Reparametrization Invariance of Chiral Kinetic Theory from High Density Effective Theory

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This talk is based on...

Chiral Kinetic Theory from Effective Field Theory Revisited

Shu Lin, Aradhya Shukla, arXiv 1901.01528 [hep-ph]

Plan of the Talk

- Introduction
  - Chiral Anomaly and Chiral Kinetic Theory
  - High Density Effective Theory
- Gauge Covariant Wigner Function and Equations of Motion (EOM)
- Reparametrization Invariance of Classical Action and EOMs
- Chiral Kinetic Theory from High Density Effective Theory
  - Transport and Constitutive Equations
- Equivalence of Chiral Kinetic Theories
- Conclusions

- Kinetic theory has many applications in nuclear physics, astrophysics, cosmology and condensed matter physics. Earlier it's relativistic version misses the effect triangle anomaly

$$\partial_\mu J_5^\mu = \frac{1}{4\pi^2}\mathbf{E}\cdot\mathbf{B}$$

(Breaking of the axial symmetry at quantum level !)

- For Fermi liquid, this deficiency has been fixed by including the effect of Berry Phase and Berry Curvature[1] $\Omega_p = \pm\frac{\hat{\mathbf{p}}}{2|\mathbf{p}|}$
- The effect of Berry phase and curvature modify the particle number current together with transport equation.

[1] Berry, Proc. R. Soc. A (1984)
Son, Yamamoto, PRD (2012) PRL (2012),
Stephanov, Yin, PRL (2012),
Gao, Liang, Pu, Q. Wang, X.-N. Wang, PRL (2012), PRD (2014),
Hidaka, Pu, Yang, PRD (2016)

For the derivation of Chiral Kinetic Theory in QFT, two different formalism are used

- Field Theory Approach[2]
- Effective Field Theory Approach
    - On-Shell Effective Field Theory (OSEFT)

      [*Carignano, Manuel, Torres-Rincol, PRD (2018)*]
    - High density Effective Theory (HDET)

      [*Son, Yamamoto, PRD (2014)*]

High density effective theory is an effective field theory valid in the vicinity of Fermi surface.

[2]Hidaka, Pu, Yang, PRD (2017), PRD (2018),
Gao, Liang, Pu, Q. Wang, X.-N. Wang, PRD (2011),
Huang, Shi, Jiang, Liao, Zhuang, PRD (2018)

- High density effective theory is useful to describe the low energy dynamics of the system.
- It can be constructed by integrating out fast modes from the theory.
- This process generates the non-local effective Lagrangian which can be expanded in terms of the large momenta.

High Density Effective Theory

- Lagrangian for right handed chiral fermions with finite density and zero temperature

$$\mathcal{L}_0 = \bar{\psi}(i\gamma^\mu D_\mu)\psi + \mu\bar{\psi}\gamma^0\psi,$$

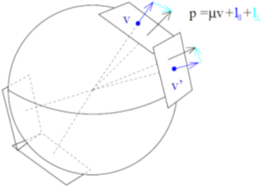
  With $E_\pm = \mu \pm |\mathbf{p}|$: Energy of (anti-)particles

- At low energy, particles with $E_+ \sim 0 \implies$ slow modes

  antiparticles with $E_- \sim -2\mu \implies$ fast modes

High Density Effective Theory

- Decomposing the energy and momentum of fermions as[3]
  $p^0 = \mu + l^0$ and $\mathbf{p} = \mu\mathbf{v} + \mathbf{l}$ with $l^0, \mathbf{l} \ll \mu$ with



$$\psi(x) = \sum_{\mathbf{v}} e^{i\mu\mathbf{v}\cdot\mathbf{x}}\left[\psi_{+v}(x) + \psi_{-v}(x)\right]$$

Projection Operator: $P_{\pm} = \frac{1}{2}(1 + \boldsymbol{\sigma}\cdot\mathbf{v})]$

with $P_{\pm}\psi_{\pm v} = \psi_{\pm v},\ P_{\pm}\psi_{\mp v} = 0$

$$\begin{aligned}\mathcal{L}_1 &= \psi^{\dagger}_{+v} iv\cdot D\psi_{+v} + \psi^{\dagger}_{-v}(2\mu + i\bar{v}\cdot D)\psi_{-v} \\ &+ \psi^{\dagger}_{+v} i\not{D}_{\perp}\psi_{-v} + \psi^{\dagger}_{-v} i\not{D}_{\perp}\psi_{+v},\end{aligned}$$

where $\not{D}_{\perp} = \sigma^{\mu}_{\perp} D_{\mu}$ and $\sigma^{\mu}_{\perp} = (0, \boldsymbol{\sigma} - \mathbf{v}(\mathbf{v}\cdot\boldsymbol{\sigma}))$.

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[3] Hong, PLB (1998), NPB (2000)
Schafer NPA (2003)
Son, Yamamoto, PRD (2013)

High Density Effective Theory

Integrating out heavy mode by using EOM

$$\psi_{-v} = \frac{1}{2\mu} \sum_n \left(\frac{-i\bar{v} \cdot D}{2\mu} \right)^n (-i\not{D}_\perp \psi_{+v})$$

Effective Lagrangian:

$$\begin{aligned} \mathcal{L}_{eff} &= \psi^\dagger_{+v} \sum_n D^{(n)} \psi_{+v} \\ &= \psi^\dagger_{+v} \left[iv \cdot D + \frac{\not{D}_\perp^2}{2\mu} + \frac{\not{D}_\perp (-i\bar{v} \cdot D) \not{D}_\perp}{4\mu^2} \right] \psi_{+v}, \end{aligned}$$

uto order $O(1/\mu^2)$

Equations of Motion

- Equation of motion emerging from the effective Lagrangian is satisfied by the two-point function

$$\mathcal{D}_x\, G_v(x,y) = 0, \qquad G_v(x,y)\,\mathcal{D}_y^{\dagger} = 0,$$
$$P_- G_v(x,y) = 0, \qquad G_v(x,y) P_- = 0 \qquad \text{[Projection Condition]}$$

- In terms of central $X = (x+y)/2$ and relative $s = (x-y)$ coordinates, corresponding Wigner function:

$$G_v(X,l) = \int d^4s\, e^{il\cdot s}\, G_v(x,y) \equiv \int_s e^{il\cdot s}\, G_v(x,y),$$

$\int_s = \int d^4 s$, and l^μ : Residual Momenta of particle

Equations of Motion

- The gauge invariant Wigner function is[4]

$$\tilde{G}_v(X,l) = \int_s e^{il\cdot s}\, G_v(X+s/2, X-s/2)U(X-s/2, X+s/2).$$

with Wilson line defined as

$$U(y,x) = P\, exp\Big[-i\int_x^y dz^\mu A_\mu(z)\Big],$$

path ordering P from y to x.

[4] Elze, NPB (1986)
Vasak, Ann. of Phys.,(1987)

Equations of Motion

- The sum and subtracted part of EOM

$$I_{\pm}^{(n)} = \int_s e^{il\cdot s}\left(\mathcal{D}_x^{(n)}\, G_v(x,y) \pm G_v(x,y)\, \mathcal{D}_y^{(n)\dagger}\right)$$

with $n = 0, 1, 2.$

- Taking gradient expansion for derivatives

$$\partial_x = \partial_s + \frac{1}{2}\partial_X, \qquad \partial_y = -\partial_s + \frac{1}{2}\,\partial_X$$

and gauge field

$$A_\mu(X \pm s/2) \approx A_\mu(X) \pm \frac{1}{2}\,(s\cdot\partial_X)\,A_\mu(X) + O(\partial_X^2).$$

neglecting the higher order terms.

Equations of Motion

With contributions from Wilson line $I_{\pm}^{(n)}$

$$I_{+}^{(0)} = 2v\cdot\bar{l}\,\tilde{G}_v, \qquad I_{-}^{(0)} = iv^{\mu}\Delta_{\mu}\tilde{G}_v,$$

$$I_{+}^{(1)} = \frac{1}{\mu}\Big[-\bar{l}_{\perp}^{2} + \mathbf{B}\cdot\mathbf{v}\Big]\tilde{G}_v, \qquad I_{-}^{(1)} = \frac{i}{\mu}\bar{l}_{\perp}^{\mu}\Delta_{\mu}\tilde{G}_v,$$

$$I_{+}^{(2)} = \frac{1}{4\mu^2}\Big[4\bar{l}_{\parallel}\bar{l}_{\perp}^{2} - 4\bar{l}_{\parallel}(\mathbf{B}\cdot\mathbf{v}) + 2\mathbf{B}\cdot\bar{\mathbf{l}}_{\perp} + 2(\mathbf{E}\times\bar{\mathbf{l}})\cdot\mathbf{v}\Big]\tilde{G}_v$$

$$I_{-}^{(2)} = -\frac{i}{4\mu^2}\Big[\Big(4\bar{l}_{\parallel}\bar{l}^{\mu} - \bar{v}^{\mu}(\bar{l}_{\perp}^{2} - \mathbf{B}\cdot\mathbf{v})\Big)\Delta_{\mu} - \Big(\varepsilon^{ijk}v^{k}\bar{v}_{\mu}F^{i\mu}\Big)\Delta_{j}\Big]\tilde{G}_v,$$

$\Delta_{\mu} = \partial_{\mu} - F_{\mu\nu}\frac{\partial}{\partial l_{\nu}}$ and $\bar{l}^{\mu} = (l^{\mu} - A^{\mu})$: kinetic momentum of particle.

Disagree with Son and Yamamoto, PRD (2013) at $O(1/\mu^2)$.

Equations of Motion

From $I_+^{(n)}$ terms

$$\begin{aligned}\tilde{G}_v &= 2\pi\, P_+\, \delta\Big(l_0 - l_\parallel - \frac{1}{2\mu}[l_\perp^2 - \mathbf{B}\cdot\mathbf{v}] + \frac{1}{2\mu^2}[l_\parallel(l_\perp^2 - \mathbf{B}\cdot\mathbf{v})] \\ &+ \frac{1}{4\mu^2}[\mathbf{B}\cdot\mathbf{l}_\perp + (\mathbf{E}\times\mathbf{l})\cdot\mathbf{v}]\Big) n_v(X,l),\end{aligned}$$

$n_v(X,l)$: distribution function.

- PUZZLE: δ-function $\Rightarrow$ dispersion relation depends on $\mathbf{v}$ and not invariant under Reparametrization!

Reparametrization Invariance and Frame Dependence

- Reparametrization is a redundancy under which the physical implications remain unchanged.
- HDET is constructed by dividing momentum[5] $p^\mu = \mu v^\mu + l^\mu,\ v^\mu = (1, \mathbf{v})$, and $v^2 = 0$.
- Decomposition is not unique:
  $v^\mu \longrightarrow v^{\mu'} = v^\mu + \delta v^\mu, \quad l^\mu \longrightarrow l^{\mu'} = l^\mu - \mu\,\delta v^\mu$,
  with $v \cdot \delta v = 0$.

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[5] Killian, Ohl, PRD (1994),
Finkemeier, Georgi, Irvin, PRD (1997)
Sundrum, PRD (1998)

Reparametrization Invariance and Frame Dependence

Under reparametrization transformation (RT)

$$\begin{aligned}\delta\psi_v &= i\mu\delta v\cdot x\psi_v - \frac{\delta\slashed{v}}{2}\left(1-\frac{1}{2\mu+i\bar{v}\cdot D}i\slashed{D}_\perp\right)\psi_v,\\ \delta\psi_v^\dagger &= -i\mu\delta v\cdot x\psi_v^\dagger - \psi_v^\dagger\left(1-i\slashed{D}_\perp\frac{1}{2\mu+i\bar{v}\cdot D}\right)\frac{\delta\slashed{v}}{2}.\end{aligned}$$

Non-local effective Lagrangian

$$\delta\mathcal{L} = \psi_v^\dagger iv\cdot D\psi_v + \psi_v^\dagger\slashed{D}_\perp\frac{1}{2\mu+i\bar{v}\cdot D}\slashed{D}_\perp\psi_v = 0.$$

remains invariant.

Reparametrization Invariance Equations of Motion

Variation of $\tilde{G}_v$

$$\delta \tilde{G}_v(X,l) = \int_s \Big[-\frac{\delta \not{v}}{2}\,\tilde{G}_v(X,l) - \tilde{G}_v(X,l)\,\frac{\delta \not{v}}{2} - \frac{1}{4\mu}\,\varepsilon_{jik}\,\delta v_j\,\Delta_i \sigma^k\,\tilde{G}_v(X,l) + \frac{1}{2\mu}\,\delta v_j\,l_j\,\Delta_{ij}\,\tilde{G}_v(X,l)\Big],$$

with $\Delta_{ij} = \delta_{ij} - v_i\,v_j$

$$\mathrm{tr}\,\delta\tilde{G}_v(X,l) = \tfrac{1}{4\mu}\delta v_j \Delta_i v^k \varepsilon^{ijk}\mathrm{tr}\tilde{G}_v(X,l) + \tfrac{1}{2\mu}\delta v_j l_i \Delta_{ij}\mathrm{tr}\tilde{G}_v(X,l)$$

Arises from antiparticle contribution

Wigner function is not invariant under RT!

Reparametrization Invariance Equations of Motion

$$\delta\, I_{\pm}^{(n)} = \int_s e^{il\cdot s}(\mathcal{D}_x^{(n)} G_v(x,y) \pm G_v(x,y)\mathcal{D}_y^{\dagger(n)}) = 0, \quad \text{Invariant under RT}$$

Contributions of Differential operators and Gauge invariant Wigner function cancels each other.

$$I_{+}^{(0)} = 2v\cdot \bar{l}\,\tilde{G}_v, \qquad I_{-}^{(0)} = iv^{\mu}\Delta_{\mu}\tilde{G}_v,$$

$$I_{+}^{(1)} = \frac{1}{\mu}\Big[-\bar{l}_{\perp}^2 + \mathbf{B}\cdot\mathbf{v}\Big]\,\tilde{G}_v, \qquad I_{-}^{(1)} = \frac{i}{\mu}\bar{l}_{\perp}^{\mu}\Delta_{\mu}\tilde{G}_v,$$

$$I_{+}^{(2)} = \frac{1}{4\mu^2}\Big[4\bar{l}_{\parallel}\bar{l}_{\perp}^2 - 4\bar{l}_{\parallel}(\mathbf{B}\cdot\mathbf{v}) + 2\mathbf{B}\cdot\bar{\mathbf{l}}_{\perp} + 2(\mathbf{E}\times\bar{\mathbf{l}})\cdot\mathbf{v}\Big]\tilde{G}_v$$

$$I_{-}^{(2)} = -\frac{i}{4\mu^2}\Big[\Big(4\bar{l}_{\parallel}\bar{l}^{\mu} - \bar{v}^{\mu}(\bar{l}_{\perp}^2 - \mathbf{B}\cdot\mathbf{v})\Big)\Delta_{\mu} - \Big(\varepsilon^{ijk}v^k\bar{v}_{\mu}F^{i\mu}\Big)\Delta_j\Big]\tilde{G}_v,$$

Dispersion relation and CKE depends on $\mathbf{v}$ and not unique

Expected: antiparticle contribution is $\mathbf{v}$ dependent!

Transport Equation

We use a natural scheme by making a choice $\mathbf{l} \parallel \mathbf{v}$, or equivalently $l_{\|} = l$, $l_{\perp} = 0$[6]

$$I_+^{(0)} = 2v \cdot l\tilde{G}_v, \qquad I_-^{(0)} = iv^\mu \Delta_\mu \tilde{G}_v,$$
$$I_+^{(1)} = \frac{\mathbf{B} \cdot \mathbf{v}}{\mu}\tilde{G}_v, \qquad I_-^{(1)} = 0,$$
$$I_+^{(2)} = -\frac{\mathbf{B} \cdot \mathbf{v}l}{\mu^2}\tilde{G}_v,$$
$$I_-^{(2)} = \frac{1}{4\mu^2}\big[- i\bar{v}^\mu \mathbf{B} \cdot \mathbf{v}\Delta_\mu + i\bar{v}^\nu \epsilon^{ijm} v^m F_{i\nu}\Delta_j\big]\tilde{G}_v.$$

[6] Hands, PRD (2004)

Transport Equation

Combining plus equations as

$$I_+^{(0)} + I_+^{(1)} + I_+^{(2)} = \left[2(l_0 - l) + \frac{\mathbf{B}\cdot\mathbf{v}}{\mu} - \frac{\mathbf{B}\cdot\mathbf{v}\,l}{\mu^2}\right]\tilde{G}_v.$$

- Results into the dispersion relation $l_0 = l - \frac{\mathbf{B}\cdot\mathbf{v}}{2\mu} + \frac{\mathbf{B}\cdot\mathbf{v}\,l}{2\mu^2}$.
- In terms of original momentum

$$p_0 = p - \frac{\mathbf{B}\cdot\hat{\mathbf{p}}}{2p}$$

Same as dispersion relation for particle in magnetic field.

Transport Equation

Using $P_+\tilde{G}_v = \tilde{G}_v P_+ = \tilde{G}_v$, $\tilde{G}_v$ can be parametrize as

$$\tilde{G}_v = 2\pi\delta\Big(l_0 - l + \frac{\mathbf{B}\cdot\mathbf{v}}{2\mu} - \frac{\mathbf{B}\cdot\mathbf{v}l}{2\mu^2}\Big)\, n_v(X,\mathbf{l})P_+.$$

n_v is the distribution function

The transport equation from the $I_-^{(n)}$ terms is

$$\Big[\Delta_0 + v^i\Big(1 + \frac{\mathbf{B}\cdot\mathbf{v}}{2\mu^2}\Big)\Delta_i + \frac{\bar{v}^\nu \epsilon^{ijm} v^m F_{i\nu}\Delta_j}{4\mu^2}\Big]\, n_v(X,l) = 0.$$

Transport Equation

In terms of full momentum $\mathbf{p} = \mu\mathbf{v} + \mathbf{l}$:

$$\left[\Delta_0 + \hat{\mathbf{p}}^i \left(1 + \frac{\mathbf{B}\cdot\hat{\mathbf{p}}}{2p^2}\right)\Delta_i - \frac{\epsilon^{ijk}\hat{\mathbf{p}}^j\mathbf{E}^k + \mathbf{B}^i_\perp}{4p^2}\Delta_i\right] n_v(X,l) = 0.$$

Agrees with OSEFT approach upon identifying a cut-off between two theory[7].

CKE from field theory approach[8]

$$\left[\Delta_0 + \hat{\mathbf{p}}^i \left(1 + \frac{\mathbf{B}\cdot\hat{\mathbf{p}}}{2p^2}\right)\Delta_i - \frac{\epsilon^{ijk}\hat{\mathbf{p}}^j\mathbf{E}^k}{2p^2}\Delta_i\right] n(X,l) = 0$$

[7] Cariganano, Manuel, Torres-Rincon, PRD (2018)
[8] Hidaka, Pu, Yang, PRD (2017), PRD (2018)

Constitutive Equation

From $j^\mu = \psi^\dagger \sigma^\mu \psi$, particle number and total current is

$$\begin{aligned} n &= \frac{1}{(2\pi)^4}\int_l \Big(1 + \frac{1}{2\mu^2}\big[\mathbf{B}\cdot\mathbf{v}\big]\Big)\,\mathrm{tr}\tilde{G}_v(X,l), \\ j^i &= \frac{1}{(2\pi)^4}\int_l \Big[v^i + \frac{1}{2\mu}\Delta_j v^k + \frac{1}{4\mu^2}\Big(- \partial_{Xj} l_\nu \varepsilon^{ijm} v^m \bar{v}^\nu \\ &- 2\mathbf{B}\cdot\mathbf{v}v^i + F_{\nu j}\bar{v}^\nu v^m \varepsilon^{ijm}\Big)\Big]\mathrm{tr}\tilde{G}_v(X,l). \end{aligned}$$

We used scheme condition $\mathbf{l} \parallel \mathbf{v}$ to simplify the expression

Agrees with [Manuel et al. PRD (2018)] after identifying a cut-off.

Equivalence of Chiral Kinetic Equation

Distribution function n and n_v are coefficients of δ-function of $\tilde{G}$ and $\tilde{G}_v$

$$\tilde{G} = \int_s e^{ip\cdot s}\psi(x)\psi^\dagger(y)U(y,x), \quad \tilde{G}_v = \int_s e^{il\cdot s}\psi_v(x)\psi_v^\dagger(y)U(y,x).$$

using $\psi(x) = e^{i\mu\mathbf{v}\cdot\mathbf{x}}\left(1 + \frac{1}{2\mu}(-i\not{D}_\perp)\right)\psi_v(x)$

ψ_v : Dressed particle

$$\mathrm{tr}\tilde{G} = \mathrm{tr}\tilde{G}_v - \frac{1}{4\mu^2} l_i \Delta_j \mathrm{tr}\tilde{G}_v \varepsilon^{ijm} v^m \quad \Rightarrow n = n_v - \frac{1}{4\mu^2} l_i \Delta_j n_v \varepsilon^{ijm} v^m$$

Equivalence of Chiral Kinetic Equations

$$\left[\Delta_0 + \hat{\mathbf{p}}^i \left(1 + \frac{\mathbf{B}\cdot\hat{\mathbf{p}}}{2p^2}\right)\Delta_i - \frac{\epsilon^{ijk}\hat{\mathbf{p}}^j\mathbf{E}^k}{2p^2}\Delta_i\right] n(X,l) = 0$$

[CKE from Field Theory formalism]

$$\Downarrow$$

$$\left[\Delta_0 + \hat{\mathbf{p}}^i \left(1 + \frac{\mathbf{B}\cdot\hat{\mathbf{p}}}{2p^2}\right)\Delta_i - \frac{\epsilon^{ijk}\hat{\mathbf{p}}^j\mathbf{E}^k + \mathbf{B}_\perp^i}{4p^2}\Delta_i\right] n_v(X,l) = 0$$

[CKE from HDET formalism]

Conclusions

- We revisit CKT and find it differs from its counterpart from field theory approach at higher order of $(1/\mu)$. It agrees with the CKE obtained by OSEFT.
- Despite the disagreement, both CKE obtained from field theory and effective field theory formalisms are equivalent with the difference being choices of degrees of freedom.
- Under Reparametrization transformation of Fermi velocity $\mathbf{v}$, distribution function and CKE both transforms. A specific choice $\mathbf{v} \parallel \mathbf{l}$ results into our CKE.

Thanks...