Overview on chiral kinetic transports

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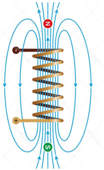

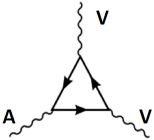

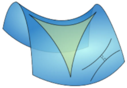

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Contents

Introduction

Chiral anomalous transports

- Chiral fermions + magnetic field $\Rightarrow$ chiral magnetic effect (CME) **(Kharzeev, McLerran, Warringa, Fukushima 2008; Son, Zhitnitsky 2004; $\cdots$)**:

$$\boldsymbol{J}_R = \frac{1}{4\pi^2}\mu_R \boldsymbol{B}, \qquad \boldsymbol{J}_L = -\frac{1}{4\pi^2}\mu_L \boldsymbol{B}$$

- Chiral fermions + fluid vorticity $\Rightarrow$ chiral vortical effect (CVE) **(Erdmenger etal 2008; Barnerjee etal 2008, Son, Surowka 2009; Landsteiner etal 2011)**:

$$\boldsymbol{J}_R = \frac{1}{4\pi^2}\mu_R^2 \boldsymbol{\omega} + \frac{T^2}{12}\boldsymbol{\omega}, \qquad \boldsymbol{J}_L = -\frac{1}{4\pi^2}\mu_L^2 \boldsymbol{\omega} - \frac{T^2}{12}\boldsymbol{\omega}$$

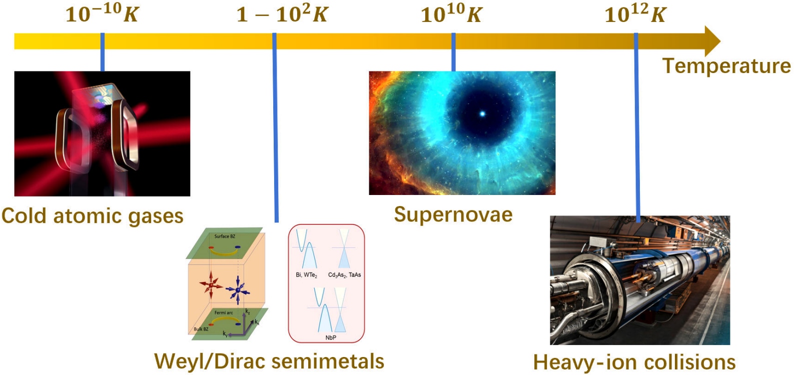
- Chiral fermions + Electric field $\Rightarrow$ anomalous Hall effect (AHE) **(see e.g.: Nagaosa 2010)**:

$$\boldsymbol{J}_R = \frac{1}{4\pi^2}\boldsymbol{b} \times \boldsymbol{E}, \qquad \boldsymbol{J}_L = \frac{1}{4\pi^2}\boldsymbol{b} \times \boldsymbol{E}$$

- $\cdots\cdots$

Chiral anomalous transports

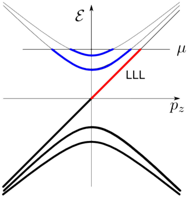
- Phenomenology: universal phenomena that may happen across a very broad hierarchy of scales.



- Many talks in the following days.

Chiral anomalous transports

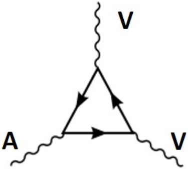
- Chiral anomalous transports are closely related to chiral anomaly.
  - For example, right-handed Weyl fermion in $\boldsymbol{B}$: The Landau Levels



$$\text{LLL} \Rightarrow \text{CME: } \boldsymbol{J} = \frac{\mu}{4\pi^2}\boldsymbol{B}$$
$$\Downarrow$$
$$\mu\frac{dn}{dt} = \boldsymbol{E}\cdot\boldsymbol{J} = \mu\frac{1}{4\pi^2}\boldsymbol{E}\cdot\boldsymbol{B}$$
$$\Downarrow$$

Chiral anomaly

(Adler 1969, Bell, Jackiw 1969, Nielsen, Ninomiya 1983)



- Symmetry and spontaneous symmetry breaking from UV to IR: Effective theories

Microscopic: Quantum field theory $\psi, \phi, A^\mu, \cdots$ $\rightarrow$ Mesoscopic: Kinetic theory $f(x, p)$ $\rightarrow$ Macroscopic: Hydrodynamics $u^\mu, T, n,$ NG modes

Chiral anomalous transports

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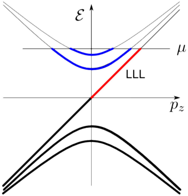
  - For example, right-handed Weyl fermion in $\boldsymbol{B}$: The Landau Levels



LLL $\Rightarrow$ CME: $\boldsymbol{J} = \dfrac{\mu}{4\pi^2}\boldsymbol{B}$

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Chiral anomaly

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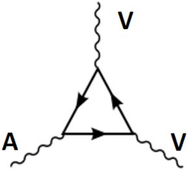


- Symmetry and anomalous symmetry breaking from UV to IR: Anomaly matching **('t Hooft 1980)**

Microscopic: Quantum field theory → Mesoscopic: Chiral kinetic theory → Macroscopic: Anomalous hydrodynamics

Chiral Kinetic Theory

- We focus on the kinetic theory: The chiral kinetic theory (CKT).
- Boltzmann-type equation with chiral anomaly encoded:

$$\partial_t f + \dot{\boldsymbol{x}} \cdot \boldsymbol{\nabla}_x f + \dot{\boldsymbol{p}} \cdot \boldsymbol{\nabla}_p f = \mathcal{C}[f]$$

  - Dynamical variable is $f(x, p)$ in phase space.
  - With background or dynamical electromagnetic (EM) fields and/or curved spacetime, $\cdots$.
  - Dilute enough to make a classical trajectory in phase space sensible. Has to contain $\hbar$ correction to encode quantum anomaly.
- Theoretically, CKT is very interesting: Berry monopole, side-jump, $\cdots$
- Phenomenologically, CKT provides an useful out-of-equilibrium framework: Quark-gluon plasma, Weyl/Dirac semimetals, electroweak gases, $\cdots$
- Incomplete list of references: **Son, Yamamoto 2012; Stephanov, Yin 2012; Gao, Liang, Pu, Wang, Wang 2012; Chen, Pu, Wang, Wang 2013; XGH 2015; Hidaka, Pu, Yang 2016; Gorbar, Miransky, Shovkovy, Sukhachov 2016; Muller, Venogopalan 2017; Huang, Shi, Jiang, Liao, Zhuang 2018; XGH, Sadofyev 2018; Carignano, Manuel, Torres-Rincon 2018; Liu, Gao, Mameda, XGH 2018; Lin, Shukla 2019;** $\cdots\cdots$

CKT from quantum mechanics

Semiclassical equations of motion in EM field

- To understand the Boltzmann-type equation, we need to first understand single-particle equations of motion (EOMs).
- Consider a Weyl fermion in EM field:

$$H = (\boldsymbol{p} - \boldsymbol{A}) \cdot \boldsymbol{\sigma} + A_0$$

  with $\boldsymbol{p}$ the canonical momentum, $\boldsymbol{\sigma}$ the Pauli matrix, and A^μ the gauge potential.
- Solve Heisenberg equations for particle branch at $O(\hbar)$ **(XGH 2015)**:

$$\sqrt{G}\dot{\boldsymbol{x}} = \boldsymbol{\nabla}_k \varepsilon + \hbar \boldsymbol{E} \times \boldsymbol{\Omega}_B + \hbar \boldsymbol{B}(\hat{\boldsymbol{k}} \cdot \boldsymbol{\Omega}_B) + O(\hbar^2)$$
$$\sqrt{G}\dot{\boldsymbol{k}} = \boldsymbol{E} + \boldsymbol{\nabla}_k \varepsilon \times \boldsymbol{B} + \hbar \boldsymbol{\Omega}_B(\boldsymbol{E} \cdot \boldsymbol{B}) + O(\hbar^2)$$

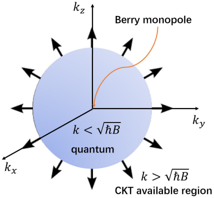
  - $\boldsymbol{k} = \boldsymbol{p} - \boldsymbol{A}$ is the kinetic momentum and $\hat{\boldsymbol{k}} = \boldsymbol{k}/|\boldsymbol{k}|$.
  - $\boldsymbol{\Omega}_B = \hat{\boldsymbol{k}}/(2|\boldsymbol{k}|^2)$ is called Berry curvature, $\sqrt{G} = 1 + \hbar \boldsymbol{B} \cdot \boldsymbol{\Omega}_B$ (see next slide).
  - $\varepsilon = |\boldsymbol{k}|(1 - \hbar \boldsymbol{B} \cdot \boldsymbol{\Omega}_B)$ is the single-particle energy.
- Similar EOMs was first derived for semiclassical motion of electron wave packet in solid, by applying adiabatic approximation to Schrodinger equation **(Chang, Niu 1995; Sundaram, Niu 1999).**

Chiral kinetic equation in EM field

- Denote $(\boldsymbol{x}, \boldsymbol{k})$ by z_a (a=1-6). The EOMs become:

$$G_{ab}\dot{z}_b = -\partial_a h,\ h = \varepsilon + A_0,\ G_{ab} = \begin{pmatrix} -\epsilon_{ijk}B_k & \delta_{ij} \\ -\delta_{ij} & \hbar\epsilon_{ijk}\Omega_{Bk} \end{pmatrix}$$

  - Semiclassical motion of a spinful fermion $\equiv$ classical motion of a spinless particle with Hamiltonian h in phase space equipped by symplectic form G_{ab}. Phase space measure $\sqrt{\mathrm{Det}G_{ab}} = 1 + \hbar\boldsymbol{B}\cdot\boldsymbol{\Omega}_B$
  - Dual roles of $\boldsymbol{B}$ and $\boldsymbol{\Omega}_B$: x-space magnetic field $\boldsymbol{B}(x) = \boldsymbol{\nabla}_x \times \boldsymbol{A}(x)$, k-space magnetic field $\boldsymbol{\Omega}_B(\boldsymbol{k}) = \boldsymbol{\nabla}_k \times a(\boldsymbol{k})$.
  - "Gauge field" $\boldsymbol{a}(\boldsymbol{k}) = -iu^\dagger\boldsymbol{\nabla}_k u$ is the Berry phase where u is positive-energy eigenfunction of H.
  - Valid when $|\boldsymbol{k}| > \sqrt{\hbar|\boldsymbol{B}|}$: the adiabatic condition.



- We can write down the Boltzmann equation:

$$\partial_t f + \dot{\boldsymbol{x}}\cdot\boldsymbol{\nabla}_x f + \dot{\boldsymbol{k}}\cdot\boldsymbol{\nabla}_k f = C[f],\ \ C[f] = C_0[f] + \hbar C_1[f]$$

- The CME:

$$\boldsymbol{J} = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\sqrt{G}\dot{\boldsymbol{x}} f \Rightarrow \boldsymbol{J}_{\mathrm{CME}} = \hbar\boldsymbol{B}\int\frac{d^3\boldsymbol{k}}{(2\pi)^3}(\hat{\boldsymbol{k}}\cdot\boldsymbol{\Omega}_B)f = \frac{\mu}{4\pi^2}\boldsymbol{B}\Big|_{\mathrm{eq},T=0}$$

Semiclassical EOMs in rotation

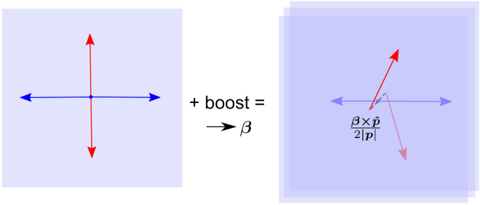
- To understand CVE in CKT, consider a Weyl fermion in a rotating frame:
$$H = \boldsymbol{p}\cdot\boldsymbol{\sigma} - \boldsymbol{\omega}\cdot\left(\boldsymbol{x}\times\boldsymbol{p} + \frac{\hbar}{2}\boldsymbol{\sigma}\right)$$
- Solve Heisenberg equations for particle branch $O(\hbar)$ **(XGH and Sadofyev 2018)**:
$$\dot{\boldsymbol{x}} = \hat{\boldsymbol{p}} + \boldsymbol{x}\times\boldsymbol{\omega} = \boldsymbol{\nabla}_p\varepsilon + \hbar\boldsymbol{p}\times\boldsymbol{\Omega}_B$$
$$\dot{\boldsymbol{p}} = \boldsymbol{p}\times\boldsymbol{\omega} = -\boldsymbol{\nabla}_p\varepsilon$$
  - $\varepsilon = |\boldsymbol{p}| - \boldsymbol{\omega}\cdot(\boldsymbol{x}\times\boldsymbol{p}) - \frac{\hbar}{2}\boldsymbol{\omega}\cdot\hat{\boldsymbol{p}}$ is energy in rotating frame, $\dot{\boldsymbol{x}}$ is velocity in rotating frame, $\boldsymbol{p}$ is momentum in inertial frame.
  - The Coriolis force and centrifugal force: $\ddot{\boldsymbol{x}} = -2\boldsymbol{\omega}\times\dot{\boldsymbol{x}} - \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{x})$
  - Surprisingly, the EOMs do not have $\hbar$ order correction. The phase space is as usual and $\sqrt{G} = 1$.
- This suggests a collisionless kinetic equation up to $O(\hbar)$:
$$\partial_t f + \dot{\boldsymbol{x}}\cdot\boldsymbol{\nabla}_x f + \dot{\boldsymbol{p}}\cdot\boldsymbol{\nabla}_p f = 0$$
- How to get CVE: the magnetization current. At equilibrium, it reproduces the correct CVE current. **(Chen, Son, Stephanov, Yin 2014)**:
$$\boldsymbol{J} = \int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\dot{\boldsymbol{x}}f + \boldsymbol{\nabla}\times\boldsymbol{M},\ \boldsymbol{M} = \hbar\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}|\boldsymbol{p}|\boldsymbol{\Omega}_B f$$

Semiclassical EOMs in rotation

- At equilibrium, at linear order of $\boldsymbol{\omega}$:

$$\boldsymbol{J}_{\text{CVE}} = -\frac{\hbar}{2}\boldsymbol{\omega}\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\left(\frac{1}{3}+\frac{2}{3}\right)f'(|\boldsymbol{p}|)$$

- The inclusion of magnetization current is also important to make J^μ a Lorentz vector: The side-jump effect



$\boldsymbol{P}_{\text{in}} = \boldsymbol{P}_{\text{out}} = 0$
$\boldsymbol{S}_{\text{in}} = \boldsymbol{S}_{\text{out}} = 0$
$\boldsymbol{L}_{\text{in}} = \boldsymbol{L}_{\text{out}} = 0$

Cf. Stephanov

$\boldsymbol{P}_{\text{in}} = \boldsymbol{P}_{\text{out}}$
$\boldsymbol{S}_{\text{in}} = 0, \quad \boldsymbol{S}_{\text{out}} = \rightarrow \mathcal{O}(\hbar)$
$\boldsymbol{L}_{\text{in}} = 0, \quad \boldsymbol{L}_{\text{out}} = \leftarrow$

"Side-jump"

- The angular momentum conservation enforces $\hbar$-correction to Lorentz boost **(Chen, Son, Stephanov, Yin 2014)**:

$$\delta_\beta \boldsymbol{x} = \boldsymbol{\beta}t + \hbar|\boldsymbol{p}|\boldsymbol{\beta}\times\boldsymbol{\Omega}_B \; , \; \delta_\beta t = \boldsymbol{\beta}\cdot\boldsymbol{x} \; , \; \delta_\beta \boldsymbol{p} = \varepsilon\boldsymbol{\beta}$$

Semiclassical EOMs in rotation

- Consider a local Lorentz transformation with $\boldsymbol{\beta} = -\boldsymbol{\omega} \times \boldsymbol{x}$, near the rotating axis:

$$\boldsymbol{x} \to \boldsymbol{y} = \boldsymbol{x} + \hbar |\boldsymbol{p}| \boldsymbol{\Omega}_B \times (\boldsymbol{\omega} \times \boldsymbol{x}) - t\boldsymbol{\omega} \times \boldsymbol{x}, \quad \boldsymbol{p} \to \boldsymbol{k} = \boldsymbol{p} + |\boldsymbol{p}| \boldsymbol{x} \times \boldsymbol{\omega}$$

- This can be considered as phase space coordinate transformation: $(\boldsymbol{x}, \boldsymbol{p}) \to (\boldsymbol{y}, \boldsymbol{k})$. The EOMs is $(\boldsymbol{y}, \boldsymbol{k})$ **(XGH and Sadofyev 2018; Dayi, Kilincarslan, Yunt 2018)**:

$$\sqrt{G}\dot{\boldsymbol{y}} = \boldsymbol{\nabla}_k \varepsilon + 2\hbar |\boldsymbol{k}| \; \boldsymbol{\omega}(\hat{\boldsymbol{k}} \cdot \boldsymbol{\Omega}_B) + O(\boldsymbol{x}, \boldsymbol{\omega}^2)$$
$$\sqrt{G}\dot{\boldsymbol{k}} = 2\boldsymbol{\nabla}_k \varepsilon \times |\boldsymbol{k}|\boldsymbol{\omega} + O(\boldsymbol{x}, \boldsymbol{\omega}^2)$$

  - Energy $\varepsilon = |\boldsymbol{k}| - \hbar \boldsymbol{\omega} \cdot \hat{\boldsymbol{k}}/2$. Phase space measure $\sqrt{G} = 1 + 2\hbar |\boldsymbol{k}| \boldsymbol{\omega} \cdot \boldsymbol{\Omega}_B$ is the transformation Jacobian.
  - Similarity with EOMs in $\boldsymbol{B}$: $\boldsymbol{B} \leftrightarrow 2|\boldsymbol{k}|\boldsymbol{\omega}$ (Lorentz force $\boldsymbol{F} = \dot{\boldsymbol{x}} \times \boldsymbol{B}$ $\leftrightarrow$ Coriolis force $\boldsymbol{F} = 2|\boldsymbol{k}|\dot{\boldsymbol{x}} \times \boldsymbol{\omega}$), except for ε where $\boldsymbol{B} \leftrightarrow |\boldsymbol{k}|\boldsymbol{\omega}$ because Landre $g = 2$.

- The collisionless kinetic equation up to $O(\hbar, \boldsymbol{\omega}, \boldsymbol{x}^0)$:

$$\partial_t f + \dot{\boldsymbol{x}} \cdot \boldsymbol{\nabla}_x f + \dot{\boldsymbol{k}} \cdot \boldsymbol{\nabla}_k f = 0$$

- The CVE current which gives the same result at equilibrium:

$$\boldsymbol{J}_{\mathrm{CVE}} = 2\hbar \; \boldsymbol{\omega} \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} |\boldsymbol{k}| (\hat{\boldsymbol{k}} \cdot \boldsymbol{\Omega}_B) f$$

CKT from quantum field theory

Wigner function and $\hbar$ expansion

- Construction from EOMs has more transparent physical meaning. Derivation from QFT is more systematic and in a sense powerful.
- Consider massless Dirac fermions in curved spacetime and EM field. The Wigner function is:

$$W_{\alpha\beta}(x,p) = \int d^4y[-g(x)]^{1/2} e^{-ip\cdot y/\hbar} \langle \left[\bar{\psi}(x) e^{y\cdot \overleftarrow{D}/2}\right]_\beta \left[e^{-y\cdot D/2}\psi(x)\right]_\alpha \rangle$$

  - The position of the spacetime manifold is x^μ, the position in the tangent space of x is y^μ, p_μ is in the cotangent space of x. The whole phase space is the cotangent bundle.
  - D_μ is the derivative in tangent or cotangent bundle and $U(1)$ bundle lifted from the usual covariant derivative ∇_μ in the spacetime manifold:

$$D_\mu = \nabla_\mu - \Gamma^\lambda_{\mu\nu} y^\nu \frac{\partial}{\partial y^\lambda} + iA_\mu/\hbar \;\text{ for tangent bundle}$$

$$D_\mu = \nabla_\mu + \Gamma^\lambda_{\mu\nu} p_\lambda \frac{\partial}{\partial p_\nu} + iA_\mu/\hbar \;\text{ for cotangent bundle}$$

  - The advantage of using D_μ: y^ν and p_ν are parallelly transported:

$$D_\mu y^\nu = 0 = D_\mu p_\nu$$

  - Flat spacetime version, put $g_{\mu\nu}$ to be $\eta_{\mu\nu}$ **(Heiz 1983; Vasak, Gyulassy, Elze 1987)**

Wigner function and $\hbar$ expansion

- Dynamics of Wigner function obtained by applying Dirac equations:

$$i\hbar\gamma\cdot\nabla\psi(x)=i\hbar\bar{\psi}(x)\overleftarrow{\nabla}\cdot\gamma=0$$

- Up to $\hbar^2$ order, it is given by: **(See talk by Yu-Chen Liu)**

$$\gamma^{\mu}\left(\frac{i\hbar}{2}\Delta_{\mu}+\Pi_{\mu}\right)W=\frac{i\hbar^2}{32}\gamma^{\mu}\left[R_{\mu\nu ab}+i\frac{\hbar}{6}\partial_p\cdot(\nabla R_{\mu\nu ab})\right]\frac{\partial}{\partial p_{\nu}}\left[W,\sigma^{ab}\right]$$

- In the above:

$$\Delta_{\mu}=\nabla_{\mu}+\left(-F_{\mu\lambda}+\Gamma^{\nu}_{\mu\lambda}p_{\nu}\right)\frac{\partial}{\partial p_{\lambda}}-\frac{\hbar^2}{12}(\nabla_{\rho}R_{\mu\nu})\partial_p^{\rho}\partial_p^{\nu}+\frac{\hbar^2}{8}R^{\rho}{}_{\sigma\mu\nu}\partial_p^{\nu}\partial_p^{\sigma}D_{\rho}$$
$$-\frac{\hbar^2}{24}(\nabla_{\lambda}R^{\rho}{}_{\sigma\mu\nu})\partial_p^{\nu}\partial_p^{\sigma}\partial_p^{\lambda}p_{\rho}+\frac{\hbar^2}{24}(\nabla_{\alpha}\nabla_{\beta}F_{\mu\nu}+4R^{\rho}{}_{\alpha\mu\nu}F_{\beta\rho})\partial_p^{\nu}\partial_p^{\alpha}\partial_p^{\beta},$$
$$\Pi_{\mu}=p_{\mu}-\frac{\hbar^2}{12}\left(\nabla_{\rho}F_{\mu\nu}\right)\frac{\partial^2}{\partial p_{\nu}\partial p_{\rho}}+\frac{\hbar^2}{24}R^{\rho}{}_{\sigma\mu\nu}\frac{\partial^2}{\partial p_{\nu}\partial p_{\sigma}}p_{\rho}+\frac{\hbar^2}{4}R_{\mu\nu}\frac{\partial}{\partial p_{\nu}}.$$

- In the above:
  - Electromagnetic tensor: $F_{\mu\nu}=2\partial_{[\mu}A_{\nu]}$
  - Riemann curvature 1: $R^{\rho}{}_{\sigma\mu\nu}=2\partial_{[\nu}\Gamma^{\rho}_{\mu]\sigma}+2\Gamma^{\rho}_{\lambda[\nu}\Gamma^{\lambda}_{\mu]\sigma}$
  - Riemann curvature 2: $R_{\mu\nu ab}=R^{\rho}{}_{\nu\sigma\lambda}g_{\rho\mu}e_a^{\sigma}e_b^{\lambda}$
  - Vierbein field: e_a^{μ}
  - Spacetime index: $\mu,\nu,\rho,\cdots$. Local Lorentz index: $a,b,c,\cdots$.

Wigner function and $\hbar$ expansion

- Dirac decomposition:

$$W(x,p) = \frac{1}{4}\left[\mathcal{F} + i\gamma^5\mathcal{P} + \gamma^\mu\mathcal{V}_\mu + \gamma^5\gamma^\mu\mathcal{A}_\mu + \frac{1}{2}\sigma^{\mu\nu}\mathcal{S}_{\mu\nu}\right]$$

- Focus on $\mathcal{V}_\mu$ and $\mathcal{A}_\mu$ or equivalently $\mathcal{R}_\mu/\mathcal{L}_\mu = (1/2)(\mathcal{V}_\mu \pm \mathcal{A}_\mu)$:

$$\Delta_\mu\mathcal{R}^\mu = \frac{\hbar^2}{24}(\nabla_\rho R_{\mu\nu})\partial_p^\rho\partial_p^\mu\mathcal{R}^\nu + O(\hbar^3)$$

$$\Pi_\mu\mathcal{R}^\mu = \frac{\hbar^2}{8}R_{\mu\nu}\partial_p^\mu\mathcal{R}^\nu + O(\hbar^3)$$

$$\hbar\Delta_{[\mu}\mathcal{R}_{\nu]} - \varepsilon_{\mu\nu\rho\sigma}\Pi^\rho\mathcal{R}^\sigma = -\frac{\hbar^2}{16}\varepsilon_{\mu\nu\alpha\beta}R^{\alpha\beta\rho\sigma}\partial_\rho^p\mathcal{R}_\sigma + O(\hbar^3)$$

  - $\mathcal{L}_\mu$ satisfies similar equations.
  - The last two equations fix $\mathcal{R}_\mu$ up to a scalar function (distribution function) and frame defining conditions.
  - The first equation then gives kinetic equation for distribution function.
  - The dynamic equations for $\mathcal{F}, \mathcal{P}, \mathcal{S}^{\mu\nu}$ can similarly obtained.

Covariant chiral kinetic equation

- The solutions to the last two equations:

$$\mathcal{R}^{\mu}=2\pi\delta(p^2)\left[p^{\mu}-\frac{\hbar}{p^2}\tilde{F}^{\mu\nu}p_{\nu}+\hbar\Sigma^{\mu\nu}\Delta_{\nu}\right]f+O(\hbar^2)$$

  - $\Delta_{\mu}=\nabla_{\mu}+\left(-F_{\mu\lambda}+\Gamma^{\nu}_{\mu\lambda}p_{\nu}\right)\frac{\partial}{\partial p_{\lambda}}$.
  - $f=f_0+f_1+O(\hbar^2)$ is the distribution function.
  - Spin tensor: $\Sigma^{\mu\nu}=\epsilon^{\mu\nu\rho\sigma}p_{\rho}n_{\sigma}/(2p\cdot n)$.
  - n^{μ} is a frame choosing vector.
  - When $n^{\mu}=g_{00}^{-1}\delta_0^{\mu}$, rest frame of n: $\Sigma_{ij}=\epsilon_{0ijk}p^k/(2p_0)$
  - Massless particle, spin is slaved to momentum.
- The collisionless kinetic equation for f up to $O(\hbar)$:

$$\delta\left(p^2-\hbar F_{\alpha\beta}\Sigma^{\alpha\beta}\right)\left[p\cdot\Delta+\hbar\left(\frac{n_{\mu}\tilde{F}^{\mu\nu}}{p\cdot n}+\Delta_{\mu}\Sigma^{\mu\nu}\right)\Delta_{\nu}+\frac{\hbar}{2}\Sigma^{\mu\nu}\left(\nabla_{\rho}F_{\mu\nu}+p_{\lambda}R^{\lambda}{}_{\rho\mu\nu}\right)\partial_p^{\rho}\right]f=0$$

  - This equation is invariant under general coordinate transformation.
  - This equation is invariant under local Lorentz transformation.
  - In Minkowski spacetime, $\Gamma^{\lambda}{}_{\mu\nu}=0=R_{\nu}{}^{\lambda\alpha\beta}$, it reduces to the kinetic equation in pure electromagnetic field. **(Hidaka, Pu, Yang 2017; Huang, Shi, Jiang, Liao, Zhuang 2018)**

Discussions

Discussions

- (1) Integrating out energy $\int d\varepsilon$, with $\varepsilon = K^\mu p_\mu$ (K^μ a timelike Killing vector), one can obtain a 3D Boltzmann equation. The CKT in EM field in last part corresponds to $K^\mu = \delta_0^\mu$ in flat spacetime with $n^\mu = K^\mu$. The two CKTs for CVE in last part correspond to $K^\mu = \delta_0^\mu$ in rotating frame, with one $n^\mu = (1, \boldsymbol{x} \times \boldsymbol{\omega})$ and another $n^\mu = (g_{00})^{-1} K^\mu$ and $k^i = g^{i\mu} p_\mu$. **(Liu, Gao, Mameda, Huang 2018)**
- (2) One can determine the equilibrium distribution by requiring f to depend on only linear combinations of collisional conserved quantities, $1, p_\mu, \Sigma^{\mu\nu}$. For Fermi-Dirac distribution, one can reproduce the correct CME, CVE currents. One can show that the physical current is independent of spin-frame vector n^μ. **(Gao, Wang 2018; Liu, Gao, Mameda, Huang 2018)**
- (3) A full treatment of the collision term is not done. However, some interesting features were discussed, e.g., the modification of side-jump current $\delta\mathcal{R}^\mu = 2\pi\hbar\delta(p^2)\Sigma^{\mu\nu}C_\nu[f]$ with C_ν determined by collision kernel. **(Chen, Stephanov, Son 2015; Hidaka, Pu, Yang 2016)**

Discussions

- (4) In the derivation with Wigner function, we were essentially treating "spin currents". For massless fermion, spin is slaved with momentum, two independent CKTs for particle number and helicity. For massive fermions, spin is independent of p^μ, we need 4 equations for particle number and 3 spin degree of freedom. **(See talks by: Jianhua Gao, Ziyue Wang, Xinli Sheng, Lixin Yang, Shu Lin)**
- (5) CKT from other approaches. The worlkine formalism**(Mueller, Venogopolan 2017)**, The high-density effective theory**(Son, Yamamoto 2012; Lin, Shukla 2019)**, The on-shell effective theory**(Carignano, Manuel, Torres-Rincon 2018) (See talks by: Shudla, Mueller)**
- (6) Applications to real-time anomalous transport phenomena and spin polarization in quark-gluon plasma, electroweak plasma, $\cdots$. **(See talks by: Cheming Ko, Qun Wang, Wenhao Zhou, Jun Xu, Shi Pu)**
- (7) Out looks: CKT beyond leading approximation?. Gravitoelectromagnetism and gravitational effects? CKT in other spacetime dimensions and for higher spins? Merge CKT with parton cascade? A full treatment of the collision term? Derivation of anomalous hydrodynamics from CKT? $\cdots$

Thank you!