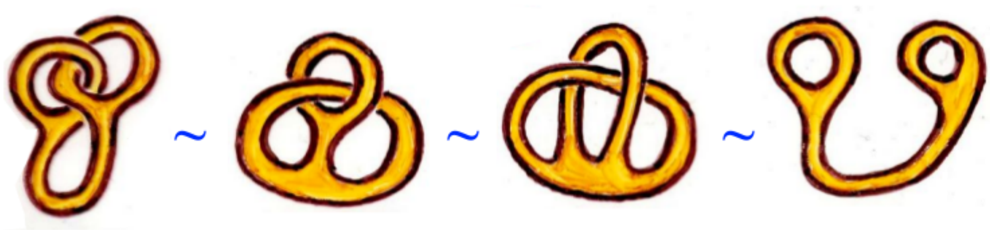


Figure 1.2: Continuous deformation of a pretzel. Tilde stands for topological equivalence between different states.

no material particle leaves or enters W during motion, a condition that is mathematically expressed by saying that there is no flux of such particles through the bounding surface ∂W.

In order to be able to make analytic progress we must consider a rather specific family of flow maps, given by

$$\varphi \in \Phi = \left\{ \mathsf{C}^{\nu} \ s.t. \ \exists \varphi^{-1} : \mathbf{x}_0 = \varphi^{-1}(\mathbf{x}, t), \ \forall t \in I \right\} \ , \tag{1.2}$$

where ν is appropriate (in general $\nu \leq 3$). Even though this restriction seems legitimate, there is a multitude of natural phenomena that cannot be captured by such a flow class.

Example 1.2.2. Consider two immiscible, fluid regions of different densities and surface tension present in a container and imagine that at some initial time the heavier fluid lies entirely on top of the lighter fluid. As time progresses the heavy fluid starts to sink through the lighter one. This leads to a continuous deformation of the two fluid regions, with the the heavy fluid gradually severing the light fluid towards the bottom. At some intermediate time — before the final deposition of all the heavy fluid at the bottom of the container — top and bottom can be connected by a thin thread of heavy fluid, that determines a topological change of the initial fluid domain (see Figure 1.3). During this stage the inverse flow map that relates the final position of each fluid particle in terms of its initial value is ill-defined.

Example 1.2.3. Another common example of topological change is represented by the production of a drop of fluid. At some intermediate time — just before the final detachment of the drop from the bulk of fluid — a fluid cusp may form. At the point of detachment the flow map is ill-defined and the whole process can be described only by reconciling the evolution before and after the detachment by a limiting process.

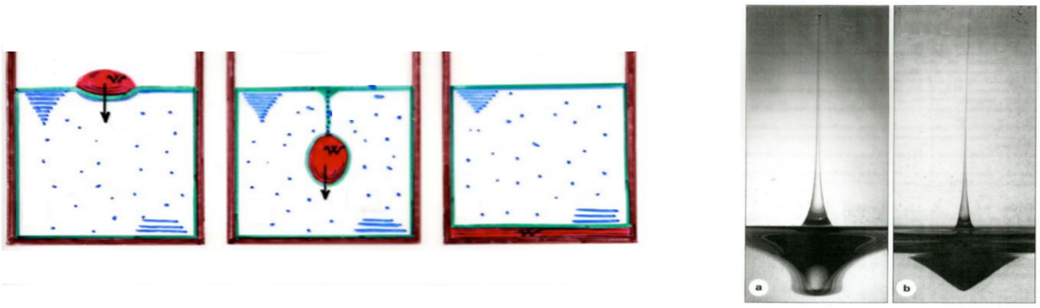


Figure 1.3: Left: topological change of the fluid domain due to a continuous rearrangement of fluid regions. Right: liquid cusp formation.

From these two examples we see that both the assumptions we have made about the flow map, regularity and invertibility, need some careful considerations and adaptations in order to tackle some unsolved, yet fundamental, problems of contemporary research.

1.3 Continuum hypothesis and basic definitions

In what follows we make use of the following standard concepts.

Continuum hypothesis: we assume that all properties are defined for elementary volumes of fluid and hold true in the limiting case of infinitesimally small volume.

Eulerian description: the observer is on a fixed reference system that does not move and everything is function of $\mathbf{x}$ and t; for instance, the *velocity* of a particle must be interpreted as $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$. Hence, we may say that Euler sees particles motion standing from the shore.

Lagrangian description: the observer is on a reference system that is fixed on a fluid particle and it moves with the velocity of the fluid particle. This means that the velocity of a particle, for instance, must be interpreted in terms of the initial position of the particle, that is $\mathbf{u} = \mathbf{u}[\mathbf{x}(\mathbf{x}_0), t] = \mathbf{u}(\mathbf{x}_0, t)$. In this case we may say that Lagrange, sitting on a particle, sees the motion by looking at the shore.

Acceleration: this is due to a change in velocity, i.e.

$$\delta\mathbf{u} = \mathbf{u}(\mathbf{x} + \mathbf{u}\delta t, t + \delta t) - \mathbf{u}(\mathbf{x}, t) \; ; \tag{1.3}$$

by taking the limit, we have

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u}\ , \tag{1.4}$$

where

$$\mathbf{u}\cdot\nabla = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\ ;$$

$(\mathbf{u}\cdot\nabla)\mathbf{u}$ is proportional to the directional derivative of $\mathbf{u}$ in the direction of $\mathbf{u}$. In general any change in time of a fluid quantity is quantified by the total derivative $\mathrm{D}/\mathrm{D}t$ that is given by the sum of the partial, time derivative $\partial/\partial t$ and the *convective derivative* $\mathbf{u}\cdot\nabla$.

Stationary (or steady) state: the condition determined by a mathematical solution that is time-independent, i.e. where there is no time. For the velocity field this means that $\mathbf{u} = \mathbf{u}(\mathbf{x})$ or $\mathbf{u} = \mathbf{u}(\mathbf{x}_0)$.

Fluid invariant: this is a quantity, say $G(\mathbf{x}, t)$, that denotes a property of fluid particles that does not change in time during fluid motion, i.e. $\mathrm{D}G/\mathrm{D}t = 0$; hence, if $\mathrm{D}G/\mathrm{D}t = 0$ then G is a conserved quantity transported by the fluid flow φ.

In general there are four important conserved quantities associated with fundamental conservation laws:

1. conservation of mass
2. conservation of linear momentum
3. conservation of angular momentum
4. conservation of helicity

Vector field lines represent a simple way to visualize the geometry of physical fields such as velocity, vorticity, electric currents or magnetic fields. In terms of velocity we have:

Definition 1.3.1. A *streamline* is the integral curve given by the solution of the system

$$\frac{\mathrm{d}x}{u} = \frac{\mathrm{d}y}{v} = \frac{\mathrm{d}z}{w}\ , \tag{1.5}$$

where $\mathbf{x} = (x, y, z)$ is the position vector of the points on the curve and $\mathbf{u} = (u, v, w)$ is the velocity field.

It's easy to understand this definition by identifing the streamline with a geometric space curve $\mathcal{C}$. By assuming $\mathcal{C}$ sufficiently regular (typically C^3) we can define a unit tangent $\hat{\mathbf{t}}$ to $\mathcal{C}$ by $\hat{\mathbf{t}} = \mathrm{d}\mathbf{x}/\mathrm{d}s$ where s is arc-length. By replacing the differentials with arc-length derivatives (1.5) becomes

$$\frac{\mathrm{d}x/\mathrm{d}s}{u} = \frac{\mathrm{d}y/\mathrm{d}s}{v} = \frac{\mathrm{d}z/\mathrm{d}s}{w} \ ,$$

that is

$$\frac{t_1}{u} = \frac{t_2}{v} = \frac{t_3}{w} \tag{1.6}$$

where $\hat{\mathbf{t}} = (t_1, t_2, t_3)$. Thus $\mathbf{u} \propto \hat{\mathbf{t}}$ so that we can identify velocity with the unit tangent to the integral curve.

1.4 Kinematic transport theorem

In continuum mechanics it is customary to calculate the rate of change of a quantity in a volume of fluid. Now different choices can be made for the control volume, such as a geometric volume fixed in space, or moving with the fluid in a prescribed manner, or as a material volume always consisting of the same particles (i.e. that never leave or enter the material volume). Calculation of the rate of change can be facilitated by the so-called Reynolds kinematic transport theorem. This can be viewed as an extension of the time-derivative of a generic function of time written in terms of an integral, whose domain is also function of time. The one-dimensional version is rooted in Leibniz's rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{a(x)}^{b(x)} f(x,t)\,\mathrm{d}t = \int_{a(x)}^{b(x)} \frac{\partial f(x,t)}{\partial x}\,\mathrm{d}t + f(x,b(x))\frac{\mathrm{d}b(x)}{\mathrm{d}x} - f(x,a(x))\frac{\mathrm{d}a(x)}{\mathrm{d}x} \ .$$

Let us now establish in some generality a theorem for any moving volume $V(t)$ bounded by the surface $S(t) = \partial V(t)$ whose elements move with a velocity $\mathbf{w}$ (not necessarily coinciding with the velocity $\mathbf{u}$ with which the fluid moves; see Figure 1.4).

Theorem 1.4.1 (Kinematic transport theorem)**.** *Let $G = G(\mathbf{x},t)$ be some fluid property per unit volume; we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} G(\mathbf{x},t)\,\mathrm{d}^3\mathbf{x} = \int_{V(t)} \frac{\partial G(\mathbf{x},t)}{\partial t}\,\mathrm{d}^3\mathbf{x} + \int_{S(t)} G(\mathbf{x},t)\,(\mathbf{w}\cdot\hat{\boldsymbol{\nu}})\,\mathrm{d}^2\mathbf{x} \ , \tag{1.7}$$

where $\mathbf{w}$ is the velocity with which the area element of the surface S of normal $\hat{\boldsymbol{\nu}}$ moves.

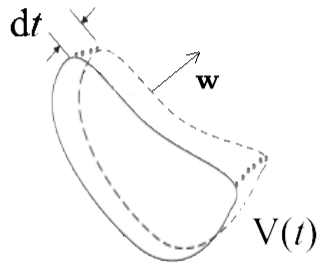


Figure 1.4: Change of fluid volume in time due to a velocity $\mathbf{v}$

Proof. By definition, we have

$$\int_V G(\mathbf{x},t)\,\mathrm{d}^3\mathbf{x}\bigg|_{t+\mathrm{d}t} = \int_{V(t+\mathrm{d}t)} G(\mathbf{x},t+\mathrm{d}t)\,\mathrm{d}^3\mathbf{x}$$

$$= \int_{V(t+\mathrm{d}t)} \left[G(\mathbf{x},t) + \frac{\partial G(\mathbf{x},t)}{\partial t}\,\mathrm{d}t + O(\mathrm{d}t)^2\right]\mathrm{d}^3\mathbf{x}\ , \tag{1.8}$$

and due to the movement of $S(t)$ the change of volume induced by the surface element motion $\mathrm{d}^2\mathbf{x} = \mathrm{d}S$ associated with $\mathbf{w}$ is given by $(\mathbf{w}\cdot\hat{\boldsymbol{\nu}})\,\mathrm{d}t\mathrm{d}S$, thus

$$\int_{V(t+\mathrm{d}t)} \mathrm{d}V = \int_{V(t)} \mathrm{d}V + \int_{\delta V(t)} \mathrm{d}V = \int_{V(t)} \mathrm{d}V + \int_{S(t)} (\mathbf{w}\cdot\hat{\boldsymbol{\nu}})\,\mathrm{d}t\,\mathrm{d}S\ . \tag{1.9}$$

Now, by using first eq. (1.8) and then the position (1.9), we have

$$\int_V G(\mathbf{x},t)\,\mathrm{d}^3\mathbf{x}\bigg|_{t+\mathrm{d}t} = \int_{V(t)} \left[G(\mathbf{x},t) + \frac{\partial G(\mathbf{x},t)}{\partial t}\,\mathrm{d}t + O(\mathrm{d}t)^2\right]\mathrm{d}^3\mathbf{x}$$

$$+ \int_{S(t)} \left[G(\mathbf{x},t) + \frac{\partial G(\mathbf{x},t)}{\partial t}\,\mathrm{d}t + O(\mathrm{d}t)^2\right](\mathbf{w}\cdot\hat{\boldsymbol{\nu}})\,\mathrm{d}t\,\mathrm{d}S$$

$$= \int_{V(t)} G(\mathbf{x},t)\,\mathrm{d}^3\mathbf{x} + \left[\int_{V(t)} \frac{\partial G(\mathbf{x},t)}{\partial t}\,\mathrm{d}^3\mathbf{x} + \int_{S(t)} G(\mathbf{x},t)(\mathbf{w}\cdot\hat{\boldsymbol{\nu}})\,\mathrm{d}S\right]\mathrm{d}t + O(\mathrm{d}t)^2\ . \tag{1.10}$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} G(\mathbf{x},t)\,\mathrm{d}^3\mathbf{x} = \lim_{\delta t\to 0}\frac{1}{\delta t}\left[\int_V G(\mathbf{x},t)\,\mathrm{d}^3\mathbf{x}\bigg|_{t+\delta t} - \int_V G(\mathbf{x},t)\,\mathrm{d}^3\mathbf{x}\bigg|_{t}\right]\ , \tag{1.11}$$

then theorem (1.4.1) is proved.

■

If $V(t)$ is a *material volume* (i.e. a parcel of fluid made always by the same material particles) that moves with its boundary with the *flow velocity* $\mathbf{u}$ (that is $\mathbf{w} \equiv \mathbf{u}$; cfr. eq. (1.7)), then we identify the ordinary derivative $\mathrm{d}/\mathrm{d}t$ with the total derivative $\mathrm{D}/\mathrm{D}t$ and we have

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_{V(t)} G(\mathbf{x},t)\,\mathrm{d}^3\mathbf{x} = \int_{V(t)} \frac{\partial G(\mathbf{x},t)}{\partial t}\,\mathrm{d}^3\mathbf{x} + \int_{S(t)} G(\mathbf{x},t)(\mathbf{u}\cdot\hat{\boldsymbol{\nu}})\,\mathrm{d}^2\mathbf{x} \ . \tag{1.12}$$

By applying the Gauss divergence theorem to the second term on the left hand side of eq. (1.12) we have, as corollary, the kinematic transport theorem for a material volume, i.e.:

Corollary 1.4.1. *If $V(t)$ is a material volume, then*

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_{V(t)} G\,\mathrm{d}^3\mathbf{x} = \int_{V(t)} \left[\frac{\partial G}{\partial t} + \nabla\cdot(G\mathbf{u})\right]\mathrm{d}^3\mathbf{x} \ . \tag{1.13}$$

Hence

Definition 1.4.1. A quantity that is invariant in a material volume $V(t)$ satisfies the *conservation law*

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_{V(t)} G\,\mathrm{d}^3\mathbf{x} = 0 \ . \tag{1.14}$$

By using (1.13) we have the conservation law

$$\int_{V(t)} \left[\frac{\partial G}{\partial t} + \nabla\cdot(G\mathbf{u})\right]\mathrm{d}^3\mathbf{x} = 0 \ . \tag{1.15}$$

If $V(t)$ is a material volume, material particles do not leave or enter the volume, thus the integral applies to any infinitesimally small region, and we have the conservation law in differentail form

$$\frac{\partial G}{\partial t} + \nabla\cdot(G\mathbf{u}) = 0 \ ,$$

or

$$\frac{\partial G}{\partial t} = -\nabla\cdot(G\mathbf{u}) \ . \tag{1.16}$$

As a special case, if we take V to be constant with respect to time, then $\mathbf{w} = 0$ (or $\mathbf{u} = 0$), and the identity reduces to

Corollary 1.4.2. *If V is a volume constant in time, then we have*

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_{V} G\,\mathrm{d}^3\mathbf{x} = \int_{V} \frac{\partial G}{\partial t}\,\mathrm{d}^3\mathbf{x} \ .$$

1.5 Conservation of mass

Let us introduce the following definition.

Definition 1.5.1. Given the *mass density* $\rho = \rho(\mathbf{x}, t)$, we define the *mass* of fluid present in $\mathbb{R}^3$ by

$$M(t) = \int_V \rho(\mathbf{x}, t)\, \mathrm{d}^3\mathbf{x}\ . \tag{1.17}$$

From a fundamental principle of nature, we have conservation of total mass, i.e.

$$\frac{\mathrm{D}}{\mathrm{D}t} M(t) = 0 \quad \Rightarrow \quad M = \text{constant}\ . \tag{1.18}$$

By taking $G = \rho$ from eq. (1.15) we have the equation of the conservation of mass in integral form, i.e.

$$\int_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \right] \mathrm{d}^3\mathbf{x} = 0\ . \tag{1.19}$$

Under the general assumptions of regularity, that is if ρ and $\mathbf{u}$ remain smooth at all scales in finite time, then we have the standard equation of conservation of mass in differential form:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0\ . \tag{1.20}$$

Since $\nabla \cdot (\rho \mathbf{u}) = \nabla \rho \cdot \mathbf{u} + \rho \nabla \cdot \mathbf{u}$ we can re-write (1.20) as

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = -\rho \nabla \cdot \mathbf{u}\ , \tag{1.21}$$

or

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho \nabla \cdot \mathbf{u}\ , \tag{1.22}$$

known as *continuity equation*.

A fluid is said to be *homogeneous* if its density ρ is constant in space and time. Conversely, a fluid is said to be *non-homogeneous* if its density $\rho = \rho(\mathbf{x}, t)$ varies in space and time. A non-homogeneous fluid is said to be *incompressible* if ρ is conserved (invariant under fluid motion), that is

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0\ ; \tag{1.23}$$

thus, from (1.22), we have

$$\nabla \cdot \mathbf{u} = 0\ . \tag{1.24}$$

known as *incompressibility condition*.

1.6 Transport of dynamical property

For the moment let $\mathbf{G}$ be an unspecified fluid dynamical property and V a material volume moving with a flow velocity $\mathbf{u}$. Then we can extend the results of Theorem 1.4.1 and Corollary 1.4.1 to vector- or tensor-value quantities. We have:

Lemma 1.6.1 (Transport of dynamical property)**.** *Let* $\mathbf{G} = \mathbf{G}(\mathbf{x}; t)$ *be a vector- or tensor-value quantity; then we have*

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_V \mathbf{G}\,\mathrm{d}^3\mathbf{x} = \int_V \left[\frac{\mathrm{D}\mathbf{G}}{\mathrm{D}t} + \mathbf{G}(\nabla\cdot\mathbf{u})\right]\mathrm{d}^3\mathbf{x}\ . \tag{1.25}$$

Proof. Let's consider a unit vector field $\hat{\mathbf{e}}$ fixed in space and time. We can define the scalar quantity $G = \mathbf{G}\cdot\hat{\mathbf{e}}$ and, by applying the divergence theorem to Corollary 1.4.1, use eq. (1.13). We have

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_V (\mathbf{G}\cdot\hat{\mathbf{e}})\,\mathrm{d}^3\mathbf{x} = \int_V \left[\frac{\partial(\mathbf{G}\cdot\hat{\mathbf{e}})}{\partial t} + \nabla\cdot[(\mathbf{G}\cdot\hat{\mathbf{e}})\mathbf{u}]\right]\mathrm{d}^3\mathbf{x}\ . \tag{1.26}$$

Since $\hat{\mathbf{e}}$ is fixed and constant in time, evidently

$$\frac{\partial(\mathbf{G}\cdot\hat{\mathbf{e}})}{\partial t} = \frac{\partial\mathbf{G}}{\partial t}\cdot\hat{\mathbf{e}}\ ;$$

moreover, by treating $\mathbf{G}\cdot\hat{\mathbf{e}} = G$ as a generic scalar quantity we have

$$\nabla\cdot[(\mathbf{G}\cdot\hat{\mathbf{e}})\mathbf{u}] = \nabla\cdot(G\mathbf{u}) = (\mathbf{u}\cdot\nabla)G + G(\nabla\cdot\mathbf{u})\ .$$

Since $\hat{\mathbf{e}}$ is constant it can be brought in and out of integration, so that direct substitution of these latter expressions into (1.26) gives

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\int_V \mathbf{G}\,\mathrm{d}^3\mathbf{x}\right)\cdot\hat{\mathbf{e}} = \frac{\mathrm{D}}{\mathrm{D}t}\int_V (\mathbf{G}\cdot\hat{\mathbf{e}})\,\mathrm{d}^3\mathbf{x} = \int_V \left[\left(\frac{\partial G}{\partial t} + (\mathbf{u}\cdot\nabla)G\right) + G(\nabla\cdot\mathbf{u})\right]\mathrm{d}^3\mathbf{x}$$

$$= \int_V \left[\frac{\mathrm{D}G}{\mathrm{D}t} + G(\nabla\cdot\mathbf{u})\right]\mathrm{d}^3\mathbf{x} = \left\{\int_V \left[\frac{\mathrm{D}\mathbf{G}}{\mathrm{D}t} + \mathbf{G}(\nabla\cdot\mathbf{u})\right]\mathrm{d}^3\mathbf{x}\right\}\cdot\hat{\mathbf{e}}\ .$$

Multiplying the first and the last term above by $\hat{\mathbf{e}}$, we prove Lemma 1.6.1. ■

Now, let's take $\mathbf{G} = \rho\mathbf{q}$, where $\rho = \rho(\mathbf{x}; t)$ is fluid density and $\mathbf{q} = \mathbf{q}(\mathbf{x}; t)$ is a fluid dynamical property per unit volume. We have

Lemma 1.6.2. *Let* $\mathbf{G} = \rho\mathbf{q}$*; we have*

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_V \rho\mathbf{q}\,\mathrm{d}^3\mathbf{x} = \int_V \rho\frac{\mathrm{D}\mathbf{q}}{\mathrm{D}t}\,\mathrm{d}^3\mathbf{x}\ . \tag{1.27}$$

Proof. Using Lemma 1.6.1, we have

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_V \rho\mathbf{q}\,\mathrm{d}^3\mathbf{x} = \int_V \left[\frac{\mathrm{D}(\rho\mathbf{q})}{\mathrm{D}t} + (\rho\mathbf{q})(\nabla\cdot\mathbf{u})\right]\mathrm{d}^3\mathbf{x}\ . \tag{1.28}$$

Now

$$\begin{aligned}\frac{\mathrm{D}(\rho\mathbf{q})}{\mathrm{D}t} + (\rho\mathbf{q})(\nabla\cdot\mathbf{u}) &= \rho\frac{\mathrm{D}\mathbf{q}}{\mathrm{D}t} + \mathbf{q}\left[\frac{\partial\rho}{\partial t} + (\mathbf{u}\cdot\nabla)\rho\right] + (\rho\mathbf{q})(\nabla\cdot\mathbf{u}) \\ &= \rho\frac{\mathrm{D}\mathbf{q}}{\mathrm{D}t} + \mathbf{q}\left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u})\right]\ . \end{aligned} \tag{1.29}$$

Substituting this last expression into eq. (1.28) and applying the mass conservation in integral form given by (1.19), we have Corollary 1.6.2.

∎

1.7 Conservation of linear momentum

Let's apply the result above when $\mathbf{q} = \mathbf{u}$ is fluid velocity. In absence of any force acting on a fluid material volume, we have (trivially)

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_V \rho\mathbf{u}\,\mathrm{d}^3\mathbf{x} = \int_V \rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}\,\mathrm{d}^3\mathbf{x}\ . \tag{1.30}$$

In presence of forces, we must consider the action of these forces on each fluid element. Typically these include *volume* (or stress) *forces* (that depend on the material volume) and *conservative forces* (i.e. forces that can be associated with a potential). In a fluid, stress is proportional to the rate of strain. For the moment we restrict our considerations to the study of an ideal fluid, according to the following definition:

Definition 1.7.1. An *ideal fluid* is a fluid that supports only normal stresses and no shear stresses.

Thus, any applied strain manifests itself as an isotropic stress (pressure) and vice-versa. Pressure forces acts solely along the normal to the fluid surface bounding a fluid element. This means that an ideal flow has no resistance for relative motion between fluid layers, with no exchange of momentum (due to viscosity) or energy (due to heat flux) between layers. Typically fluid pressure is the standard volume force that acts normally on the bounding surface $\partial V = S$ of a fluid element. Such a force is given by

$$\mathbf{\Pi}_\perp = -\int_S p\hat{\boldsymbol{\nu}}\,\mathrm{d}^2\mathbf{x}\ , \tag{1.31}$$

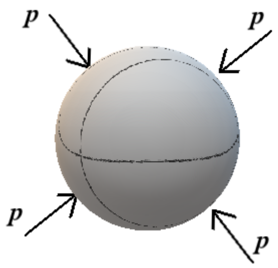


Figure 1.5: An elementary fluid volume subject to pressure force p.

where $-p\hat{\boldsymbol{\nu}}$ denotes ordinary pressure (a force per unit surface) that acts inwardly on S (see Figure 1.5). Among conservative forces we have gravity — associated with gravitational potential — electric forces, and magnetic (Lorentz) forces — associated with a magnetic vector potential. Let's denote them (generically) by $\mathbf{F}$, so that

$$\mathbf{F} = \int_V \rho \mathbf{f} \, \mathrm{d}^3\mathbf{x} \ , \tag{1.32}$$

where $\mathbf{f}$ is the force per unit mass. In this case by the second Newton's law eq. (1.30) becomes

$$\frac{\mathrm{D}}{\mathrm{D}t} \int_V \rho \mathbf{u} \, \mathrm{d}^3\mathbf{x} = \int_V \rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} \, \mathrm{d}^3\mathbf{x} = \mathbf{\Pi}_\perp + \mathbf{F} \ . \tag{1.33}$$

Let's consider a unit vector $\hat{\mathbf{e}}$ fixed in space and time. By using (1.31) and (1.32), the equation above becomes

$$\int_V \rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} \cdot \hat{\mathbf{e}} \, \mathrm{d}^3\mathbf{x} = -\int_S p\hat{\boldsymbol{\nu}} \cdot \hat{\mathbf{e}} \, \mathrm{d}^2\mathbf{x} + \int_V \rho \mathbf{f} \cdot \hat{\mathbf{e}} \, \mathrm{d}^3\mathbf{x} \ . \tag{1.34}$$

Now, by applying the divergence theorem to the pressure term, $\nabla \cdot \hat{\mathbf{e}} = 0$ and (1.34) becomes

$$\int_V \rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} \cdot \hat{\mathbf{e}} \, \mathrm{d}^3\mathbf{x} = \int_V (-\nabla p \cdot \hat{\mathbf{e}} + \rho \mathbf{f} \cdot \hat{\mathbf{e}}) \, \mathrm{d}^3\mathbf{x} \ . \tag{1.35}$$

By eliminating $\hat{\mathbf{e}}$, we can write the equation of the conservation of linear momentum in integral form, given by

$$\int_V \rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} \, \mathrm{d}^3\mathbf{x} = \int_V (-\nabla p + \rho \mathbf{f}) \, \mathrm{d}^3\mathbf{x} \ . \tag{1.36}$$

Under the general assumption of *regularity*, that is if ρ and $\mathbf{u}$ remain smooth at all scales in time, then we have the standard equation of *conservation of linear momentum* in differential form:

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p + \rho \mathbf{f} \ . \tag{1.37}$$

In absence of conservative forces ($\mathbf{f} = 0$) we recover the celebrated *Euler equations*, given by

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p \ . \tag{1.38}$$

1.8 Decomposition of fluid motion

First, let us introduce a new vector field $\boldsymbol{\omega} = \boldsymbol{\omega}(\mathbf{x}; t)$ derived from the velocity $\mathbf{u} = (u, v, w)$ at a point in space $\mathbf{x} = (x, y, z)$.

Definition 1.8.1. *Vorticity* $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ is given by $\boldsymbol{\omega} = \nabla \times \mathbf{u}$, that is

$$\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3) = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \ . \tag{1.39}$$

Let's consider the *strain rate tensor*, a purely kinematic concept that describes the macroscopic motion of the material. In an arbitrary reference frame this is identified with the jacobian matrix $\mathsf{J} = \mathsf{J}(\mathbf{u}; \mathbf{x})$ given by

$$\mathsf{J}(\mathbf{u}; \mathbf{x}) = \frac{\partial(\mathbf{u})}{\partial(\mathbf{x})} = \begin{pmatrix} \partial_x u & \partial_y u & \partial_z u \\ \partial_x v & \partial_y v & \partial_z v \\ \partial_x w & \partial_y w & \partial_z w \end{pmatrix} \ . \tag{1.40}$$

Let's denote the jacobian simply by $\mathsf{J} = \mathsf{J}(\mathbf{x})$; from standard linear algebra we can always write

$$\mathsf{J}(\mathbf{x}) = \mathsf{D}(\mathbf{x}) + \mathsf{S}(\mathbf{x}) \ , \tag{1.41}$$

where $\mathsf{D}(\mathbf{x})$ and $\mathsf{S}(\mathbf{x})$ denote respectively the symmetric and anti-symmetric part of J given by

$$\mathsf{D}(\mathbf{x}) = \frac{1}{2} \left(\mathsf{J}(\mathbf{x}) + \mathsf{J}^{\mathrm{T}}(\mathbf{x}) \right) \ , \qquad \mathsf{S}(\mathbf{x}) = \frac{1}{2} \left(\mathsf{J}(\mathbf{x}) - \mathsf{J}^{T}(\mathbf{x}) \right) \ ,$$

where J^{T} is the transpose of J. Since D is real symmetric we can always reduce D to diagonal form by choosing an appropriate orthonormal coordinate system given by the principal reference system $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$. In this system $\mathsf{D}(\mathbf{x})$ and $\mathsf{S}(\mathbf{x})$ reduce to

$$\mathsf{D}(\mathbf{x}) = \begin{pmatrix} \partial_x u & 0 & 0 \\ 0 & \partial_y v & 0 \\ 0 & 0 & \partial_z w \end{pmatrix} = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix} \ , \tag{1.42}$$

and

$$\mathsf{S}(\mathbf{x}) = \frac{1}{2} \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \ . \tag{1.43}$$

From Lagrange's original analysis, we have the following fundamental result.

Lemma 1.8.1 (Decomposition of motion)**.** *Let $P = P(\mathbf{x})$ and $Q = Q(\mathbf{y})$ be two neighbouring material points in the fluid and let $\mathbf{h} = \mathbf{y} - \mathbf{x}$ be the displacement vector $\overrightarrow{PQ}$. We have*

$$\mathbf{u}(\mathbf{y}) = \mathbf{u}(\mathbf{x}) + \mathsf{D}(\mathbf{x}) \cdot \mathbf{h} + \frac{1}{2}\boldsymbol{\omega}(\mathbf{x}) \times \mathbf{h} + O(|\mathbf{h}|^2) \ . \tag{1.44}$$

Proof. Let us consider the Taylor expansion

$$\mathbf{u}(\mathbf{y}) = \mathbf{u}(\mathbf{x}) + \mathsf{J}(\mathbf{x}) \cdot \mathbf{h} + O(|\mathbf{h}|^2) \ , \tag{1.45}$$

and the decomposition (1.41), so that

$$\mathbf{u}(\mathbf{y}) = \mathbf{u}(\mathbf{x}) + [\mathsf{D}(\mathbf{x}) + \mathsf{S}(\mathbf{x})] \cdot \mathbf{h} + O(|\mathbf{h}|^2) \ . \tag{1.46}$$

We can write

$$\mathsf{S}(\mathbf{x}) \cdot \mathbf{h} = \frac{1}{2}\boldsymbol{\omega} \times \mathbf{h} \ . \tag{1.47}$$

By combining (1.46) with (1.47) the Lemma is proved.

■

1.8.1 Physical interpretation in terms of local flows

The background flow $\mathbf{u}(\mathbf{x})$ can be interpreted in terms of a rigid translation of fluid particles. We can then decompose the motion as the sum of two other components. Let's examine them separately.

I Case: $\mathbf{u}(\mathbf{x}) = 0$ and $\boldsymbol{\omega}(\mathbf{x}) = 0$. Let's assume that locally (i.e. at a given $\mathbf{x}$) we move with the fluid ($\mathbf{u}(\mathbf{x}) = 0$) and we have no rotation ($\boldsymbol{\omega}(x) = 0$); in this case equation (1.44) reduces to

$$\mathbf{u}(\mathbf{y}) = \frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = \mathsf{D}(\mathbf{x}) \cdot \mathbf{h} + O(|\mathbf{h}|^2) \ . \tag{1.48}$$

Since $\mathbf{y} = \mathbf{h}$ (here $\mathbf{x} = 0$), to a first approximation this can be re-written as

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = \mathsf{D}(\mathbf{x}) \cdot \mathbf{h} \ , \tag{1.49}$$

that is a linear, first-order, ordinary differential equation in $\mathbf{h}$, whose solution clearly depends on D. Note that

$$\mathrm{Tr}(\mathsf{D}) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla \cdot \mathbf{u} \ ; \tag{1.50}$$

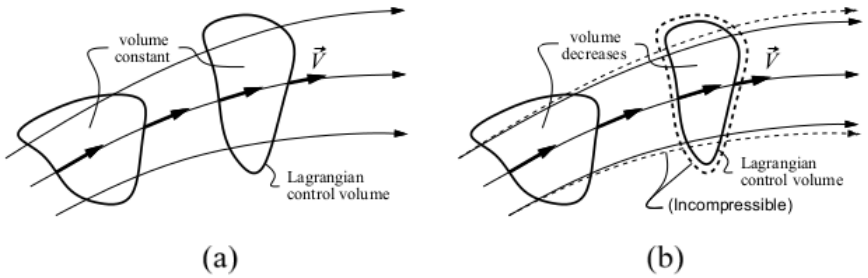


Figure 1.6: Advection of a fluid element under the flow map: (a) incompressible case; (b) compressible case. The change of volume is shown by the dashed region.

by using eq. (1.22) we see that Tr(D) admits physical interpretation in terms of compressibility (note that Tr(D) is invariant under change of coordinates).

In the principal coordinate system the elementary volume is given by the product of the cube sizes h_1, h_2, h_3 in the directions $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$. The change of $\delta V = h_1 h_2 h_3$ in time (see Figure 1.6b) is given by

$$\frac{\mathrm{d}(h_1 h_2 h_3)}{\mathrm{d}t} = \frac{\mathrm{d}h_1}{\mathrm{d}t} h_2 h_3 + \frac{\mathrm{d}h_2}{\mathrm{d}t} h_1 h_3 + \frac{\mathrm{d}h_3}{\mathrm{d}t} h_1 h_2 = (d_1 + d_2 + d_3)(h_1 h_2 h_3) \ , \quad (1.51)$$

where $\mathrm{d}_i h_i = \mathrm{d}h_i/\mathrm{d}t$ $(i = 1, 2, 3)$. For the elementary displacement $\mathbf{h} = \delta\mathbf{x}$ we have

$$\frac{\mathrm{d}(\delta\mathbf{x})}{\mathrm{d}t} \frac{1}{\delta\mathbf{x}} = \frac{\delta}{\delta\mathbf{x}} \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \frac{\delta}{\delta\mathbf{x}} \mathbf{u} = \nabla \cdot \mathbf{u} \ . \quad (1.52)$$

This means that in the limit $h_i \to 0$ we have

$$d_1 + d_2 + d_3 = \sum_i \frac{\mathrm{d}h_i}{\mathrm{d}t} \frac{1}{h_i} = \nabla \cdot \mathbf{u} \ , \quad (1.53)$$

and

$$\frac{\mathrm{d}(\delta V)}{\mathrm{d}t} = (d_1 + d_2 + d_3)\, \delta V = [\mathrm{Tr}(\mathsf{D})]\, \delta V = (\nabla \cdot \mathbf{u})\, \delta V \ , \quad (1.54)$$

that relates the change in time of the volume with the divergence of $\mathbf{u}$. Since D is responsible for the geometric change of the elementary volume by distortion, expansion or contraction, D can be seen as a *deformation tensor*.

II Case: $\mathbf{u}(\mathbf{x}) = 0$ and $\mathsf{D}(\mathbf{x}) \cdot \mathbf{h} = 0$. Let's now assume that at a given point $\mathbf{x}$ we have $\mathbf{u}(\mathbf{x}) = 0$ and no rigid translation and deformation i.e. $\mathsf{D}(\mathbf{x}) \cdot \mathbf{h} = 0$. In this case equation (1.44) reduces to

$$\mathbf{u}(\mathbf{y}) = \frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = \frac{1}{2} \boldsymbol{\omega} \times \mathbf{h} + O(|\mathbf{h}|^2) \ . \quad (1.55)$$

Again, since $\mathbf{y} = \mathbf{h}\ (\mathbf{x} = 0)$ to a first approximation this can be expressed in terms of a rotation matrix Q

$$\frac{\mathrm{d}\mathbf{h}}{\mathrm{d}t} = \mathsf{Q}(\mathbf{x}) \cdot \mathbf{h} \ , \tag{1.56}$$

that is formally equivalent to eq. (1.49). $\mathsf{Q}(\mathbf{x})$ can be interpreted in terms of a rotation of the fluid volume of an 'angle' t around the direction of $\boldsymbol{\omega}(\mathbf{x})$. Thus, $(1/2)\,\boldsymbol{\omega} \times \mathbf{h}$ can be interpreted in terms of a rigid rotation of the fluid volume through the *rotation tensor* $\mathsf{Q}(\mathbf{x})$.

In summary we can interpret eq. (1.44) as a linear combination of a rigid translation due to the flow $\mathbf{u}_{\mathrm{T}}$, a volume deformation due to $\mathbf{u}_{\mathrm{D}}$ and a rigid rotation due to $\mathbf{u}_{\mathrm{R}}$:

Lemma 1.8.2 (Decomposition of fluid motion)**.** *The velocity field* $\mathbf{u}$ *at a point* $\mathbf{y}$ *can always be decomposed by*

$$\mathbf{u}(\mathbf{y}) \approx \mathbf{u}_{\mathrm{T}} + \mathbf{u}_{\mathrm{D}} + \mathbf{u}_{\mathrm{R}} \ . \tag{1.57}$$

that is

$$\mathbf{u}(\mathbf{y}) \approx \underbrace{\mathbf{u}(\mathbf{x})}_{\substack{\textit{rigid} \\ \textit{translation}}} + \underbrace{\mathsf{D}(\mathbf{x}) \cdot \mathbf{h}}_{\substack{\textit{volume} \\ \textit{deformation}}} + \underbrace{\frac{1}{2}\boldsymbol{\omega}(\mathbf{x}) \times \mathbf{h}}_{\substack{\textit{rigid} \\ \textit{rotation}}} \ .$$

1.9 Analytic solutions to local flows

From the local decomposition of fluid motion (1.57), we can determine the analytic solution for the velocity field $\mathbf{u}$ at any point of $\mathbb{R}^3$ as a linear combination of the solutions for the local flows $\mathbf{u}_{\mathrm{T}}$, $\mathbf{u}_{\mathrm{D}}$ and $\mathbf{u}_{\mathrm{R}}$. Let us consider each contribution separately.

I Case: determine $\mathbf{u}_{\mathrm{T}}$ by assuming $\mathbf{u}_{\mathrm{D}} = \mathbf{u}_{\mathrm{R}} = 0$. In this case there is no deformation or rotation of the fluid element, but simple rigid translation. This implies

$$\nabla \cdot \mathbf{u}_{\mathrm{T}} = 0 \ , \quad \nabla \times \mathbf{u}_{\mathrm{T}} = 0 \ . \tag{1.58}$$

Since there is no rotation of $\mathbf{u}_{\mathrm{T}}$, by Stokes' theorem we have

$$\int_{\Sigma} (\nabla \times \mathbf{u}_{\mathrm{T}}) \cdot \hat{\boldsymbol{\nu}}\, \mathrm{d}^2\mathbf{x} = \Gamma_{\mathcal{C}} = \oint_{\mathcal{C}} \mathbf{u}_{\mathrm{T}} \cdot \mathrm{d}\mathbf{l} = 0 \ , \tag{1.59}$$

which means that circulation of $\mathbf{u}_{\mathrm{T}}$ is always zero for any closed path $\mathcal{C}$ in $\mathbb{R}^3$. Since $\mathbf{u}_{\mathrm{T}}$ is non-zero and $\mathcal{C}$ is arbitrary, the vanishing of the last integral is only possible by assuming a vanishing integration path through its continuous deformation to a point (see Figure 1.7). This implies a *simply connected* fluid domain.

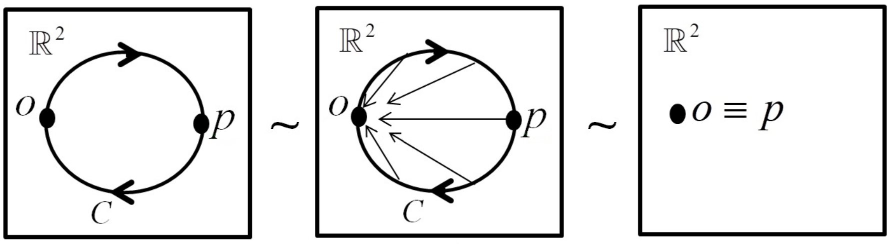


Figure 1.7: Reduction of a closed path to a point in a simply connected region of $\mathbb{R}^2$

Now, let's split $\mathcal{C}$ into two contiguous arcs ($\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$) joint at the two points O (origin of the parametrization of $\mathbf{l}$ on $\mathcal{C}_1$) and P (see Figure 1.8a). By reverting the parametrization on the arc $\mathcal{C}_2$ (denoted by $\bar{\mathbf{l}}$; see in Figure 1.8b) we must have $\bar{\mathbf{l}} = -\mathbf{l}$ so that both arcs have same common origin and endpoint. Thus, from (1.59) we have

$$\oint_{\mathcal{C}} \mathbf{u}_{\mathrm{T}} \cdot \mathrm{d}\mathbf{l} = 0 = \int_{\mathcal{C}_1} \mathbf{u}_{\mathrm{T}} \cdot \mathrm{d}\mathbf{l} + \int_{\mathcal{C}_2} \mathbf{u}_{\mathrm{T}} \cdot \mathrm{d}\bar{\mathbf{l}} = \int_{\mathcal{C}_1} \mathbf{u}_{\mathrm{T}} \cdot \mathrm{d}\mathbf{l} - \int_{\mathcal{C}_2} \mathbf{u}_{\mathrm{T}} \cdot \mathrm{d}\mathbf{l} \ . \tag{1.60}$$

Since $\mathcal{C}$ is arbitrary, from the last equation we have

$$\int_{\mathcal{C}_1} \mathbf{u}_{\mathrm{T}} \cdot \mathrm{d}\mathbf{l} = \int_{\mathcal{C}_2} \mathbf{u}_{\mathrm{T}} \cdot \mathrm{d}\mathbf{l} = \int_{\mathcal{C}^*} \mathbf{u}_{\mathrm{T}} \cdot \mathrm{d}\mathbf{l} \tag{1.61}$$

for any arbitrary path $\mathcal{C}^*$ (Figure 1.8c). Since the integral does not depend on the specific path, but it depends only on the value at the two endpoints O to P, we can define a potential $\phi_{\mathrm{T}} = \phi_{\mathrm{T}}(\mathbf{x})$ such that

$$\int_{\mathcal{C}^*} \mathbf{u}_{\mathrm{T}} \cdot \mathrm{d}\mathbf{l} = \int_O^P \mathbf{u}_T \cdot \mathrm{d}\mathbf{x} = \phi_{\mathrm{T}}(\mathbf{x}_P) - \phi_{\mathrm{T}}(\mathbf{x}_O) \ . \tag{1.62}$$

by setting

$$\mathbf{u}_{\mathrm{T}} = \nabla \phi_{\mathrm{T}} \ . \tag{1.63}$$

Since $\mathbf{u}_{\mathrm{T}}$ is divergence-less, we have $\nabla \cdot (\nabla \phi_{\mathrm{T}}) = 0$, with potential satisfying the *Laplace's equation*

$$\nabla^2 \phi_{\mathrm{T}}(\mathbf{x}) = 0 \ , \tag{1.64}$$

in a simply connected region, with appropriate boundary conditions. The potential ϕ_{T} is therefore a harmonic function in $\mathbb{R}^3$. From the solution of (1.64) by using (1.63), we have the velocity $\mathbf{u}_{\mathrm{T}}$.

II Case: determine $\mathbf{u}_{\mathrm{D}}$ by assuming $\mathbf{u}_{\mathrm{T}} = \mathbf{u}_{\mathrm{R}} = 0$. In this case there is no background translation or rotation of the fluid element, but simple deformation

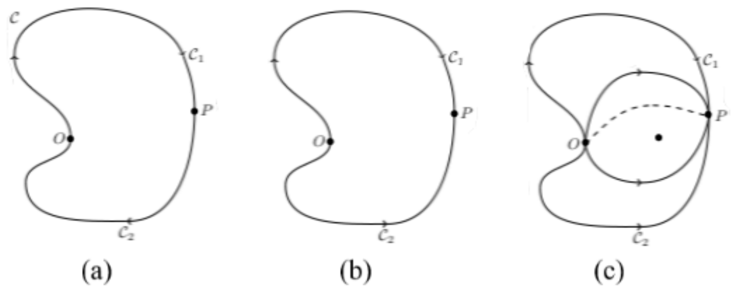


Figure 1.8: (a) A closed, oriented path $\mathcal{C}$ can be seen as the joint union of two paths given by the oriented arcs $\mathcal{C}_1$ and $\mathcal{C}_2$, originating and ending at the points O and P. (b) By re-defining parametrization we can revert orientation of $\mathcal{C}_2$ so that both curves have same origin O and endpoint P. (c) The path integral from O to P is then independent of the specific path.

due to contraction or expansion of the fluid volume. This means that the velocity field is no longer divergence-less due to the presence of sources or sinks at some point $\mathbf{x}^*$ through a function $\eta = \eta(\mathbf{x}^*)$, i.e.

$$\nabla \cdot \mathbf{u}_{\mathrm{D}} = \eta \ , \quad \nabla \times \mathbf{u}_{\mathrm{D}} = 0 \ . \tag{1.65}$$

As in the previous case, since there is no rotation we can introduce a potential $\phi_{\mathrm{D}} = \phi_{\mathrm{D}}(\mathbf{x})$ for $\mathbf{u}_{\mathrm{D}}$ such that

$$\mathbf{u}_{\mathrm{D}} = \nabla \phi_{\mathrm{D}} \ ; \tag{1.66}$$

since $\nabla \cdot \mathbf{u}_{\mathrm{D}} = \nabla \cdot (\nabla \phi_{\mathrm{D}}) = \eta$, we have

$$\nabla^2 \phi_{\mathrm{D}}(\mathbf{x}) = \eta(\mathbf{x}^*) \tag{1.67}$$

known as *Poisson's equation for contraction or expansion*. From Green's potential theory, (1.67) has solution given by

$$\phi_{\mathrm{D}}(\mathbf{x}) = -\frac{1}{4\pi} \int_W \frac{\eta(\mathbf{x}^*)}{|\mathbf{x} - \mathbf{x}^*|} \mathrm{d}^3\mathbf{x}^* \ , \tag{1.68}$$

where $G(\mathbf{x}, \mathbf{x}^*) = -1/4\pi|\mathbf{x} - \mathbf{x}^*|$ is Green's fundamental solution (*Green's function*) to Laplace's operator and $W = W(\mathbf{x}^*)$ is the volume of the source or sink points at $\mathbf{x}^*$. As usual by assigning appropriate boundary conditions we can determine $\mathbf{u}_{\mathrm{D}}$ from the gradient of $\phi_{\mathrm{D}}(\mathbf{x})$, so that

$$\mathbf{u}_{\mathrm{D}}(\mathbf{x}) = -\frac{1}{4\pi} \int_W \eta(\mathbf{x}^*) \nabla \left(\frac{1}{|\mathbf{x} - \mathbf{x}^*|} \right) \mathrm{d}^3\mathbf{x}^* \ , \tag{1.69}$$

or

$$\mathbf{u}_{\mathrm{D}}(\mathbf{x}) = \frac{1}{4\pi}\int_W \frac{\eta(\mathbf{x}^*)(\mathbf{x}-\mathbf{x}^*)}{|\mathbf{x}-\mathbf{x}^*|^3}\,\mathrm{d}^3\mathbf{x}^* \ . \tag{1.70}$$

III Case: determine $\mathbf{u}_{\mathrm{R}}$ by assuming $\mathbf{u}_{\mathrm{T}} = \mathbf{u}_{\mathrm{D}} = 0$. In this case there is no translation and deformation of the fluid. In presence of pure rotation at points $\mathbf{x}^*$, we have

$$\nabla\cdot\mathbf{u}_{\mathrm{R}} = 0 \ , \quad \nabla\times\mathbf{u}_{\mathrm{R}} = \boldsymbol{\omega} \ , \tag{1.71}$$

where $\boldsymbol{\omega} = \boldsymbol{\omega}(\mathbf{x}^*)$ is vorticity. Since vorticity is a vector field and is divergenceless (the divergence of a curl is always zero), we may introduce a vector potential $\mathbf{A} = \mathbf{A}_{\mathrm{R}}(\mathbf{x}^*)$, such that

$$\mathbf{u}_{\mathrm{R}} = \nabla\times\mathbf{A}_{\mathrm{R}} \ ; \tag{1.72}$$

so

$$\boldsymbol{\omega} = \nabla\times\mathbf{u}_{\mathrm{R}} = \nabla\times(\nabla\times\mathbf{A}_{\mathrm{R}}) \ . \tag{1.73}$$

Now, from the standard vector identity

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = (\nabla\cdot\mathbf{B}+\mathbf{B}\cdot\nabla)\mathbf{A} - (\nabla\cdot\mathbf{A}+\mathbf{A}\cdot\nabla)\mathbf{B} \ , \tag{1.74}$$

by taking $\mathbf{A} = \nabla$ and $\mathbf{B} = \mathbf{A}_{\mathrm{R}}$, we have

$$\boldsymbol{\omega} = \nabla\times(\nabla\times\mathbf{A}_{\mathrm{R}}) = (\nabla\cdot\mathbf{A}_{\mathrm{R}}+\mathbf{A}_{\mathrm{R}}\cdot\nabla)\nabla - (\nabla\cdot\nabla+\nabla\cdot\nabla)\mathbf{A}_{\mathrm{R}} \ ,$$

and since $(\mathbf{A}_{\mathrm{R}}\cdot\nabla)\nabla = (\nabla\cdot\nabla)\mathbf{A}_{\mathrm{R}}$, we have the standard relation for the curl of a curl:

$$\boldsymbol{\omega} = \nabla(\nabla\cdot\mathbf{A}_{\mathrm{R}}) - \nabla^2\mathbf{A}_{\mathrm{R}} \ . \tag{1.75}$$

It is customary to take $\nabla\cdot\mathbf{A}_{\mathrm{R}} = 0$ (*Coulomb gauge*), so that the equation above becomes

$$\nabla^2\mathbf{A}_{\mathrm{R}}(\mathbf{x}) = -\boldsymbol{\omega}(\mathbf{x}^*) \ , \tag{1.76}$$

known as *Poisson's equation for vorticity* $\boldsymbol{\omega}$. By using Green's fundamental solution again, we have

$$\mathbf{A}_{\mathrm{R}}(\mathbf{x}) = \frac{1}{4\pi}\int_W \frac{\boldsymbol{\omega}(\mathbf{x}^*)}{|\mathbf{x}-\mathbf{x}^*|}\,\mathrm{d}^3\mathbf{x}^* \ , \tag{1.77}$$

and by taking the curl

$$\begin{aligned}\mathbf{u}_{\mathrm{R}}(\mathbf{x}) &= \frac{1}{4\pi}\int_W \nabla\times\left(\frac{\boldsymbol{\omega}(\mathbf{x}^*)}{|\mathbf{x}-\mathbf{x}^*|}\right)\mathrm{d}^3\mathbf{x}^* \\ &= -\frac{1}{4\pi}\int_W \frac{\boldsymbol{\omega}(\mathbf{x}^*)\times(\mathbf{x}-\mathbf{x}^*)}{|\mathbf{x}-\mathbf{x}^*|^3}\,\mathrm{d}^3\mathbf{x}^* \ , \end{aligned} \tag{1.78}$$

known as *Biot-Savart induction law* for vorticity. Note that this is a global functional of vorticity whose solution $\mathbf{u}_{\mathrm{R}} = \mathbf{u}_{\mathrm{R}}(\mathbf{x})$ depends not only on $\boldsymbol{\omega}$, but also on the domain of definition $W = W(\mathbf{x}^*)$ of vorticity.

1.10 Conservation of energy

Since an ideal fluid is a fluid deprived of dissipation (i.e. *inviscid*) and heat transfers (i.e. *adiabatic*), there is no mechanism responsible for energy depletion. So total energy can be converted from one form to another, while being conserved. Thus, ideal motion is a constant entropy process. For this ideal fluid is also called *isentropic*. In general total energy is given by the sum of kinetic and internal energy, i.e.

$$E_{\mathrm{tot}} = \int_V \rho\epsilon \,\mathrm{d}^3\mathbf{x} + \frac{1}{2}\int_V \rho|\mathbf{u}|^2 \,\mathrm{d}^3\mathbf{x} \ , \tag{1.79}$$

where the first integral on the right-hand-side of (1.79) represents internal energy E_{int} (ϵ is internal energy per unit mass), while the second integral the kinetic term E_{kin}. The sum of the energies of all the molecules of the system is represented by E_{int}, while E_{kin} is evidently associated with fluid motion. From a thermodynamic viewpoint it is customary to refer to the *specific enthalpy* $h = h(\mathbf{x}, t)$ given by the sum of specific internal energy $\epsilon = \epsilon(\mathbf{x}, t)$ and specific pressure, i.e.

$$h = \epsilon + \frac{p}{\rho} \ . \tag{1.80}$$

From the principle of conservation of energy, and in absence of volume or conservative forces, we have

$$\frac{\mathrm{D}}{\mathrm{D}t}E_{\mathrm{tot}} = 0 \ . \tag{1.81}$$

In presence of forces, eq. (1.79) must be amended as follows

$$\frac{\mathrm{D}}{\mathrm{D}t}E_{\mathrm{tot}} = -\int_S p\mathbf{u}\cdot\hat{\boldsymbol{\nu}}\,\mathrm{d}^2\mathbf{x} + \int_V \rho\mathbf{u}\cdot\mathbf{f}\,\mathrm{d}^3\mathbf{x} \ . \tag{1.82}$$

In ideal conditions the internal energy term is usually neglected and in absence of forces acting on the material volume total energy reduces to the sole contribution due to the kinetic term.

1.10.1 Conservation of kinetic energy of an incompressible, ideal fluid

For simplicity we assume that the fluid is ideal and incompressible and occupies an infinite domain. Moreover, we neglect conservative forces and assume vanishing pressure forces at infinity. Under these assumptions we have:

Theorem 1.10.1 (Conservation of kinetic energy). *The kinetic energy E_{kin} of an ideal, incompressible fluid is conserved, that is*

$$\frac{\mathrm{D}}{\mathrm{D}t}E_{\mathrm{kin}}(t) = 0 \ . \tag{1.83}$$

Proof. By applying the result of Lemma 1.6.2, we have

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_V \frac{1}{2}\rho|\mathbf{u}|^2\mathrm{d}^3\mathbf{x} = \frac{1}{2}\int_V \rho\frac{\mathrm{D}}{\mathrm{D}t}|\mathbf{u}|^2\,\mathrm{d}^3\mathbf{x} = \int_V \rho\mathbf{u}\cdot\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}\,\mathrm{d}^3\mathbf{x}\ . \tag{1.84}$$

By taking the potential $\chi = -p + \phi$ for pressure and conservative force $= \nabla\phi$ and substituting eq. (1.37) into the last integral, we have

$$\frac{\mathrm{D}E_{\mathrm{kin}}}{\mathrm{D}t} = \int_V \mathbf{u}\cdot\nabla\chi\,\mathrm{d}^3\mathbf{x} = \int_V \mathbf{u}\cdot\nabla(\chi-\chi_\infty)\,\mathrm{d}^3\mathbf{x}\ , \tag{1.85}$$

where χ_∞ denotes the value at infinity. By applying the divergence theorem to an incompressible fluid ($\nabla\cdot\mathbf{u} = 0$), we have

$$\frac{\mathrm{D}E_{\mathrm{kin}}}{\mathrm{D}t} = \int_V \nabla\left(\mathbf{u}\left(\chi-\chi_\infty\right)\right)\,\mathrm{d}^3\mathbf{x} = \int_S \mathbf{u}\cdot\hat{\boldsymbol{\nu}}(\chi-\chi_\infty)\,\mathrm{d}^2\mathbf{x} = 0\ , \tag{1.86}$$

since in the limit $\mathbf{x}\to\infty$ the potential difference vanishes.

■

1.11 Governing equations of an ideal fluid

Incompressible ideal fluids play an important role in the study of fluid motion by offering an idealised and simplified model useful for mathematical analysis. The jacobian determinant $J = J(\mathbf{x};\mathbf{x}_0)$ associated with the deformation of fluid elements under the flow map φ is given by

$$J(\mathbf{x};\mathbf{x}_0) = \left|\frac{\partial(\mathbf{x})}{\partial(\mathbf{x}_0)}\right|\ . \tag{1.87}$$

From linear algebra one can directly demonstrate that

$$\frac{\mathrm{D}J}{\mathrm{D}t} = J\,\nabla\cdot\mathbf{u}\ . \tag{1.88}$$

Note the similarity between the form of this equation and the continuity equation (1.22) in absence of the convective term. If $\mathbf{x} = \mathbf{x}(t)$ denotes a particle trajectory from $t = 0$ to some time t, eq. (1.88) becomes an ordinary differential equation with solution

$$J(\mathbf{x};\mathbf{x}_0) = \exp\left(\int_{t=0}^{t}\nabla\cdot\mathbf{u}\,\mathrm{d}t\right)\ ,$$

that shows how the ratio of the new to the original volume element (expressed by $J(\mathbf{x};\mathbf{x}_0)$) is governed by the degree of expansion or contraction of a bundle of trajectories (measured by $\nabla \cdot \mathbf{u}$). The action of the jacobian on the fluid density is made explicit by showing that the product ρJ is constant along a particle trajectory, i.e.

$$\frac{\mathrm{d}(\rho J)}{\mathrm{d}t} = \frac{\mathrm{d}\rho}{\mathrm{d}t} J + \rho \frac{\mathrm{d}J}{\mathrm{d}t} = (-\rho \nabla \cdot \mathbf{u}) J + \rho (J \nabla \cdot \mathbf{u}) = 0 \ .$$

Hence, we have:

Lemma 1.11.1. *The following statements are equivalent:*

(i) the fluid is incompressible $\Leftrightarrow \dfrac{\mathrm{D}\rho}{\mathrm{D}t} = 0$*;*

(ii) the velocity field is solenoidal $\Leftrightarrow \nabla \cdot \mathbf{u} = 0$*;*

(iii) the fluid flow is volume-preserving $\Leftrightarrow J(\mathbf{x}; \mathbf{x}_0) = 1$.

It is now possible to provide an alternative proof of the (Reynolds) transport theorem (1.4.1) for a material volume by direct use of the jacobian of the transformation from the initial volume $V_0 = V(t = 0)$ to the volume $V = V(t)$. Remember that since V is a material volume there is no flux of particles through its bounding surface. Since $\mathbf{x} = J(\mathbf{x}; \mathbf{x}_0)\mathbf{x}_0$ (and thus $V = JV_0$), we evidently have

$$\begin{aligned}
\frac{\mathrm{D}}{\mathrm{D}t} \int_V G \, \mathrm{d}^3\mathbf{x} &= \frac{\mathrm{D}}{\mathrm{D}t} \int_{V_0} GJ \, \mathrm{d}^3(\mathbf{x}_0) = \frac{\mathrm{D}}{\mathrm{D}t} \int_{V_0} G \, \mathrm{d}^3(J\mathbf{x}_0) = \int_{V_0} \frac{\mathrm{D}}{\mathrm{D}t}(GJ) \, \mathrm{d}^3\mathbf{x}_0 \\
&= \int_{V_0} \left[\frac{\mathrm{D}G}{\mathrm{D}t} J + \frac{\mathrm{D}J}{\mathrm{D}t} G \right] \mathrm{d}^3\mathbf{x}_0 = \int_V \frac{\mathrm{D}G}{\mathrm{D}t} \, \mathrm{d}^3\mathbf{x} + \int_{V_0} GJ \nabla \cdot \mathbf{u} \, \mathrm{d}^3\mathbf{x}_0 \\
&= \int_V \left[\frac{\partial G}{\partial t} + \nabla \cdot (G\mathbf{u}) \right] \mathrm{d}^3\mathbf{x} = \int_V \frac{\partial G}{\partial t} \, \mathrm{d}^3\mathbf{x} + \int_S G(\mathbf{u} \cdot \hat{\boldsymbol{\nu}}) \, \mathrm{d}^2\mathbf{x} \ .
\end{aligned}$$

Hence we recover (1.12) and (1.13).

In summary, the governing equations of an incompressible, ideal fluid are given by

$$\mathrm{E1}: \quad \begin{cases} \nabla \cdot \mathbf{u} = 0 \ , \\ \rho \dfrac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p + \rho \mathbf{f} \ . \end{cases} \tag{1.89}$$

If we relax incompressibility, the ideal fluid is governed by the alternative set of equations

$$\mathrm{E2}: \quad \begin{cases} \dfrac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho \nabla \cdot \mathbf{u} \ , \\ \rho \dfrac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p + \rho \mathbf{f} \ . \end{cases} \tag{1.90}$$

Depending on the situation, solution to (E1) or (E2) are given by assuming one of the two boundary conditions:

- for a finite volume of fluid, the *Neumann condition* at the boundary: $\mathbf{u}\cdot\hat{\boldsymbol{\nu}} = 0$ on ∂V;
- for a fluid in an infinite volume, the *Dirichlet condition* at infinity: $\mathbf{u} = 0$ as $\mathbf{x} \to \infty$.

Chapter 2

Vortex dynamics

2.1 Kinematic transport of circulation

The concept of circulation plays a fundamental role in many aspects of field theory and fluid dynamics in particular.

Definition 2.1.1. The *circulation* $\Gamma = \Gamma(t)$ of a vector field $\mathbf{u}$ along a simple, closed curve $\mathcal{C}(t)$ in $\mathbb{R}^3$ is defined by

$$\Gamma(t) = \oint_{\mathcal{C}(t)} \mathbf{u} \cdot \mathrm{d}\mathbf{l} = \oint \mathbf{u}_s \cdot \frac{\partial \mathbf{l}_s}{\partial s} \mathrm{d}s \ , \tag{2.1}$$

where $\mathrm{d}\mathbf{l}$ represents the elementary length of $\mathcal{C}(t)$ and the last integral is between fixed (cyclic) values of the arc-length s on $\mathcal{C}$.

Note that $\mathcal{C}(t)$ is an arbitrary material curve lying entirely in the fluid..

Let us consider the transport of Γ under the action of a fluid flow. In analogy with the result of Lemma (1.30) one can prove the following result.

Theorem 2.1.1 (Kinematic transport of circulation)**.**

$$\frac{\mathrm{D}}{\mathrm{D}t} \oint_{\mathcal{C}(t)} \mathbf{u} \cdot \mathrm{d}\mathbf{l} = \oint_{\mathcal{C}(t)} \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} \cdot \mathrm{d}\mathbf{l} \ . \tag{2.2}$$

Proof. Let $\mathcal{C}(t)$ be a simple, closed, space curve given by the vector equation $\mathbf{x} = \mathbf{x}(s, t)$. Note that

$$\mathrm{d}\mathbf{l} = \frac{\partial \mathbf{x}}{\partial s} \mathrm{d}s = \hat{\mathbf{t}} \, \mathrm{d}s \ , \tag{2.3}$$

where $\hat{\mathbf{t}}$ is the unit tangent to $\mathcal{C}$. Then

$$\frac{\mathrm{D}}{\mathrm{D}t} \oint_{\mathcal{C}} \mathbf{u} \cdot \mathrm{d}\mathbf{l} = \frac{\mathrm{D}}{\mathrm{D}t} \oint_{\mathcal{C}} \frac{\partial \mathbf{x}}{\partial t} \cdot \frac{\partial \mathbf{x}}{\partial s} \mathrm{d}s = \oint_{\mathcal{C}} \left(\frac{\partial^2 \mathbf{x}}{\partial t^2} \cdot \frac{\partial \mathbf{x}}{\partial s} + \frac{\partial \mathbf{x}}{\partial t} \cdot \frac{\partial}{\partial t} \frac{\partial \mathbf{x}}{\partial s} \right) \mathrm{d}s$$

$$= \oint_{\mathcal{C}} \left(\frac{\partial \mathbf{u}}{\partial t} \cdot \hat{\mathbf{t}} + \mathbf{u} \cdot \frac{\partial}{\partial s} \frac{\partial \mathbf{x}}{\partial t} \right) \mathrm{d}s = \oint_{\mathcal{C}} \left(\frac{\partial \mathbf{u}}{\partial t} \cdot \hat{\mathbf{t}} + \mathbf{u} \cdot \frac{\partial}{\partial s} \mathbf{u} \right) \mathrm{d}s$$

$$= \oint_{\mathcal{C}} \left(\frac{\partial \mathbf{u}}{\partial t} \cdot \hat{\mathbf{t}} + (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \hat{\mathbf{t}} \right) \mathrm{d}s = \oint_{\mathcal{C}} \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} \cdot \hat{\mathbf{t}}\, \mathrm{d}s = \oint_{\mathcal{C}} \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} \cdot \mathrm{d}\mathbf{l} \; . \tag{2.4}$$

■

2.2 Kelvin's circulation theorem

An immediate consequence of the kinematic transport of circulation is Kelvin's circulation theorem.

Theorem 2.2.1 (Kelvin's circulation theorem)**.**

$$\frac{\mathrm{D}}{\mathrm{D}t}\Gamma(t) = 0 \quad \Rightarrow \quad \Gamma = \textit{constant} \; . \tag{2.5}$$

Proof. Let's apply Theorem 2.1.1 and substitute eq. (1.38) into the right-hand-side of eq. (2.2). By neglecting conservative forces, we have

$$\oint_{\mathcal{C}} \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} \cdot \mathrm{d}\mathbf{l} = -\oint_{\mathcal{C}} \frac{1}{\rho} \nabla p \cdot \mathrm{d}\mathbf{l} = 0 \; , \tag{2.6}$$

since the integral of the gradient is taken over a closed path.

■

This means that as long as Γ is a material curve under Euler equations the circulation is constant in time ('frozen' in the fluid) and it is invariant of the flow map. Moreover, if the fluid domain is simply connected we know that we can always reduce any given closed path $\mathcal{C}$ to a point; hence, simply-connectedness implies reducibility and zero circulation. The reverse, however, is not true.

2.3 Vorticity transport equations

Theorem 2.3.1 (Vorticity transport theorem)**.** *For isentropic flow, we have*

$$\frac{\mathrm{D}}{\mathrm{D}t} \left(\frac{\boldsymbol{\omega}}{\rho} \right) = \left(\frac{\boldsymbol{\omega}}{\rho} \cdot \nabla \right) \mathbf{u} \; . \tag{2.7}$$

Proof. By re-writing the vector identity (**??**) in the form

$$(\mathbf{u} \cdot \nabla)\, \mathbf{u} = \frac{1}{2} \nabla \left(|\mathbf{u}|^2 \right) - \mathbf{u} \times (\nabla \times \mathbf{u}) \; , \tag{2.8}$$

we substitute the convective derivative above in the momentum equations (1.37), so to have

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2}\nabla\left(|\mathbf{u}|^2\right) - \mathbf{u}\times(\nabla\times\mathbf{u}) = -\frac{\nabla p}{\rho} + \nabla\phi \ , \tag{2.9}$$

where $\mathbf{f} = \nabla\phi$ because forces are conservative. By taking the curl of (2.9), since time and space derivatives commute and the curl of a gradient is identically zero, we have the *vorticity induction law*, given by

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla\times(\mathbf{u}\times\boldsymbol{\omega}) \ . \tag{2.10}$$

Moreover, by using the vector identity (1.74) with $\mathbf{A} = \mathbf{u}$ and $\mathbf{B} = \boldsymbol{\omega}$, we have

$$\nabla\times(\mathbf{u}\times\boldsymbol{\omega}) = (\nabla\cdot\boldsymbol{\omega} + \boldsymbol{\omega}\cdot\nabla)\mathbf{u} - (\nabla\cdot\mathbf{u} + \mathbf{u}\cdot\nabla)\boldsymbol{\omega} \ ; \tag{2.11}$$

By substituting the second term on the right-hand-side of (2.10) and recalling that $\nabla\cdot\boldsymbol{\omega} = \nabla\cdot(\nabla\times\mathbf{u}) = 0$, we have

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla)\mathbf{u} - \boldsymbol{\omega}(\nabla\cdot\mathbf{u}) \ . \tag{2.12}$$

Thus

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = (\boldsymbol{\omega}\cdot\nabla)\mathbf{u} - \boldsymbol{\omega}(\nabla\cdot\mathbf{u}) \ . \tag{2.13}$$

From the continuity equation (1.22), we have that

$$\nabla\cdot\mathbf{u} = -\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} \ ,$$

and noting that

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{\rho}\right) = -\frac{1}{\rho^2}\frac{\mathrm{D}\rho}{\mathrm{D}t} \ ,$$

we have

$$\nabla\cdot\mathbf{u} = \rho\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{\rho}\right) \ ;$$

then, eq. (2.13) becomes

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = (\boldsymbol{\omega}\cdot\nabla)\,\mathbf{u} - \boldsymbol{\omega}\rho\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{\rho}\right) \ . \tag{2.14}$$

By dividing everything by ρ, we have

$$\frac{1}{\rho}\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} + \boldsymbol{\omega}\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{\rho}\right) = \left(\frac{\boldsymbol{\omega}}{\rho}\cdot\nabla\right)\mathbf{u}$$

or

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \left(\frac{\boldsymbol{\omega}}{\rho}\cdot\nabla\right)\mathbf{u}\ , \tag{2.15}$$

known as the *vorticity transport equations* in lagrangian form.

■

2.4 Vortex tubes and filaments

In analogy with the definition of streamline we have the concept of vortex-line:

Definition 2.4.1. A *vortex-line* is the integral curve given by the solution of the system

$$\frac{\mathrm{d}x}{\omega_1} = \frac{\mathrm{d}y}{\omega_2} = \frac{\mathrm{d}z}{\omega_3}\ , \tag{2.16}$$

where $\mathbf{x} = (x, y, z)$ is the position vector of the points on the curve and $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ is vorticity.

Consider a tubular region centered on a space curve C and of cross-sectional dimensions small compared to the local radius of curvature R of C. If the tube is made by a coherent bundle of vortex lines running parallel to the tube centreline, then we have a *vortex tube* (see Figure 2.1). Let Σ denote the tube cross-section; we have

Definition 2.4.2. The *flux of vorticity* Φ through $\Sigma \subset \mathbb{R}^2$ is defined by

$$\Phi = \int_\Sigma \boldsymbol{\omega}\cdot\hat{\boldsymbol{\nu}}\,\mathrm{d}^2\mathbf{x}\ , \tag{2.17}$$

where Σ is a fluid plane region.

It is useful to assume a tube cross-section of circular shape, so that $\Sigma = \pi\sigma^2$; a *vortex filament* is a tube of negligible cross-section with $\sigma \ll R$. The flux Φ of such concentrated vorticity is referred to as the *strength* of the vortex.

In general a vortex tube is a localized region W of vorticity where $\boldsymbol{\omega} \neq 0$. Since the tube is embedded in an irrotational fluid domain where $\boldsymbol{\omega} = 0$ — and the fluid is assumed to be ideal, without shear forces — we can think of the fluid domain as made of two complementary regions, union of an irrotational region V and a rotational region W. Hence, a closed path $\mathcal{C}$ of the irrotational velocity in V is either reducible to a point (as shown in Figure 1.7) or, if $\mathcal{C}$ is chosen to encircle (singly) a vortex region W, it cannot be reducible (see Figure 2.2); thus for such

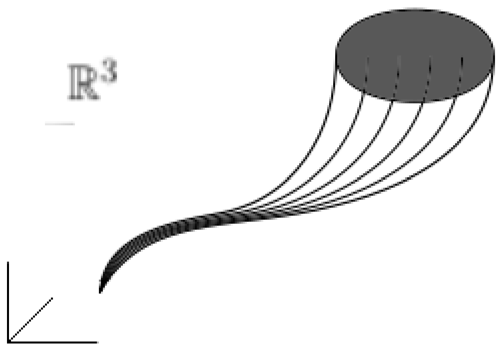


Figure 2.1: A vortex tube embedded in an ideal, irrotational fluid.

a path $\Gamma \neq 0$. To avoid confusion, let $\mathbf{x}$ denote a point in the irrotational domain V and $\mathbf{x}^*$ a rotational point where $\boldsymbol{\omega} \neq 0$. By using Stokes theorem, we have

$$\Gamma = \oint_{\mathcal{C}} \mathbf{u}(\mathbf{x}) \cdot \mathrm{d}\mathbf{l} = \int_{\Sigma} \boldsymbol{\omega}(\mathbf{x}^*) \cdot \hat{\boldsymbol{\nu}}\, \mathrm{d}^2\mathbf{x}^* = \Phi \ , \tag{2.18}$$

where Σ denotes a generic vortex cross-section. Since $\boldsymbol{\omega} = 0$ for any point $\mathbf{x}$ (outside Σ), there is no contribution to the flux outside Σ. Now, suppose $\Phi = \Phi(t)$. As direct application of Kelvin's circulation theorem 2.2.1 to eq. (2.18), we have:

Corollary 2.4.1 (Flux conservation)**.** *In an ideal fluid, the flux of vorticity is constant in time, that is*

$$\frac{\mathrm{D}}{\mathrm{D}t}\Phi(t) = 0 \quad \Rightarrow \quad \Phi = constant \ . \tag{2.19}$$

This means that in an ideal conditions the flux is 'frozen' in the fluid. In presence of localized vorticity (as in vortex tubes) circulation is evidently given by concentrated flux.

2.5 Cauchy's solutions to the transport equations and topological equivalence class

In ideal conditions initial vorticity $\boldsymbol{\omega}_0 = \boldsymbol{\omega}(\mathbf{x}_0, 0)$ is continuously deformed by the jacobian associated with the flow map φ and transported by the flow. This was already known to Cauchy in 1815; we have:

Lemma 2.5.1 (Cauchy's solutions)**.** *For isentropic flow, vorticity is transported by the flow according to*

$$\frac{\boldsymbol{\omega}}{\rho} = \frac{\partial(\mathbf{x})}{\partial(\mathbf{x}_0)} \cdot \left(\frac{\boldsymbol{\omega}}{\rho}\right)_0 \ , \tag{2.20}$$

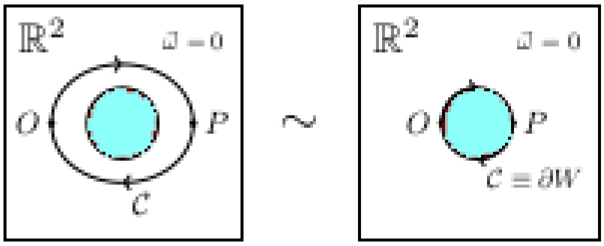


Figure 2.2: A vortex patch W bounded by the circuit $\mathcal{C} = \partial W$ in $\mathbb{R}^2$.

where the index 0 *denotes initial condition; the equations above are solutions to the vorticity transport equations* (2.15).

Proof. Since $\varphi : \mathbf{x}_0 \to \mathbf{x}$ any material element $\delta\mathbf{l}_0$ of a vortex line at time $t = 0$ is transported by the flow map φ to the new position $\delta\mathbf{l}$. If $\mathbf{x} = \mathbf{x}(s,t)$ denotes the position vector of a vortex line, we have

$$\delta\mathbf{l} = \hat{\mathbf{t}}\delta s = \frac{\partial\mathbf{x}}{\partial s}\delta s = \frac{\partial(\mathbf{x})}{\partial(\mathbf{x}_0)} \cdot \frac{\partial\mathbf{x}_0}{\partial s}\delta s = \frac{\partial(\mathbf{x})}{\partial(\mathbf{x}_0)} \cdot \delta\mathbf{l}_0 \ . \tag{2.21}$$

where $\partial(\mathbf{x})/\partial(\mathbf{x}_0) = \mathsf{J}(\mathbf{x};\mathbf{x}_0)$ is the jacobian of the transformation. By definition of vortex line we can identify $\delta\mathbf{l}_0$ with $(\boldsymbol{\omega}/\rho)_0$, so that we have proved (2.20).

Now let's prove the second statement. By moving lagrangianly on the line, we can derive (2.20) with respect to time, so to have

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial(\mathbf{x})}{\partial(\mathbf{x}_0)} \cdot \frac{\boldsymbol{\omega}_0}{\rho_0}\right] = \frac{\partial(\mathbf{u})}{\partial(\mathbf{x}_0)} \cdot \left(\frac{\boldsymbol{\omega}}{\rho}\right)_0$$

$$= \left(\frac{\partial(\mathbf{u})}{\partial(\mathbf{x})}\frac{\partial(\mathbf{x})}{\partial(\mathbf{x}_0)}\right) \cdot \left(\frac{\boldsymbol{\omega}}{\rho}\right)_0 = \frac{\partial(\mathbf{u})}{\partial(\mathbf{x})}\left[\frac{\partial(\mathbf{x})}{\partial(\mathbf{x}_0)} \cdot \left(\frac{\boldsymbol{\omega}}{\rho}\right)_0\right] = \frac{\partial(\mathbf{u})}{\partial(\mathbf{x})} \cdot \frac{\boldsymbol{\omega}}{\rho} \ . \tag{2.22}$$

By re-arranging the last term of (2.22) we have

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \frac{\partial(\mathbf{u})}{\partial(\mathbf{x})} \cdot \frac{\boldsymbol{\omega}}{\rho} = \left(\frac{\boldsymbol{\omega}}{\rho} \cdot \nabla\right)\mathbf{u} \ , \tag{2.23}$$

that is equation (2.15). Thus, we have proved Lemma (2.5.1).

■

Equations (2.20) are also known as *Cauchy's invariants*. Since vorticity is transported by continuous deformation of the initial vortex configuration to the final vortex configuration, vortex topology is conserved. Hence, we have

Corollary 2.5.1 (Topological equivalence class)**.** *For isentropic flow, Cauchy's solutions given by* (2.20) *establish a topological equivalence class between vortex configurations at different times; we write*

$$\left(\frac{\boldsymbol{\omega}}{\rho}\right)_{t=0} \sim \left(\frac{\boldsymbol{\omega}}{\rho}\right)_{t} . \tag{2.24}$$

In establishing a topological equivalence between vortex regions the Corollary 2.5.1 provides the foundations to Helmholtz's conservation laws (see Section 2.6) and for a topological approach to vortex dynamics.

2.6 Helmholtz's conservation laws

Vortex motion in ideal conditions is governed by the vorticity transport equations and, through the topological interpretation of Cauchy's solutions, by the following laws:

Theorem 2.6.1 (Helmholtz's conservation laws)**.** *Under isentropic flow, vortex motion is subject to the following laws:*

(i) Irrotational fluid particles at time $t = 0$ remain irrotational at any subsequent time $t > 0$.

(ii) A vortex line is a material line made by the same rotational elements at all times.

(iii) The flux of vorticity of any portion of vortex tube is constant.

Proof of (i). First consider a material line in $\mathbb{R}^3$ made of particles carrying a scalar-like quantity θ (for example temperature) and consider the time variation of this quantity evaluated at the lagrangian position ξ along the line; since the reference is fixed on the line, the material derivative coincides with the ordinary derivative:

$$\frac{\mathrm{D}\theta}{\mathrm{D}t} = \frac{\mathrm{d}\theta_\xi}{\mathrm{d}t} . \tag{2.25}$$

We can thus interpret the vector equation (2.15) as a set of equations for the scalar quantities $\omega_1, \omega_2, \omega_3$ evaluated lagrangianly at the position ξ, so that

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{\omega}}{\rho}\right)_{\xi} = \left(\frac{\boldsymbol{\omega}}{\rho}\cdot\nabla\right)\mathbf{u} . \tag{2.26}$$

This allows us to interpret the equation above as an ordinary differential equation for the components of $\boldsymbol{\omega}/\rho$. If $\mathbf{u}$ remains smooth at all times, so does its gradient and curl. If $(\boldsymbol{\omega}/\rho)_\xi = 0$ at $t = 0$, then by the theory of existence and uniqueness of smooth solutions $(\boldsymbol{\omega}/\rho)_\xi = 0$ at any subsequent time $t > 0$. The first law is proved.

Proof of (ii). We can now show that a material line element $\delta\mathbf{l}_\xi = \hat{\mathbf{t}}_\xi \delta s$ at the lagrangian position ξ satisfies the same equation (2.26):

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta\mathbf{l}_\xi = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathbf{x}}{\partial s}\right)_\xi \delta s = \left(\frac{\partial}{\partial s}\right)_\xi \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\delta s = (\delta\mathbf{l}_\xi \cdot \nabla)\,\mathbf{u}\ . \tag{2.27}$$

By combining (2.26) and (2.27), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{\omega}}{\rho} - \delta\mathbf{l}\right)_\xi = \left[\left(\frac{\boldsymbol{\omega}}{\rho} - \delta\mathbf{l}\right)_\xi \cdot \nabla\right]\mathbf{u}\ . \tag{2.28}$$

Now, if $\delta\mathbf{l}_\xi$ is a vortex element at $t = 0$, we have a unique, smooth solution at all subsequent times that ensures $\delta\mathbf{l}_\xi$ to remain a vortex element at all subsequent times. The second law is therefore proved.

Proof of (iii). Consider a vortex tube and the vortex region confined between two generic cross-sections Σ_1 and Σ_2. Since by assumption the tube is made by a bundle of vortex lines running parallel to the tube centreline, then $\boldsymbol{\omega}\cdot\hat{\boldsymbol{\nu}} = 0$ on the bounding tubular surface. The total flux between Σ_1 and Σ_2 is given by

$$\int_{\Sigma_1} \boldsymbol{\omega}\cdot\hat{\boldsymbol{\nu}}\,\mathrm{d}^2\mathbf{x} - \int_{\Sigma_2} \boldsymbol{\omega}\cdot\hat{\boldsymbol{\nu}}\,\mathrm{d}^2\mathbf{x} = \int_W \nabla\cdot\boldsymbol{\omega}\,\mathrm{d}^3\mathbf{x} = 0\ , \tag{2.29}$$

where the minus sign takes care of the prescribed centreline orientation and W denotes here the tube volume between the two cross-sections; the last integral is simply due to the application of the divergence theorem on W that vanishes since the divergence of a curl is always zero.

From mass conservation, which is true for any elementary vortex volume, we have

$$(\rho\,\delta\mathbf{l}\cdot\hat{\boldsymbol{\nu}}\delta\Sigma)_\xi = \text{constant}\,. \tag{2.30}$$

If the material line element is an element of a vortex line, i.e. if $\delta\mathbf{l}_\xi = (\boldsymbol{\omega}/\rho)_\xi$, then

$$\boldsymbol{\omega}_\xi\cdot\hat{\boldsymbol{\nu}}\,\delta\Sigma = \text{constant}\,, \tag{2.31}$$

and this can be integrated over the whole cross-section, so that

$$\Phi = \int_\Sigma \boldsymbol{\omega}\cdot\hat{\boldsymbol{\nu}}\,\mathrm{d}^2\mathbf{x} = \text{constant}\,. \tag{2.32}$$

Hence, the third law is also proved.

■

Some remarks are in order:

1. The first and the second Helmholtz's law state that an irrotational region at a given time remains irrotational at all subsequent times and vorticity on a vortex line is a material property of that line, i.e. that the vortex line is 'frozen' in the fluid. Since these two laws state purely qualitative properties of the fluid these two laws are purely topological in character.

2. Since a vortex tube can be thought of as a bundle of vortex lines, the second law can equally be extended to vortex tubes, or to more general vortex regions. By applying Lemma 2.5.1 to generic vortex configurations we can state that in ideal conditions rotational and irrotational regions do not mix and vortex topology is 'frozen' in the fluid. This means that if vortex knots and links are present in the fluid, they cannot change topology while changing shape during motion. Thus, a change in topology can only be achieved by the presence of shear forces due to, for example, viscosity.

3. The third Helmholtz's law provides quantitative information by stating that the flux through any section of a vortex tube is constant. In particular, if the fluid is uniform, then $\rho = \text{constant}$, and from (2.30) we have that at any station $s = \xi$ we have
$$\delta\Sigma \propto \frac{1}{|\delta\mathbf{l}|} . \tag{2.33}$$
This means that if vortex stretching is present, it must be accompanied by a corresponding shrinking of the average vortex cross-section, and vice-versa. Moreover, since flux is conserved, from (2.31) and (2.33) any stretching must be accompanied by an increase of vorticity since $\boldsymbol{\omega} \propto \delta\mathbf{l}$.

2.7 Conservation of kinetic helicity

Kinetic helicity is a fundamental invariant of fluid mechanics. Let us start with its definition.

Definition 2.7.1. *Kinetic helicity* $H = H(t)$ is defined by
$$H(t) = \int_W \mathbf{u} \cdot \boldsymbol{\omega} \, \mathrm{d}^3\mathbf{x}^* , \tag{2.34}$$
where $W = W(\mathbf{x}^*)$ is the volume of vorticity, with $\boldsymbol{\omega} \cdot \hat{\boldsymbol{\nu}} = 0$ on the boundary ∂W.

We conclude this chapter with the proof of the conservation of kinetic helicity for fluids in ideal conditions. We have:

Theorem 2.7.1 (Conservation of kinetic helicity)**.** *For isentropic flow, kinetic helicity is conserved , that is*

$$\frac{\mathrm{D}}{\mathrm{D}t}H(t) = 0 \quad \Rightarrow \quad H = \mathit{constant}\ . \tag{2.35}$$

Proof. By applying the kinematic result (1.27) we have

$$\begin{aligned}\frac{\mathrm{D}H}{\mathrm{D}t} &= \frac{\mathrm{D}}{\mathrm{D}t}\int_W \mathbf{u}\cdot\boldsymbol{\omega}\,\mathrm{d}^3\mathbf{x}^* = \int_W \frac{\mathrm{D}}{\mathrm{D}t}(\mathbf{u}\cdot\boldsymbol{\omega})\,\mathrm{d}^3\mathbf{x}^* \\ &= \int_W \left(\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}\cdot\boldsymbol{\omega} + \mathbf{u}\cdot\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t}\right)\mathrm{d}^3\mathbf{x}^*\ . \end{aligned} \tag{2.36}$$

From Euler's equations (1.38) and vorticity transport equations (2.15), we have

$$\frac{\mathrm{D}H}{\mathrm{D}t} = \int_W \left[-\frac{1}{\rho}\nabla p\cdot\boldsymbol{\omega} + \mathbf{u}\cdot(\boldsymbol{\omega}\cdot\nabla\mathbf{u})\right]\mathrm{d}^3\mathbf{x}^*\ . \tag{2.37}$$

Since $\nabla\cdot\boldsymbol{\omega} = 0$, we have

$$\frac{\mathrm{D}H}{\mathrm{D}t} = \int_W (\boldsymbol{\omega}\cdot\nabla)\left(-\frac{p}{\rho} + \frac{1}{2}\left|\mathbf{u}\right|^2\right)\mathrm{d}^3\mathbf{x}^* = \int_W \nabla\cdot\left[\boldsymbol{\omega}\left(-\frac{p}{\rho} + \frac{1}{2}\left|\mathbf{u}\right|^2\right)\right]\mathrm{d}^3\mathbf{x}^*\ . \tag{2.38}$$

By using the divergence theorem, we have

$$\frac{\mathrm{D}H}{\mathrm{D}t} = \int_{\partial W}\left(\frac{p}{\rho} - \frac{1}{2}\left|\mathbf{u}\right|^2\right)\boldsymbol{\omega}\cdot\hat{\boldsymbol{\nu}}\,\mathrm{d}^2\mathbf{x}^* = 0\ . \tag{2.39}$$

because of the assumption on $\boldsymbol{\omega}$ at the boundary ∂W. Hence, H is 'frozen' in the fluid.

∎

Chapter 3

Elements of magnetohydrodynamics

3.1 Maxwell's equations

We briefly recall the fundamental equations that govern the motion of a charged fluid (a plasma) in presence of electric currents and magnetic fields. For this we must complete the set of fundamental quantities necessary to provide a comprehensive description of the physical state of the charged fluid. A fluid continuum is prescribed by specifying the following physical quantities.

(i) For a classical fluid, these are:

- density ρ;
- kinetic viscosity ν;
- ratio of specific heats $\Gamma = c_p/c_v$, where we assume that all the molecules have 3 degrees of freedom (c_p and c_v are respectively the specific heat at constant pressure and at constant volume).

(ii) For an electrically charged fluid, these are:

- electron density ρ_e;
- electrical conductivity σ_0;
- magnetic permeability of free space μ_0.

For an electrically charged fluid the analogue of the kinetic Reynolds number Re is the *magnetic Reynolds number* Re_m, given by

$$\mathrm{Re}_m = \frac{UL}{\eta} \;, \tag{3.1}$$

that provides the ratio between advection and magnetic diffusion, where $\eta = (\sigma_0\mu_0)^{-1}$ (analogue to viscous diffusivity ν) is *magnetic diffusivity*. In presence

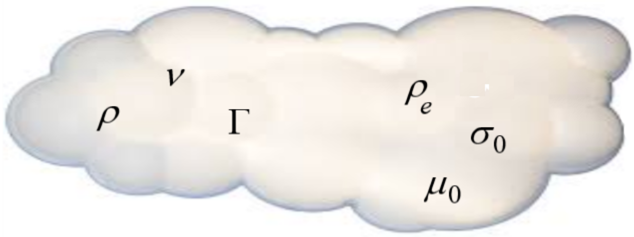


Figure 3.1: Fundamental physical constants in a charged fluid.

of electrically charged particles transported by the flow with velocity $\mathbf{u}$, the fluid experience electric currents, denoted by $\mathbf{J}$, usually decomposed into two contributions:

$$\mathbf{J} = \mathbf{J}_{\text{conduction}} + \mathbf{J}_{\text{advection}} \ , \tag{3.2}$$

where

$$\mathbf{J}_{\text{conduction}} = \mathbf{J} + \rho_e \mathbf{u} \ , \qquad \mathbf{J}_{\text{advection}} = -\rho_e \mathbf{u} \ .$$

The conduction current is related to the combined action of the electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ through the constitutive *Ohm's law*, given by

$$\mathbf{J}_{\text{conduction}} = \sigma_0 \left(\mathbf{E} + \mathbf{u} \times \mathbf{B} \right) \ . \tag{3.3}$$

The interplay between electric currents and magnetic fields is cast in the celebrated set of equations known as (pre-)*Maxwell's equations*, given by

$$\begin{cases} \nabla \cdot \mathbf{E} = \rho_e \ , & \dfrac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{B} - \mathbf{J} \ , \\ \nabla \cdot \mathbf{B} = 0 \ , & \dfrac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \ , \end{cases} \tag{3.4}$$

where for simplicity we set all physical constants equal to 1. These equations provide the conditions for the source and transport of $\mathbf{E}$ and $\mathbf{B}$. The emergence of electric fields is essentially associated with the presence of charged particles through ρ_e, with variation governed by the interplay of a rotational magnetic field $\mathbf{B}$ and a current $\mathbf{J}$. The magnetic field is considered to be divergenceless (no magnetic poles have been found so far) and its variation is due to the rotational action of an electric field $\mathbf{E}$.

3.2 Ideal magnetohydrodynamics (MHD)

As we did for a classical fluid, it is useful to introduce some simplifications to reduce Maxwell's equations to a set of equations amenable to mathematical analysis. For this we make the following assumptions:

(i) the charged fluid is *electrically neutral*. For this we take $\mathbf{J}_{\text{conduction}} = 0$, so that Ohm's law can be reduced to

$$\mathbf{E} = -\mathbf{u} \times \mathbf{B} \; ; \tag{3.5}$$

(ii) the electric field $\mathbf{E}$ is *steady*, that is

$$\frac{\partial \mathbf{E}}{\partial t} = 0 \; ; \tag{3.6}$$

(iii) the propagation velocity of the perturbations of the magnetic field, that travel with the *Alfvén speed* v_A, is much smaller than the speed of light c, i.e.

$$v_A = \frac{|\mathbf{B}|}{\sqrt{\rho_e \mu_0}} \ll \frac{1}{\sqrt{\epsilon_0 \mu_0}} = c \; , \tag{3.7}$$

where ε_0 is the electric constant of the vacuum (that measures the capability of the vacuum to permit electric fields).

Under these assumptions we can re-write the second of (3.4) as

$$\mathbf{J} = \nabla \times \mathbf{B} \; , \tag{3.8}$$

known as *Ampère's law*, and the fourth of (3.4) as

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) \; , \tag{3.9}$$

known as *Faraday's equation*.

Charged particles transported with a fluid velocity $\mathbf{u}$ are responsible for the combined action of electric currents and magnetic fields. These, in turn, generate a *Lorentz force* that acts on magnetic fields, given by

$$\mathbf{f}_{\text{Lorentz}} = \mathbf{J} \times \mathbf{B} \; . \tag{3.10}$$

In summary, if we set for simplicity $\rho =$ constant and consider magnetic diffusivity η, the governing equations for *incompressible magnetohydrodynamics* (MHD) become

$$\text{MHD}: \begin{cases} \dfrac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{J} \times \mathbf{B} & \text{linear momentum} \\ \dfrac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times (\mathbf{u} \times \boldsymbol{\omega}) + \nu \nabla^2 \boldsymbol{\omega} + \nabla \times (\mathbf{J} \times \mathbf{B}) & \text{vorticity transport} \\ \dfrac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B} & \text{magnetic field transport} \end{cases}$$

By setting dissipative effects (viscous and resistive) equal to zero, i.e. by taking $\nu = \eta = 0$, the equations above reduce to the standard equations of *ideal magnetohydrodynamics*.

3.3 Alfvén's conservation law of ideal MHD

Let us consider the ideal MHD equations. First of all notice that by neglecting dissipative effects equations of ideal MHD conserve topology. This is because we can re-interpret Faraday's law (3.9) as the equation governing the advection of an initial magnetic field to its final configuration by continuous deformation through the flow map φ, exactly as we did for vorticity. Indeed, by noting the formal analogy with the vorticity induction law (2.10), we can re-write Faraday's law in terms of a material derivative (following the same derivation as for eq. 2.15), so to have the *magnetic induction equation*

$$\frac{\mathrm{D}\mathbf{B}}{\mathrm{D}t} = (\mathbf{B} \cdot \nabla)\,\mathbf{u} \ , \tag{3.11}$$

with solutions formally analogue to the Cauchy's solutions given by eqs. (2.20). Hence, similarly to what is done for (2.24), we say that eqs. (3.11) govern the evolution of a class of topologically equivalent magnetic field configurations, i.e.

$$(\mathbf{B})_{t=0} \sim (\mathbf{B})_t \ . \tag{3.12}$$

By identifying magnetic field lines as material lines and following the same arguments as is done for Helmholtz's conservation laws, we have an analogue interpretation for magnetic lines, filaments and tubes. In this context the following result holds true.

Theorem 3.3.1 (Alfvén's theorem)**.** *For ideal magnetohydrodynamic flow, the magnetic flux*

$$\Phi(t) = \int_\Sigma \mathbf{B} \cdot \hat{\boldsymbol{\nu}} \, \mathrm{d}^2\mathbf{x} \ , \tag{3.13}$$

is conserved in the fluid, i.e.

$$\frac{\mathrm{D}}{\mathrm{D}t}\Phi(t) = 0 \quad \Rightarrow \quad \Phi = \mathit{constant} \ . \tag{3.14}$$

Proof. By direct application of the transport theorem (1.7) in two dimensions (where Σ replaces V), we have

$$\frac{\mathrm{D}}{\mathrm{D}t}\Phi(t) = \int_\Sigma \frac{\partial \mathbf{B}}{\partial t} \cdot \hat{\boldsymbol{\nu}} \, \mathrm{d}^2\mathbf{x} + \oint_C \mathbf{B} \cdot \mathbf{u} \times \mathrm{d}\mathbf{l} \ . \tag{3.15}$$

By substituting the induction equation (3.9) and by using the vector identity $\mathbf{B} \cdot \mathbf{u} \times \mathrm{d}\mathbf{l} = -(\mathbf{u} \times \mathbf{B}) \cdot \mathrm{d}\mathbf{l}$, we have

$$\frac{\mathrm{D}}{\mathrm{D}t}\Phi(t) = \int_\Sigma \nabla \times (\mathbf{u} \times \mathbf{B}) \cdot \hat{\boldsymbol{\nu}} \, \mathrm{d}^2\mathbf{x} - \oint_{\mathcal{C}} (\mathbf{u} \times \mathbf{B}) \cdot \mathrm{d}\mathbf{l} \ , \tag{3.16}$$

By applying Stokes' theorem to the first integral, the two integrals above cancel out and the statement is thus proved.

■

3.4 Conservation of magnetic helicity

In ideal MHD we can define a quantity analogue to kinetic helicity.

Definition 3.4.1. *Magnetic helicity* $H_m = H_m(t)$ is defined by

$$H_m(t) = \int_{W_m} \mathbf{A} \cdot \mathbf{B} \, \mathrm{d}^3\mathbf{x}^* \ , \tag{3.17}$$

where $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{A}$ is the vector potential of $\mathbf{B}$, with Coulomb gauge $\nabla \cdot \mathbf{A} = 0$, $W_m = W_m(\mathbf{x}^*)$ volume of the magnetic field, and $\mathbf{B} \cdot \hat{\boldsymbol{\nu}} = 0$ on the boundary ∂W_m (magnetic surface).

Similarly to kinetic helicity, also magnetic helicity is 'frozen' in ideal conditions.

Theorem 3.4.1. *Magnetic helicity $H_m = H_m(t)$ is conserved in ideal MHD, i.e.*

$$\frac{\mathrm{D}}{\mathrm{D}t} H_m(t) = 0 \quad \Rightarrow \quad H_m = \textit{constant} \ . \tag{3.18}$$

Proof. Let's un-curl Faraday's induction law (3.9); we have

$$\frac{\partial \mathbf{A}}{\partial t} = \mathbf{u} \times \mathbf{B} + \nabla \phi \ , \tag{3.19}$$

where ϕ is an arbitrary scalar field (since $\nabla \times (\nabla \phi) = 0$). Because of the Coulomb condition on $\mathbf{A}$, ϕ is defined by taking the divergence of (3.19), i.e.

$$\nabla^2 \phi = -\nabla \cdot (\mathbf{u} \times \mathbf{B}) \ . \tag{3.20}$$

By using (3.19), Faraday's eq. (3.9) and $\nabla \cdot \mathbf{B} = 0$, we have

$$\frac{\partial}{\partial t}(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \cdot [\nabla \times (\mathbf{u} \times \mathbf{B})] + \nabla \cdot (\phi \mathbf{B}) \ , \tag{3.21}$$

which, with the help of the vector identity

$$\mathbf{A} \cdot [\nabla \times (\mathbf{u} \times \mathbf{B})] = \nabla \cdot [\mathbf{A} \times (\mathbf{B} \times \mathbf{u})] = \nabla \cdot [(\mathbf{A} \cdot \mathbf{u})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{u}] \ , \tag{3.22}$$

becomes

$$\frac{\mathrm{D}(\mathbf{A} \cdot \mathbf{B})}{\mathrm{D}t} = \nabla \cdot [(\phi + \mathbf{A} \cdot \mathbf{u})\mathbf{B}] \ , \tag{3.23}$$

since $\nabla\cdot(\mathbf{A}\cdot\mathbf{B})\mathbf{u} = \mathbf{u}\cdot\nabla(\mathbf{A}\cdot\mathbf{B})$. Now, by using the above equation and integration over the *material* magnetic volume W_m, we have

$$\frac{\mathrm{D}\mathcal{H}_m}{\mathrm{D}t} = \int_{W_m} \frac{\mathrm{D}(\mathbf{A}\cdot\mathbf{B})}{\mathrm{D}t}\,\mathrm{d}^3\mathbf{x}^* = \int_{W_m} \nabla\cdot[(\phi + \mathbf{A}\cdot\mathbf{u})\mathbf{B}]\,\mathrm{d}^3\mathbf{x}^* = 0 \ , \tag{3.24}$$

by using the divergence theorem and the condition that ∂W_m is a magnetic surface. ∎

3.5 Analogous Euler flows

It is useful to compare the ideal equations of a classical fluid with those of magnetohydrodynamics. Two different types of analogies are relevant. Consider first the following case.

Non-perfect analogy: $\boldsymbol{\omega} \leftrightarrow \mathbf{B}$

$$\text{Euler's equations:} \quad \begin{cases} \boldsymbol{\omega} = \nabla\times\mathbf{u} \ , \\ \dfrac{\partial\boldsymbol{\omega}}{\partial t} = \nabla\times(\mathbf{u}\times\boldsymbol{\omega}) \ , \\ H = \int_W \mathbf{u}\cdot\boldsymbol{\omega}\,\mathrm{d}^3\mathbf{x}^* \ ; \end{cases} \tag{3.25}$$

and

$$\text{ideal MHD equations:} \quad \begin{cases} \mathbf{B} = \nabla\times\mathbf{A} \ , \\ \dfrac{\partial\mathbf{B}}{\partial t} = \nabla\times(\mathbf{u}\times\mathbf{B}) \ , \\ H_m = \int_W \mathbf{A}\cdot\mathbf{B}\,\mathrm{d}^3\mathbf{x}^* \ . \end{cases} \tag{3.26}$$

The analogy between $\boldsymbol{\omega}$ and $\mathbf{B}$ is non-perfect since the relationship between $\boldsymbol{\omega}$ and $\mathbf{u}$ does not carry through $\mathbf{B}$ and $\mathbf{u}$: the magnetic field is simply transported by $\mathbf{u}$, but it is not related to $\mathbf{u}$. In this sense $\mathbf{B}$ behaves like a passive scalar. Contrary, in the case of vorticity there is a direct interplay between $\boldsymbol{\omega}$ and induced velocity due to the Biot-Savart induction law.

A stricter analogy occurs when we consider steady conditions. Indeed, from the vorticity induction law we have

$$\nabla\times(\mathbf{u}\times\boldsymbol{\omega}) = 0 \ , \tag{3.27}$$

but since $\nabla\times(\nabla h) = 0$ for any arbitrary h, we have

$$\mathbf{u}\times\boldsymbol{\omega} = \nabla h \ . \tag{3.28}$$

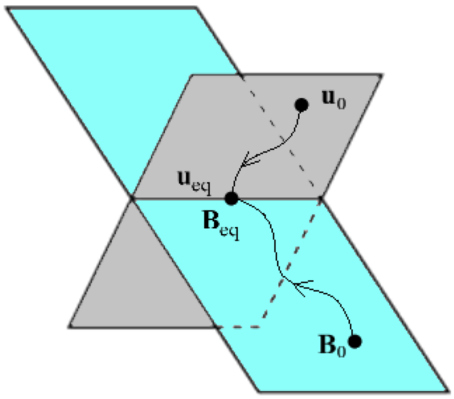


Figure 3.2: Analogous solutions via steady equilibria.

From the steady equations for ideal MHD, we have

$$-\nabla p + \mathbf{J} \times \mathbf{B} = 0 \ . \tag{3.29}$$

Thus, in steady conditions we have a perfect analogy between Euler's equations and ideal MHD equations:

Perfect analogy: $\boldsymbol{\omega} \Leftrightarrow \mathbf{J}$

$$\text{steady Euler's equations:} \quad \begin{cases} \boldsymbol{\omega} = \nabla \times \mathbf{u} \ , \\ \mathbf{u} \times \boldsymbol{\omega} = \nabla h \ , \\ H = \int_W \mathbf{u} \cdot \boldsymbol{\omega} \, \mathrm{d}^3\mathbf{x}^* \ ; \end{cases} \tag{3.30}$$

and

$$\text{steady, ideal MHD equations:} \quad \begin{cases} \mathbf{J} = \nabla \times \mathbf{B} \ , \\ \mathbf{J} \times \mathbf{B} = \nabla p \ , \\ H_{\rm cross} = \int_W \mathbf{J} \cdot \mathbf{B} \, \mathrm{d}^3\mathbf{x}^* \ . \end{cases} \tag{3.31}$$

In this case we have a direct correspondence between the following quantities:

$$\boldsymbol{\omega} \Leftrightarrow \mathbf{J} \ , \qquad \mathbf{u} \Leftrightarrow \mathbf{B} \ , \qquad h \Leftrightarrow -p \ .$$

In agreement with Arnold's strategy, this suggests to look for possible solutions of the velocity field, given by $\mathbf{u}_{\rm eq}$, by considering the analogous problem in terms of magnetic field equilibria: starting from the initial condition $\mathbf{B}_0$ we first look for steady state equilibria of ideal MHD configurations, given by $\mathbf{B}_{\rm eq}$, and then, by the analogy (3.32) above, we have $\mathbf{u}_{\mathrm{eq}}$. This strategy is particularly appropriate when we want to investigate topologically complex configurations, because the MHD equations are linear in $\mathbf{B}$ (since $\mathbf{u}$ is de-coupled from $\mathbf{B}$). Topologically complex solution of Euler flows can be therefore studied through magnetic equilibria. Toroidal solutions to the so-called Grad-Shafranov equations for confined plasma in Tokamaks correspond indeed to the toroidal vortex ring solutions of the Euler equations.

Chapter 4

Topological interpretation of helicity

4.1 The Gauss linking number

An important result rooted in the seminal work of Moffatt (1969) establishes a deep connection between the helicity of a field , which is a well-known conserved quantity of ideal fluid flows and linking number, which is a fundamental invariant of the topology of links. This result is of general validity, but it is particularly illuminating in the case of a discrete distribution of linked flux tubes in ideal conditions. To present this result and discuss its consequences we introduce first the concept of linking number as introduced by Gauss (1833).

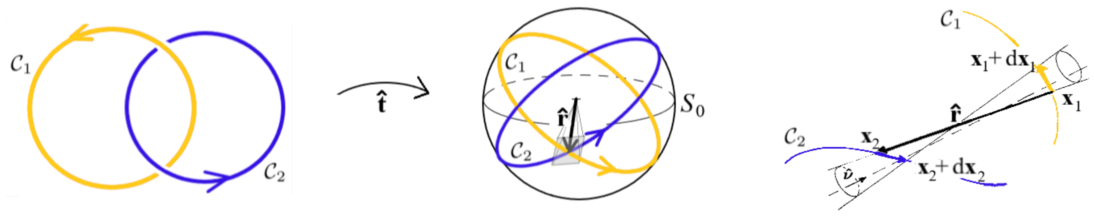
In low-dimensional topology a *knot* is just a simple, closed curve in $\mathbb{R}^3$. A knot is said to be *trivial* if it can be continuously deformed to the standard circle, the *unknot*. A *link* is a disjoint, inseparable union of knots (the components of the link) in $\mathbb{R}^3$. One of the most fundamental topological invariants of links is the Gauss linking number.

Definition 4.1.1. Given a link of two closed curves $\mathcal{C}_1$ and $\mathcal{C}_2$ in $\mathbb{R}^3$ the *linking number* $Lk_{1,2} = Lk(\mathcal{C}_1, \mathcal{C}_2)$ is defined by

$$Lk_{1,2} = \frac{1}{4\pi} \oint_{\mathcal{C}_1} \oint_{\mathcal{C}_2} \frac{(\mathbf{x}_1 - \mathbf{x}_2) \cdot \mathrm{d}\mathbf{x}_1 \times \mathrm{d}\mathbf{x}_2}{|\mathbf{x}_1 - \mathbf{x}_2|^3} \ , \tag{4.1}$$

where $\mathbf{x}_1$ and $\mathbf{x}_2$ denote points on $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively.

The linking number of two closed curves computes the number of times one curve, say $\mathcal{C}_1$, winds around the other, say $\mathcal{C}_2$ (see Figure (4.1)a). $Lk_{1,2}$ is a topological invariant of the link, it is a pure integer and is symmetric, i.e. $Lk_{1,2} = Lk_{2,1}$.



(a) (b) (c)

Figure 4.1: A Hopf link of two curves $\mathcal{C}_1$ and $\mathcal{C}_2$ in $\mathbb{R}^3$.

By taking $\mathbf{r} = (\mathbf{x}_1 - \mathbf{x}_2)$, we can introduce a unit vector $\hat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}|$ and re-write the above integral as

$$Lk_{1,2} = \oint_{\mathcal{C}_1} \oint_{\mathcal{C}_2} \frac{(\mathrm{d}\mathbf{x}_1 \times \mathrm{d}\mathbf{x}_2) \cdot \hat{\mathbf{r}}}{4\pi |\mathbf{r}|^2} = \frac{1}{2} \int_{\Omega} \mathrm{d}\varpi \, , \tag{4.2}$$

where the first integrand can be interpreted as the ratio between the elementary area given by the triple product at the numerator and the area of the (Gauss) sphere of radius $|\mathbf{r}|$ at the denominator. Since this ratio is a pure number, we can take $|\mathbf{r}| = 1$ so that it can be interpreted via the tangent map $\hat{\mathbf{t}}$ from the link components (see Figure 4.1b) as the elementary solid angle contribution $\mathrm{d}\varpi$ for the apparent intersections of the two curves on the Gauss unit sphere Ω (see Ricca & Nipoti, 2011). From the solid angle interpretation we can re-write the linking number formula in algebraic form in terms of apparent intersections of strands. To see this let us consider the over-pass or under-pass in a diagram projection of the link (see Figure 4.2), and assign to each apparent intersection a ± 1 according to a convention rule. Now consider the diagram of Figure 4.1(c): the elements $\mathrm{d}\mathbf{x}_1$ of $\mathcal{C}_1$ and $\mathrm{d}\mathbf{x}_2$ of $\mathcal{C}_2$ will contribute to the double integration in (4.2) if and only if they will intersect in projection along the direction of sight $\hat{\boldsymbol{\nu}}$. This happens if and only if $\hat{\boldsymbol{\nu}}$ is directed along $\pm(\mathbf{r} + \lambda \, \mathrm{d}\mathbf{x}_1 - \mu \, \mathrm{d}\mathbf{x}_2)$, with $\lambda \in (0,1)$ and $\mu \in (0,1)$, i.e. if and only if $\hat{\boldsymbol{\nu}}$ lies within the elementary solid angle

$$\mathrm{d}\varpi = 2 \frac{(\mathrm{d}\mathbf{x}_1 \times \mathrm{d}\mathbf{x}_2) \cdot \hat{\mathbf{r}}}{4\pi |\mathbf{r}|^2} \, ,$$

the factor 2 resulting from double counting the intersection sites. Thus, when we average over all directions of $\hat{\boldsymbol{\nu}}$, take account of crossing signs and then integrate over all pairs of elements $\mathrm{d}\mathbf{x}_1$, $\mathrm{d}\mathbf{x}_2$, the double integration over $\mathcal{C}_1$ and $\mathcal{C}_2$ reduces to the discrete sum of $\pm 4\pi$ coming from each apparent intersection of strands; hence, we have

$$Lk_{1,2} = \frac{1}{2} \sum_{r \in \{\mathcal{C}_1 \sqcap \mathcal{C}_2\}} \varepsilon_r \, , \tag{4.3}$$

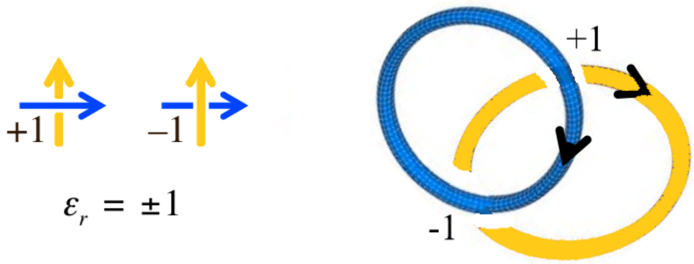


Figure 4.2: Sign convention for under-passes and over-passes of knot strands.

where r is the number of apparent intersections in the link diagram $\{\mathcal{C}_1 \sqcap \mathcal{C}_2\}$ and $\varepsilon_r = \pm 1$ according to the right-hand rule for over-passes or under-passes.

Since the linking number is a topological invariant of the link, it does not depend on the link projection. So, eq. (4.3) allows to compute $Lk_{1,2}$ by using any convenient projection of a given link without knowing anything analytic about the link components. This algebraic computation of the linking number proves to be very useful indeed for applications.

4.2 Derivation of the Gauss linking number from the helicity of a link

Now suppose $\mathcal{C}_1$ and $\mathcal{C}_2$ are the centrelines of tubular rings and consider these rings to be physical flux tubes given by a discrete distribution of field lines within the tubes. To fix ideas let us consider magnetic fields and flux tubes embedded in ideal fluid. We can then prove that the total helicity of the system admits a topological interpretation in terms of linking number. Indeed, under ideal magnetohydrodynamic conditions we know that the magnetic flux Φ_i $(i = 1, 2)$ is conserved by Alfvén's theorem (3.3.1) and that magnetic helicity is also conserved by (3.4.1). Thus, the ratio of these two quantities is evindently conserved and the proof that this ratio is actually a topological invariant cast in the Cauchy solutions to Faraday's equations (see Section 3.3). By assuming (for simplicity) that fluid density is either uniform or a function of pressure only, and that all body forces are conservative, we have:

Theorem 4.2.1. *In ideal conditions the helicity H_m of a magnetic link of two (or more) flux tubes in the shape of planar rings of centreline $\mathcal{C}_i$ $(i = 1, 2, \ldots)$, tubular boundary a magnetic surface and flux Φ_i, is given by*

$$H_m = \sum_{i \neq j} \Phi_i \Phi_j Lk_{i,j} \ , \tag{4.4}$$

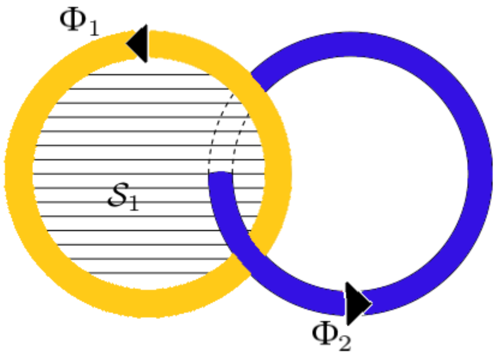


Figure 4.3: A simple link of two, planar flux tubes.

where $Lk_{i,j} = Lk(\mathcal{C}_i, \mathcal{C}_j)$ is the Gauss linking number of $\mathcal{C}_i$ with $\mathcal{C}_j$.

Proof. First, let us consider 2 rings of axes $\mathcal{C}_1$ and $\mathcal{C}_2$ and small cross-section. We identify the flux Φ_1 with the field $\mathbf{B} = \nabla \times \mathbf{A}$ (such that $\nabla \cdot \mathbf{A} = 0$) within the tube of axis $\mathcal{C}_1$ and interpret Φ_1 as the circulation of $\mathbf{A}$ along $\mathcal{C}_2$, that is $\Phi_1 = \oint_{\mathcal{C}_2} \mathbf{A} \cdot \mathrm{d}\mathbf{l} = K_2 = \text{constant}$ (see Figure 4.3). From Stokes theorem applied to $\mathcal{C}_1$ we also have

$$K_1 = \oint_{\mathcal{C}_1} \mathbf{A} \cdot \mathrm{d}\mathbf{l} = \int_{\mathcal{S}_1} \mathbf{B} \cdot \hat{\boldsymbol{\nu}}\, \mathrm{d}^2 x \ , \tag{4.5}$$

where $\mathcal{C}_1 = \partial \mathcal{S}_1$.

Since the flux of the magnetic field across S_1 is simply that due to the second flux tube of axis $\mathcal{C}_2$, we have

$$K_1 = \begin{cases} \pm\Phi_2 & \text{if } \mathcal{C}_1 \text{ and } \mathcal{C}_2 \text{ are linked} \ , \\ 0 & \text{if } \mathcal{C}_1 \text{ and } \mathcal{C}_2 \text{ are not linked} \ , \end{cases} \tag{4.6}$$

where $\pm$ refers to the two possible relative orientations of $\mathcal{C}_1$ and $\mathcal{C}_2$. If $\mathcal{C}_2$ winds an integer number of times around $\mathcal{C}_1$ (see Figure 4.4), then

$$K_1 = Lk_{1,2}\Phi_2 \ , \tag{4.7}$$

where $Lk_{1,2}$ denotes the degree of linking of $\mathcal{C}_1$ and $\mathcal{C}_2$. In general, if we have N rings $\mathcal{C}_j$ all linked with $\mathcal{C}_i$ an obvious generalization of (4.7) is

$$K_i = \oint_{\mathcal{C}_i} \mathbf{A} \cdot \mathrm{d}\mathbf{l} = \sum_j Lk_{i,j}\Phi_j \ , \tag{4.8}$$

where now $Lk_{i,j}$ denotes the degree of linking of the N rings of axes $\mathcal{C}_j$ with $\mathcal{C}_i$. The quantity $\Phi_i K_i$ may be written in the form of an integral over the volume V_i

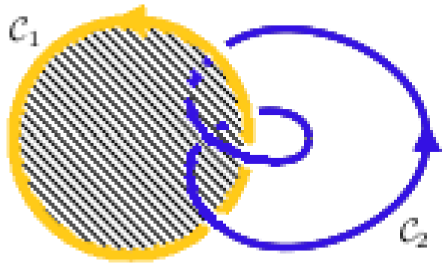


Figure 4.4: The curve $\mathcal{C}_2$ is multiply linked with the curve $\mathcal{C}_1$.

occupied by the tube centred on $\mathcal{C}_i$. Since the rings have small cross-section and $\mathrm{d}\mathbf{l}$ denotes an elementary length of $\mathcal{C}_i$, $\Phi_i\,\mathrm{d}\mathbf{l}$ may be replaced by $\mathbf{B}\,\mathrm{d}^3\mathbf{x}^*$, so that

$$\Phi_i K_i = \oint_{\mathcal{C}_i} \Phi_i\,\mathbf{A}\cdot\mathrm{d}\mathbf{l} = \int_{V_i} \mathbf{A}\cdot\mathbf{B}\,\mathrm{d}^3\mathbf{x} = H_{m,i}\,, \tag{4.9}$$

where V_i is the volume of the i-th ring. If we sum over all the tubes present, we obtain an invariant integral over the entire distribution of magnetic field, given by

$$H_{m,i} = \sum_i \int_{V_i} \mathbf{A}\cdot\mathbf{B}\,\mathrm{d}^3\mathbf{x}^* = \sum_i \Phi_i K_i = \sum_{i\neq j} Lk_{i,j}\Phi_i\Phi_j\ . \tag{4.10}$$

It should be noted that $H_{m,i}$ is determined solely by the magnetic field within each ring; this dependence may be made explicit by writing

$$\mathbf{A} = \mathbf{A}_{\mathrm{BS}} + \nabla\phi$$

where $\mathbf{A}_{\mathrm{BS}}$ is the field induced by the Biot-Savart induction law given by

$$\mathbf{A}_{\mathrm{BS}}(\mathbf{x}) = -\frac{1}{4\pi}\int_V \frac{(\mathbf{x}-\mathbf{x}^*)\times\mathbf{B}(\mathbf{x}^*)}{|\mathbf{x}-\mathbf{x}^*|^3}\,\mathrm{d}^3\mathbf{x}^*\ . \tag{4.11}$$

If the tubular boundary $S = \partial V$ of each ring is a magnetic surface, that is if $\mathbf{B}\cdot\hat{\boldsymbol{\nu}} = 0$ on S, the potential contribution to $\mathbf{A}$ makes no contribution to $H_{m,i}$, i.e.

$$\int_V \nabla\phi\cdot\mathbf{B}\,\mathrm{d}^3\mathbf{x}^* = \int_V \nabla\cdot(\phi\mathbf{B})\,\mathrm{d}^3\mathbf{x}^* = \int_S (\phi\mathbf{B})\cdot\hat{\boldsymbol{\nu}}\,\mathrm{d}^2\mathbf{x}^* = 0\ .$$

Substitution (4.11) into (4.10) and re-labelling $\mathbf{x}^*$ by $\mathbf{x}_i$ and $\mathbf{x}_j$ where appropriate, we have

$$H_m = \frac{1}{4\pi}\int_{V_i}\int_{V_j} \frac{(\mathbf{x}_i-\mathbf{x}_j)\cdot\mathbf{B}(\mathbf{x}_i)\times\mathbf{B}(\mathbf{x}_j)}{|\mathbf{x}_i-\mathbf{x}_j|^3}\,\mathrm{d}^3\mathbf{x}_i\,\mathrm{d}^3\mathbf{x}_j\ . \tag{4.12}$$

Since flux tubes have small cross-sections we can re-write (4.12) in terms of fluxes and reduce the volume integrals to line integrals, thus obtaining

$$H_m = \int_V \mathbf{A} \cdot \mathbf{B} \, \mathrm{d}^3\mathbf{x} = \sum_{i,j} Lk_{i,j} \Phi_i \Phi_j \ ,$$

in terms of linking numbers

$$Lk_{i,j} = \frac{1}{4\pi} \oint_{\mathcal{C}_i} \oint_{\mathcal{C}_j} \frac{(\mathbf{x}_i - \mathbf{x}_j) \cdot \mathrm{d}\mathbf{x}_i \times \mathrm{d}\mathbf{x}_j}{|\mathbf{x}_i - \mathbf{x}_j|^3} \ .$$

extended over N linked rings. Thus (4.4) is proved.

■

The result of Theorem 4.2.1 is of general validity and can be equally extended to vorticity fields (in terms of kinetic helicity) or electric currents (in terms of cross-helicity). This result can be even generalized to vector fields chaotically dense in a volume region by extending the concept of linking number to the asymptotic formulation of linking number of a dense set of open space curves, as given by Arnol'd (1972). These further extensions may find useful applications in the theory of dynamical systems as well as in applied sciences as, for example, in the study of a collection of open chains in polymer physics.

4.3 The Călugăreanu-White invariant

Consider now a physical knot made by a single, knotted flux tube in isolation embedded in an ideal fluid. A fundamental topological invariant of physical knots in $\mathbb{R}^3$ is the so-called Călugăreanu-White invariant, introduced by Călugăreanu in 1961 and extended to high dimensions by White in 1969.

An intuitive introduction to this invariant can be made by considering as a starting point a simple link of two disjoint curves $\mathcal{C}_1$ and $\mathcal{C}_2$ as shown in Figure 4.5(a). Suppose to keep fixed in space $\mathcal{C}_1 \equiv \mathcal{C}$ and gradually deform $\mathcal{C}_2 \equiv \mathcal{C}^*$ in a continuous fashion so to place $\mathcal{C}^*$ parallely as close as possible to $\mathcal{C}$, all along $\mathcal{C}$ (see Figure 4.5b). If the two curves are kept an $\epsilon =$constant apart we can define a *mathematical ribbon* $\mathcal{R} = \mathcal{R}_\epsilon(\mathcal{C}, \mathcal{C}^*)$ by taking the two curves as the edges of $\mathcal{R}$ of width ϵ (see Figure 4.5c). By construction the two curves are defined by

$$\mathcal{C}: \ \mathbf{x} = \mathbf{x}(s) \ , \qquad \mathcal{C}^*: \ \mathbf{x}^* = \mathbf{x}(s) + \epsilon\hat{\mathbf{N}}(s) \ ,$$

where $\hat{\mathbf{N}}$ is a spanwise unit vector from $\mathcal{C}$ to $\mathcal{C}^*$. Since the two edges are two disjoint, linked, closed curves obtained by continuous deformation of the original link, the Gauss linking number $Lk(\mathcal{C}, \mathcal{C}^*)$ is still a well-defined quantity. We have

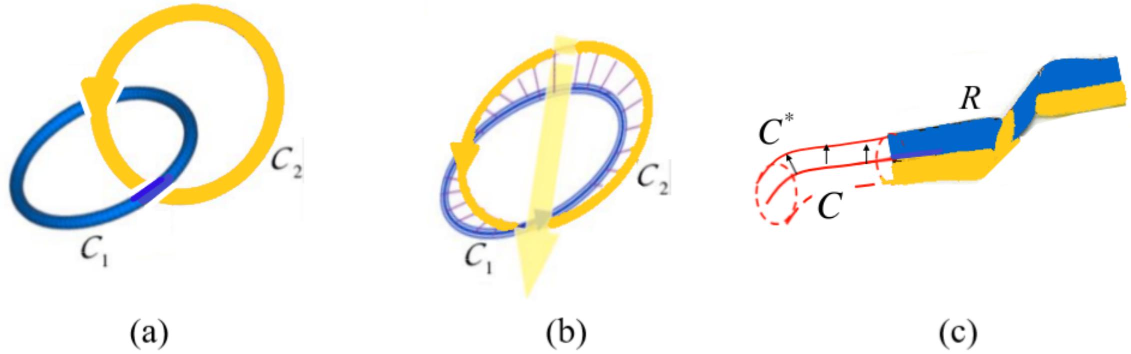


Figure 4.5: Construction of a mathematical ribbon $\mathcal{R}$ from (a) a simple link made by $\mathcal{C}_1$ and $\mathcal{C}_2$; (b) juxtaposition of $\mathcal{C}_1$ and $\mathcal{C}_2$; (c) identification of $\mathcal{C}_1$ and $\mathcal{C}_2$ with the edges of the ribbon.

Definition 4.3.1. Given a ribbon $\mathcal{R} = \mathcal{R}_\epsilon(\mathcal{C}, \mathcal{C}^*)$ of edges $\mathcal{C}$ and $\mathcal{C}^*$, the *self-linking number* $SL = SL(\mathcal{R})$ is the Călugăreanu-White invariant given by

$$SL = \lim_{\epsilon \to 0} Lk(\mathcal{C}, \mathcal{C}^*) \ . \tag{4.13}$$

The Călugăreanu-White invariant is a fundamental topological invariant of physical knots through the reference ribbon $\mathcal{R}$ placed on the knot. Moreover, this topological invariant admits decomposition in terms of two global geometric quantities, as established by the following theorem of Călugăreanu (1969).

Theorem 4.3.1. *Given a ribbon $\mathcal{R} = \mathcal{R}_\epsilon(\mathcal{C}, \mathcal{C}^*)$ of edges $\mathcal{C}$ and $\mathcal{C}^*$, the self-linking number $SL = SL(\mathcal{R})$ admits a geometric decomposition in terms of writhing number $Wr = Wr(\mathcal{C})$ and total twist number $Tw = Tw(\mathcal{R})$, given by*

$$SL(\mathcal{R}) = Wr(\mathcal{C}) + Tw(\mathcal{R}) \ , \tag{4.14}$$

where Wr is a global geometric property of $\mathcal{C}$ and Tw is a global geometric property of $\mathcal{R}$.

This remarkable result shows that a fundamental topological invariant of physical knots can be decomposed into global geometric term: the writhing number $Wr = Wr(\mathcal{C})$, given by

$$Wr = \frac{1}{4\pi} \oint_{\mathcal{C}} \oint_{\mathcal{C}} \frac{(\mathbf{x} - \mathbf{x}^*) \cdot \mathrm{d}\mathbf{x} \times \mathrm{d}\mathbf{x}^*}{|\mathbf{x} - \mathbf{x}^*|^3} \ , \tag{4.15}$$

where $\mathbf{x}$ and $\mathbf{x}^*$ denote two points on $\mathcal{C}$; the total twist number $Tw = Tw(\mathcal{R})$, given by

$$Tw = \frac{1}{2\pi} \oint_{\mathcal{C}_\mathcal{R}} (\hat{\mathbf{N}}' \times \hat{\mathbf{N}}) \cdot \hat{\mathbf{t}} \, \mathrm{d}s \ , \tag{4.16}$$

Figure 4.6: Geometric decomposition of the self-linking SL of a closed ribbon: (a) writhe, (b) localized torsion and (c) intrinsic twist.

where the notation $\mathcal{C}_{\mathcal{R}}$ denotes a line integration that depends also on the ribbon definition through $\hat{\mathbf{N}}$.

It is useful to note that the analytic expression of the writhing number of a knot is formally identical to that of the linking number of a link, the only difference being that computation of writhe is performed through a double integration over the same curve $\mathcal{C}$. This formal analogy with the linking number formula allows also to express the writhing number in terms of algebraic count of over-passes and under-passes, now limited to the strands of the same knot; hence, we have

$$Wr = \langle \sum_{r \in \{\mathcal{C} \sqcap \mathcal{C}\}} \varepsilon_r \rangle \ , \tag{4.17}$$

where the angular brackets denote averaging over all directions of projection. Since Wr is *not* a topological invariant, its exact value cannot be established by a single projection, and so we need averaging over all directions of projections.

For applications it is useful to consider a further decomposition (see Figure 4.6) given by the following lemma.

Lemma 4.3.1. *The total twist number $Tw = Tw(\mathcal{R})$ admits the following decomposition*

$$Tw = \frac{1}{2\pi} \oint_{\mathcal{C}} \tau(s)\, \mathrm{d}s + \frac{[\theta]_{\mathcal{R}}}{2\pi} = T + N \ , \tag{4.18}$$

where the integral represents the normalized total torsion $T = T(\mathcal{C})$ and $[\theta]_{\mathcal{R}}/2\pi$ defines the intrinsic twist $N = N(\mathcal{R})$.

Proof. Consider the cross-sectional plane of the ribbon given by the local principal unit normal $\hat{\mathbf{n}}$ and binormal $\hat{\mathbf{b}}$ (see Figure 4.7); using polar coordinates (r, θ) the ribbon spanwise unit normal $\hat{\mathbf{N}}$ along r is written as

$$\hat{\mathbf{N}} = \hat{\mathbf{n}} \cos\theta + \hat{\mathbf{b}} \sin\theta \ ,$$

where $\theta = \theta(s)$ is the angle between $\hat{\mathbf{N}}$ and $\hat{\mathbf{n}}$. By using Frenet-Serret equations, we have

$$\begin{aligned}\hat{\mathbf{N}}' &= \hat{\mathbf{n}}'\cos\theta + \hat{\mathbf{b}}'\sin\theta - \hat{\mathbf{n}}\sin\theta\,\theta' + \hat{\mathbf{b}}\cos\theta\,\theta' \\ &= -c\cos\theta\,\hat{\mathbf{t}} + (\tau + \theta')\,\hat{\mathbf{e}}_\theta\end{aligned}$$

where $\hat{\mathbf{e}}_\theta = -\hat{\mathbf{n}}\sin\theta + \hat{\mathbf{b}}\cos\theta$. Hence,

$$\begin{aligned}Tw &= \frac{1}{2\pi}\oint_{\mathcal{C}_\mathcal{R}} \left(\hat{\mathbf{N}}' \times \hat{\mathbf{N}}\right)\cdot\hat{\mathbf{t}}\,\mathrm{d}s \\ &= \frac{1}{2\pi}\oint_{\mathcal{C}_\mathcal{R}} \left\{\left[-c\cos\theta\,\hat{\mathbf{t}} + (\tau+\theta')\,\hat{\mathbf{e}}_\theta\right]\times\hat{\mathbf{N}}\right\}\cdot\hat{\mathbf{t}}\,\mathrm{d}s \\ &= \frac{1}{2\pi}\oint_{\mathcal{C}}\tau(s)\,\mathrm{d}s + \frac{1}{2\pi}\oint_{\mathcal{C}_\mathcal{R}}\mathrm{d}\theta = T + \frac{[\theta]_\mathcal{R}}{2\pi} = T + N\,,\end{aligned}$$

where $[\theta]_\mathcal{R}/2\pi$ represents the number of complete rotations of $\hat{\mathbf{N}}$ around $\mathcal{C}$ over the period L. The statement (4.18) is thus proved.

■

It should be noted that while T can take any real value, $N \in \mathbb{Z}$. The main properties of SL, Wr and Tw can thus be summarized as follows.

1. SL: (i) self-linking number is a topological invariant of a physical knot endowed with a reference ribbon $\mathcal{R}$; (ii) SL is an integer, i.e. $SL \in \mathbb{Z}$; (iii) by exchanging an under-pass with an over-pass: $\mp 1 \to \pm 1 \;\Rightarrow\; \Delta SL = \pm 2$.

2. Wr: (i) writhing number is a global geometric property of the base curve $\mathcal{C}$; (ii) Wr is a conformational invariant and $Wr \in \mathbb{R}$; (iii) by exchanging an under-pass with an over-pass: $\mp 1 \to \pm 1 \;\Rightarrow\; \Delta Wr = \pm 2$.

3. Tw: (i) total twist number is a global geometric property of the ribbon $\mathcal{R}$; (ii) Tw is a conformational invariant and $Tw \in \mathbb{R}$; (iii) $Tw(A) + Tw(B) = Tw(A+B)$, where A and B denote two pieces of ribbon.

4.3.1 Passage through an inflexional configuration

If a curve $\mathbf{x} = \mathbf{x}(s)$ has an inflexion point at $s = s_c$, then $c = 0$ and $\hat{\mathbf{t}}' = 0$ at $s = s_c$, so that near $s = s_c$ we have the Taylor expansion

$$\hat{\mathbf{t}}(s) = \hat{\mathbf{t}}_c + \frac{1}{2}(s - s_c)^2\hat{\mathbf{t}}''_c + \dots\ ,$$

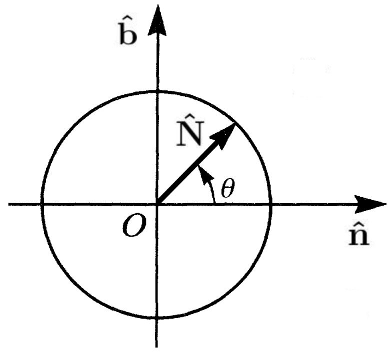


Figure 4.7: Polar coordinates on the tube cross-section.

that is

$$\mathbf{x}(s) = \mathbf{x}_c + (s - s_c)\hat{\mathbf{t}}_c + \frac{1}{6}(s - s_c)^3\hat{\mathbf{t}}''_c + \dots .$$

Moreover, since $|\hat{\mathbf{t}}| = 1$,

$$(\hat{\mathbf{t}}'' \cdot \hat{\mathbf{t}})_{s=s_c} = \frac{d^2}{ds^2}\hat{\mathbf{t}}^2\bigg|_{s=s_c} = 0 \quad ,$$

so that $\hat{\mathbf{t}}''_c$ is perpendicular to $\hat{\mathbf{t}}_c$. We may therefore choose the origin at the inflexion point ($\mathbf{x}_c = 0$, $s_c = 0$) and axes $Oxyz$ with Ox parallel to $\hat{\mathbf{t}}_c$ and Oz parallel to $\hat{\mathbf{t}}''_c$. The form of the curve near the inflexion point is then given by

$$\mathbf{x}(s) = (s, 0, \alpha s^3) \ , \tag{4.19}$$

where $\alpha = \frac{1}{6}|\hat{\mathbf{t}}''_c|$, i.e. it is the plane cubic curve $y = 0$, $z = \alpha x^3$. By simple rescaling, we may take $\alpha = 1$.

We now wish to consider a time-dependent curve $\mathbf{x} = \mathbf{x}(s, t)$ passing through the inflexional configuration (4.19) at $t = 0$, but having $\partial\hat{\mathbf{t}}/\partial s \neq 0$ when $t \neq 0$. Since

$$\hat{\mathbf{t}}' \cdot \hat{\mathbf{t}} = \frac{1}{2}\frac{\partial}{\partial s}(\hat{\mathbf{t}}^2) = 0 \ ,$$

we may always, by rigid rotation, ensure that at $s = 0$, $\hat{\mathbf{t}}$ remains parallel to Ox and $\hat{\mathbf{t}}'$ remains parallel to Oy. These conditions are satisfied by the time-dependent twisted cubic

$$\mathbf{x}(s, t) = \left(s - \frac{2}{3}t^2 s^3, ts^2, s^3\right) \ , \tag{4.20}$$

for which

$$\hat{\mathbf{t}} = \frac{\partial \mathbf{x}}{\partial s} = (1 - 2t^2 s^2, 2ts, 3s^2) \ , \tag{4.21}$$

and $|\hat{\mathbf{t}}| = 1 + O(s^4)$, so that near $s = 0$ $\hat{\mathbf{t}}$ is indeed the unit tangent vector.

From (4.21), to leading order in $|t|$ and $|s|$,

$$\frac{\partial \hat{\mathbf{t}}}{\partial s} \sim 2(0, t, 3s) \ ,$$

so that

$$c(s,t) = \left|\frac{\partial \hat{\mathbf{t}}}{\partial s}\right| \sim 2(t^2 + 9s^2)^{1/2} \ ,$$

and

$$\hat{\mathbf{n}}(s,t) = \frac{1}{c}\frac{\partial \hat{\mathbf{t}}}{\partial s} \sim \frac{(0, t, 3s)}{(t^2 + 9s^2)^{1/2}} \ .$$

Note here that for very small t, $\hat{\mathbf{n}}$ rotates through an angle π about the direction $\hat{\mathbf{t}}_c = (1, 0, 0)$ as s increases from $-s_0$ to $+s_0$ where $s_0 \gg |t|$; and that this rotation is clockwise (right-handed) for $t < 0$, and anticlockwise (left-handed) for $t > 0$; thus the number of rotations of the pair $(\hat{\mathbf{n}}, \hat{\mathbf{b}})$ about the tangent direction $\hat{\mathbf{t}}$ in the anticlockwise sense increases by $+1$ as t increases through zero (at the instant $t = 0$, this number is undefined).

Now the binormal is given by $\hat{\mathbf{b}} = \hat{\mathbf{t}} \times \hat{\mathbf{n}}$, and the torsion is obtained from the Frenet-Serret equations: for $|t|$ and $|s|$ small, we gave

$$\tau(s,t) \sim \frac{3t}{t^2 + 9s^2} \ .$$

As expected, c vanishes only at $t = s = 0$, and τ is singular at this inflexion. However the *singularity is integrable*; the contribution to the normalised total torsion T from any small interval $[-s_0, s_0]$ is

$$\frac{1}{2\pi}\int_{-s_0}^{s_0} \tau(s,t)\,\mathrm{d}s = \frac{1}{\pi}\int_0^{s_0} \frac{3t}{t^2 + 9s^2}\,\mathrm{d}s = \frac{1}{\pi}\tan^{-1}\left(\frac{3s_0}{t}\right) \ ,$$

and, irrespective of the value of s_0, this jumps from $-1/2$ to $+1/2$ as t increases through zero, i.e. as the curve passes through the inflexional configuration. Hence T is discontinuous as C passes through the inflexion, with discontinuity $[T] = +1$. The reverse passage (or equivalently replacement of t by $-t$ in (4.20)) gives a jump $[T] = -1$. This behaviour, recognised by Călugăreanu (1961) for a particular example, appears to be generic.

4.4 Derivation of the Călugăreanu-White invariant from the helicity of a knot

Consider now a physical knot made by a flux tube centred on a knotted curve $\mathcal{C}$ in $\mathbb{R}^3$. For the moment suppose $\mathcal{C}$ has no inflexion points (i.e. points of zero curvature). If the curve is closed, then $\mathbf{x} = \mathbf{x}(s)$ represents the parametric equation of $\mathcal{C}$; $\mathbf{x}(s) = \mathbf{x}(s+L)$, being a periodic function of the length L of $\mathcal{C}$. An alternative expression for the helicity of a knotted flux tube is given by the following result (Moffatt & Ricca, 1992).

Theorem 4.4.1. *In ideal conditions the helicity H_m of a magnetic knot given by a flux tube centred on the knot $\mathcal{C}$, with tubular boundary a magnetic surface and flux Φ is given by*

$$H_m = \Phi^2 SL \ , \tag{4.22}$$

where $SL = SL(\mathcal{R})$ is the Călugăreanu-White self-linking invariant of the reference ribbon $\mathcal{R}$ on $\mathcal{C}$.

Proof. Let us consider a flux tube of magnetic field $\mathbf{B}$ given by

$$\mathbf{B} = \mathbf{B}_a + \mathbf{B}_m \ , \tag{4.23}$$

where $\mathbf{B}_a$ is the axial field parallel to the tube axis and $\mathbf{B}_m$ is the meridional field in the meridian planes perpendicular to the tube axis. In the tube cross-section we adopt a local cylindrical coordinate system (r, θ, z) (see Figure 4.7) and suppose

$$\mathbf{B}_a = (0, 0, B_z(r)) \ , \quad \mathbf{B}_m = (0, B_\theta(r), 0) \ . \tag{4.24}$$

Evidently $\nabla \cdot \mathbf{B}_a = 0$ and $\nabla \cdot \mathbf{B}_m = 0$, so that we may introduce separate vector potentials:

$$\mathbf{B}_a = \nabla \times \mathbf{A}_a \ , \quad \mathbf{B}_m = \nabla \times \mathbf{A}_m \ , \tag{4.25}$$

with $\nabla \cdot \mathbf{A}_a = 0$ and $\nabla \cdot \mathbf{A}_m = 0$. Since the lines of force of the $\mathbf{B}_m$-field are unlinked circles, we have

$$\int_V \mathbf{A}_m \cdot \mathbf{B}_m \, \mathrm{d}^3\mathbf{x}^* = 0 \ ,$$

where V is the tube volume. The total field helicity is thus given by

$$\begin{aligned} H_m &= \int_V \mathbf{A}_a \cdot \mathbf{B}_a \, \mathrm{d}^3\mathbf{x}^* + \int_V \mathbf{A}_a \cdot \mathbf{B}_m \, \mathrm{d}^3\mathbf{x}^* + \int_V \mathbf{A}_m \cdot \mathbf{B}_a \, \mathrm{d}^3\mathbf{x}^* \\ &= \int_V \mathbf{A}_a \cdot \mathbf{B}_a \, \mathrm{d}^3\mathbf{x}^* + 2 \int_V \mathbf{A}_a \cdot \mathbf{B}_m \, \mathrm{d}^3\mathbf{x}^* \ , \end{aligned} \tag{4.26}$$

using integration by parts and the divergence theorem.

(i) Writhe contribution from helicity. Consider first the axial contribution $H_{ma} = \int_V \mathbf{A}_a \cdot \mathbf{B}_a \, \mathrm{d}^3\mathbf{x}^*$. Here we may use the Biot-Savart expression in the limiting form

$$\mathbf{A}_{\mathrm{BS}} = -\frac{\Phi}{4\pi} \oint_{\mathcal{C}} \frac{(\mathbf{x} - \mathbf{x}^*) \times \mathrm{d}\mathbf{x}^*}{|\mathbf{x} - \mathbf{x}^*|^3} \ .$$

Although this expression diverges when $\mathbf{x} \in \mathcal{C}$, its axial component remains finite, and the limiting expression

$$H_{ma} = \frac{\Phi^2}{4\pi} \oint_{\mathcal{C}} \oint_{\mathcal{C}} \frac{(\mathbf{x} - \mathbf{x}^*) \cdot \mathrm{d}\mathbf{x} \times \mathrm{d}\mathbf{x}^*}{|\mathbf{x} - \mathbf{x}^*|^3} = \Phi^2 Wr \ , \tag{4.27}$$

is finite. This is the writhe contribution from helicity.

(ii) Twist contribution from helicity. Consider now the second contribution in equation (4.26),

$$H_{mm} = 2 \int_V \mathbf{A}_a \cdot \mathbf{B}_m \, \mathrm{d}^3\mathbf{x}^* = 2 \int_V A_\theta(r) B_\theta(r) \, \mathrm{d}^3\mathbf{x}^* \ . \tag{4.28}$$

Note that from the first of (4.24) and from the first of (4.25), $\mathbf{A}_a = (0, A_\theta(r), 0)$ where

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(rA_\theta) = B_z(r) \ . \tag{4.29}$$

Let us consider the change in H_m under a virtual displacement $\delta\boldsymbol{\xi}(s)$ of the flux tube due to instantaneous changes $\delta c(s)$, $\delta\tau(s)$ in curvature and torsion. We have

$$\boldsymbol{\xi} = r\hat{\mathbf{e}}_r = r(\hat{\mathbf{n}}\cos\theta + \hat{\mathbf{b}}\sin\theta) \ ,$$

and

$$\hat{\mathbf{e}}_\theta = -\hat{\mathbf{n}}\sin\theta + \hat{\mathbf{b}}\cos\theta \ ,$$

so that, assuming

$$\delta\boldsymbol{\xi} = r\cos\theta\delta\hat{\mathbf{n}} + r\sin\theta\delta\hat{\mathbf{b}}$$

to be the same for all (r, θ), we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\delta\boldsymbol{\xi} = r\cos\theta\frac{\mathrm{d}}{\mathrm{d}s}\delta\hat{\mathbf{n}} + r\sin\theta\frac{\mathrm{d}}{\mathrm{d}s}\delta\hat{\mathbf{b}} \ . \tag{4.30}$$

Since it is only the variation of $\delta\boldsymbol{\xi}$ with arc-length s that contributes to the distortion of the field, we may suppose that at $s = s_1$, $\delta\boldsymbol{\xi}(s_1) = 0$, i.e. $\delta\hat{\mathbf{n}}(s_1) = \delta\hat{\mathbf{b}}(s_1) = 0$. Then, from the Frenet relations at $s = s_1$ we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\delta\hat{\mathbf{n}} = -\delta c\,\hat{\mathbf{t}} + \delta\tau\,\hat{\mathbf{b}} \ , \quad \frac{\mathrm{d}}{\mathrm{d}s}\delta\hat{\mathbf{b}} = -\delta\tau\,\hat{\mathbf{n}} \ . \tag{4.31}$$

Now under the assumed virtual displacement $\delta\boldsymbol{\xi}(s)$, the axial field $\mathbf{B}_a$ (and so $A_\theta(r)$) is unchanged, but the meridional field $\mathbf{B}_m$ at $s = s_1$ is changed by an amount

$$\delta\mathbf{B}_m = (\mathbf{B}_a \cdot \nabla)\delta\boldsymbol{\xi} = B_z(r)\frac{\mathrm{d}}{\mathrm{d}s}\delta\boldsymbol{\xi} \ , \tag{4.32}$$

due to the variation of $\delta\boldsymbol{\xi}$ with arc-length; as we see this change resembles a convective contribution from a process that in MHD is known as 'generation of toroidal field by differential rotation'; see Moffatt 1984). Hence,

$$\begin{aligned}\delta B_\theta &= \delta\mathbf{B}_m\cdot\hat{\mathbf{e}}_\theta = B_z(r)\left(\frac{d}{ds}\delta\boldsymbol{\xi}\right)\cdot\hat{\mathbf{e}}_\theta \\ &= B_z(r)\left[-\left(\frac{d}{ds}\delta\boldsymbol{\xi}\right)\cdot\sin\theta\,\hat{\mathbf{n}}+\left(\frac{d}{ds}\delta\boldsymbol{\xi}\right)\cdot\cos\theta\,\hat{\mathbf{b}}\right]_{s=s_1}\ . \end{aligned}\tag{4.33}$$

Substituting from (4.30) and (4.31), we have

$$\delta B_\theta = B_z(r)r\delta\tau(s)\ ,\quad \text{at } s=s_1\ . \tag{4.34}$$

Since the same argument may be used at any section, (4.34) gives the field perturbation due to the virtual displacement for all s_1, and the resulting change in the H_{mm} is therefore

$$H_{mm} = 2\int_V A_\theta(r)\delta B_\theta(r)\,\mathrm{d}^3\mathbf{x}^* = 2\int_V A_\theta(r)B_z(r)r\delta\tau(s)\,\mathrm{d}^3\mathbf{x}^*\ . \tag{4.35}$$

If we integrate first over the cross-section, using (4.29) and the result

$$\int_0^\infty A_\theta\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(rA_\theta)r2\pi r\,\mathrm{d}r = \frac{2\pi}{2}\left[(rA_\theta)^2\right]_0^\infty = \pi\left(\frac{\Phi}{2\pi}\right)^2\ ,$$

then from (4.35)

$$\delta H_{mm} = \Phi^2\delta T\ .$$

If we consider a time-dependent deformation of $\mathcal{C}$ which does not pass through any inflexional configuration, then (4.35) may be written

$$\frac{\mathrm{d}H_{mm}}{\mathrm{d}t} = \Phi^2\frac{\mathrm{d}T}{\mathrm{d}s}\ ,$$

or equivalently

$$H_{mm} = \Phi^2(T+T_0)\ , \tag{4.36}$$

where T_0 is a constant.

Under continuous deformation of the flux tube combining (4.27) with (4.36) total helicity can thus be written as

$$\frac{H_{mm}}{\Phi^2} = Wr + T + T_0 = \text{ constant}\ .$$

When $\mathcal{C}$ goes through an inflexional state Wr varies continuously, but T is known to jump by ±1 (see previous Section 4.3.1). Hence, as T jumps by ±1, the term T_0

must jump by a compensating amount ∓ 1, to maintain the overriding invariance of helicity; the jump by ∓ 1 is the jump in the intrinsic twist N when $\mathcal{C}$ is deformed through an isolated inflexion point, hence $T_0 = N$. The statement is thus proved. ■

4.5 Helicity of a tangle of knots and links

Previous results apply to any system of physical knots and links made by localized distributions of vorticity, magnetic or electric fields of strength ϕ. If the discrete field is confined to N filamentary structures we can combine and extend results of Theorems 4.2.1 and 4.4.1 to state the following Corollary.

Corollary 4.5.1. *In ideal conditions the total helicity H of a tangle $\mathcal{T}$ of N physical knots and links of total volume V, each carrying a flux Φ_i ($i = 1, \ldots N$), is given by*

$$
\begin{aligned}
H = \int_V \mathbf{A} \cdot \mathbf{B} \, \mathrm{d}^3\mathbf{x} &= \sum_i \Phi_i^2 SL_i + \sum_{i \neq j} \Phi_i \Phi_j Lk_{ij} \\
&= \sum_i \Phi_i^2 (Wr_i + T_i + N_i) + \sum_{i \neq j} \Phi_i \Phi_j Lk_{ij} \ . \qquad (4.37)
\end{aligned}
$$

This formula has great potentials for applications since from the left-hand side we see that there is no need to know the distribution of fields analytically and it allows to estimate integral helicity from the computation of the terms on the right-hand side of (4.37). On the other hand, from experimental measurements of integral helicity and fluxes Φ_i, direct computation of Wr_i and T_i from the geometry of the centrelines and $Lk_{i,j}$ from diagram projections, one can estimate intrinsic twist N_i, which is one of the most difficult and interesting terms for the energetics of flux tubes.

4.6 Helicity estimates from crossing number information

The algebraic interpretation of linking and writhing in terms of crossing number information can be extended naturally to another useful quantity introduced by Freedman and He (1992).

Definition 4.6.1. Given a tangle of closed curves $\mathcal{C}_i$ $(i = 1, \ldots, N)$ in space, we can define the *average crossing number* $\bar{C}$ of the tangle the quantity

$$\bar{C} = \frac{1}{4\pi} \oint_{\mathcal{C}_i} \oint_{\mathcal{C}_j} \frac{|(\mathbf{x}_i - \mathbf{x}_j) \cdot \mathrm{d}\mathbf{x}_i \times \mathrm{d}\mathbf{x}_j|}{|\mathbf{x}_i - \mathbf{x}_j|^3} = \langle \sum_{r \in \{\mathcal{C}_i \sqcap \mathcal{C}_j\}} |\epsilon_r| \rangle \ , \tag{4.38}$$

where $\mathbf{x}_i$ and $\mathbf{x}_j$ denote points on $\mathcal{C}_i$ and $\mathcal{C}_j$, respectively.

By definition $\bar{C}$ (sometimes denoted by ACN) counts all the unsigned apparent crossings of the tangle averaged over all direction of sights. Evidently $\bar{C}$ is a pure integer; this quantity is neither a geometric nor a topological property of the tangle, providing a purely algebraic information of structural complexity. It is therefore a useful information to quantify the complexity of a network of filaments in space.

Linking, writhing and average crossing number can thus be estimated directly from crossing number information. Since linking is a topological quantity its value does not depend on directions of projection and so its computation is based on crossing counts of a single diagram projection. In case of a complex tangle of filaments we have total linking Lk_{Tot}, given by

$$Lk_{\mathrm{Tot}} = \frac{1}{2} \sum_{i \neq j} \sum_{r \in \{\mathcal{C}_i \sqcap \mathcal{C}_j\}} \varepsilon_r \ . \tag{4.39}$$

Exact values of Wr and $\bar{C}$, however, depend on the entire solid angle integration. For practical purposes approximate values can be simply obtained by considering crossing counts averaged over only 3 mutually orthogonal directions of sight, say along x, y and z directions. By referring to these 3 directions of projection, we introduce the *estimated writhing number* $Wr_{\perp}$ and the *estimated crossing number* $\bar{C}_{\perp}$, given respectively by

$$Wr_{\perp} = \frac{1}{3} \sum_i \Big[\sum_{r \in \{\mathcal{C}_i \sqcap \mathcal{C}_j\}} \epsilon_r \Big]_i \ , \qquad (i = x, y, z) \ , \tag{4.40}$$

and

$$\bar{C}_{\perp} = \frac{1}{3} \sum_i \Big[\sum_{r \in \{\mathcal{C}_i \sqcap \mathcal{C}_j\}} |\epsilon_r| \Big]_i \ , \qquad (i = x, y, z) \ . \tag{4.41}$$

These measures have been successfully applied to relate geometric and topological information to physical properties of superfluid tangles, energy and helicity in particular using eq. (4.37) and energy estimates (Barenghi *et al.* 2001).

Chapter 5

Magnetic relaxation and energy spectrum of knots and links

5.1 Magnetic knots and Lorentz force

Consider tubular knots and links as tubular embeddings of the magnetic field in an ideal, incompressible, perfectly conducting fluid in S^3 (i.e. $\mathbb{R}^3 \cup \{\infty\}$, simply connected). A magnetic knot $\mathbf{B}_K$ is given by the embedding of the magnetic field in a regular tubular neighbourhood $\mathcal{T}_a$ of radius $a > 0$, centred on the knot axis $\mathcal{C}$ of local radius of curvature $\rho > 0$. The field is actually embedded onto nested tori $\mathcal{T}_i$ $(i = 1, \ldots, n)$ in $\mathcal{T}_a$ (see Figure 5.1), and regularity is ensured by taking $a \leq \rho$ pointwise along $\mathcal{C}$. The existence of non-self-intersecting nested tori in $\mathcal{T}_a$ is guaranteed by the tubular neighbourhood theorem (Spivak, 1979). $\mathcal{C}$ is assumed to be a C^3-smooth, closed loop (submanifold of S^3 homeomorphic to S^1), simple (i.e. non-self-intersecting) and parametrized by arc-length. The total length of $\mathcal{C}$ is $L = L(\mathcal{C})$. Evidently $\mathbf{B}_K$ has the knot type of $\mathcal{C}$, being either a *trivial knot*, if $\mathcal{C}$ is equivalent to the unknot that bounds a smoothly embedded disk, or an *essential knot*. For simplicity we take $\mathcal{T}_a = \mathcal{C} \otimes \mathcal{S}$ given by the product of $\mathcal{C}$ with the solid circular disk $\mathcal{S}$ of area $A = \pi a^2$, taken in the cross-sectional plane to $\mathcal{C}$. The total volume $V = V(\mathcal{T}_a) = \pi a^2 L$. To a first approximation we neglect deviations from cylindrical geometry and we regard them as effects of higher-order analysis. We therefore assume that the tubular boundary $\partial\mathcal{T}_a = \partial\mathcal{T}$ (dropping the suffix) remains a magnetic circular, cylindrical surface at all times, of uniform cross-section all along $\mathcal{C}$; denoting by $\hat{\boldsymbol{\nu}}$ the unit normal to $\partial\mathcal{T}$, we have:

Definition 5.1.1. A *magnetic knot* is a smooth immersion into R^3 of finitely many disjoint standard solid tori $\mathcal{T}_i$, such that

$$\cup_i \mathcal{T}_i \mapsto \mathbf{B}_K \equiv \mathrm{supp}(\mathbf{B}) \qquad (i = 1, \ldots, n)\ . \tag{5.1}$$

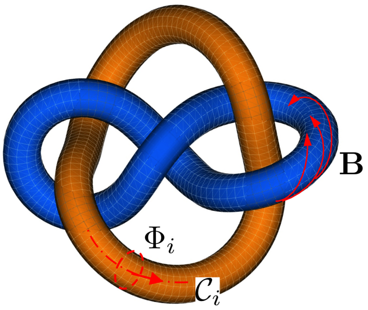


Figure 5.1: Magnetic link given by the tubular embedding of the magnetic field in two knots. In ideal conditions knot volume V and magnetic flux Φ are conserved quantities.

with $\mathbf{B} \cdot \hat{\boldsymbol{\nu}} = 0$ on $\partial \mathcal{T}$.

5.1.1 Flux tube coordinates and field decomposition

As mentioned earlier the magnetic field $\mathbf{B}$ is subject to the action of a Lorentz force $\mathbf{F} = \mathbf{J} \times \mathbf{B}$ and we want to determine its effect on $\mathbf{B}_K$. For this we need to introduce a proper orthonormal reference system on the knot axis $\mathcal{C}$ to take account of the internal geometry of the field lines. Let (r, ϑ_R, s) be orthogonal, curvilinear coordinates centered on $\mathcal{C}$ (Mercier, 1963). A point P on $\mathcal{S}$ (see Figure 5.2a) is given by

$$\mathbf{x} = \mathbf{X}(s) + r \cos\theta(s)\,\hat{\mathbf{n}}(s) + r \sin\theta(s)\,\hat{\mathbf{b}}(s) \ , \tag{5.2}$$

where $\theta(s)$ is the polar angle referred to the unit normal $\hat{\mathbf{n}} = \hat{\mathbf{n}}(s)$ in the intrinsic reference frame (Frenet frame), given by unit tangent $\hat{\mathbf{t}} = \hat{\mathbf{t}}(s) = d\mathbf{x}/ds$, normal $\hat{\mathbf{n}}$ and binormal $\hat{\mathbf{b}} = \hat{\mathbf{b}}(s)$ to $\mathcal{C}$. This angle varies according to the geometry of the tube-axis. The polar angle ϑ_R, however, is an independent coordinate and is related to the former by the equation

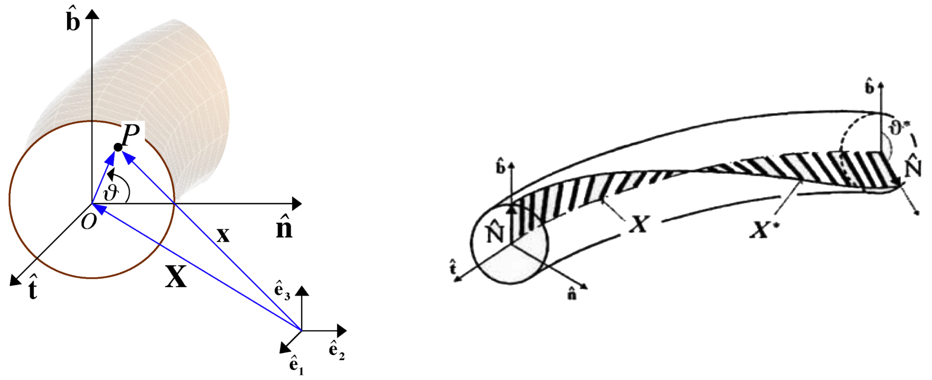
$$\theta(s) = \vartheta_R + \gamma(s) \ , \tag{5.3}$$

where

$$\gamma(s) = -\int_0^s \tau(\xi)\, d\xi \tag{5.4}$$

takes into account the accumulative contribution from torsion $\tau = \tau(s)$ of $\mathcal{C}$. By using Frenet-Serret formulae

$$\hat{\mathbf{t}}' = c\hat{\mathbf{n}} \ , \quad \hat{\mathbf{n}}' = -c\hat{\mathbf{t}} + \tau\hat{\mathbf{b}} \ , \quad \hat{\mathbf{b}}' = -\tau\hat{\mathbf{n}} \ , \tag{5.5}$$



(a) (b)

Figure 5.2: (a) Relationship between fixed and moving frame for a point P on $\mathcal{S}$: the angle $\theta(s)$ varies with the torsion of $\mathcal{C}$. (b) Twist of field lines visualized by means of a ribbon, whose edges are given by the tube axis $\mathbf{X} = \mathbf{X}(s)$ and a neighbouring field line $\mathbf{X}^*$, placed at a distance r in the spanwise direction $\mathbf{N}$ on the ribbon relative to the Frenet pair $(\hat{\mathbf{n}}, \hat{\mathbf{b}})$.

where $c = c(s)$ is curvature ($c = \rho^{-1}$) and prime denotes derivative with respect to s, we define the orthonormal metric

$$\mathrm{d}\mathbf{x} \cdot \mathrm{d}\mathbf{x} = (\mathrm{d}r)^2 + r^2(\mathrm{d}\vartheta_R)^2 + K^2(\mathrm{d}s)^2 \ , \tag{5.6}$$

which is orthogonal; here $K = K(s) = 1 - c(s)\, r \cos\theta(s)$. The orthogonal basis $(\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{t}})$ is given by $\hat{\mathbf{e}}_r = \hat{\mathbf{n}}\cos\theta + \hat{\mathbf{b}}\sin\theta$, $\hat{\mathbf{e}}_\theta = -\hat{\mathbf{n}}\sin + \hat{\mathbf{b}}\cos$ and $\hat{\mathbf{t}}$, in the radial, meridian and longitudinal direction, respectively.

Field lines inside magnetic flux tubes may generally twist about the knot axis. Twist can be visualized by means of a ribbon (see Figure 5.2b), whose edges are the tube axis $\mathbf{X} = \mathbf{X}(s)$ and a neighbouring field line $\mathbf{X}^*(s)$. For simplicity we consider the field decomposition given by

$$\mathbf{B} = \mathbf{B}_m + \mathbf{B}_a \ , \tag{5.7}$$

with $\mathbf{B}_m = B_\theta \hat{\mathbf{e}}_\theta$ the meridian (poloidal) component in $\mathcal{S}$ and $\mathbf{B}_a = B_s\hat{\mathbf{t}}$ the longitudinal (toroidal) component along $\mathcal{C}$. The field is chosen so as to have no radial component, so that the tube boundary is a magnetic surface ($\mathbf{B}\cdot\mathbf{N} = 0$). We take

$$\mathbf{B}_m = [0, B_\theta(r, \theta(s)), 0], \quad \mathbf{B}_a = [0, 0, B_s(r)] \ , \tag{5.8}$$

where everything is a smooth function of radius and arc-length. Since $\mathbf{B}$ is divergenceless, we have

$$r\nabla\cdot\mathbf{B} = \frac{\partial B_\theta}{\partial\vartheta_R} + B_\theta \frac{cr}{K}\sin\theta + \frac{r}{K}\frac{\partial B_s}{\partial s} = 0 \ . \tag{5.9}$$

From the second of (5.8) we have $\partial B_s/\partial s = 0$, so that (5.9) reduces to

$$\frac{\partial B_\theta}{\partial \vartheta_R} = -B_\theta \frac{cr}{K} \sin\theta \;, \tag{5.10}$$

a relation that will be used below to derive the Lorentz force.

It is worth mentioning the special case of *uniform twist*. From the definition of magnetic line, we have

$$\frac{r\delta\vartheta_R}{B_\theta} = \frac{K\delta s}{B_s} \;. \tag{5.11}$$

In case of uniform twist we must have

$$\frac{\delta\vartheta_R}{\delta s} = \frac{\oint_C d\vartheta_R}{\oint_C ds} = \frac{2\pi Tw}{L} \;. \tag{5.12}$$

Hence, by using eq. (5.11), we have

$$2\pi Tw = \frac{KL}{r}\frac{B_\theta}{B_s} \;, \tag{5.13}$$

that puts in relation geometric and magnetic quantities. From this equation we note that the condition of uniform twist implies $B_\theta/B_s \approx r$, that poses a restriction on the relationship between toroidal and poloidal components of the magnetic field.

5.1.2 Lorentz force

The Lorentz force is given by $\mathbf{F} = \mathbf{J} \times \mathbf{B}$, where $\mathbf{J}$ is the current density. Since $\mathbf{J} = \nabla \times \mathbf{B}$ we have:

$$\mathbf{F} = (\nabla \times \mathbf{B}) \times \mathbf{B} = (\mathbf{B} \cdot \nabla)\mathbf{B} - \frac{1}{2}\nabla(\mathbf{B}^2) \;. \tag{5.14}$$

The r.h.s. of this equation can be made explicit in terms of the magnetic field prescribed. By using (5.7), the first of (5.5) and (5.8), we have

$$(\mathbf{B} \cdot \nabla)\mathbf{B} = \frac{{B_\theta}^2}{r}\frac{\partial \hat{\mathbf{e}}_\theta}{\partial \vartheta_R} + \left(\frac{B_\theta}{r}\frac{\partial B_\theta}{\partial \vartheta_R} + \frac{B_s}{K}\frac{\partial B_\theta}{\partial s}\right)\hat{\mathbf{e}}_\theta + \frac{B_\theta B_s}{K}\frac{\partial \hat{\mathbf{e}}_\theta}{\partial s} + \frac{{B_s}^2}{K}c\hat{\mathbf{n}} \;. \tag{5.15}$$

By (5.3) and (5.4) we have

$$\frac{\partial B_\theta}{\partial s} = \left.\frac{\partial B_\theta}{\partial \theta}\right|_s \frac{\partial \theta}{\partial s} = \left.\frac{\partial B_\theta}{\partial \vartheta_R}\right|_s \frac{\partial \gamma}{\partial s} = -\tau \frac{\partial B_\theta}{\partial \vartheta_R} \;, \tag{5.16}$$

so that by (5.10), we have

$$\frac{\partial B_\theta}{\partial s} = B_\theta \frac{cr\tau}{K} \sin\theta \;. \tag{5.17}$$

Equation (5.15) reduces to

$$
\begin{aligned}
(\mathbf{B}\cdot\nabla)\mathbf{B} &= {B_s}^2\frac{c}{K}\hat{\mathbf{n}} - \frac{{B_\theta}^2}{r}\hat{\mathbf{e}}_r + B_\theta\frac{c}{K}\sin\theta\left(B_s\frac{r\tau}{K} - B_\theta\right)\hat{\mathbf{e}}_\theta \\
&\quad + B_\theta B_s\frac{c}{K}\sin\theta\hat{\mathbf{t}}\ . \qquad (5.18)
\end{aligned}
$$

Similarly for the second term in the r.h.s. of (5.14):

$$
\frac{1}{2}\nabla(\mathbf{B}^2) = \frac{1}{2}\left[\frac{\partial}{\partial r}({B_\theta}^2 + {B_s}^2)\hat{\mathbf{e}}_r - 2{B_\theta}^2\frac{c}{K}\sin\theta\hat{\mathbf{e}}_\theta + 2{B_\theta}^2\frac{cr\tau}{K^2}\sin\theta\hat{\mathbf{t}}\right]\ . \qquad (5.19)
$$

Substituting (5.18) and (5.19) into (5.14), we have $\mathbf{F} = \mathbf{F}_r + \mathbf{F}_m + \mathbf{F}_a$, where

$$
\begin{aligned}
\mathbf{F}_r &= F_r\hat{\mathbf{e}}_r = \left[{B_s}^2\frac{c}{K}\cos\theta - \frac{{B_\theta}^2}{r} - \frac{1}{2}\frac{\partial}{\partial r}\left({B_\theta}^2 + {B_s}^2\right)\right]\hat{\mathbf{e}}_r\ , \qquad (5.20) \\
\mathbf{F}_m &= F_\theta\hat{\mathbf{e}}_\theta = B_s\frac{c}{K}\sin\theta\left(B_\theta\frac{r\tau}{K} - B_s\right)\hat{\mathbf{e}}_\theta\ , \qquad (5.21) \\
\mathbf{F}_a &= F_s\hat{\mathbf{t}} = B_\theta\frac{c}{K}\sin\theta\left(B_s - B_\theta\frac{r\tau}{K}\right)\hat{\mathbf{t}}\ , \qquad (5.22)
\end{aligned}
$$

are the components of the Lorentz force $\mathbf{F}$ in the orthogonal coordinates (r, θ, s). In physical applications, it is customary to combine radial and meridian components in two contributions, one given by $\mathbf{F}_\perp$, perpendicular to the tube axis, and one given by $\mathbf{F}_p$, in the meridian plane:

$$
\begin{aligned}
\mathbf{F}_\perp &= {B_s}^2\frac{c}{K}\hat{\mathbf{n}} - \left[\frac{{B_\theta}^2}{r} + \frac{1}{2}\frac{\partial}{\partial r}\left({B_\theta}^2 + {B_s}^2\right)\right]\hat{\mathbf{e}}_r\ , \qquad (5.23) \\
\mathbf{F}_\parallel &= B_\theta B_s\frac{cr\tau}{K^2}\sin\theta\hat{\mathbf{e}}_\theta\ . \qquad (5.24)
\end{aligned}
$$

Evidently this decomposition is not unique, because now the term in $\hat{\mathbf{n}}$ includes part of the contribution in $\hat{\mathbf{e}}_\theta$. However, eqs. (5.23) and (5.24) help to understand the dynamics associated with the Lorentz force: The magnetic flux-tube moves in the fluid thanks to the action of $\mathbf{F}_\perp$, with a term proportional to curvature along the principal normal, corrected by the scale factor K, and a term in the radial direction, that takes account of magnetic pressure. The curvature term is responsible for the natural shortening of flux-tubes in free space, and contributes to the bending energy of the magnetic tube, in analogy with elastic systems. The radial term controls the confinement of the magnetic field in the tubular region, through the magnetic pressure generated by the poloidal and toroidal components of $\mathbf{B}$. Note that for relatively thick flux-tubes, the scale factor $K = 1 - cr\cos\theta$ may enhance considerably the curvature force.

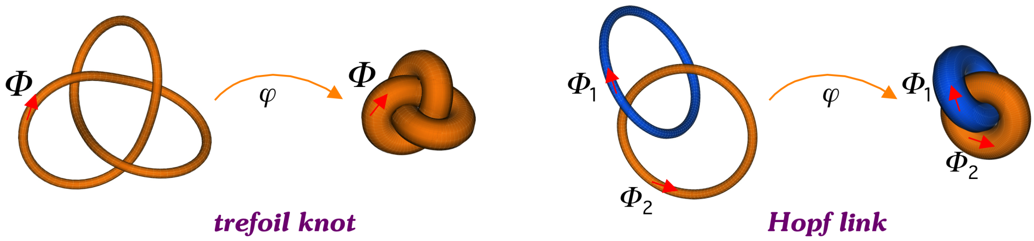


Figure 5.3: Magnetic relaxation of a loose trefoil knot and Hopf link under flux- and volume-preserving diffeomorphism governs their evolution towards a tight end-state.

The contributions $\mathbf{F}_a$ and $\mathbf{F}_\parallel$ induce internal re-organization of the magnetic field, without affecting the shape of the flux-tube; the axial component $\mathbf{F}_a$ contributes to internal stretching and generates longitudinal magnetic tension in $\mathcal{T}$, while the poloidal component (5.24) induces a meridian flow around the tube cross-section, that brings fluid from the concave to the convex region of the tube, hence modifying the twist distribution of the field-lines. In case of uniform twist (see condition (5.12)), by using eq. (5.13) we have

$$\frac{|\mathbf{F}_a|}{|\mathbf{F}_m|} = -\frac{B_\theta}{B_s} = -\frac{2\pi Tw}{L}\frac{r}{K} \ . \tag{5.25}$$

5.2 Relaxation under helicity conservation

The total energy of the magnetic system (knot or link) is given by the sum of kinetic and magnetic energy and in ideal conditions total energy and helicity are conserved quantities. Here we want to consider the magnetic relaxation under which a loose magnetic knot (or link) is brought to a minimum energy state given by its *tight* configuration. To first approximation this is because the Lorentz force $\mathbf{F}_\perp \approx c\hat{\mathbf{n}}$ which, as for elastic bands, induces continuous shrinking of the field lines and of the tube axis. Since the process is governed by volume- and flux-preserving diffeomorphisms, this leads to an average increase of the tube cross-section (since volume remains constant during evolution), accompanied by an average decrease of magnetic field intensity (since flux remains also constant during evolution). All this brings a loose knot to get gradually tighter through a continuous reduction of magnetic energy at the expense of kinetic energy. The process continues until the knot reaches the tightest possible configuration (see Figure 5.3). The process must come to a complete stop before magnetic surfaces in contact develop possible singularities.

It is useful to introduce the following:

Definition 5.2.1. The *signature* of a magnetic knot $\mathbf{B}_K(\mathbf{x})$ is the pair $\{V, \Phi\}$, which under ideal conditions is preserved during evolution.

Let us consider *magnetic energy* $M(t)$; this is given by

$$M(t) = \frac{1}{2}\int_{V(\mathbf{B}_K)} \|\mathbf{B}\|^2 \, \mathrm{d}^3\mathbf{x}^* \ . \tag{5.26}$$

The conserved magnetic helicity places a lower bound on the magnetic energy $M(t)$ of a localized magnetic field, as recognized by Arnold (1974). Suppose that the magnetic field is located in a fixed domain of given length-scale L. Then first, by the Schwarz inequality,

$$|H| \leq \left[\int_{V(\mathbf{B}_K)} \|\mathbf{A}\|^2 \, \mathrm{d}V \int_{V(\mathbf{B}_K)} \|\mathbf{B}\|^2 \, \mathrm{d}V\right]^{\frac{1}{2}} \ . \tag{5.27}$$

Second, we have a Poincarè inequality

$$\int_{V(\mathbf{B}_K)} \|\mathbf{B}\|^2 \, \mathrm{d}V \geq q^2 \int_{V(\mathbf{B}_K)} \|\mathbf{A}\|^2 \, \mathrm{d}V \ , \tag{5.28}$$

where $q = O(L^{-1})$ is a positive constant determined by the geometry of $\mathbf{B}_K$. Hence, combining these inequalities, we have the *Arnold inequality*

$$\int_{V(\mathbf{B}_K)} \|\mathbf{B}\|^2 \, \mathrm{d}^3\mathbf{x}^* \geq q^2 |H(\mathbf{B}_K)| \ . \tag{5.29}$$

This simply means that, if $|H| = 0$, then under any distortion of the field $\mathbf{B}$ by a fluid flow, the magnetic energy $M(t)$ has a positive lower bound. This result was generalised by Freedman (1988) to cover fields whose helicity is zero, but which nevertheless have nontrivial topology, in the sense that there exist closed field lines which cannot be shrunk to a point without cutting other field lines. We assume such nontrivial topology in considering magnetic relaxation as proposed by Moffatt (1985, 1990). One can prove the following statement.

Theorem 5.2.1. *Under magnetic relaxation, the magnetic energy $M = M(t)$ is a monotonic decreasing function of time.*

Proof. To assess the relaxation process discussed above and to approach the end-state (a steady state), consider a perfectly conducting but viscous fluid. Viscosity is here invoked as a mechanism to cancel out end-state fluctuations. Let's consider the change in magnetic energy:

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \int_{V(K)} \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} \, \mathrm{d}^3\mathbf{x}^* = \int_{V(K)} \mathbf{B} \cdot [\nabla \times (\mathbf{u} \times \mathbf{B})] \, \mathrm{d}^3\mathbf{x}^*$$

$$= -\int_{V(K)} \mathbf{u}\cdot[(\nabla\times\mathbf{B})\times\mathbf{B}]\,\mathrm{d}^3\mathbf{x}^* = -\int_{V(K)} \mathbf{u}\cdot[\mathbf{J}\times\mathbf{B}]\mathrm{d}^3\mathbf{x}^* \ . \tag{5.30}$$

As initial condition we may use $\mathbf{B}(\mathbf{x},0) = \mathbf{B}_K(\mathbf{x})$ given by a loose magnetic knot of signature $\{V,\Phi\}$, and assume that $\mathbf{u}(\mathbf{x},t)$ be itself instantaneously determined (near equilibrium) by the 'modified' Navier-Stokes equation (Darcy model)

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} \approx k\mathbf{u} = -\nabla p + \mathbf{J}\times\mathbf{B} \ , \tag{5.31}$$

where k is a positive constant and p is a 'pressure' field determined by the condition that $\nabla\cdot\mathbf{u} = 0$ for all t. Both $\mathbf{u}$ and ∇p are at most of the order of $|\mathbf{x}|^{-3}$ as $|\mathbf{x}|\to\infty$. Combining equations (5.30) and (5.32), we have

$$\frac{\mathrm{D}M}{\mathrm{D}t} = -k\int_{V(K)} |\mathbf{u}|^2\,\mathrm{d}^3\mathbf{x}^* \ , \tag{5.32}$$

so that $M(t)$ is indeed monotonic decreasing if $\mathbf{u}$ is not identically zero. $M(t)$ being bounded below must tend to a constant as $t\to\infty$.

■

The remarkable thing is that we can start with an arbitrary field topology containing arbitrarily knotted and linked configurations at time $t = 0$ and conclude that, under the relaxion process above, a field must exist that contains these knots and links and also satisfies the magnetostatic equation. By using the relaxation result above and by dimensional analysis Moffatt (1990) was led to the fundamental result.

Theorem 5.2.2. *Under signature-preserving flow the minimal magnetic energy* $M_{\min}$ *of* $\mathbf{B}_K$ *is given by*

$$M_{\min} = m(h)\Phi^2 V^{-1/3} \ , \tag{5.33}$$

where $m(h)$ *is a positive dimensionless function of the dimensionless twist parameter* h.

Here below we shall see that when $h = 0$ $m(h)$ has a minimum and this can be identified with the topological crossing number of the knot type.

5.3 From inflexional knots to closed braids

During evolution magnetic fields may develop twist, as sometimes occur for loops in the Solar corona (see Figure 5.4a). Twisting of field lines builds up and eventually saturates at about 2 full turns, when the flux-tube reaches a critical threshold

(a) (b)

Figure 5.4: (a) *TRACE* satellite observation of filament destabilization in the Solar photosphere (Active Region 9957) on May 27, 2002. (b) A Reidemeister type I move (twist move) on a tube strand.

for kink instability. Under conservation of helicity a full twist of a tube strand is removed when its contribution to helicity (twist helicity) is entirely converted to writhe helicity (Moffatt & Ricca 1992), a mechanism that corresponds to perform a *Reidemeister type I move* on the tube strand (see Figure 5.4b). This is clearly a fundamental process for energy re-distribution in active regions. Relaxation of twist to writhe seems seems generically associated with the relaxation of torsional energy to bending energy and it is often accompanied by a generic passage through an *inflexional state*. Inflexional configurations are states defined by the presence of an inflexion point associated with a change of concavity, as in an S-shaped plane curve. Since inflexional states are places of local change of curvature, the perpendicular component of the Lorentz force flips direction through an inflexion point. Explicit computation for a generic passage through an inflexional state can prove (Ricca 2005) the following statement.

Theorem 5.3.1. *Let* $\mathbf{B}_K$ *be a loose magnetic knot of signature* (V, Φ) *and field given by* (5.8)*, in ideal conditions. Let* $\tilde{K}_t$ *denote the generic, time-dependent passage of* K *through an inflexional state at time* $t = t_0$*. Then, the knot* $\mathbf{B}_{\tilde{K},t_0}$ *is in inflexional disequilibrium.*

Thus, inflexional flux tubes being in disequilibrium evolve to inflexion-free configurations. Since in ideal conditions topology is preserved, a magnetic knot with inflexions must change shape to get rid of inflexions. This is done by a deformation to a *closed braid* configuration, as shown in the example of Figure 5.5. Inflexion-free closed braids are called *spiral knots*. Hence, we have:

Corollary 5.3.1. *A loose inflexional magnetic knot* $\mathbf{B}_{\tilde{K},t_0}$ *evolves to a spiral knot* $\mathbf{B}_{K_o,t}$ *for* $t > t_0$*.*

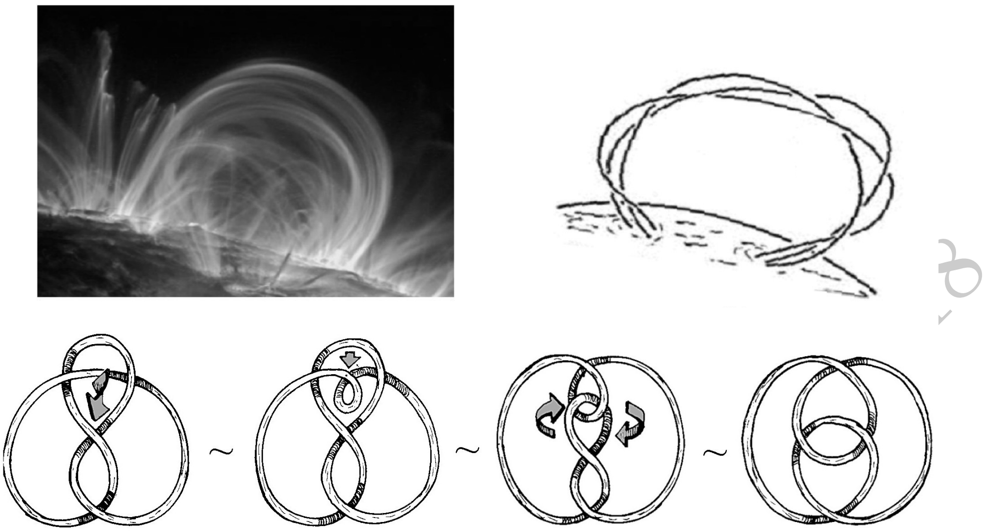


Figure 5.5: Top: (left) plasma loops in the solar corona observed by the TRACE mission; (right) a diagram of braided magnetic lines. Bottom: any knot, such as the figure-of-eight knot on the left-hand-side, can be transformed to a closed braid configuration. A spiral knot is a closed braid without inflexion points.

5.4 Goundstate energy spectrum of knots and links

A fundamental problem is to establish relationships between energy and topological complexity of magnetic systems. Some progress has been done for *zero-framed* knots: these are magnetic knots whose field lines have zero internal linking number ($h = 0$). By assuming that each link component has same flux Φ and zero-framing, one can prove the following statement.

Theorem 5.4.1. *Let* $\mathbf{B}_L$ *be a zero-framed magnetic link. Under signature-preserving flow, we have*

$$(i) \quad M(t) \geq \left(\frac{2}{\pi V}\right)^{1/3} |H| \ ; \qquad (ii) \quad M_{\min} = \left(\frac{2}{\pi V}\right)^{1/3} \Phi^2 c_{\min} \ . \tag{5.34}$$

where $c_{\min}$ *denotes the topological crossing number of the link.*

Proof of these results is based on works of Arnold (1974), Freedman & He (1991) and Ricca (2008).

The fundamental problem to determine the value of $m(h)$ for which $M = M_{\min}$ ($m_{\min}$) is solved by applying Theorem 5.4.1–(ii) to (5.33). We have:

Corollary 5.4.1. *For zero-framed magnetic knots under signature-preserving flow we have*

$$m_{\min} = m(0) = (2/\pi)^{1/3} c_{\min} \ . \tag{5.35}$$

For a signature-preserving flow (5.42) establishes a correspondence between minimum energy levels and topology, since $M_{\min} \propto c_{\min}$, a result of general validity, but rather loose. From a direct inspection of the knot table (see, for instance, the standard knot tabulation of Rolfsen 1976) for $c_{\min} > 4$ there are several topologically distinct knot types of same $c_{\min}$, whose number gets exponentially large for increasing values of $c_{\min}$. A natural question here is to determine whether different knot types of the same $c_{\min}$-family have the same minimum energy level or not.

The functional dependence of M on h, then, is also of interest for applications. By using the field decomposition (5.8) and the orthonormal Mercier coordinates (r, ϑ_R, s) introduced earlier, we can express the magnetic field in terms of poloidal and toroidal flux, Φ_P and Φ_T, respectively; we have (Maggioni & Ricca 2009)

$$\mathbf{B} = \left(0, \frac{1}{L}\frac{\mathrm{d}\Phi_P}{\mathrm{d}r}, \frac{1}{2\pi r}\frac{\mathrm{d}\Phi_T}{\mathrm{d}r}\right) + \left(0, \frac{\partial\widetilde{\psi}}{\partial s}, -\frac{\partial\widetilde{\psi}}{\partial\vartheta_R}\right), \tag{5.36}$$

where the total field is given by the sum of an average field plus a fluctuating field (represented by the function $\widetilde{\psi}$) with zero net flux. The twist parameter (knot framing) is given by $h = \Phi_P/\Phi_T$. We shall use this result in the following.

5.4.1 Knot framing and standard flux tube

Let $V_r = \pi r^2 L$ be the partial volume of the tubular neighbourhood of radius r; the ratio of the partial to total volume is given by $V_r/V(\mathcal{T}) = (r/a)^2$. Now, let $f(r/a)$ be a monotonic increasing function of r/a; we can take

$$f(r/a) = \left(\frac{r}{a}\right)^{\gamma}, \qquad (\gamma > 0) . \tag{5.37}$$

Denoting by $\Phi \equiv \Phi_T(a)$ the total flux, we have

$$\Phi_T(r) = \left(\frac{r}{a}\right)^{\gamma}\Phi , \qquad \Phi_P(r) = h\left(\frac{r}{a}\right)^{\gamma}\Phi , \tag{5.38}$$

where h, the twist parameter, denotes the *magnetic field framing*, given by $(2\pi)^{-1}$ times the turns of twist required to generate the poloidal field from the toroidal field, starting from $\Phi_P = 0$. A direct calculation of helicity in terms of fluxes shows that h is indeed the linking number of the embedded field. The choice $\gamma = 2$ provides the prescription for a *standard flux tube*.

5.4.2 Constrained relaxation to groundstate energy

By applying standard variational methods the following result holds true (Maggioni & Ricca 2009):

Theorem 5.4.2. *Let $\mathbf{B}_K$ be a magnetic knot of field given by* (5.8)*. Minimization of magnetic energy is subject to the following constraints:*
(i) (V, Φ) invariant;
(ii) flux tube cross-section $\mathcal{S}$ circular and uniform along K;
(iii) $\widetilde{\psi}$ independent of arc-length s;
(iv) knot length L independent of h.
Then we have:

$$M^* = \left(\frac{\gamma^2 L^{*2}}{8(\gamma - 1)V} + \frac{\gamma \pi h^2}{2L^*} \right) \Phi^2 \ , \tag{5.39}$$

where * denotes the constrained minimum value.

Without loss of generality we can set $V = \Phi = 1$. Since the magnetic knot in the relaxed state is in tight configuration, we can introduce a non-dimensional parameter given by the aspect ratio of the tight knot, defined by the ratio of the minimal knot length L^* to the radius R^* of the maximal circular cross-section of the tight configuration; this ratio is the so-called *ropelength* $\lambda = L^*/R^*$, a good measure of physical knot complexity. In the case of the unknot, the least possible value of λ (say λ_0) is that given by the tight torus (when the hole disappears); hence $\lambda \geq \lambda_0 = 2\pi$.

In order to investigate the relation between energy and knot topology, let us consider standard flux tubes ($\gamma = 2$); it is useful to rewrite eq. 5.39 in terms of ropelength. By using $V = \pi R^{*2} L^* = $ cst., after some straightforward algebra, we have

Corollary 5.4.2. *Constrained minimization of magnetic energy of $\mathbf{B}_K$ given by* (5.8) *gives*

$$M^* = \left(\frac{\lambda^{4/3}}{2\pi^{2/3}} + \frac{\pi^{4/3} h^2}{\lambda^{2/3}} \right) \Phi^2 V^{-1/3} \ . \tag{5.40}$$

Under the assumptions above, by comparing (5.33) with (5.40), we can determine $m(h)$:

$$m_\lambda(h) = \frac{\lambda^{4/3}}{2\pi^{2/3}} + \frac{\pi^{4/3} h^2}{\lambda^{2/3}} \ , \tag{5.41}$$

showing explicitly the effect of ropelength and framing on energy levels.

5.4.3 Energy spectra of knots and links

Let us first investigate the minima $m_{\min} = m_{\min}(h)$ by plotting (5.41) against λ for $h = 0, 1, 2, 3, \ldots$ (see Figure 5.6). The absolute minimum $m_\circ$ corresponds to the zero-framed unknot (tight torus), given by $h = 0$ and $\lambda = \lambda_0 = 2\pi$: $m_\circ = (2\pi^2)^{1/3} \approx 2.70$. The groundstate energy of zero-framed flux tubes provides

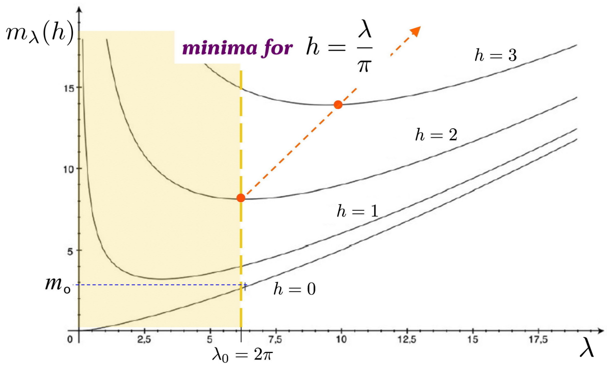


Figure 5.6: Influence of twist h on the energy function $m_\lambda(h)$, plotted against the ropelength λ, according to equation (5.41). The absolute minimum is given by the tight torus, for which $\lambda = \lambda_0 = 2\pi$ and $m_\circ \approx 2.70$.

the absolute minimum energy level; $m(h)$ remains a monotonic increasing function of λ for $h \leq 2$: at $\lambda_0 = 2\pi$ we have $m(h = 1) = 4.05$ and $m(h = 2) = 8.11$. For $h \geq 2$ the energy minima are attained for $h = \lambda/\pi$; thus, by substituting the optimal value $\lambda = \pi h$ in (5.41), we have

$$m_{\min}(h) = \frac{3}{2}\pi^{2/3}h^{4/3} \quad (h \geq 2) \ . \tag{5.42}$$

For $h > 2$ (and $\lambda \geq \lambda_0$) the functional dependence of $m(h)$ on λ ceases to be monotonic.

The minimum energy spectra of the first prime knots and links is determined by setting $h = 0$ in (5.41) and by using ropelength data (given by λ_K) obtained by the `RIDGERUNNER` tightening algorithm (Ashton, Cantarella, Piatek & Rawdon 2011) for each knot/link type K. A particularly simple expression is obtained by normalizing $m(\lambda_K, 0)$ with respect to the minimum energy value $m_\circ$ of the tight torus; thus, we have

$$\tilde{m}(K) = \frac{m(\lambda_K, 0)}{m_\circ} = \left(\frac{\lambda_K}{2\pi}\right)^{4/3} \ , \tag{5.43}$$

that gives a one-to-one relationship between minimum energy level and knot ropelength. Since the relation $\lambda_K = \lambda(K)$ is not known analytically, it must be reconstructed from numerical data. We take $\lambda_K = \lambda(\#_K)$, where $\#_K$ denotes the position of the knot/link K listed according to the increasing value of ropelength as given by `RIDGERUNNER`. Hence, instead of tabulating energy levels as function of the knot/link position given by standard knot tabulation, by taking $\lambda_K = \lambda(\#_K)$ in (5.43) we plot $\tilde{m} = \tilde{m}(\#_K)$, according to increasing ropelength data. The en-

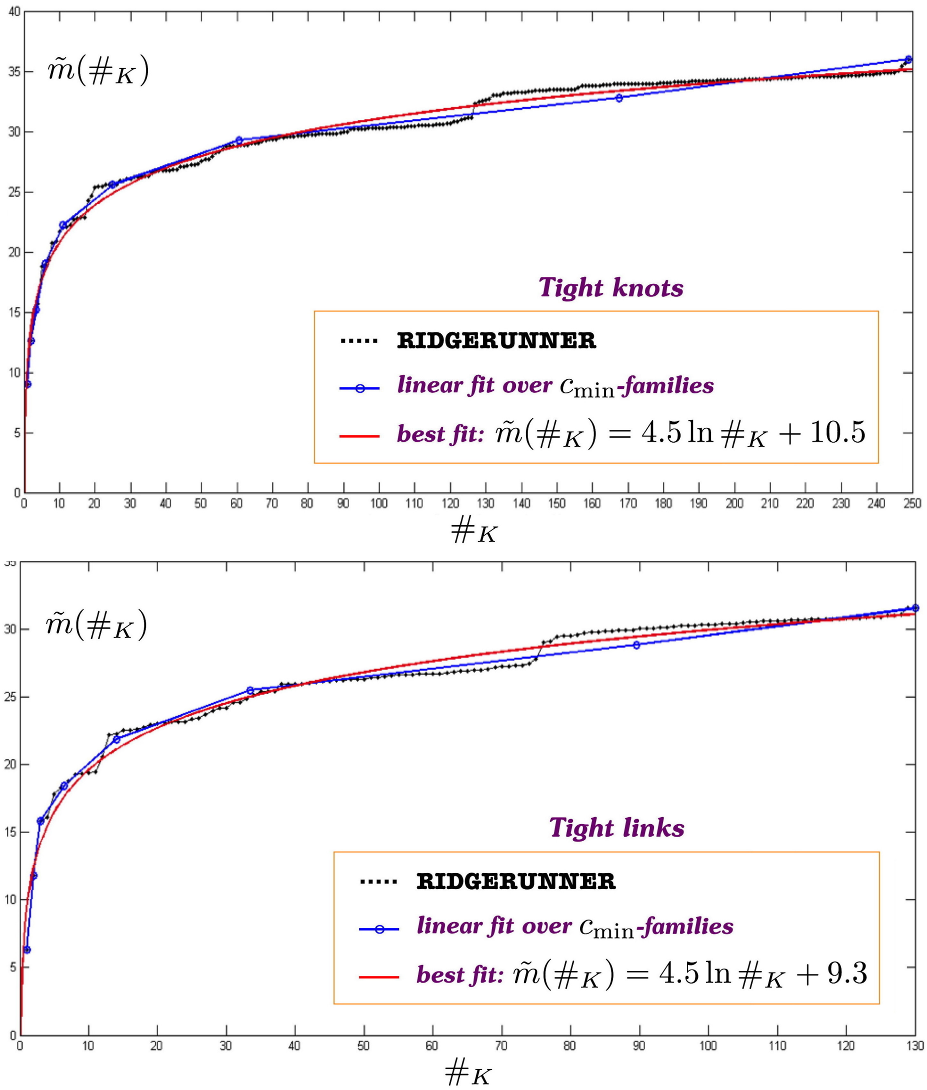


Figure 5.7: Top diagram: $\tilde{m}(\#_K)$ of tight knots plotted against the knot number $\#_K$ given by the position of the knot K listed according to increasing value of ropelength $\lambda_K = \lambda(\#_K)$. Best fit goodness: 95% confidence bounds, summed square of residuals *(SSE)* $= 80.27$, $R^2 = 0.98$, root mean squared error *(RMSE)* $= 0.57$. Bottom diagram: $\tilde{m}(\#_K)$ of tight links. Best fit goodness: 95% confidence bounds, summed square of residuals *(SSE)* $= 55.9$, $R^2 = 0.98$, root mean squared error *(RMSE)* $= 0.66$.

ergy spectra are shown in Figure 5.7 for the first 250 prime knots up to 10 crossings (top diagram) and 130 prime links up to 9 crossings (bottom diagram).

The curve dotted by circles results from a linear fit made over each $c_{\min}$–family, while the continuous curve is the best-fit interpolation over all available data. To the first decimal digit, we find that best-fit interpolations follow an almost identical logarithmic law, given by

$$\tilde{m}(\#_K) = a \ln \#_K + b \ , \tag{5.44}$$

where $a = 4.5$ and $b = b_K = 10.5$ for knots, $b = b_L = 9.3$ for links.

This unexpected result is quite remarkable and calls for some justification. Ropelength is certainly an increasing function of topological complexity (given by $c_{\min}$), simply because an increasing number of crossings implies an increasing minimal length necessary to tie a flux tube into a knot or a link. Results on ropelength bounds show that

$$O(c_{\min}^{3/4}) \leq \lambda_K \leq O(c_{\min} \ln^5 c_{\min}) \ , \tag{5.45}$$

where $O(\cdot)$ denotes order of magnitude. From (5.43) we have that $\tilde{m}(\#_K) \propto [\lambda(\#_K)]^{4/3}$; by combining this with (5.44), we have

$$[\lambda(\#_K)]^{4/3} \propto a \ln \#_K + b \ . \tag{5.46}$$

Now, if we assume that the number of knots grows exponentially with $c_{\min}$ (a plausible assumption), then $\#_K \sim C^{c_{\min}}$ for some constant C. Hence, by (5.46) we have $[\lambda(\#_K)]^{4/3} \propto c_{\min}$, or

$$\lambda(\#_K) \propto c_{\min}^{3/4} \ , \tag{5.47}$$

a result that is in good agreement with other analytical estimates.

Furthermore, if we take $V = \Phi = 1$ and define

$$\overline{m}(c_{\min}) \equiv \frac{M_{\min}}{m_\circ} = \frac{1}{\pi} c_{\min} \ . \tag{5.48}$$

We can then relate (5.34)(ii) to (5.43), and write

$$\langle \tilde{m}(K) \rangle_{c_{\min}} \geq \overline{m}(c_{\min}) = \frac{1}{\pi} c_{\min} \ , \tag{5.49}$$

since for any given K $\tilde{m}(K)$ could be further decreased to the actual minimum by relaxing the constraints (i)-(iv) of Theorem 5.4.2. By writing (5.43) in terms of $\#_K$ and substituting this latter into the above equation, we have

$$\langle \lambda(\#_K) \rangle_{c_{\min}} \geq 2\pi^{1/4} c_{\min}^{3/4} \ , \tag{5.50}$$

that gives a new relation between ropelength, averaged over each $c_{\min}$–family, and $c_{\min}$. Note that the coefficient $2\pi^{1/4} \approx 2.66$ (still subject to further optimization) is independent of the knot family and is valid for *any* $c_{\min}$.

Bibliography

References to Background Material

[1] Arnol'd, V.I. & Khesin, B.A. 1998 *Topological Methods in Hydrodynamics*. Applied Mathematical Sciences **125**. Springer–Verlag.

[2] Batchelor, G. 1967 *An Introduction to Fluid Dynamics*. Cambridge University Press.

[3] Chorin, A. & Marsden, J. 1990 *A Mathematical Introduction to Fluid Mechanics*. Springer–Verlag.

[4] Davidson, P.A. 2001 *An Introduction to Magnetohydrodynamics*. Cambridge University Press.

[5] Majda, A.J. & Bertozzi, A.L. 2001 *Vorticity and Incompressible Flow*. Cambridge University Press.

[6] Moffatt, H.K. 1983 *Magnetic Field Generation in Electrically Conducting Fluids*. Cambridge University Press.

[7] Ricca, R.L. (Editor) 2001 *An Introduction to the Geometry and Topology of Fluid Flows*. NATO ASI Series II **47**. Kluwer Academic Press.

[8] Ricca, R.L. (Editor) 2009 *Lectures on Topological Fluid Mechanics*. CIME Lecture Notes in Mathematics **1973**. Springer–Verlag.

[9] Saffman, P. 1991 *Vortex Dynamics*. Cambridge University Press.

[10] Truesdell, C. 1954 *The Kinematics of Vorticity*. Indiana University Press.

[11] Wu, J.Z., Ma, H.Y. & Zhou, M.D. 2006 *Vorticity and Vortex Dynamics*. Springer–Verlag.