Relativistic hydrodynamics with spin and spin polarization evolution in a Bjorken flow Background

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References: **Phys. Rev. C 98 (2018) 044906**
and
arXiv:1901.09655 [hep-ph], accepted in PRC

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Motivation

Global angular momentum $J \approx 10^4 \hbar$ (RHIC Au-Au 200 GeV, b=2.5 fm) [F. Becattini, F. Piccinini and J. Rizzo, Phys. Rev. C77, 024906 (2008)].

Global rotation of the matter created in the non-central collisions can induce spin polarization, similarly to magnetomechanical Barnett effect and Einstein and de Haas effect.

Emerging particles are expected to be globally polarized with their spins on average pointing along the system angular momentum.

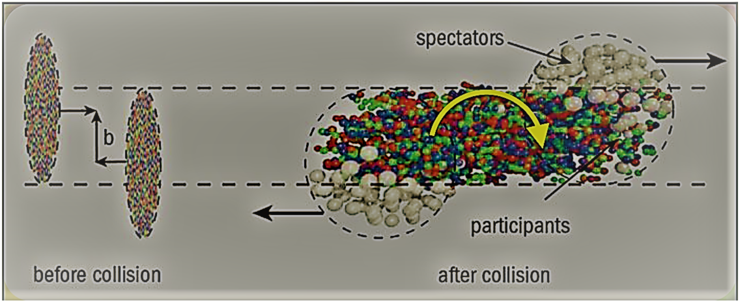


Figure: Geometry of a non-central heavy ion collision

Global Λ polarization in RHIC experiment

The average polarization $\bar{P}_H$ (where $H = \Lambda$ or $\bar{\Lambda}$) from $20 - 50\%$ central Au+Au collisions [L. Adamczyk *et al.* (STAR), Nature 548 (2017) 62-65, arXiv:1701.06657 [nucl-ex]].

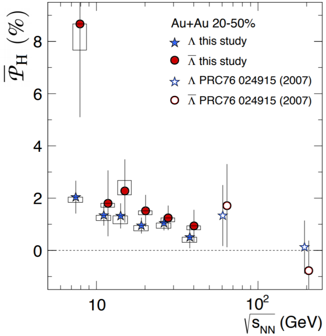


Figure: The average polarization versus collision energy

Present hydrodynamical prescription used to describe the spin polarization of particles at freeze-out makes use of equality between the thermal vorticity and spin polarization tensor.

[F. Becattini, I. Karpenko, M. Lisa, I. Upsal, S. Voloshin, Phys. Rev. C 95, 054902 (2017)]

→ Talk by F. Becattini

It has problem in explaining the correct quadrupole structure of longitudinal spin polarization.

One of the reasons may be that in global equilibrium thermal vorticity is not same as spin polarization tensor.

Note that the fact that thermal vorticity and spin polarization are same in global equilibrium holds if energy-momentum tensor $T^{\mu\nu}$ is asymmetric.

[D. Zubarev, Nonequilibrium Statistical Thermodynamics (Springer, 1974)]

→ Talk by E. Speranza

A natural framework for dealing simultaneously with spin polarization and vorticity would be relativistic hydrodynamics of polarized fluids.

[Wojciech Florkowski, Bengt Friman, Amaresh Jaiswal, Enrico Speranza, Phys. Rev. C97, 041901 (2018)]

In this talk

1. Using the equilibrium distribution functions for particles of spin 1/2 as an input to the Wigner function and its semi-classical expansion we show how a kinetic approach can lead us to the fact that in global equilibrium thermal vorticity and spin polarization tensors are constant, however, not necessarily equal (β_μ and $\omega_{\mu\nu}$ are independent).

2. A procedure to construct the hydrodynamic framework that can deal with the spin polarization phenomena.

3. We use relativistic hydrodynamics with spin to determine the space-time evolution of the spin polarization in a boost-invariant and transversely homogeneous background.

4. In our approach we use the forms of the energy-momentum and spin tensors proposed by de Groot, van Leeuwen, and van Weert.

Global and local equilibrium – spinless particles

Boltzmann equation

$$p^\mu \partial_\mu f(x,p) = C[f(x,p)]$$

1. Satisfied exactly for free streaming.
2. Satisfied in global equilibrium:

via some constraint equations on the hydrodynamic parameters (μ, T, u_μ) used to specify the form of $feq(x,p)$.

3. Does not vanish in the local thermodynamic equilibrium:

constraints on the hydrodynamic parameters are found by adding further assumptions (for instance momentum moments, $\int dPp_\mu \ldots f(x,p)$, yielding the conservation laws for energy and momentum etc)

For free streamimg $= 0$

Global equilbrium $= 0$

Local equailibrium $= 0$

Global equilibrium and the Killing equation

The equilibrium distribution function has the form

$$f_{\mathrm{eq}}(x,p) = \exp\left[\xi(x) - \beta_\mu(x)p^\mu\right] \qquad \text{with} \qquad \beta_\mu = u_\mu(x)/T(x) \qquad \xi = \mu(x)/T(x).$$

It satisfies the LHS of Boltzmann equation *i.e.* $p^\mu \partial_\mu f_{\mathrm{eq}}(x,p) = 0$ via,

$$p^\mu \partial_\mu \xi + p^\mu p^\nu \partial_\mu \beta_\nu = p^\mu \partial_\mu \xi + \frac{1}{2} p^\mu p^\nu (\partial_\mu \beta_\nu + \partial_\nu \beta_\mu) = 0$$

One arrives at constraints

$$\partial_\mu \xi = 0$$
$$\partial_\mu \beta_\nu + \partial_\nu \beta_\mu = 0 \quad \text{Killing equation}$$

giving

$$\xi = \text{constant}$$
$$\beta_\mu(x) = \beta^0_\mu + \omega^0_{\mu\nu} x^\nu \qquad \text{with} \qquad \beta^0_\mu = \text{const}, \quad \omega^0_{\mu\nu} = -\omega^0_{\nu\mu} = \text{constant}.$$

Thermal vorticity is given by

$$\varpi_{\mu\nu} = -\frac{1}{2}\left(\partial_\mu \beta_\nu - \partial_\nu \beta_\mu\right) \equiv \omega^0_{\mu\nu}.$$

Global and local thermodynamic equilibrium - particles with spin

The treatment of the collisionless kinetic equation for the Wigner function $\mathcal{W}(x,k)$ that includes spin degrees of freedom has many features in common with the simple spinless system discussed above.

Wigner function depends on β_μ, parameter $\xi = \frac{\mu}{T}$ and spin polarization tensor $\omega_{\mu\nu}$.

We can distinguish four rather than two different types of equilibrium.

1. global equilibrium — β_μ field is a Killing vector, $\varpi_{\mu\nu} = \omega_{\mu\nu} = \text{const}$, in addition $\xi = \text{const}$.

2. extended global equilibrium — β_μ field is a Killing vector, $\varpi_{\mu\nu} = \text{const}$, $\omega_{\mu\nu} = \text{const}$, but $\varpi_{\mu\nu} \neq \omega_{\mu\nu}$, in addition $\xi = \text{const}$.

3. local equilibrium — β_μ field is not a Killing vector but we still have $\omega_{\mu\nu}(x) = \varpi_{\mu\nu}(x)$, $\xi = \xi(x)$.

4. extended local equilibrium — β_μ field is not a Killing vector and $\omega_{\mu\nu}(x) \neq \varpi_{\mu\nu}(x)$, $\xi = \xi(x)$.

Spin dependent phase space distribution functions

We take $f^+_{rs}(x,p)$ and $f^-_{rs}(x,p)$ [F. Becattini et al. Annals Phys. 338 (2013) 32]

$$f^+_{rs}(x,p) = \frac{1}{2m}\bar{u}_r(p)X^+u_s(p), \qquad f^-_{rs}(x,p) = -\frac{1}{2m}\bar{v}_s(p)X^-v_r(p).$$

m is the (anti)particle mass, while $u_r(p)$ and $v_r(p)$ are Dirac bispinors.

$$X^\pm = \exp\left[\pm\xi(x) - \beta_\mu(x)p^\mu\right] M^\pm, \qquad M^\pm = \exp\left[\pm\frac{1}{2}\omega_{\mu\nu}(x)\Sigma^{\mu\nu}\right].$$

- $\beta^\mu(x) = u^\mu(x)/T(x)$ and $\xi(x) = \mu(x)/T(x)$, with $\mu(x)$ being the chemical potential.
- The quantity $\omega_{\mu\nu}(x)$ is the spin polarization tensor, while $\Sigma^{\mu\nu} = (i/4)[\gamma^\mu, \gamma^\nu]$.

Spin polarization tensor $\omega_{\mu\nu}$ satisfies the two conditions [Wojciech Florkowski, Bengt Friman, Amaresh Jaiswal, Enrico Speranza, Phys. Rev. D 97, 116017 (2018)]

$$\omega_{\mu\nu}\omega^{\mu\nu} \geq 0, \quad \omega_{\mu\nu}\tilde{\omega}^{\mu\nu} = 0, \quad \text{where } \tilde{\omega}^{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\omega^{\alpha\beta}$$

$$M^\pm = \cosh(\zeta) \pm \frac{\sinh(\zeta)}{2\zeta}\omega_{\mu\nu}\Sigma^{\mu\nu}.$$

ζ is defined by the expression

$$\zeta = \frac{\Omega}{T} = \frac{1}{2}\sqrt{\frac{1}{2}\omega_{\mu\nu}\omega^{\mu\nu}}.$$

Ω plays a role of the spin chemical potential.

Equilibrium Wigner Functions

The equilibrium Wigner functions
[S. de Groot, W. van Leeuwen, and C. van Weert, Relativistic Kinetic Theory: Principles and Applications (1980)]

$$\mathcal{W}^{+}_{\rm eq}(x,k) = \frac{1}{2}\sum_{r,s=1}^{2}\int dP\,\delta^{(4)}(k-p)u^{r}(p)\bar{u}^{s}(p)f^{+}_{rs}(x,p),$$

$$\mathcal{W}^{-}_{\rm eq}(x,k) = -\frac{1}{2}\sum_{r,s=1}^{2}\int dP\,\delta^{(4)}(k+p)v^{s}(p)\bar{v}^{r}(p)f^{-}_{rs}(x,p).$$

The Wigner functions $\mathcal{W}^{\pm}_{\rm eq}(x,k)$, being four-by-four matrices satisfying the relations $\mathcal{W}^{\pm}_{\rm eq}(x,k) = \gamma_0 \mathcal{W}^{\pm}_{\rm eq}(x,k)^{\dagger}\gamma_0$, can always be expanded in terms of the 16 independent generators of the Clifford algebra

[D. Vasak, M. Gyulassy, H. T. Elze, Annals of Physics, 173, 462 (1987)]
[P. Zhuang, U. Hienz, Annals of physics, 245, 311 (1996)]
[Jian-Hua Gao, Zuo-Tang Liang, Shi Pu, Qun Wang, Xin-Nian Wang, PRL 109, 232301 (2012)]
[Ren-hong Fang, Long-gang Pang, Qun Wang, and Xin-nian Wang Phys. Rev. C 94, 024904 (2016)]

$$\mathcal{W}^{\pm}_{\rm eq}(x,k) = \frac{1}{4}\left[\mathcal{F}^{\pm}_{\rm eq}(x,k) + i\gamma_5\mathcal{P}^{\pm}_{\rm eq}(x,k) + \gamma^{\mu}\mathcal{V}^{\pm}_{{\rm eq},\mu}(x,k) + \gamma_5\gamma^{\mu}\mathcal{A}^{\pm}_{{\rm eq},\mu}(x,k) + \Sigma^{\mu\nu}\mathcal{S}^{\pm}_{{\rm eq},\mu\nu}(x,k)\right].$$

Any other Wigner function

$$\mathcal{W}^{\pm}(x,k) = \frac{1}{4}\left[\mathcal{F}^{\pm}(x,k) + i\gamma_5\mathcal{P}^{\pm}(x,k) + \gamma^{\mu}\mathcal{V}^{\pm}_{\mu}(x,k) + \gamma_5\gamma^{\mu}\mathcal{A}^{\pm}_{\mu}(x,k) + \Sigma^{\mu\nu}\mathcal{S}^{\pm}_{\mu\nu}(x,k)\right].$$

The total Wigner function

$$\mathcal{W}(x,k) = \mathcal{W}^{+}(x,k) + \mathcal{W}^{-}(x,k).$$

Semi-classical expansion

The Wigner function satisfies the equation of the form

$$(\gamma_\mu K^\mu - m)\,\mathcal{W}(x,k) = 0; \qquad K^\mu = k^\mu + \frac{i\hbar}{2}\,\partial^\mu.$$

The above equation holds in global equilibrium (similar to the case of spin-less particles where $p^\mu \partial_\mu f_{eq} = 0$) and should give the constraints on hydrodynamic variables μ, T, u^μ and $\omega_{\mu\nu}$.

The real parts:

$$k^\mu \mathcal{V}_\mu - m\mathcal{F} = 0,$$
$$\frac{\hbar}{2}\partial^\mu \mathcal{A}_\mu + m\mathcal{P} = 0,$$
$$k_\mu \mathcal{F} - \frac{\hbar}{2}\partial^\nu \mathcal{S}_{\nu\mu} - m\mathcal{V}_\mu = 0,$$
$$-\frac{\hbar}{2}\partial_\mu \mathcal{P} + k^\beta \tilde{\mathcal{S}}_{\mu\beta} + m\mathcal{A}_\mu = 0,$$
$$\frac{\hbar}{2}(\partial_\mu \mathcal{V}_\nu - \partial_\nu \mathcal{V}_\mu) - \epsilon_{\mu\nu\alpha\beta} k^\alpha \mathcal{A}^\beta - m\mathcal{S}_{\mu\nu} = 0,$$

The imaginary parts:

$$\hbar\partial^\mu \mathcal{V}_\mu = 0,$$
$$k^\mu \mathcal{A}_\mu = 0,$$
$$\frac{\hbar}{2}\partial_\mu \mathcal{F} + k^\nu \mathcal{S}_{\nu\mu} = 0,$$
$$k_\mu \mathcal{P} + \frac{\hbar}{2}\partial^\beta \tilde{\mathcal{S}}_{\mu\beta} = 0,$$
$$(k_\mu \mathcal{V}_\nu - k_\nu \mathcal{V}_\mu) + \frac{\hbar}{2}\epsilon_{\mu\nu\alpha\beta}\partial^\alpha \mathcal{A}^\beta = 0.$$

Solutions in the form

$$\mathcal{X} = \mathcal{X}^{(0)} + \hbar\mathcal{X}^{(1)} + \hbar^2\mathcal{X}^{(2)} + \cdots. \qquad \mathcal{X} \in \{\mathcal{F}, \mathcal{P}, \mathcal{V}_\mu, \mathcal{A}_\mu, \mathcal{S}_{\nu\mu}\}$$

[D. Vasak, M. Gyulassy, H. T. Elze, Annals of Physics, 173, 462 (1987)]
[P. Zhuang, U. Hienz, Annals of physics, 245, 311 (1996)]
[Ren-hong Fang, Long-gang Pang, Qun Wang, and Xin-nian Wang Phys. Rev. C 94, 024904 (2016)]

In the NLO in $\hbar$ one gets,

$$k^\mu \partial_\mu \mathcal{F}_{(0)}(x,k) = 0.$$

$$k^\mu \partial_\mu \mathcal{A}^\nu_{(0)}(x,k) = 0, \quad k_\nu \mathcal{A}^\nu_{(0)}(x,k) = 0.$$

Taking

$$\mathcal{F}^{(0)} = \mathcal{F}_{\rm eq}, \quad \mathcal{P}^{(0)} = 0,$$
$$\mathcal{V}^{(0)}_{\mu} = \mathcal{V}_{{\rm eq},\mu}, \quad \mathcal{A}^{(0)}_{\mu} = \mathcal{A}_{{\rm eq},\mu},$$
$$\mathcal{S}^{(0)}_{\mu\nu} = \mathcal{S}_{{\rm eq},\mu\nu}.$$

We can get:

$$k^\mu \partial_\mu \mathcal{F}_{\rm eq}(x,k) = 0,$$

$$k^\mu \partial_\mu \, \mathcal{A}^\nu_{\rm eq}(x,k) = 0, \quad k_\nu \, \mathcal{A}^\nu_{\rm eq}(x,k) = 0.$$

These equations will be exactly fulfilled if,

$$\begin{aligned} \partial_\mu \beta_\nu(x) + \partial_\nu \beta_\mu(x) &= 0 \qquad \text{(Killing equation)} \\ \omega_{\mu\nu} &= \text{constant} \\ \xi = \frac{\mu}{T} &= \text{constant.} \end{aligned}$$

Solution to the Killing equation

$$\beta_\mu(x) = \beta^0_\mu + \omega^0_{\mu\nu} x^\nu, \qquad \beta^0_\mu = \text{constant}, \;\; \omega^0_{\mu\nu} = -\tfrac{1}{2}\left(\partial_\mu \beta_\nu - \partial_\nu \beta_\mu\right) = \varpi_{\mu\nu} = \text{constant}$$

• It does not constrain the spin polarization tensor $\omega_{\mu\nu}$ to be equal to thermal vorticity $\varpi_{\mu\nu}$.

⇒**Case of extended global equilibrium**

Obtaining the conservation laws from the kinetic equations

Charge conservation:

$$\int d^4k k^\mu \partial_\mu \mathcal{F}_{\rm eq}(x,k) = 0 \qquad \Rightarrow \partial_\mu N^\mu_{\rm GLW}(x) = 0.$$

Energy-momentum conservation:

$$\int d^4k k^\mu k^\nu \partial_\mu \mathcal{F}_{\rm eq}(x,k) = 0 \qquad \Rightarrow \partial_\mu T^{\mu\nu}_{\rm GLW}(x) = 0.$$

Spin conservation:

$$\int d^4k \epsilon^{\mu\gamma\delta\eta} k_\eta k^\lambda \partial_\lambda \mathcal{A}^\nu_{\rm eq}(x,k) = 0 \qquad \Rightarrow \partial_\lambda S^{\lambda,\mu\nu}_{\rm GLW}(x) = 0$$

Conservation of total angular momentum

$$\partial_\mu J^{\mu,\alpha\beta}(x) = 0, \qquad J^{\mu,\alpha\beta}(x) = -J^{\mu,\beta\alpha}(x)$$

Total angular momentum is the sum of orbital and spin parts:

$$J^{\mu,\alpha\beta}(x) = L^{\mu,\alpha\beta}(x) + S^{\mu,\alpha\beta}(x),$$

$$L^{\mu,\alpha\beta}(x) = x^\alpha T^{\mu\beta}(x) - x^\beta T^{\mu\alpha}(x),$$

Conservation of energy momentum and total angular momentum implies

$$\partial_\mu T^{\mu\nu}(x) = 0, \quad \partial_\lambda J^{\lambda,\mu\nu}(x) = 0, \Rightarrow \partial_\lambda S^{\lambda,\mu\nu}(x) = T^{\nu\mu}(x) - T^{\mu\nu}(x).$$

GLW expressions for charge current, energy-momentum tensor and Spin tensor

Charge current:

$$N^\alpha = nU^\alpha, \quad n = 4\,\sinh(\xi)\,n_{(0)}(T) = 2\left(e^{\xi} - e^{-\xi}\right)\,n_{(0)}(T).$$

Energy-momentum tensor:

$$T^{\alpha\beta}_{\rm GLW} = (\varepsilon + P)U^\alpha U^\beta - Pg^{\alpha\beta}$$

Energy density and pressure is given by,

$$\varepsilon = 4\,\cosh(\xi)\,\varepsilon_{(0)}(T) \quad P = 4\,\cosh(\xi)\,P_{(0)}(T).$$

The quantities $n_{(0)}(T)$, $\varepsilon_{(0)}(T)$ and $P_{(0)}(T)$ are number density, energy density and pressure of spinless and neutral Boltzmann particles at temperature and are given by following expreesions

$$n_{(0)}(T) = \frac{1}{2\pi^2}\,T^3\,\hat{m}^2 K_2\left(\hat{m}\right),$$

$$\varepsilon_{(0)}(T) = \frac{1}{2\pi^2}\,T^4\,\hat{m}^2\Big[3K_2\left(\hat{m}\right) + \hat{m}K_1\left(\hat{m}\right)\Big],$$

$$P_{(0)}(T) = T\,n_{(0)}(T).$$

GLW expressions for charge current, energy-momentum tensor and Spin tensor

Spin tensor:

The GLW spin tensor is given by

$$\boxed{S^{\alpha,\beta\gamma}_{\rm GLW} = \cosh(\xi) n_{(0)}(T) U^\alpha \omega^{\beta\gamma} + \frac{2\cosh(\xi)}{m^2} S^{\alpha\beta\gamma}_{\Delta\rm GLW}}$$

Here, $\omega^{\beta\gamma}$ is the spin polarization tensor.
The auxiliary tensor $S^{\alpha,\beta\gamma}_{\Delta\rm GLW}$ is defined as

$$S^{\alpha,\beta\gamma}_{\Delta\rm GLW} = \mathcal{A}\, U^\alpha U^\delta U^{[\beta} \omega^{\gamma]}{}_{\delta} + \mathcal{B}\left(U^{[\beta} \Delta^{\alpha\delta} \omega^{\gamma]}{}_{\delta} + U^\alpha \Delta^{\delta[\beta} \omega^{\gamma]}{}_{\delta} + U^\delta \Delta^{\alpha[\beta} \omega^{\gamma]}{}_{\delta}\right),$$

$$\mathcal{B} = -T\left(\varepsilon_{(0)}(T) + P_{(0)}(T)\right), \quad \mathcal{A} = T\left[3\varepsilon_{(0)}(T) + \left(3 + \frac{m^2}{T^2}\right) P_{(0)}(T)\right] = -3\mathcal{B} + \frac{m^2}{T} P_{(0)}(T).$$

NLO corrections in $\hbar$ and canonical energy-momentum tensor

$$\mathcal{P}^{(1)} = -\frac{1}{2m}\partial^{\mu}\mathcal{A}_{\mathrm{eq},\mu}, \quad \mathcal{V}^{(1)}_{\mu} = -\frac{1}{2m}\partial^{\nu}\mathcal{S}_{\mathrm{eq},\nu\mu}$$

$$\mathcal{S}^{(1)}_{\mu\nu} = \frac{1}{2m}(\partial_{\mu}\mathcal{V}_{\mathrm{eq},\nu} - \partial_{\nu}\mathcal{V}_{\mathrm{eq},\mu})$$

The canonical forms of the energy-momentum tensor $T^{\mu\nu}_{\mathrm{can}}$ can be obtained directly from the Dirac Lagrangian by applying the Noether theorem

$$T^{\mu\nu}_{\mathrm{can}}(x) = \int d^4k\, k^{\nu}\mathcal{V}^{\mu}(x,k)$$

[F. Becattini, V. Chandra, L. Del Zanna, and E. Grossi, Annals Phys. 338 (2013) 32-49, arXiv:1303.3431]
[Ren-hong Fang, Long-gang Pang, Qun Wang, and Xin-nian Wang Phys. Rev. C 94, 024904 (2016)]

Writing $\mathcal{V}^{\mu}(x,k)$ up to first order we can get,

$$T^{\mu\nu}_{\mathrm{can}}(x) = T^{\mu\nu}_{\mathrm{GLW}}(x) + \delta T^{\mu\nu}_{\mathrm{can}}(x)$$

where

$$\delta T^{\mu\nu}_{\mathrm{can}}(x) = -\frac{\hbar}{2m}\int d^4k k^{\nu}\partial_{\lambda}\mathcal{S}^{\lambda\mu}_{\mathrm{eq}}(x,k) = -\partial_{\lambda}S^{\nu,\lambda\mu}_{\mathrm{GLW}}(x).$$

The canonical energy-momentum tensor is exactly conserved

$$\partial_{\mu}T^{\mu\nu}_{\mathrm{can}}(x) = 0.$$

It is interesting to observe that the conservation laws for $T^{\mu\nu}_{\mathrm{GLW}}(x)$ and $T^{\mu\nu}_{\mathrm{can}}(x)$ are consistent, since $\partial_{\mu}\,\delta T^{\mu\nu}_{\mathrm{can}}(x) = 0$.

Canonical spin tenor

The canonical form of spin tensor is [F. Becattini, V. Chandra, L. Del Zanna, and E. Grossi, Annals Phys. 338 (2013) 32–49].

$$S^{\lambda,\mu\nu}_{\rm can}(x) = \frac{\hbar}{4}\int d^4k \, {\rm tr}\left[\left\{\sigma^{\mu\nu},\gamma^{\lambda}\right\}\mathcal{W}(x,k)\right] = \frac{\hbar}{2}\epsilon^{\kappa\lambda\mu\nu}\int d^4k \, \mathcal{A}_{\kappa}(x,k)$$

For equilibrium it is enough to consider $\mathcal{A}_{\kappa}(x,k) = \mathcal{A}^0_{\kappa}(x,k) = \mathcal{A}_{{\rm eq},\kappa}(x,k)$, therefore

$$\begin{aligned} S^{\lambda,\mu\nu}_{\rm can}(x) &= \frac{\hbar}{2}\epsilon^{\kappa\lambda\mu\nu}\int d^4k \, \mathcal{A}_{{\rm eq},\kappa}(x,k) = \frac{\hbar w}{4\zeta}\left(u^{\lambda}\omega^{\mu\nu} + u^{\mu}\omega^{\nu\lambda} + u^{\nu}\omega^{\lambda\mu}\right), \\ &= S^{\lambda,\mu\nu}_{\rm GLW} + S^{\mu,\nu\lambda}_{\rm GLW} + S^{\nu,\lambda\mu}_{\rm GLW}, \end{aligned}$$

One can show

$$\partial_{\lambda} S^{\lambda,\mu\nu}_{\rm can}(x) = T^{\nu\mu}_{\rm can} - T^{\mu\nu}_{\rm can} = -\partial_{\lambda} S^{\mu,\nu\lambda}_{\rm GLW} + \partial_{\lambda} S^{\nu,\lambda\mu}_{\rm GLW} \neq 0.$$

If we define,

$$\Phi^{\lambda,\mu\nu} \equiv S^{\mu,\lambda\nu}_{\rm GLW} - S^{\nu,\lambda\mu}_{\rm GLW},$$

we can write

$$S^{\lambda,\mu\nu}_{\rm can} = S^{\lambda,\mu\nu}_{\rm GLW} - \Phi^{\lambda,\mu\nu},$$

$$T^{\mu\nu}_{\rm can} = T^{\mu\nu}_{\rm GLW} + \frac{1}{2}\partial_{\lambda}\left(\Phi^{\lambda,\mu\nu} + \Phi^{\mu,\nu\lambda} + \Phi^{\nu,\mu\lambda}\right).$$

Boost-invariant flow

Basis for transversely homogeneous longitudinal expansion

$$
\begin{aligned}
U^\alpha &= \frac{1}{\tau}(t,0,0,z) = (\cosh\eta, 0, 0, \sinh\eta)\,,\\
X^\alpha &= (0,1,0,0)\,,\\
Y^\alpha &= (0,0,1,0)\,,\\
Z^\alpha &= \frac{1}{\tau}(z,0,0,t) = (\sinh\eta, 0, 0, \cosh\eta)\,.
\end{aligned}
$$

Here $\tau = \sqrt{t^2 - z^2}$ is the longitudinal proper time, while $\eta = \frac{1}{2}\ln((t+z)/(t-z))$ is the space-time rapidity.

$$U \cdot U = 1$$

$$
\begin{aligned}
X \cdot X &= Y \cdot Y = Z \cdot Z = -1,\\
X \cdot U &= Y \cdot U = Z \cdot U = 0,\\
X \cdot Y &= Y \cdot Z = Z \cdot X = 0.
\end{aligned}
$$

Spin Polarization tensor:

The spin polarization tensor $\omega_{\mu\nu}$ is antisymmetric and can be decomposed in terms of four-vectors κ^μ and ω^μ,

$$\omega_{\mu\nu} = \kappa_\mu U_\nu - \kappa_\nu U_\mu + \epsilon_{\mu\nu\alpha\beta} U^\alpha \omega^\beta,$$

$$\kappa \cdot U = 0, \quad \omega \cdot U = 0.$$

Using above conditions we can write

$$\kappa_\mu = \omega_{\mu\alpha} U^\alpha, \quad \omega_\mu = \frac{1}{2}\epsilon_{\mu\alpha\beta\gamma}\omega^{\alpha\beta} U^\gamma.$$

Boost-invariant flow and spin polarization tensor

One can introduce the following representation of the vectors κ^μ and ω^μ

$$
\begin{aligned}
\kappa^\alpha &= C_{\kappa X} X^\alpha + C_{\kappa Y} Y^\alpha + C_{\kappa Z} Z^\alpha, \\
\omega^\alpha &= C_{\omega X} X^\alpha + C_{\omega Y} Y^\alpha + C_{\omega Z} Z^\alpha.
\end{aligned}
$$

Here, the scalar coefficients $C_{\kappa X}$, $C_{\kappa Y}$, $C_{\kappa Z}$, $C_{\omega X}$, $C_{\omega Y}$, and $C_{\omega Z}$ are functions of the proper time τ only.

It is important to note that due to the orthogonality connditions $\kappa \cdot U = 0, \; \omega \cdot U = 0$, there are no terms proportional U^μ.

The following boost-invariant expression for the spin polarization tensor $\omega_{\mu\nu}$ can be obtained by using above decomposition of vectors κ^μ and ω^μ,

$$
\begin{aligned}
\omega_{\mu\nu} &= C_{\kappa Z}(Z_\mu U_\nu - Z_\nu U_\mu) + C_{\kappa X}(X_\mu U_\nu - X_\nu U_\mu) + C_{\kappa Y}(Y_\mu U_\nu - Y_\nu U_\mu) \\
&\quad + \epsilon_{\mu\nu\alpha\beta} U_\alpha (C_{\omega Z} Z^\beta + C_{\omega X} X^\beta + C_{\omega Y} Y^\beta).
\end{aligned}
$$

In the plane $z = 0$ we find

$$
\omega_{\mu\nu} = \begin{bmatrix}
0 & C_{\kappa X} & C_{\kappa Y} & C_{\kappa Z} \\
-C_{\kappa X} & 0 & -C_{\omega Z} & C_{\omega Y} \\
-C_{\kappa Y} & C_{\omega Z} & 0 & -C_{\omega X} \\
-C_{\kappa Z} & -C_{\omega Y} & C_{\omega X} & 0
\end{bmatrix}
$$

Boost-invariant form of fluid dynamics with spin

Charge conservation:

$$\dot{n} + \frac{n}{\tau} = 0.$$

Energy-momentum conservation:

$$\dot{\varepsilon} + \frac{(\varepsilon + P)}{\tau} = 0.$$

Spin conservation:

$$\begin{bmatrix} \mathcal{L}(\tau) & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{L}(\tau) & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{L}(\tau) & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{P}(\tau) & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{P}(\tau) & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathcal{P}(\tau) \end{bmatrix} \begin{bmatrix} \dot{C}_{\kappa X} \\ \dot{C}_{\kappa Y} \\ \dot{C}_{\kappa Z} \\ \dot{C}_{\omega X} \\ \dot{C}_{\omega Y} \\ \dot{C}_{\omega Z} \end{bmatrix} = \begin{bmatrix} \mathcal{Q}_1(\tau) & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{Q}_1(\tau) & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{Q}_2(\tau) & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{R}_1(\tau) & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{R}_1(\tau) & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathcal{R}_2(\tau) \end{bmatrix} \begin{bmatrix} C_{\kappa X} \\ C_{\kappa Y} \\ C_{\kappa Z} \\ C_{\omega X} \\ C_{\omega Y} \\ C_{\omega Z} \end{bmatrix},$$

$$\begin{aligned}
\mathcal{L}(\tau) &= \mathcal{A}_1 - \frac{1}{2}\mathcal{A}_2 - \mathcal{A}_3, \\
\mathcal{P}(\tau) &= \mathcal{A}_1, \\
\mathcal{Q}_1(\tau) &= -\left(\dot{\mathcal{A}}_1 - \frac{1}{2}\dot{\mathcal{A}}_2 - \dot{\mathcal{A}}_3 + \frac{\mathcal{A}_1}{\tau} - \frac{1}{2}\frac{\mathcal{A}_2}{\tau} - \frac{1}{2}\frac{\mathcal{A}_3}{\tau}\right), \\
\mathcal{Q}_2(\tau) &= -\left(\dot{\mathcal{A}}_1 - \frac{1}{2}\dot{\mathcal{A}}_2 - \dot{\mathcal{A}}_3 + \frac{\mathcal{A}_1}{\tau} - \frac{1}{2}\frac{\mathcal{A}_2}{\tau} - \frac{\mathcal{A}_3}{\tau}\right), \\
\mathcal{R}_1(\tau) &= -\left(\dot{\mathcal{A}}_1 + \frac{\mathcal{A}_1}{\tau} - \frac{1}{2}\frac{\mathcal{A}_3}{\tau}\right), \\
\mathcal{R}_2(\tau) &= -\left(\dot{\mathcal{A}}_1 + \frac{\mathcal{A}_1}{\tau}\right).
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}_1 &= \mathcal{C}\left(n_{(0)} - \frac{2\mathcal{B}}{m^2}\right), \\
\mathcal{A}_2 &= \frac{2\mathcal{C}}{m^2}\left(\mathcal{A} - 3\mathcal{B}\right), \\
\mathcal{A}_3 &= \frac{2\mathcal{C}\mathcal{B}}{m^2},
\end{aligned}$$

$$\mathcal{C} = \cosh(\xi).$$

Initial baryon chemical potential $\mu_0 = 800$ MeV, initial temperature $T_0 = 155$ MeV. The particle mass, $m = 1116$ MeV. The initial proper time is $\tau_0 = 1$ fm and final time $\tau_f = 10$ fm.
$C_{\kappa X0} = C_{\kappa Z0} = C_{\omega X0} = C_{\omega Z0} = 0.1$.

Numerical Solutions

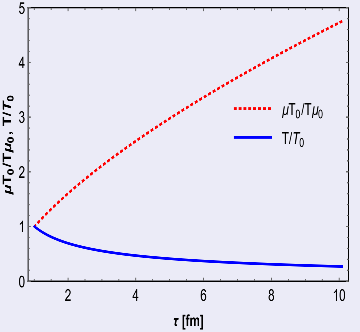


Figure: Proper-time dependence of T divided by its initial value T_0 and the ratio of baryon chemical potential μ and temperature T rescaled by the initial ratio μ_0/T_0.

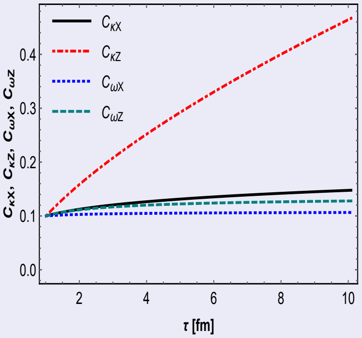


Figure: Proper-time dependence of the coefficients $C_{\kappa X}$, $C_{\kappa Z}$, $C_{\omega X}$ and $C_{\omega Z}$.

Spin polarization of particles at freeze-out

The average spin polarization is given by

$$\langle \pi_\mu \rangle = \frac{E_p \frac{d\Pi_\mu(p)}{d^3p}}{E_p \frac{d\mathcal{N}(p)}{d^3p}}$$

where for GLW form of spin tensor, total Pauli-Lubański (PL) vector for particles with momentum p is given by,

$$E_p \frac{d\Pi_\mu(p)}{d^3p} \quad = \quad -\frac{\cosh(\xi)}{(2\pi)^3 m} \int \Delta\Sigma_\lambda p^\lambda \, e^{-\beta \cdot p} \, \tilde{\omega}_{\mu\beta} p^\beta .$$

while,

$$E_p \frac{d\mathcal{N}(p)}{d^3p} \quad = \quad \frac{4\cosh(\xi)}{(2\pi)^3} \int \Delta\Sigma_\lambda p^\lambda \, e^{-\beta \cdot p} .$$

We can carry out the integration very easily by assuming that freeze-out takes place at a constant value of the proper time τ, in this case

$$\Delta\Sigma_\lambda \quad = \quad U_\lambda dx dy \tau d\eta.$$

We can parametrize of the particle four-momentum p^λ in terms of the transverse mass m_T and rapidity y_p,

$$p^\lambda \quad = \quad (m_T \cosh(y_p), p_x, p_y, m_T \sinh(y_p))$$
$$\Delta\Sigma_\lambda p^\lambda = m_T \cosh(y_p - \eta) \, dx dy \tau d\eta.$$

Boost to local rest frame (LRF) of the particle

In the local rest frame of the particle, polarization vector $\langle \pi^{\star}_{\mu} \rangle$ can be obtained by using the canonical boost [E. Leader, "Spin in Particle Physics," Cambridge University Press (2001)]

$$\Lambda^{\mu}_{\ \nu}(-\mathbf{v_p}) = \begin{bmatrix} \frac{E_p}{m} & -\frac{p_x}{m} & -\frac{p_y}{m} & -\frac{p_z}{m} \\ -\frac{p_x}{m} & 1+\alpha_p p_x^2 & \alpha_p p_x p_y & \alpha_p p_x p_z \\ -\frac{p_y}{m} & \alpha_p p_y p_x & 1+\alpha_p p_y^2 & \alpha_p p_y p_z \\ -\frac{p_z}{m} & \alpha_p p_z p_x & \alpha_p p_z p_y & 1+\alpha_p p_z^2 \end{bmatrix}$$

where, $\mathbf{v_p} = \boldsymbol{p}/E_p$ and $\alpha_p = 1/(m(E_p+m))$.

$$\langle \pi^{\star}_{\mu} \rangle = -\frac{1}{8mK_1(\hat{m}_T)} \begin{pmatrix} 0 \\ \left(\left(\frac{\sinh(y_p)p_x}{m_T\cosh(y_p)+m}\right)\left((K_0(\hat{m}_T)+K_2(\hat{m}_T))(C_{\kappa X}p_y - C_{\kappa Y}p_x) + 2C_{\omega Z}m_T K_1(\hat{m}_T)\right) + \frac{(K_0(\hat{m}_T)+K_2(\hat{m}_T))\cosh(y_p)p_x(C_{\omega X}p_x + C_{\omega Y}p_y)}{m_T\cosh(y_p)+m} + 2C_{\kappa Z}p_y K_1(\hat{m}_T) - (K_0(\hat{m}_T)+K_2(\hat{m}_T))C_{\omega X}m_T \right) \\ \left(\left(\frac{\sinh(y_p)p_y}{m_T\cosh(y_p)+m}\right)\left((K_0(\hat{m}_T)+K_2(\hat{m}_T))(C_{\kappa X}p_y - C_{\kappa Y}p_x) + 2C_{\omega Z}m_T K_1(\hat{m}_T)\right) + \frac{(K_0(\hat{m}_T)+K_2(\hat{m}_T))\cosh(y_p)p_y(C_{\omega X}p_x + C_{\omega Y}p_y)}{m_T\cosh(y_p)+m} - 2C_{\kappa Z}p_x K_1(\hat{m}_T) - (K_0(\hat{m}_T)+K_2(\hat{m}_T))C_{\omega Y}m_T \right) \\ \left(-\left(\frac{m\cosh(y_p)+m_T}{m_T\cosh(y_p)+m}\right)\left((K_0(\hat{m}_T)+K_2(\hat{m}_T))(C_{\kappa X}p_y - C_{\kappa Y}p_x) + 2C_{\omega Z}m_T K_1(\hat{m}_T)\right) - \frac{m(K_0(\hat{m}_T)+K_2(\hat{m}_T))\sinh(y_p)(C_{\omega X}p_x + C_{\omega Y}p_y)}{m_T\cosh(y_p)+m} \right) \end{pmatrix}$$

Here, $\hat{m}_T = \frac{m_T}{T}$.

Approximate expression for spin polarization

We consider particles with $y_p = 0$.
Mass of the Λ hyperon is much larger than temperature *i.e.*, $\hat{m}_T \gg 1$.
In this case, we may use the approximation $\left(K_0\left(\hat{m}_T\right) + K_2\left(\hat{m}_T\right)\right)/K_1\left(\hat{m}_T\right) \approx 2$.

$$\langle \pi^\star_\mu \rangle \quad = \quad -\frac{1}{4m} \begin{pmatrix} 0 \\ \frac{p_x(C_{\omega X} p_x + C_{\omega Y} p_y)}{m_T + m} + C_{\kappa Z} p_y - C_{\omega X} m_T \\ \frac{p_y(C_{\omega X} p_x + C_{\omega Y} p_y)}{m_T + m} - C_{\kappa Z} p_x - C_{\omega Y} m_T \\ -\left(C_{\kappa X} p_y - C_{\kappa Y} p_x\right) - C_{\omega Z} m_T \end{pmatrix}.$$

$\langle \boldsymbol{\pi}^\star \rangle = (\langle \pi^{\star 1} \rangle, \langle \pi^{\star 2} \rangle, \langle \pi^{\star 3} \rangle) \equiv (\langle \pi^\star_x \rangle, \langle \pi^\star_y \rangle, \langle \pi^\star_z \rangle)$,
If we write coefficient functions C as,

$$\begin{aligned} \boldsymbol{C}_\kappa &= (C_{\kappa X}, C_{\kappa Y}, C_{\kappa Z}), \\ \boldsymbol{C}_\omega &= (C_{\omega X}, C_{\omega Y}, C_{\omega Z}). \end{aligned}$$

We can write,

$$\boxed{\langle \boldsymbol{\pi}^* \rangle = -\frac{1}{4m}\left[E_p \boldsymbol{C}_\omega - \boldsymbol{p} \times \boldsymbol{C}_\kappa - \frac{\boldsymbol{p} \cdot \boldsymbol{C}_\omega}{E_p + m}\boldsymbol{p}\right]}$$

where, $\boldsymbol{p} = (p_x, p_y, 0)$

Spin angular momentum of the fire cylinder

$$S^{\mu\nu}_{\rm FC} = \int \Delta\Sigma_\lambda S^{\lambda,\mu\nu}_{\rm GLW} = \int dxdy\, \tau d\eta\, U_\lambda S^{\lambda,\mu\nu}_{\rm GLW}$$

$$= \pi R^2 \tau \int_{-\eta_{\rm FC}/2}^{+\eta_{\rm FC}/2} d\eta\, U_\lambda S^{\lambda,\mu\nu}_{\rm GLW}.$$

$$\begin{aligned}
S^{\rm FC}_{01} &= 2\pi R^2 \tau\, \mathcal{A}_\kappa C_{\kappa X} \sinh(\eta_{\rm FC}/2),\\
S^{\rm FC}_{02} &= 2\pi R^2 \tau\, \mathcal{A}_\kappa C_{\kappa Y} \sinh(\eta_{\rm FC}/2),\\
S^{\rm FC}_{03} &= \pi R^2 \tau\, \mathcal{A}_\kappa C_{\kappa Z}\, \eta_{\rm FC},\\
S^{\rm FC}_{23} &= -2\pi R^2 \tau\, \mathcal{A}_1 C_{\omega X} \sinh(\eta_{\rm FC}/2),\\
S^{\rm FC}_{13} &= 2\pi R^2 \tau\, \mathcal{A}_1 C_{\omega Y} \sinh(\eta_{\rm FC}/2),\\
S^{\rm FC}_{12} &= -\pi R^2 \tau\, \mathcal{A}_1 C_{\omega Z}\, \eta_{\rm FC}.
\end{aligned}$$

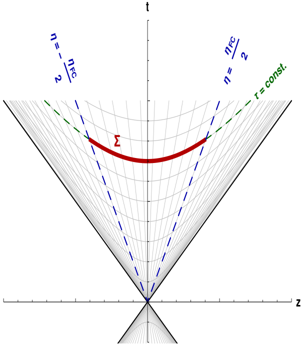


Figure: Hypersurface of the boost invariant fire-cylinder.

If spin angular momentum is in the direction of total angular momentum we must have, $\boldsymbol{C}_\kappa = (0,0,0)$, and $\boldsymbol{C}_\omega = (0, \boldsymbol{C}_{\omega y}, 0)$.

Momentum dependence of polarization

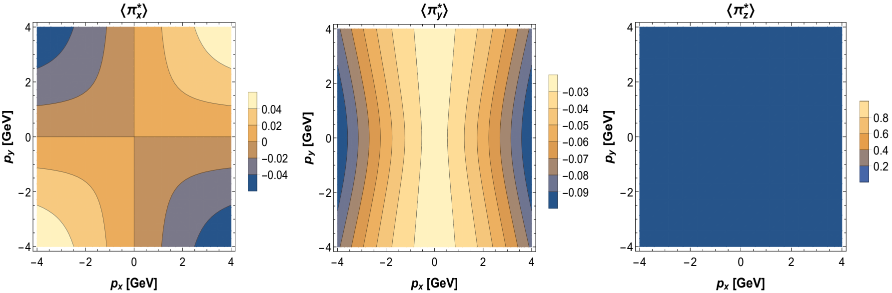


Figure: Components of the PRF mean polarization three-vector of Λ's. The results obtained with the initial conditions $\mu_0 = 800$ MeV, $T_0 = 155$ MeV, $\boldsymbol{C}_{\kappa,0} = (0.0, 0, 0.0)$, and $\boldsymbol{C}_{\omega,0} = (0.0, 0.1, 0.0)$ for $y_p = 0$.

Summary

1. We have discussed about the Wigner function (constructed from the local equilibrium phase space distribution functions for spin-1/2) and it's spinor decomposition.

2. We have found, in contrast to many earlier claims found in the literature, Wigner function approach does not imply a direct relation between the thermal vorticity and spin polarization, except for the fact that the two should be constant in global equilibrium.

3. We have also outlined procedures to formulate hydrodynamics with spin from the kinetic equations derived from Wigner function.

4. Using the simple transversely homogeneous longitudinal expansion we show that GLW formulation of hydrodynamics with spin can be used to determine the spin polarization observed in heavy ion collisions.

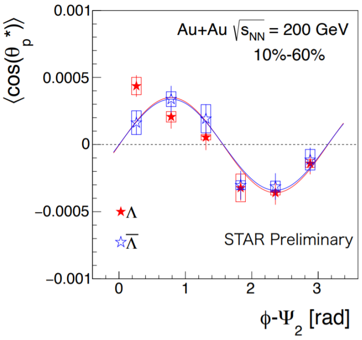
5. Numerical results obtained by us can not be compared with the experimental results [This is because we have a simple 1+0 dimensional expansion].

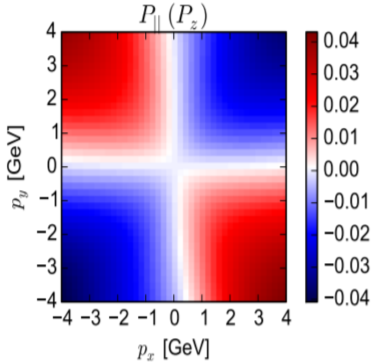
Outlook: Study of the polarization to a more realistic scenario *i.e.* for 3+1 dimensional expansion (Work in progress).

Note: With full 3+1 dimensional simulation of hydrodynamics with spin we hope to resolve the sign problem (sign of quadrupole structure of spin polarization).

THANK YOU

Polarization using the thermal vorticity model





Conservation laws

"Conservation laws of the currents are associated with the microscopic symmetries of the system (Noether's theorem)"

Internal symmetries:

Conservation of charge (baryon number, electric charge)

$$\partial_\mu \hat{N}^\mu(x) = 0, \qquad \text{1 equation}$$

Poincaré symmetry:

Conservation of energy and momentum

$$\partial_\mu \hat{T}^{\mu\nu}(x) = 0, \qquad \text{4 equations}$$

Conservation of total angular momentum

$$\partial_\mu \hat{J}^{\mu,\alpha\beta}(x) = 0, \qquad \hat{J}^{\mu,\alpha\beta}(x) = -\hat{J}^{\mu,\beta\alpha}(x) \qquad \text{6 equations}$$

Total angular momentum is the sum of orbital and spin parts:

$$\hat{J}^{\mu,\alpha\beta}(x) = \hat{L}^{\mu,\alpha\beta}(x) + \hat{S}^{\mu,\alpha\beta}(x),$$

$$\hat{L}^{\mu,\alpha\beta}(x) = x^\alpha \hat{T}^{\mu\beta}(x) - x^\beta \hat{T}^{\mu\alpha}(x),$$

Conservation of energy momentum and total angular momentum implies

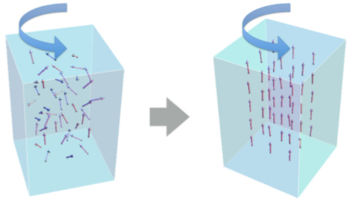
$$\partial_\mu T^{\mu\nu}(x) = 0, \quad \partial_\lambda J^{\lambda,\mu\nu}(x) = 0, \Rightarrow \partial_\lambda S^{\lambda,\mu\nu}(x) = T^{\nu\mu}(x) - T^{\mu\nu}(x) \neq 0.$$

Thus spin tensor $\hat{S}^{\mu,\alpha\beta}(x)$ is in general not conserved.

Rotation and Polarization

Barnett Effect

S. J. Barnett, Rev. Mod. Phys. 7, 129 (1935)



$$B_\Omega = \Omega/\rho$$

Figure: Mechanical rotation of an unmagnetized metallic object induces magnetization, an effective magnetic field emerges.

Einstein-de Haas Effect

A. Einstein and W. de Haas, Deutsche Physikalische Gesellschaft, Verhandlungen 17, 152 (1915)

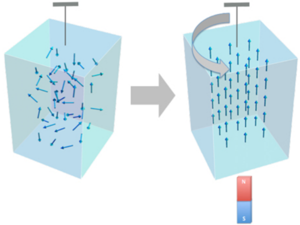


Figure: Application of magnetic field on an unmagnetized metallic object induces magnetization, body start rotating (mechanical angular momentum emerges)

Connection between spin polarization and thermal vorticity

The density operator [D. Zubarev, Nonequilibrium Statistical Thermodynamics (Springer, 1974); F. Becattini, Phys. Rev. Lett. 108, 244502 (2012)],

$$\hat{\rho}(t) = \exp\left[-\int d^3\Sigma_\mu(x)\left(\hat{T}^{\mu\nu}(x)b_\nu(x) - \frac{1}{2}\hat{J}^{\mu,\alpha\beta}(x)\omega_{\alpha\beta}(x) - \hat{N}^\mu(x)\xi(x)\right)\right].$$

$d^3\Sigma_\mu$ is an element of a space-like, three-dimensional hypersurface Σ_μ. We can take it as, $d^3\Sigma_\mu = (dV, 0, 0, 0)$. The operators $\hat{T}^{\mu\nu}(x)$, $\hat{J}^{\mu,\alpha\beta}(x)$ and $\hat{N}^\mu(x)$ are the energy-momentum, angular momentum and charge operators respectively.

In global thermodynamic equilibrium the operator $\hat{\rho}(t)$ should be independent of time.

$$\partial_\mu\left(\hat{T}^{\mu\nu}(x)b_\nu(x) - \frac{1}{2}\hat{J}^{\mu,\alpha\beta}(x)\omega_{\alpha\beta}(x) - \hat{N}^\mu(x)\xi(x)\right)$$
$$= \hat{T}^{\mu\nu}(x)\left(\partial_\mu b_\nu(x)\right) - \frac{1}{2}\hat{J}^{\mu,\alpha\beta}(x)\left(\partial_\mu\omega_{\alpha\beta}(x)\right) - \hat{N}^\mu(x)\partial_\mu\xi(x) = 0.$$

From above equation we can conclude that $\omega_{\alpha\beta} = \omega^0_{\alpha\beta}$, $\xi = \xi^0$, But

For asymmetric energy momentum tensor, $b_\nu = b^0_\nu$.

For symmetric energy momentum tensor, $b_\nu = b^0_\nu + \delta\omega^0_{\nu\rho}x^\rho$.

Global equilibrium; particle with spin

Total angular momentum

$$\hat{J}^{\mu,\alpha\beta}(x) = \hat{L}^{\mu,\alpha\beta}(x) + \hat{S}^{\mu,\alpha\beta}(x).$$

Using above equation, we can write two cases discussed above can be expressed by a single form of the density operator

$$\hat{\rho}_{\rm EQ} = \exp\left[-\int d^3\Sigma_\mu(x)\left(\hat{T}^{\mu\nu}(x)\beta_\nu(x) - \frac{1}{2}\hat{S}^{\mu,\alpha\beta}(x)\omega^0_{\alpha\beta} - \hat{N}^\mu(x)\xi^0\right)\right].$$

For asymmetric energy-momentum tensor $\beta_\mu(x) = b^0_\mu + \omega^0_{\mu\gamma}x^\gamma$.
$\beta_\mu(x)$ is a Killing vector, $\omega_{\mu\gamma} = \omega^0_{\mu\gamma} = \varpi_{\mu\nu}$.
global equilibrium

For symmetric energy-momentum tensor $\beta_\mu(x) = b^0_\mu + (\delta\omega^0_{\mu\gamma} + \omega^0_{\mu\gamma})x^\gamma$.
$\beta_\mu(x)$ is again a Killing vector, $\omega_{\mu\gamma} = \omega^0_{\mu\gamma} \neq \varpi_{\mu\nu}(= \delta\omega^0_{\mu\gamma} + \omega^0_{\mu\gamma})$.
extended global equilibrium

Local thermodynamic equilibrium; particle with spin

We define the statistical operator for local equilibrium by the same form as

$$\hat{\rho}_{\rm eq} \quad = \quad \exp\left[-\int d^3\Sigma_\mu(x)\left(\hat{T}^{\mu\nu}(x)\beta_\nu(x) - \frac{1}{2}\hat{S}^{\mu,\alpha\beta}(x)\omega_{\alpha\beta}(x) - \hat{N}^\mu(x)\xi(x)\right)\right].$$

We allow for arbitrary form of $\beta_\mu(x)$ [not a killing vector] and $\xi = \xi(x)$ and two cases for $\omega_{\mu\nu}$.

$\omega_{\mu\nu} = \varpi_{\mu\nu}$.

local equilibrium

$\omega_{\mu\nu} \neq \varpi_{\mu\nu}$.

extended local equilibrium