Relativistic fluid dynamics for polarized media and the classical treatment of spin

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[1811.04409], to appear in *Prog. Part. Nucl. Phys*

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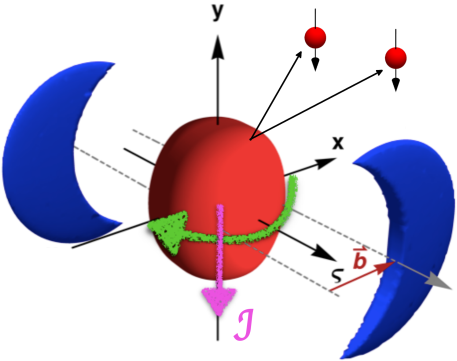
Motivation

- Noncentral HIC create fireballs with large global angular momenta $\sim 10^5\hbar$
  F. Becattini, F. Piccinini, J. Rizzo, PRC 77 (2008) 024906, [0711.1253]

- Vorticity of the system may produce a spin polarization of the produced particles
  S. J. Barnett, Rev. Mod. Phys. 7, 129 (1935)
  A. Einstein and W. de Haas, Deutsche Physikalische Gesellschaft, Verhandlungen 17, 152 (1915)



Motivation

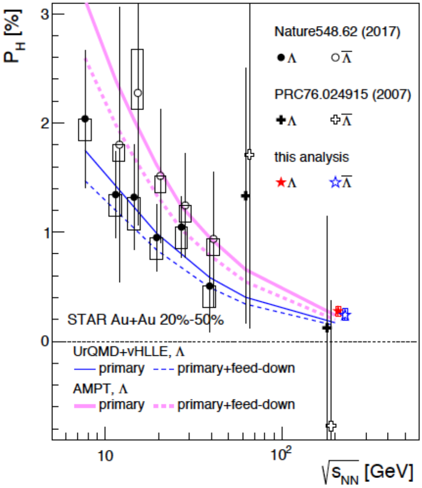
- The first positive measurement of global polarization of $\Lambda/\bar{\Lambda}$ hyperons
  STAR , Nature 548 (2017) 62-65, [1701.06657]

spin polarization
$\updownarrow$?
vorticity

- Statistical formalism describes the global polarization
  D. N. Zubarev, A. V. Prozorkevich, S. A. Smolyanskii, Theoret. and Math. Phys. 40 (1979), 821
  F. Becattini, L. Bucciantini, E. Grossi, L. Tinti, Eur. Phys. J. C 75 (2015) 191 [1403.6265]

spin polarization tensor $\omega^{\mu\nu}$
$\Updownarrow$ global equilibrium
thermal vorticity $\varpi^{\mu\nu}$

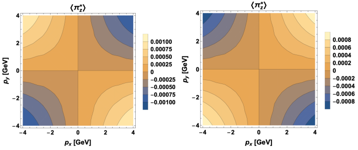
$\rightarrow$ **talk by F. Becattini**



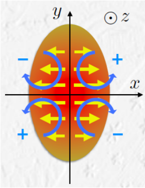
STAR, PRC 98 (2018) 014910, [1805.04400]
UrQMD+vHLLE: I. Karpenko, F. Becattini, EPJC 77, 213 (2017), [1610.04717]
AMPT: H. Li, L. Pang, Q. Wang, and X. Xia, PRC 96, 054908 (2017), [704.01507]

Motivation

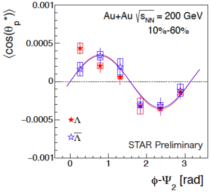
- The quadrupole structure of longitudinal polarization is not described within current approach
- Some solution of the sign problem found within thermal models
  S. Voloshin, EPJ Web Conf. 17 (2018) 10700, [1710.08934]
  W.Florkowski, A. Kumar, R.R., [1904.00002]

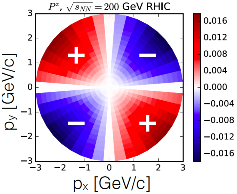


(a) (b)



S. Voloshin, SQM2017





(LEFT) T. Niida, NPA 982 (2019) 511514 [1808.10482]
(RIGHT) F. Becattini, I. Karpenko, PRL 120 (2018) no.1, 012302, [1707.07984]

→ **talk by T. Niida**

Motivation

- Present works limited to the calculation of polarization at freeze-out

  $\rightarrow$ space-time dynamics of spin polarization?

- HIC evolution best described within relativistic hydrodynamics

  $\rightarrow$ inclusion of polarization effects in hydrodynamics?

Motivation

- Relativistic hydrodynamics with spin formulated recently
  W. Florkowski, B. Friman, A. Jaiswal, E. Speranza, PRC 97 (4) (2018) 041901. [1705.00587]
  W. Florkowski, B. Friman, A. Jaiswal, R. R., E. Speranza, PRD 97 (11) (2018) 116017. [1712.07676]

  follow up study: K. Hattori, M. Hongo, X-G. Huang, M. Matsuo, H.Taya [1901.06615]

  $\rightarrow$ particular choice of the forms of energy-momentum $T^{\mu\nu}$ and spin tensors $S^{\lambda\mu\nu}$

  $\rightarrow$ recent works clarified the use of Groot, Leeuwen, Weert (GLW) forms of $T^{\mu\nu}$ and $S^{\lambda\mu\nu}$
  W. Florkowski, A. Kumar, R. R., PRC 98 (2018) 044906. [1806.02616]
  F. Becattini, W. Florkowski, E. Speranza, PLB 789, (2019) 419-425 [1807.10994],

- Realistic applications of the GLW-based hydrodynamics with spin to the HIC in the small polarization limit is in progress
  W. Florkowski, A. Kumar, R. R. and R. Singh, [arXiv:1901.09655], accepted to PRC
  W. Florkowski, A. Kumar, R. R. and R. Singh, forthcoming

$\rightarrow$ **talk by A. Kumar**

Motivation

- However, in this approach **for large momenta** of particles one encounters problems with normalization of the mean polarization three-vector

$$\langle \boldsymbol{P}(x,p) \rangle = -\frac{1}{4\zeta}\tanh(\zeta)\boldsymbol{P}.$$

$$\boldsymbol{P} = \frac{1}{m}\left[E_p\,\boldsymbol{b} - \boldsymbol{p}\times\boldsymbol{e} - \frac{\boldsymbol{p}\cdot\boldsymbol{b}}{E_p+m}\boldsymbol{p}\right] = \boldsymbol{b}_*$$

$$\omega_{\mu\nu} = \begin{bmatrix} 0 & e^1 & e^2 & e^3 \\ -e^1 & 0 & -b^3 & b^2 \\ -e^2 & b^3 & 0 & -b^1 \\ -e^3 & -b^2 & b^1 & 0 \end{bmatrix}.$$

- We (in parallel) study the polarization effects in the framework which treats spin classically in order to improve the quantum description based on the spin density matrices

Internal angular momentum tensor

Lets introduce **internal angular momentum tensor**
M. Mathisson, APPB 6 (1937) 163–2900

$$s^{\alpha\beta} = \frac{1}{m}\epsilon^{\alpha\beta\gamma\delta}p_\gamma s_\delta$$

It satisfies the **Frenkel (or Weyssenhoff) condition**

$$s^{\alpha\beta}p_\alpha = 0$$

Since $s \cdot p = 0$ the spin four-vector is

$$s^\alpha = \frac{1}{2m}\epsilon^{\alpha\beta\gamma\delta}p_\beta s_{\gamma\delta} \qquad \overset{PRF\ [p^\mu=(m,0,0,0)]}{\Longrightarrow} \qquad s^\alpha = (0, \boldsymbol{s}_*)$$

For spin-1/2 particles the length of $\boldsymbol{s}_*$ is given by the value of the Casimir operator,

$$|\boldsymbol{s}_*|^2 = \mathfrak{s}^2 = \frac{1}{2}\left(1+\frac{1}{2}\right) = \frac{3}{4}$$

Angular momentum conservation

To construct the equilibrium function we have to identify the collisional invariants of the Boltzmann equation (BE)

In addition to four-momentum and conserved charges one can include the **total angular momentum**

$$j_{\alpha\beta} = l_{\alpha\beta} + s_{\alpha\beta} = x_\alpha p_\beta - x_\beta p_\alpha + s_{\alpha\beta}$$

The **locality of the standard BE** suggests that the **orbital part can be eliminated**, and **the spin part can be considered separately**.

For elastic binary collisions of particles 1 and 2 going to $1'$ and $2'$, this suggests that
C. G. van Weert, Henkes- Holland N.V. – Haarlem, 1970.

$$s_1^{\alpha\beta} + s_2^{\alpha\beta} = s_{1'}^{\alpha\beta} + s_{2'}^{\alpha\beta}$$

These equations admit two types of simple solutions: either the **sum of two spin three-vectors or their difference (before and after the collision) vanishes**.
They may be interpreted as collisions in the spin **singlet** and **triplet** states.

If the collision integral allows for processes such as discussed above, the spin angular momentum conservation law should be included among the other conservation laws.

Non-local effects ?

Locality of the collision kernel suggests that we cannot describe the Barnett effect - the fact realized in 1966.

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Kinetic Theory for a Dilute Gas of Particles with Spin

S. HESS and L. WALDMANN

Institut für Theoretische Physik der Universität Erlangen-Nürnberg, Erlangen

(Z. Naturforschg. **21 a**, 1529—1546 [1966]; received 6 April 1966)

conserved. There is another effect which we cannot describe with a local collision operator (even in thermal equilibrium): the orientation of the spin by a local or uniform rotation of the system (BARNETT effect).

Further developments require the use of non-local collisional kernels.

A. Jaiswal, R. S. Bhalerao, S. Pal, PLB 720 (2013) 347–351

Spin-dependent distribution function and invariant measure

We introduce a **spin-dependent equilibrium distribution functions for particles and antiparticles**

$$f^{\pm}_{\rm eq}(x,p,s) = \exp\left(\pm\xi(x) - p\cdot\beta(x) + \frac{1}{2}\omega_{\alpha\beta}(x)s^{\alpha\beta}\right)$$

with $\beta^\mu = u^\mu/T$ and $\xi^\mu = \mu/T$.

Different orientations of spin can be integrated out with the help of a **covariant measure**

$$\int dS \ldots = \frac{m}{\pi\,\mathfrak{s}} \int d^4s\, \delta(s\cdot s + \mathfrak{s}^2)\, \delta(p\cdot s) \ldots$$

The prefactor $m/(\pi\,\mathfrak{s})$ is chosen to obtain the **normalization**

$$\int dS = \frac{m}{\pi\,\mathfrak{s}} \int d^4s\, \delta(s\cdot s + \mathfrak{s}^2)\, \delta(p\cdot s) = 2$$

Charge current

The **charge current** is obtained from the generalization of the standard definition

$$N^\mu_{\rm eq} = \int dP \int dS \; p^\mu \left[f^+_{\rm eq}(x,p,s) - f^-_{\rm eq}(x,p,s)\right]$$

which after using the forms of the equilibrium functions leads to

$$N^\mu_{\rm eq} = 2\sinh(\xi) \int dP \, p^\mu e^{-p\cdot\beta} \int dS \; \exp\left(\frac{1}{2}\omega_{\alpha\beta}s^{\alpha\beta}\right)$$

For **small values of** ω one gets

$$N^\mu_{\rm eq} = 2\sinh(\xi) \int dP \, p^\mu \, e^{-p\cdot\beta} \int dS \left(1 + \frac{1}{2}\omega_{\alpha\beta}s^{\alpha\beta}\right) = 4\sinh(\xi) \int dP \, p^\mu \, e^{-p\cdot\beta}$$

Agrees up to the first order in ω with the charge current obtained by taking zeroth moment of the kinetic equation for scalar coefficient function of the Wigner function obtained at NLO from the semi-classical expansion of the kinetic equation in $\hbar$

Energy-momentum tensor

The **energy-momentum tensor** is given by

$$T^{\mu\nu}_{\rm eq} = \int dP \int dS \; p^\mu p^\nu \left[f^+_{\rm eq}(x,p,s) + f^-_{\rm eq}(x,p,s) \right]$$

which leads to

$$T^{\mu\nu}_{\rm eq} = 2\cosh(\xi) \int dP \, p^\mu p^\nu \, e^{-p\cdot\beta} \int dS \; \exp\left(\frac{1}{2}\omega_{\alpha\beta}s^{\alpha\beta}\right)$$

For **small values of** ω one gets

$$T^{\mu\nu}_{\rm eq} = 2\cosh(\xi) \int dP \, p^\mu p^\nu \, e^{-p\cdot\beta} = 4\cosh(\xi) \int dP \, p^\mu p^\nu \, e^{-p\cdot\beta}$$

Agrees up to the first order in ω with the $T^{\mu\nu}_{\rm eq}$ obtained by taking zeroth moment of the kinetic equation for scalar coefficient function of the Wigner function obtained at NLO from the semi-classical expansion of the kinetic equation in $\hbar$ and with the GLW energy-momentum tensor

$N^{\mu\nu}$ and $T^{\mu\nu}$ - case of arbitrarily large polarization

Let us consider the integral for **arbitrary values of** ω

$$\int dS\, \exp\left(\frac{1}{2}\omega_{\alpha\beta}s^{\alpha\beta}\right)$$

Using definition of the dual polarization tensor $\tilde{\omega}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\,\omega^{\alpha\beta}$ one may write

$$\frac{1}{2}\omega_{\alpha\beta}s^{\alpha\beta} = \frac{p_\gamma \tilde{\omega}^{\gamma\delta}}{m}\, s_\delta \qquad \overset{PRF}{\Rightarrow} \qquad \frac{1}{2}\omega_{\alpha\beta}s^{\alpha\beta} = \boldsymbol{b}_* \cdot \boldsymbol{s}_* = \boldsymbol{P} \cdot \boldsymbol{s}_*$$

For **arbitrary values of** ω one gets

$$\int dS\, \exp\left(\frac{1}{2}\omega_{\alpha\beta}s^{\alpha\beta}\right) = \int_{-1}^{+1} e^{\mathfrak{s} P x} dx = \frac{2\sinh(\mathfrak{s} P)}{\mathfrak{s} P} \qquad \text{with} \qquad P = |\boldsymbol{P}| = |\boldsymbol{b}_*|$$

Since P depends on momentum $T^{\mu\nu}$ has no longer perfect fluid form.

Large values of ω induces momentum anisotropy $\rightarrow$ anisotropic hydrodynamics methods

W. Florkowski, R. R., PRC 83 (2011) 034907. [1007.0130]

M. Martinez, M. Strickland, NPA 848 (2010) 183–197. [1007.0889]

Spin tensor

The **spin tensor** is defined as an expectation value of the internal angular momentum tensor,

$$S_{\rm eq}^{\lambda\mu\nu} = \int dP \int dS \; p^\lambda \, s^{\mu\nu} \left[f^+_{\rm eq}(x,p,s) + f^-_{\rm eq}(x,p,s)\right]$$

$$S_{\rm eq}^{\lambda\mu\nu} = 2\cosh(\xi) \int dP \, p^\lambda \, e^{-p\cdot\beta} \int dS \, s^{\mu\nu} \; \exp\left(\frac{1}{2}\omega_{\alpha\beta}s^{\alpha\beta}\right)$$

For **small values of** ω one gets

$$\int dS \, s^{\mu\nu} \left(1 + \frac{1}{2}\omega_{\alpha\beta}s^{\alpha\beta}\right) = \frac{\omega_{\alpha\beta}}{2m^2}\epsilon^{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\gamma\delta}p_\rho p_\gamma \int dS \, s_\delta s_\sigma = \frac{2}{3m^2}\,\mathfrak{s}^2 \left(m^2\omega^{\mu\nu} + 2p^\alpha p^{[\mu}\omega^{\nu]}_{\ \ \alpha}\right)$$

$$S_{\rm eq}^{\lambda\mu\nu} \quad = \quad \frac{4\,\mathfrak{s}^2}{3m^2}\cosh(\xi) \int dP \, p^\lambda \, e^{-p\cdot\beta} \left(m^2\omega^{\mu\nu} + 2p^\alpha p^{[\mu}\omega^{\nu]}_{\ \ \alpha}\right)$$

Agrees up to the first order in ω with the $S_{\rm eq}^{\lambda\mu\nu}$ obtained by taking moment of the kinetic equation for axial vector coefficient function of the Wigner function obtained at NLO from the semi-classical expansion of the kinetic equation in $\hbar$ and with the GLW spin tensor

$S^{\lambda\mu\nu}$ - case of arbitrarily large polarization

Let us consider the integral for arbitrary value of ω

$$\int dS\, s^{\mu\nu}\ \exp\left(\frac{1}{2}\omega_{\alpha\beta}s^{\alpha\beta}\right)$$

In this case one can find

$$\int dS\, s^{\mu\nu}\ \exp\left(\frac{1}{2}\omega_{\alpha\beta}s^{\alpha\beta}\right) = \frac{\chi(P\mathfrak{s})}{m^2}\left(m^2\omega^{\mu\nu} + 2p^\alpha p^{[\mu}\omega^{\nu]}_{\ \ \alpha}\right)$$

with

$$\chi(P\mathfrak{s}) = \mathfrak{s}^2\,\frac{2\sinh(P\mathfrak{s})}{P\mathfrak{s}}\,\frac{L(P\mathfrak{s})}{P\mathfrak{s}} \quad \text{where} \quad L(x) = \coth(x) - \frac{1}{x} \quad \text{is the Langevin function}$$

For **arbitrary values of** ω one gets

$$S^{\lambda,\mu\nu}_{\rm eq} \quad = \quad \frac{2}{m^2}\cosh(\xi)\int dP\, p^\lambda\, e^{-p\cdot\beta}\chi(P\mathfrak{s})\left(m^2\omega^{\mu\nu} + 2p^\alpha p^{[\mu}\omega^{\nu]}_{\ \ \alpha}\right)$$

Pauli-Lubański vector (I)

We introduce **phase-space density of the Pauli-Lubański (PL) vector**

F. Becattini, V. Chandra, L. Del Zanna, E. Grossi, Annals Phys. 338 (2013) 32–49. [1303.3431]

$$E_p \frac{d\Delta\Pi_\mu(x,p)}{d^3p} = -\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\left(\Delta\Sigma_\lambda(x)\, E_p \frac{dS^{\lambda,\nu\alpha}(x,p)}{d^3p}\right)\frac{p^\beta}{m}$$

where

$$E_p \frac{d\Delta\mathcal{S}^{\mu\nu}}{d^3p} = \frac{\cosh(\xi)}{4\pi^3}\Delta\Sigma\cdot p\, e^{-p\cdot\beta}\int dS\, s^{\mu\nu}\, \exp\left(\frac{1}{2}\omega_{\alpha\beta}s^{\alpha\beta}\right)$$

We define the **particle number current** for both particles and antiparticles as

$$\mathcal{N}^\mu_{\rm eq} = \int dP \int dS\; p^\mu \left[f^+_{\rm eq}(x,p,s) + f^-_{\rm eq}(x,p,s)\right].$$

Using this expression we obtain the momentum density of the total number of particles

$$E_p \frac{d\Delta\mathcal{N}}{d^3p} = \frac{\cosh(\xi)}{4\pi^3}\Delta\Sigma\cdot p\, e^{-p\cdot\beta}\int dS\, \exp\left(\frac{1}{2}\omega_{\alpha\beta}s^{\alpha\beta}\right)$$

The **mean PL vector** is obtained as the ratio

$$\pi_\mu(x,p) = -\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\frac{\int dS\, s^{\nu\alpha}\, \exp\left(\frac{1}{2}\omega_{\rho\sigma}s^{\rho\sigma}\right)}{\int dS\, \exp\left(\frac{1}{2}\omega_{\rho\sigma}s^{\rho\sigma}\right)}\frac{p^\beta}{m}$$

Pauli-Lubański vector (II)

For **arbitrary values of the polarization** one gets

$$\pi_\mu = -\mathfrak{s}^2 \frac{\tilde{\omega}_{\mu\beta} p^\beta}{m} \frac{L(P\,\mathfrak{s})}{P\,\mathfrak{s}}$$

In PRF

$$\pi_*^0 = 0, \quad \boldsymbol{\pi}_* = -\mathfrak{s}^2 \boldsymbol{P} \frac{L(P\,\mathfrak{s})}{P\,\mathfrak{s}}$$

For small and large P Langevin function is having the form

$$L \approx 1 \quad \text{for} \quad x \gg 1 \quad \text{and} \quad L \approx \frac{x}{3} \quad \text{for} \quad x \ll 1$$

respectively, thus we obtain two important results:

$$\boldsymbol{\pi}_* = -\mathfrak{s}\,\frac{\boldsymbol{P}}{P}, \quad |\boldsymbol{\pi}_*| = \mathfrak{s} = \sqrt{\frac{3}{4}}, \quad \text{if} \quad P \gg 1$$

The normalization of the PL vector cannot exceed the value of $\mathfrak{s}$.

$$\boldsymbol{\pi}_* = -\mathfrak{s}^2\,\frac{\boldsymbol{P}}{3}, \quad |\boldsymbol{\pi}_*| = \mathfrak{s}^2\,\frac{P}{3} = \frac{P}{4}, \quad \text{if} \quad P \ll 1$$

For small values of P the classical treatment of spin reproduces the quantum mechanical result

Entropy conservation

Classical treatment of spin allows for explicit derivation of the **entropy current conservation**. We adopt the Boltzmann definition

$$H^\mu = -\int dP \int dS\, p^\mu \left[f^+_{\rm eq}\left(\ln f^+_{\rm eq} - 1\right) + f^-_{\rm eq}\left(\ln f^-_{\rm eq} - 1\right)\right]$$

Using form of $f^\pm$ and the conservation laws for energy, linear and angular momentum, and charge, we obtain

$$H^\mu = \beta_\alpha T^{\mu\alpha}_{\rm eq} - \frac{1}{2}\omega_{\alpha\beta} S^{\mu,\alpha\beta}_{\rm eq} - \xi N^\mu_{\rm eq} + \mathcal{N}^\mu_{\rm eq}$$

$$\partial_\mu H^\mu = \left(\partial_\mu \beta_\alpha\right) T^{\mu\alpha}_{\rm eq} - \frac{1}{2}\left(\partial_\mu \omega_{\alpha\beta}\right) S^{\mu,\alpha\beta}_{\rm eq} - \left(\partial_\mu \xi\right) N^\mu_{\rm eq} + \partial_\mu \mathcal{N}^\mu_{\rm eq}$$

With the help of the relation $\mathcal{N}^\mu_{\rm eq} = \frac{\cosh(\xi)}{\sinh(\xi)} N^\mu_{\rm eq}$ and the conservation of charge one can easily show that

$$\partial_\mu H^\mu = 0$$

Contributions to H^μ, connected with the polarization tensor, start with **quadratic terms in** ω.

If we restrict ourselves to **linear terms in** ω, all thermodynamic quantities become independent of ω, while the conservation of the angular momentum determines the polarization evolution in a given hydrodynamic background.

Conclusions and Summary

We show that the classical approach to spin:

- allows for arbitrarily large values of spin chemical potential leading to the anisotropic hydrodynamics framework with spin
- agrees with the quantum GLW results for small values of the spin chemical potential
- indicates how to avoid problems with the normalization of the average polarization three-vector for large values of the spin potential
- helps to define microscopic conditions that validate the use of the proposed equilibrium functions
- can be used to define entropy current and prove its conservation within the perfect-fluid approach with spin

Thank you for your attention!