Spin tensor and its role in non-equilibrium thermodynamics

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Outline

- What is the spin tensor?
- Why and when do we need the spin tensor in hydrodynamics?
- Is hydrodynamics invariant under different choices of energy-momentum and spin tensor?
- Can we find observables which are sensitive to different choices of energy-momentum and spin tensor?

Canonical energy-momentum and spin tensors

Lagrangian $\Rightarrow$ Poincaré symmetry $\Rightarrow$ Noether's th. $\Rightarrow$ Conservation laws

- **Conservation of energy and momentum:**
  Canonical energy-momentum tensor $\widehat{T}_C^{\mu\nu}(x)$

$$\partial_\mu \widehat{T}_C^{\mu\nu}(x) = 0$$

- **Conservation of total angular momentum:**
  Canonical total angular momentum tensor ("orbital"+"spin")

$$\widehat{J}_C^{\lambda,\mu\nu}(x) = x^\mu \widehat{T}_C^{\lambda\nu}(x) - x^\nu \widehat{T}_C^{\lambda\mu}(x) + \widehat{S}_C^{\lambda,\mu\nu}(x)$$

$$\partial_\lambda \widehat{J}_C^{\lambda,\mu\nu}(x) = 0 \Longrightarrow \partial_\lambda \widehat{S}_C^{\lambda,\mu\nu}(x) = \widehat{T}_C^{\nu\mu}(x) - \widehat{T}_C^{\mu\nu}(x)$$

Pseudo-gauge transformations

- Total energy-momentum and angular momentum must be fixed

$$\widehat{P}^{\mu} = \int d^3\Sigma_\lambda \, \widehat{T}^{\lambda\mu}(x) \qquad \widehat{J}^{\mu\nu} = \int d^3\Sigma_\lambda \, \widehat{J}^{\lambda,\mu\nu}(x)$$

- Densities are not uniquely defined
  $\Longrightarrow$ **Pseudo-gauge transformations:**
  (F. W. Hehl, Rep. Mat. Phys. 9, 55 (1976))

$$\widehat{T}'^{\,\mu\nu}(x) = \widehat{T}^{\mu\nu}(x) + \tfrac{1}{2}\partial_\lambda \left[\widehat{\Phi}^{\lambda,\mu\nu}(x) + \widehat{\Phi}^{\mu,\nu\lambda}(x) + \widehat{\Phi}^{\nu,\mu\lambda}(x)\right]$$

$$\widehat{S}'^{\,\lambda,\mu\nu} = \widehat{S}^{\lambda,\mu\nu}(x) - \widehat{\Phi}^{\lambda,\mu\nu}(x)$$

Leave $\widehat{P}^{\mu}$ and $\widehat{J}^{\mu\nu}$ invariant

- Belinfante's case ($\widehat{\Phi}^{\lambda,\mu\nu}(x) = \widehat{S}_C^{\lambda,\mu\nu}(x)$)

$$\widehat{T}_B^{\mu\nu}(x) = \widehat{T}_C^{\mu\nu}(x) + \tfrac{1}{2}\partial_\lambda \left[\widehat{S}_C^{\lambda,\mu\nu}(x) + \widehat{S}_C^{\mu,\nu\lambda}(x) + \widehat{S}_C^{\nu,\mu\lambda}(x)\right]$$

$$\widehat{S}_B^{\lambda,\mu\nu}(x) = 0$$

Example - Dirac theory

- Dirac Lagrangian

$$\mathcal{L}(x) = \frac{i}{2}\bar{\widehat{\psi}}(x)\gamma^\mu \overleftrightarrow{\partial}_\mu \widehat{\psi}(x) - m\bar{\widehat{\psi}}(x)\widehat{\psi}(x)$$

- Canonical case

$$\widehat{T}^{\mu\nu}_C(x) = \frac{i}{2}\bar{\widehat{\psi}}(x)\gamma^\mu \overleftrightarrow{\partial}^\nu \widehat{\psi}(x) - g^{\mu\nu}\mathcal{L}(x)$$

$$\widehat{S}^{\lambda,\mu\nu}_C(x) = \frac{1}{4}\bar{\widehat{\psi}}(x)(\gamma^\lambda \sigma^{\mu\nu} + \sigma^{\mu\nu}\gamma^\lambda)\widehat{\psi}(x)$$

with $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$

- Belinfante case

$$\widehat{T}^{\mu\nu}_B(x) = \frac{i}{4}\bar{\widehat{\psi}}(x)(\gamma^\mu \overleftrightarrow{\partial}^\nu + \gamma^\nu \overleftrightarrow{\partial}^\mu)\widehat{\psi}(x) - g^{\mu\nu}\mathcal{L}(x)$$

$$\widehat{S}^{\lambda,\mu\nu}_B(x) = 0$$

Local equilibrium - Belinfante

- Maximization of entropy

$$S = -\mathrm{tr}(\widehat{\rho}_B \log \widehat{\rho}_B)$$

- Constraints on energy, momentum and charge

$$n_\mu \mathrm{tr}\left[\widehat{\rho}_B\, \widehat{T}_B^{\mu\nu}(x)\right] = n_\mu T_B^{\mu\nu}(x), \qquad n_\mu \mathrm{tr}\left[\widehat{\rho}_B\, \widehat{j}^{\mu}(x)\right] = n_\mu j^{\mu}(x)$$

n^μ - vector orthogonal to hypersurface Σ

- Density operator

$$\widehat{\rho}_B = \frac{1}{Z} \exp\left[-\int_\Sigma d\Sigma_\mu \left(\widehat{T}_B^{\mu\nu}(x)\beta_\nu(x) - \zeta(x)\widehat{j}^{\mu}(x)\right)\right],$$

β_μ - Lagrange multiplier for conservation of energy and momentum
ζ - Lagrange multiplier for conservation of charge

Do we need a constraint for the conservation of total angular momentum?

$$n_\mu \mathrm{tr}\left[\widehat{\rho}_B\, \widehat{J}_B^{\mu,\lambda\nu}(x)\right] = n_\mu \mathrm{tr}\left[\widehat{\rho}_B \left(x^\lambda\, \widehat{T}_B^{\mu\nu}(x) - x^\nu\, \widehat{T}_B^{\mu\lambda}(x)\right)\right] = n_\mu J_B^{\mu,\lambda\nu}(x)$$

No, it is redundant (it follows from constraint on energy and momentum)

Global equilibrium - Belinfante

- Density operator must be stationary

$$\partial_\mu \left(\widehat{T}_B^{\mu\nu} \beta_\nu - \zeta \widehat{j}^\mu \right) = \widehat{T}_B^{\mu\nu} (\partial_\mu \beta_\nu) - (\partial_\mu \zeta) \widehat{j}^\mu = 0$$

- Global equilibrium conditions

$$\zeta = \text{constant}$$
$$\partial_\mu \beta_\nu + \partial_\nu \beta_\mu = 0$$
$$\beta_\nu = b_\nu + \Omega_{\nu\lambda} x^\lambda$$
$$\Omega_{\mu\nu} = \text{constant}$$

Local equilibrium - Canonical

- Maximization of entropy

$$S = -\mathrm{tr}(\widehat{\rho}_C \log \widehat{\rho}_C)$$

- Constraints on energy, momentum and charge

$$n_\mu \mathrm{tr}\left[\widehat{\rho}_C\, \widehat{T}^{\mu\nu}_C(x)\right] = n_\mu T^{\mu\nu}_C(x), \qquad n_\mu \mathrm{tr}\left[\widehat{\rho}_C\, \widehat{j}^{\mu}(x)\right] = n_\mu j^{\mu}(x)$$

Spin tensor $\Longrightarrow$ Constraint on total angular momentum

$$n_\mu \mathrm{tr}\left(\widehat{\rho}_C\, \widehat{J}^{\mu,\lambda\nu}_C\right) = n_\mu \mathrm{tr}\left[\widehat{\rho}_C\left(x^\lambda\, \widehat{T}^{\mu\nu}_C - x^\nu\, \widehat{T}^{\mu\lambda}_C + S^{\mu,\lambda\nu}_C\right)\right] = n_\mu J^{\mu,\lambda\nu}_C$$

- Density operator

$$\begin{aligned}\widehat{\rho}_C =& \frac{1}{Z}\exp\left[-\int_\Sigma d\Sigma_\mu\left(\widehat{T}^{\mu\nu}_C(x) b_\nu(x) - \frac{1}{2}\widehat{J}^{\mu,\lambda\nu}_C(x)\Omega_{\lambda\nu}(x) - \zeta(x)\widehat{j}^{\mu}(x)\right)\right]\\ =& \frac{1}{Z}\exp\left[-\int_\Sigma d\Sigma_\mu\left(\widehat{T}_C(x)^{\mu\nu}\beta_\nu(x) - \frac{1}{2}\widehat{S}^{\mu,\lambda\nu}_C(x)\Omega_{\lambda\nu}(x) - \zeta(x)\widehat{j}^{\mu}(x)\right)\right]\end{aligned}$$

$\Omega_{\lambda\nu}$ - (Antisymmetric) Lagrange multiplier for conservation of total angular momentum

Global equilibrium - Canonical

- Asymmetric EM tensor $\widehat{T}_C^{\mu\nu} = \widehat{T}_S^{\mu\nu} + \widehat{T}_A^{\mu\nu}$ with $\widehat{T}_S^{\mu\nu} = \widehat{T}_S^{\nu\mu}$, $\widehat{T}_A^{\mu\nu} = -\widehat{T}_A^{\nu\mu}$
- Density operator must be stationary

$$\frac{1}{2}\widehat{T}_S^{\mu\nu}(\partial_\mu\beta_\nu+\partial_\nu\beta_\mu)+\frac{1}{2}\widehat{T}_A^{\mu\nu}(\partial_\mu\beta_\nu-\partial_\nu\beta_\mu)-\frac{1}{2}(\partial_\mu\widehat{S}_C^{\mu,\lambda\nu})\Omega_{\lambda\nu}-\frac{1}{2}\widehat{S}_C^{\mu,\lambda\nu}(\partial_\mu\Omega_{\lambda\nu}) = 0$$

- Global equilibrium conditions:

$$\zeta = \text{constant}$$
$$\Omega_{\mu\nu} = \text{constant}$$
$$\partial_\mu\beta_\nu+\partial_\nu\beta_\mu = 0$$
$$\Omega_{\mu\nu} = -\frac{1}{2}(\partial_\mu\beta_\nu - \partial_\nu\beta_\mu) \quad \text{thermal vorticity}$$
$$\beta_\nu = b_\nu + \Omega_{\nu\lambda}x^\lambda$$

We used $\partial_\mu\widehat{S}_C^{\mu,\lambda\nu} = -2\widehat{T}_A^{\lambda\nu}$

Observables

- Consequences on some measurable quantities?

$$\mathrm{tr}(\widehat{\rho}_B\, \widehat{O}) = \mathrm{tr}(\widehat{\rho}_C\, \widehat{O})?$$

Question: Is $\widehat{\rho}_B = \widehat{\rho}_C$?

In global equilibrium yes!
In local equilibrium in general not

Local equilibrium - Canonical vs Belinfante

- Start with Canonical

$$\widehat{\rho}_C = \frac{1}{Z}\exp\left[-\int_\Sigma d\Sigma_\mu \left(\widehat{T}_C^{\mu\nu}\beta_\nu - \frac{1}{2}\widehat{S}_C^{\mu,\lambda\nu}\Omega_{\lambda\nu} - \zeta\widehat{j}^\mu\right)\right]$$

- PS to Belinfante: $\widehat{T}_B^{\mu\nu} = \widehat{T}_C^{\mu\nu} + \frac{1}{2}\partial_\lambda(\widehat{S}_C^{\lambda,\,\mu\nu} + \widehat{S}_C^{\mu,\,\nu\lambda} + \widehat{S}_C^{\nu,\,\mu\lambda}), \quad \widehat{S}_B^{\lambda,\,\mu\nu} = 0$
- Canonical density operator becomes

$$\widehat{\rho}_C = \frac{1}{Z}\exp\left[-\int_\Sigma d\Sigma_\mu \left(\widehat{T}_B^{\mu\nu}\beta_\nu - \frac{1}{2}(\Omega_{\lambda\nu} - \varpi_{\lambda\nu})\widehat{S}_C^{\mu,\lambda\nu} + \frac{1}{2}\xi_{\lambda\nu}\left(\widehat{S}_C^{\lambda,\mu\nu} + \widehat{S}_C^{\nu,\mu\lambda}\right) - \zeta\widehat{j}^\mu\right)\right]$$

with $\varpi_{\lambda\nu} \equiv \frac{1}{2}(\partial_\nu\beta_\lambda - \partial_\lambda\beta_\nu), \qquad \xi_{\lambda\nu} \equiv \frac{1}{2}(\partial_\nu\beta_\lambda + \partial_\lambda\beta_\nu)$

- When is $\rho_C = \rho_B$?

$$\widehat{\rho}_B = \frac{1}{Z}\exp\left[-\int_\Sigma d\Sigma n_\mu \left(\widehat{T}_B^{\mu\nu}\beta_\nu - \zeta\widehat{j}^\mu\right)\right]$$

1. β_μ is the same in both cases
2. $\Omega_{\lambda\nu}$ coincides with thermal vorticity, $\Omega_{\lambda\nu} = \varpi_{\lambda\nu} \equiv \frac{1}{2}(\partial_\nu\beta_\lambda - \partial_\lambda\beta_\nu)$
3. $\xi_{\lambda\nu} \equiv \frac{1}{2}(\partial_\nu\beta_\lambda + \partial_\lambda\beta_\nu) = 0$ or $\widehat{S}_C^{\lambda,\mu\nu} + \widehat{S}_C^{\nu,\mu\lambda} = 0$

Equivalence in global equilibrium!

Polarization in relativistic heavy-ion collisions

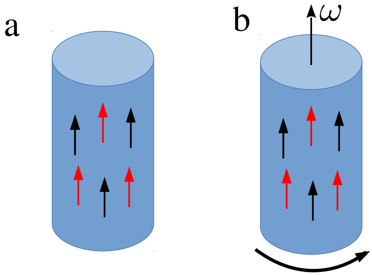
- Observable: single-particle polarization matrix

$$\Theta(p)_{\sigma,\sigma'} = \mathrm{tr}(\widehat{\rho}\, a^\dagger(p)_\sigma a(p)_{\sigma'}),$$

  Only $\widehat{\rho}$ can depend on pseudo-gauge!

- Local equilibrium: $\Theta(p)_C \neq \Theta(p)_B$
- Global equilibrium: $\Theta(p)_C = \Theta(p)_B$

Polarized neutral system - Physical meaning of $\Omega_{\mu\nu}$



- a) Fluid at rest with constant temperature with particles and antiparticles polarized in the same direction $\beta^\mu = (1/T)(1, \mathbf{0}) \Rightarrow \varpi = 0$

> Belinfante's "gauge" does not imply that polarization vanishes, but rather it is locked to thermal vorticity

> In the Canonical "gauge" one needs the spin potential to describe hydro evolution, but only if spin density relaxes "slowly" to equilibrium

- In general in local equilibrium $\Omega_{\lambda\nu} \neq \varpi_{\lambda\nu} = \frac{1}{2}(\partial_\nu \beta_\lambda - \partial_\lambda \beta_\nu)$

Hydrodynamics with spin tensor

- 11 equations of motion (5 usual hydro + 6 due to total angular momentum conservation)

$$\partial_\mu T^{\mu\nu} = 0 \qquad \partial_\mu j^\mu \qquad \partial_\lambda S^{\lambda,\mu\nu} = T^{\nu\mu} - T^{\mu\nu}$$

- 11 unknowns

$$\beta^\mu \qquad \zeta \qquad \Omega^{\mu\nu} \quad \text{(6 additional independent components)}$$

Different sets of $T^{\mu\nu}$, $S^{\lambda,\mu\nu}$
$\Longrightarrow$ Different explicit forms of hydrodynamic EOM

$\Omega^{\mu\nu}$ and β^μ evolve separately

Summary

- Local equilibrium or non-equilibrium thermodynamics is sensitive to the choice of different sets of energy-momentum and spin tensors
- Particle polarization may be sensitive to different choices of energy-momentum and spin tensors
- Hydrodynamics with spin tensor is needed if spin density relaxes "slowly" to equilibrium
- Hydrodynamics with spin tensor: 6 additional fields $\Omega_{\mu\nu}$ (spin potential) to be evolved