Kinetic theory for massive spin-1/2 particles
from Wigner-function formalism

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Outline

Spin effects in HICs

Spin

$\Downarrow$

Chiral Magnetic Effect

D.E.Kharzeev, et al. Nucl.Phys.A803,227 (2008)

$$J^\mu = \xi_B B^\mu + \xi \omega^\mu$$
$$J_5^\mu = \xi_{B5} B^\mu + \xi_5 \omega^\mu$$

Chiral Vortical Effect

D.T.Son and P.Surowka, Phys.Rev.Lett.103,191601 (2009)

Chiral Separation Effect

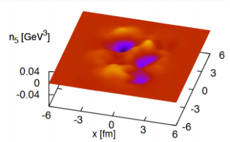
D.E.Kharzeev, et al. Prog.Part.Nucl.Phys.88,1 (2016)

Axial Chiral Vortical Effect

D.E.Kharzeev, et al. Prog.Part.Nucl.Phys.88,1 (2016)

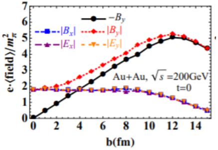
- In heavy-ion collisions:

Spin imbalance



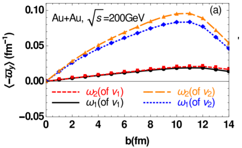
Y.Hirono, et al. arXiv:1412.0311.

Magnetic field



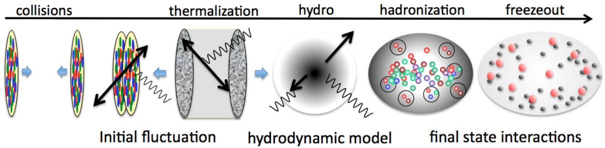
W.-T.Deng, X.-G.Huang, Phys.Rev.C85,044907 (2012).

Flow vorticity



W.-T.Deng, et al. J.Phys.Conf.Ser.779,012070 (2017)

Quantum transport



- Relativistic Hydrodynamics: massive, w/o spin. see D.H. Rischke's talk
- Chiral Kinetic Theory: massless, spin effects included. see Xuguang Huang's talk
- Massless Chiral Magnetohydrodynamics:
  D. T. Son and P. Surowka, Phys. Rev. Lett.103,191601 (2009),
  K. Hattori, Y. Hirono, H.-U. Yee, and Y. Yin, arXiv:1711.08450.

- Kinetic description for massive spin-1/2 particles?
  - Hydrodynamics based on local spin-dependent equilibrium distribution:
    F. Becattini, V. Chandra, L. Del Zanna, and E. Grossi, Annals Phys. 338, 32-49 (2013),
    W. Florkowski, R. Ryblewski, arXiv:1811.04409. see Radoslaw Ryblewski's talk
  - Kinetic theory from Wigner function:
    J.-H. Gao and Z.-T. Liang, arXiv:1902.06510, see Jianhua Gao's talk
    N. Weickgenannt, XLS, E. Speranza, Q. Wang, and D.H. Rischke, arXiv:1902.06513,
    K. Hattori, Y. Hidaka, and D.-L. Yang, arXiv:1903.01653,
    Z. Wang, X. Guo, S. Shi, and P. Zhuang, arXiv:1903.03461. see Ziyue Wang's talk

Definition of Wigner function

- Covariant Wigner function

$$W_{\alpha\beta}(x,p) = \int \frac{d^4y}{(2\pi)^4} e^{-\frac{i}{\hbar}p\cdot y} \left\langle : \bar{\psi}_\beta \left(x+\frac{y}{2}\right) U\left(x+\frac{y}{2}, x-\frac{y}{2}\right) \psi_\alpha\left(x-\frac{y}{2}\right) : \right\rangle . \tag{1}$$

- Gauge link

$$U\left(x+\frac{y}{2}, x-\frac{y}{2}\right) = \exp\left[-\frac{i}{\hbar} y^\mu \int_{-1/2}^{1/2} ds\ \mathbb{A}_\mu(x+sy)\right] . \tag{2}$$

- Kinetic equation

$$\left[\gamma_\mu \left(\Pi^\mu + i\frac{\hbar}{2}\nabla^\mu\right) - m\right] W(x,p) = 0\,, \tag{3}$$

where $\nabla_\mu \equiv \partial_\mu - j_0(\frac{\hbar}{2}\triangle)F_{\mu\nu}\partial_p^\nu$, $\Pi_\mu = p_\mu - \frac{\hbar}{2}j_1(\frac{\hbar}{2}\triangle)F_{\mu\nu}\partial_p^\nu$ and $\Delta \equiv \partial_x \cdot \partial_p$. Spherical Bessel functions j_0, j_1.

H. T. Elze, M. Gyulassy, and D. Vasak, Nucl. Phys. B276, 706 (1986)

Semi-classical expansion

- Decomposed in terms of generators of Clifford algebra

$$W = \frac{1}{4}\left\{\mathcal{F} + i\gamma^5\mathcal{P} + \gamma^\mu\mathcal{V}_\mu + \gamma^5\gamma^\mu\mathcal{A}_\mu + \frac{1}{2}\sigma^{\mu\nu}\mathcal{S}_{\mu\nu}\right\}, \tag{4}$$

  where the 16 coefficients are real functions of $\{x, p\}$.
- Semi-classical expansion in $\hbar$, valid if (i) $\hbar\gamma^\mu\nabla_\mu W \ll mW$ and (ii) $\hbar \ll \Delta R \Delta P$.
- Zeroth-order solutions derived from first-principle calculation,

$$\begin{aligned}
\mathcal{F}^{(0)} &= m\, V^{(0)} \delta(p^2 - m^2)\,, \\
\mathcal{P}^{(0)} &= 0\,, \\
\mathcal{V}^{(0)}_\mu &= p_\mu\, V^{(0)} \delta(p^2 - m^2)\,, \\
\mathcal{A}^{(0)}_\mu &= m\, n^{(0)}_\mu A^{(0)} \delta(p^2 - m^2)\,, \\
\mathcal{S}^{(0)}_{\mu\nu} &= m\, \Sigma^{(0)}_{\mu\nu} A^{(0)} \delta(p^2 - m^2)\,.
\end{aligned} \tag{5}$$

where spin direction $n^{(0)}_\mu$ and dipole-moment tensor $\Sigma^{(0)}_{\mu\nu}$ satisfy

$$n^{(0)}_\mu = -\frac{1}{2m}\epsilon_{\mu\nu\alpha\beta}p^\nu \Sigma^{(0)\alpha\beta}, \quad \Sigma^{(0)}_{\mu\nu} = -\frac{1}{m}\epsilon_{\mu\nu\alpha\beta}p^\alpha n^{(0)\beta}\,. \tag{6}$$

Solutions up to order $\hbar$

- Higher-order solutions can be derived by expanding the kinetic equation of Wigner function into series of $\hbar$ and solve them order by order.

$$
\begin{aligned}
\mathcal{F} &= m\left[V\,\delta(p^2-m^2)-\frac{\hbar}{2}F^{\mu\nu}\Sigma_{\mu\nu}A\,\delta'(p^2-m^2)\right]+\mathcal{O}(\hbar^2)\,,\\
\mathcal{P} &= \frac{\hbar}{4m}\epsilon^{\mu\nu\alpha\beta}\nabla_\mu\left[p_\nu\Sigma_{\alpha\beta}A\,\delta(p^2-m^2)\right]+\mathcal{O}(\hbar^2)\,,\\
\mathcal{V}_\mu &= \delta(p^2-m^2)\left[p_\mu V+\frac{\hbar}{2}\nabla^\nu(\Sigma_{\mu\nu}A)\right]+\mathcal{O}(\hbar^2)\\
&\quad -\hbar\left(\frac{1}{2}p_\mu F^{\alpha\beta}\Sigma_{\alpha\beta}+\Sigma_{\mu\nu}F^{\nu\alpha}p_\alpha\right)A\,\delta'(p^2-m^2)+\mathcal{O}(\hbar^2)\,,\\
\mathcal{A}_\mu &= -\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}p^\nu\Sigma^{\alpha\beta}A\,\delta(p^2-m^2)+\hbar\tilde{F}_{\mu\nu}p^\nu V\,\delta'(p^2-m^2)+\mathcal{O}(\hbar^2)\,,\\
\mathcal{S}_{\mu\nu} &= m\left[\Sigma_{\mu\nu}A\,\delta(p^2-m^2)-\hbar F_{\mu\nu}V\,\delta'(p^2-m^2)\right]+\mathcal{O}(\hbar^2)\,.
\end{aligned}
\tag{7}
$$

- Tensor $\Sigma_{\mu\nu}$ is properly normalized so that one can separate distribution A.

Kinetic equations and classical limit

- Vector distribution

$$\delta(p^2-m^2)\left[p\cdot\nabla V+\frac{\hbar}{4}(\partial_{x\alpha}F_{\mu\nu})\partial_p^\alpha\left(\Sigma^{\mu\nu}A\right)\right] -\delta'(p^2-m^2)\frac{\hbar}{2}F_{\mu\nu}p\cdot\nabla\left(\Sigma^{\mu\nu}A\right)=0\,. \quad (8)$$

- Dipole-moment tensor

$$\delta(p^2-m^2)\left[p\cdot\nabla(\Sigma_{\mu\nu}A)-F^{\alpha}_{\ [\mu}\Sigma_{\nu]\alpha}A+\frac{\hbar}{2}\left(\partial_{x\alpha}F_{\mu\nu}\right)\partial_p^\alpha V\right] +\delta'(p^2-m^2)\hbar F_{\mu\nu}p\cdot\nabla V=0\,. \quad (9)$$

where $\nabla^\mu=\partial_x^\mu-F^{\mu\nu}\partial_{p\nu}+\mathcal{O}(\hbar^2)$.

- Modified on-shell condition.
- Effect of inhomogeneous fields, Mathisson force.
- Agrees with collisionless Boltzmann-Vlasov equation.
- Recovers second Mathisson-Papapetrou-Dixon (MPD) equation (equivalent to Bargmann-Michel-Telegdi (BMT) equation).

Generalized Boltzmann equation

- Meaning of functions V and A can be obtained by comparing our solutions with the ones from first-principle calculation,

$$\begin{aligned} V &= \frac{2}{(2\pi\hbar)^3}\sum_s \left[\theta(p^0)f_s^+(x,\mathbf{p})+\theta(-p^0)f_s^-(x,-\mathbf{p})\right], \\ A &= \frac{2}{(2\pi\hbar)^3}\sum_s s\left[\theta(p^0)f_s^+(x,\mathbf{p})+\theta(-p^0)f_s^-(x,-\mathbf{p})\right], \end{aligned} \tag{10}$$

where s denotes chirality (massless) or spin-up/spin-down along given quantization direction (massive).

- Generalized Boltzmann equation

$$\sum_s \delta\left(p^2-m^2-s\frac{\hbar}{2}F^{\mu\nu}\Sigma_{\mu\nu}\right)\left[p\cdot\nabla f_s+s\frac{\hbar}{4}(\partial_{x\alpha}F_{\beta\gamma})\partial_p^\alpha\left(\Sigma^{\beta\gamma}f_s\right)\right]=0\,. \tag{11}$$

with $f_\pm=(V\pm A)/2$.

Recover massless limit

- Massive case: recovers non-relativistic magnetic dipole-moment tensor in local rest frame

$$\Sigma^{\mu\nu} = -\frac{1}{m}\epsilon^{\mu\nu\alpha\beta}p_\alpha n_\beta \, . \tag{12}$$

- Spin-vector n^μ satisfies $p^\mu n_\mu = 0$, $n^\mu n_\mu = -1$, can be decomposed as

$$m\, n^\mu = n_\parallel \sqrt{\frac{(p\cdot u)^2}{(p\cdot u)^2 - m^2}} \left(p^\mu - \frac{m^2}{p\cdot u} u^\mu \right) + m\, n_\perp^\mu \, , \tag{13}$$

where $p_\mu n_\perp^\mu = u_\mu n_\perp^\mu = 0$ and $n_\parallel^2 - n_\perp^\mu n_{\perp\mu} = 1$.

- $n_\parallel / n_\perp^\mu$ $\Rightarrow$ logitudinal/transverse polarization
  $\Rightarrow$ spin-components parallel/perpendicular to momentum in frame u^μ.

- Massless limit:

$$m\, n^\mu \to p^\mu, \quad \Sigma^{\mu\nu} \to \frac{1}{p\cdot u}\epsilon^{\mu\nu\alpha\beta}p_\alpha u_\beta \, . \tag{14}$$

- Results coincide with previously massless solutions:
  Y. Hidaka, S. Pu, and D.-L. Yang, Phys. Rev. D95, 091901 (2017),
  A. Huang, S. Shi, Y. Jiang, J. Liao, and P. Zhuang, Phys. Rev. D98, 036010 (2018),
  J.-H. Gao, Z.-T. Liang, Q. Wang, and X.-N. Wang, Phys. Rev. D98, 036019 (2018).

Thermal equilibrium

- Distribution function

$$f_s^{eq} = \left[\exp\left(\pi\cdot\beta - \beta\mu_s + s\frac{\hbar}{4}\Sigma^{\mu\nu}\omega_{\mu\nu}\right) + 1\right]^{-1} . \tag{15}$$

where π^μ is canonical momentum, β^μ is thermal velocity, β is inverse temperature, μ_s is spin-dependent chemical potential and $\omega_{\mu\nu}$ is thermal vorticity

- Contains spin-vorticity coupling.

- Constraints for the variables:

$$\begin{aligned} &\partial_{x\mu}\beta_\nu + \partial_{x\nu}\beta_\mu = 0 \,, \\ &\omega_{\mu\nu} = \text{const.}\,, \\ &\beta\mu_s = \text{const.}\,, \\ &p\cdot\nabla\Sigma^{\mu\nu} - F^{\alpha[\mu}\Sigma^{\nu]}_{\ \alpha} = 0 \,. \end{aligned} \tag{16}$$

- Agrees with previous work if $\mu_+ = \mu_- = \mu$ and EM fields vanish:
F. Becattini, V. Chandra, L. Del Zanna, and E. Grossi, Annals Phys. 338, 32 (2013).

Vector current

- Vector current up to first order in $\hbar$:

$$\mathcal{V}^\mu = \frac{2}{(2\pi\hbar)^3}\sum_s \Big[\delta(p^2-m^2)\left(p^\mu - s\,m\,\frac{\hbar}{2}\tilde{\omega}^{\mu\nu}n_\nu\partial_{\pi\cdot\beta}\right) + s\,m\,\hbar\tilde{F}^{\mu\nu}n_\nu\delta'(p^2-m^2) + s\frac{\hbar}{2m}\delta(p^2-m^2)\epsilon^{\mu\nu\alpha\beta}p_\nu\left(\nabla_\alpha n_\beta\right)\Big] f_s^{(0)}. \tag{17}$$

- Zeroth-order equilibrium distribution:

$$f_s^{(0)} = \left[\exp\left(\pi\cdot\beta - \beta\mu_s\right)+1\right]^{-1}. \tag{18}$$

- Analogue of Chiral Vortical Effect (CVE).

  D. T. Son and P. Surowka, Phys. Rev. Lett. 103, 191601 (2009).

- Analogue of Chiral Magnetic Effect (CME).

  D. E. Kharzeev, L. D. McLerran, and H. J. Warringa, Nucl. Phys. A803, 227 (2008).

- Current induced by inhomogeneity of magnetization direction.

Axial-vector current

- Axial-vector current up to first order in $\hbar$:

$$\mathcal{A}^{\mu} = \frac{2}{(2\pi\hbar)^3} \sum_s \Big[\delta(p^2 - m^2) \Big(s\, m\, n^{\mu} - \frac{\hbar}{2} \tilde{\omega}^{\mu\nu} p_{\nu} \partial_{\pi \cdot \beta} \Big) + \hbar \tilde{F}^{\mu\nu} p_{\nu} \delta'(p^2 - m^2) \Big] f_s^{(0)} - \frac{\hbar}{2} \epsilon^{\mu\nu\alpha\beta} p_{\nu} \Xi_{\alpha\beta} \delta(p^2 - m^2). \qquad (19)$$

- Analogue of Axial Chiral Vortical Effect (ACVE).
- Analogue of Chiral Separation Effect (CSE).

  D. E. Kharzeev, J. Liao, S. A. Voloshin and G. Wang, Prog. Part. Nucl. Phys. 88, 1 (2016).

- Undetermined n^{μ} and $\Xi^{\mu\nu}$: possible polarization from other source.

Fluid-dynamical quantities

- Net particle-number current:

$$J^\mu = \int dP\ p^\mu V + \frac{\hbar}{2}\partial_{x\nu} \int dP\ \Sigma^{\mu\nu} A + \frac{\hbar}{4} F^{\alpha\beta} \int dP\ \partial_p^\mu (\Sigma_{\alpha\beta} A) + \mathcal{O}(\hbar^2). \tag{20}$$

- Canonical energy-momentum tensor:

$$T^{\mu\nu}_{mat} = \int dP\ p^\mu p^\nu V + \frac{\hbar}{2}\partial_{x\alpha} \int dP\ p^\nu \Sigma^{\mu\alpha} A + \frac{\hbar}{4} g^{\mu\nu} F^{\alpha\beta} \int dP\ \Sigma_{\alpha\beta} A - \frac{\hbar}{2} F^\nu{}_\alpha \int dP\ \Sigma^{\mu\alpha} A + \frac{\hbar}{4} F^{\alpha\beta} \int dP\ p^\nu \partial_p^\mu (\Sigma_{\alpha\beta} A) + \mathcal{O}(\hbar^2). \tag{21}$$

Canonical energy-momentum tensor is in general not symmetric.
see E. Speranza's talk for further discussion.

- $dP \equiv d^4p\, \delta(p^2 - m^2)$.

Conclusion and outlook

- Conclusion
  - Solved covariant Wigner function up to order $\hbar$.
  - Derived kinetic equations for distribution and dipole-moment tensor.
  - Recovered previously known results in classical and massless limits.
  - Derived vector and axial-vector currents in thermal equilibrium and found analogues of massless chiral effects.
  - Derived particle-number current and energy-momentum tensor.

- Outlook
  - Include collisions between particles with spin.
  - Solve kinetic equations $\rightarrow$ method of moments.