An introduction to Large Momentum Effective Theory

Yizhuang Liu

TDLI

August 24, 2019

Parton Distribution Function

QCD Factorization

- Asymptotic Freedom of QCD: non-perturbative at large distance, perturbative at short distance.

Parton Distribution Function

QCD Factorization

- Asymptotic Freedom of QCD: non-perturbative at large distance, perturbative at short distance.
- High energy process: hard collision at short distance + time dilated bound state effect at large distance $\rightarrow$ factorization.

$$\sigma = \hat{\sigma} \otimes f \,. \tag{1}$$

Parton Distribution Function

QCD Factorization

- Asymptotic Freedom of QCD: non-perturbative at large distance, perturbative at short distance.
- High energy process: hard collision at short distance + time dilated bound state effect at large distance → factorization.

$$\sigma = \hat{\sigma} \otimes f \, . \tag{1}$$

- Non-pertubative function f : PDFs, DAs, GPDs, TMD PDFs etc.

Parton Distribution Function

Light-cone PDF

$$f(x) = \int \frac{dx^-}{2\pi} e^{-iP^+x^-x} \langle P|\bar{\psi}(0,x^-)\gamma^+ \mathcal{W}(0;x^-,0)\psi(0,0)|P\rangle \ . \tag{2}$$

$$x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^1) \ . \tag{3}$$

1. Time dependent; light-cone time x^+ independent.
2. Light-cone quantization with $A^+ = 0$:

$$f(x) \propto \int \frac{d^2k_\perp}{(2\pi)^2} \langle P^+|a^\dagger(xP^+,k_\perp)a(xP^+,k_\perp)|P^+\rangle \ . \tag{4}$$

3. Occupation number.

Parton Distribution Function

Hard for model calculation

- Correct Spectrum/Hilbert Space Structure required
- Explicitly Lorentz Invariance required
- Original quark-gluon operators required

Hard for direct lattice simulation

- Time dependent in equal-time quantization.
- Light cone quantization on lattice suffer from sign problem due to the term $iF^a_{+i}F^a_{-i}$.
- Moments of PDF calculable but suffer from serious power divergence .

PDF again

A close look at PDF

1. $\psi(0, x^-)$: $x^+ = 0, t = -z$. The quark operator travel at speed of light in $-z$ direction.
2. **Infinite rapidity separation** ($v = c$) between the state $|P\rangle$ and the operator $\bar{\psi}(0, x^-)\gamma^+\mathcal{W}(x^-, 0)\psi(0, 0)$.
3. Infinite rapidity replaced by **large but finite rapidity** should lead to the same physics.

The quasi-PDF

Two ways to assign the finite but large rapidity due to **boost invariance**:

1. Carried by the operator: almost equal to the light-cone PDF
2. Carried by the state: **the large momentum effective theory.**

- The operator must have a longitudinal (t or z) separation.

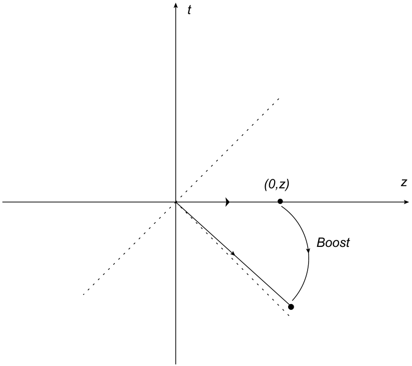

Figure: Under Lorentz boost space-like direction $(0, z, 0, 0)$ approaches light-cone direction

The quasi-PDF

The quasi-PDF:

$$\tilde{f}(x, P^z) = \int \frac{dz}{2\pi} e^{ixP^z z} \langle P^z | \bar{\psi}(z) \Gamma \mathcal{W}(z, 0) \psi(0) | P^z \rangle \ . \qquad (5)$$

1. Time independent: ready for lattice simulation.
2. $x^- \to z$
3. Light cone quantization $\to$ Equal time quantization.
4. In $A^z = 0$ gauge and $\Gamma = \gamma^0$, an occupation number interpretation:

$$\tilde{f}(x, P^z) \propto \int \frac{dk_T^2}{(2\pi)^2} \langle P^z | a^\dagger(xP^z, k_T) a(xP^z, k_T) | P^z \rangle \ . \qquad (6)$$

The quasi-PDF

Spacetime picture:

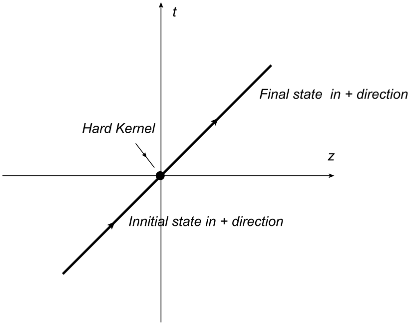

Figure: The spacetime picture for DIS/PDF/quasi-PDF

The large momentum effective theory

The large momentum effective theory

1. Light-cone limit = Infinite momentum limit.
2. Light-cone correlation functions: non-perturbative matrix elements
3. Matching kernel: Wilson coefficients

Momentum region for quasi-PDF

Momentum region analysis

1. Soft region $k = (\Lambda, \Lambda, \Lambda, \Lambda)$: non-perturbative but cancels after k_T integration.
2. Collinear region $k = (P^z, \frac{\Lambda^2}{P^z}, \Lambda)$: good and non-perturbative.
3. UV region $k \propto (P^z, P^z, P^z, P^z)$ or larger: not good, **may be dangerous.**

Momentum region for quasi-PDF

Super-renormalizable theory

1. For Super-renormalizable theory, UV contribution vanish as $P^z \to \infty$. This indicates:

$$\lim_{P^z \to \infty} \tilde{f}(x, P^z) = f(x) \,. \tag{7}$$

2. Verified for 1+1 large N_c spinor and scalar QCD.

Momentum region for quasi-PDF

Renormalizable and asymptotic free theory

1. For QCD: $\Lambda_{QCD} \ll P^z \ll \frac{1}{a}$.

Momentum region for quasi-PDF

Renormalizable and asymptotic free theory

1. For QCD: $\Lambda_{QCD} \ll P^z \ll \frac{1}{a}$.
2. $k_T \approx \Lambda_{QCD}$: non-perturbative contribution, PDF=quasi-PDF+power correction.

Momentum region for quasi-PDF

Renormalizable and asymptotic free theory

1. For QCD: $\Lambda_{QCD} \ll P^z \ll \frac{1}{a}$.
2. $k_T \approx \Lambda_{QCD}$: non-perturbative contribution, PDF=quasi-PDF+power correction.
3. $\Lambda_{QCD} \ll k_T \ll P^z$: the perturbative DGLAP evolution region, PDF=quasi-PDF+power correction.

Momentum region for quasi-PDF

Renormalizable and asymptotic free theory

1. For QCD: $\Lambda_{QCD} \ll P^z \ll \frac{1}{a}$.
2. $k_T \approx \Lambda_{QCD}$: non-perturbative contribution, PDF=quasi-PDF+power correction.
3. $\Lambda_{QCD} \ll k_T \ll P^z$: the perturbative DGLAP evolution region, PDF=quasi-PDF+power correction.
4. $P^z \leq k_T \leq \frac{1}{a}$: the extra **perturbative** UV contribution, PDF $\neq$ quasi-PDF, **need to be subtracted out.**

Momentum region for quasi-PDF

- If one first takes $P^z \to \infty$ then $a \to 0$, then quasi-PDF =PDF.
- $x \ll 1$ or $x \approx 1$, future work.

The matching

For lattice calculation, one always have $P^z \leq \frac{1}{a}$, then how to subtract out unwanted UV contributions? The answer is through factorization :

The factorization formula

$$\tilde{f}(x, P^z, \tilde{\mu}) = \int_{-1}^{1} \frac{dy}{|y|} C(\frac{x}{y}, \frac{\tilde{\mu}}{P^z y}, \frac{\tilde{\mu}}{\mu}) f(y, \mu) + \mathcal{O}(\frac{\Lambda^2_{QCD}}{x^2 (P^z)^2}, \frac{M^2}{(P^z)^2}) . \tag{8}$$

- $C(\frac{x}{y}, \frac{\tilde{\mu}}{P^z y}, \frac{\tilde{\mu}}{\mu})$ is the perturbative matching kernel.
- $\tilde{\mu}$, μ are renormalization scales for quasi-PDF and PDF.
- **Proved** by both diagrammatic method and OPE.

The matching

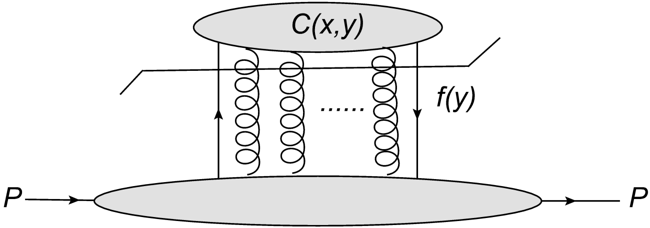

Figure: The matching in diagrammatic language

Renormalization and Evolution

Renormalization

- $\tilde{f}(x, P^z, \tilde{\mu})$ is multiplicative renormalizable, UV divergences being **local** and independent of x and P^z.

Renormalization and Evolution

Renormalization

- $\tilde{f}(x, P^z, \tilde{\mu})$ is multiplicative renormalizable, UV divergences being **local** and independent of x and P^z.
- The gauge link contains linear divergence $e^{-\delta m|z|}$.

Renormalization and Evolution

Renormalization

- $\tilde{f}(x, P^z, \tilde{\mu})$ is multiplicative renormalizable, UV divergences being **local** and independent of x and P^z.
- The gauge link contains linear divergence $e^{-\delta m|z|}$.
- $f(x, \mu)$ is renomalizable by **non-local** counter-terms depending on x: unique feature of light-cone quantity.

Renormalization and Evolution

Renormalization Group Equations

$$\tilde{\mu}\frac{d\tilde{f}(x,P^z,\tilde{\mu})}{d\tilde{\mu}} = \gamma(\alpha_s)\tilde{f}(x,P^z,\tilde{\mu}) \ . \quad (9)$$

$$\mu\frac{df(x,\mu)}{d\mu} = \int_{-1}^{1}\frac{dy}{|y|}K_{\rm DGLAP}(\frac{x}{y},\alpha_s)f(y,\mu) \ . \quad (10)$$

Renormalization and Evolution

Evolution

- DGLAP evolution from $f(x,\mu)$: RG evolution with μ.
- DGLAP evolution from $\tilde{f}(x,P^z,\tilde{\mu})$: Dynamical evolution with P^z.
- $$\frac{d}{d\ln P^z}\tilde{f}(x,P^z,\tilde{\mu}) \approx \int_0^1 dy K_{\mathrm{DGLAP}}(\frac{x}{y},\alpha_s)\tilde{f}(y,P^z,\tilde{\mu})\,. \quad (11)$$
- **Emergence of light-cone feature from large momentum evolution.**

RI/MOM Renormalization

- UV contribution subject to lattice artifact $\mathcal{O}(aP^z)$.
- Regularization Independent(RI) renormalization scheme can be adopted to reduce such lattice artifacts.

RI/MOM

$$\tilde{f}_{\mathrm{RI}}(x, P^z, \mu_R) = \lim_{a\to 0} \frac{\tilde{f}(x, P^z, a)}{Z^{\mathrm{RI}}(\mu_R, a)} = \frac{\tilde{f}(x, P^z, \tilde{\mu})}{Z^{\mathrm{RI}}(\mu_R, \tilde{\mu})}\Big|_{\overline{MS}}\,, \tag{12}$$

$$\tilde{f}_{\mathrm{RI}}(x, P^z, \mu_R) = \int_{-1}^{1} \frac{dy}{|y|} C_{\mathrm{RI}}\left(\frac{x}{y}, \frac{\mu_R}{P^z y}, \frac{\mu_R}{\mu}\right) f(y, \mu)\,. \tag{13}$$

Power divergence

- Moments of PDF:

$$\langle P|\bar{\psi}\gamma^{+}(D^{+})^{n}\psi|P\rangle = a_n(\mu)\ , \tag{14}$$

$$a_n(\mu) = \int_{-1}^{1} x^n f(x,\mu)\ . \tag{15}$$

- Moments of quasi-PDF $\langle P|\bar{\psi}\gamma^{z}(D^{z})^{n}\psi|P\rangle$ contain power divergences.
- After matching, power divergences are **removed** by the matching kernel C.

Application

The LaMET have been applied successfully to:

- Proton unpolarized/helicity/transversity PDF.
- Pion unpolarized PDF and DA.

The matching kernel have been worked out for GPDs as well.

Proton Unpolarized

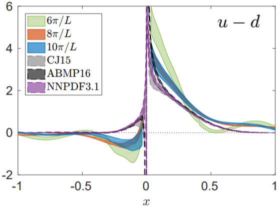

FIG. 4: Comparison of unpolarized PDF at momenta $\frac{6\pi}{L}$ (green band), $\frac{8\pi}{L}$ (orange band), $\frac{10\pi}{L}$ (blue band), and ABMP16 [39] (NNLO), NNPDF [40] (NNLO) and CJ15 [38] (NLO) phenomenological curves.

Figure: The proton unpolarized PDF at $P^z = 0.83, 1.11, 1.38$GeV (ETMC,1803.02685)

Proton transversity

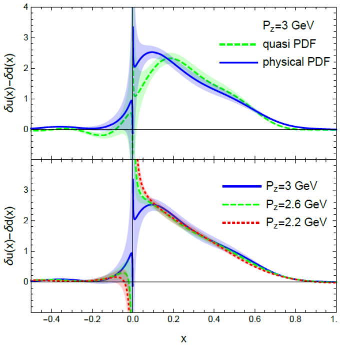

Figure: The proton transversity PDF at $P^z = 3$ GeV, 1810.05043(LP^3)

Proton transversity

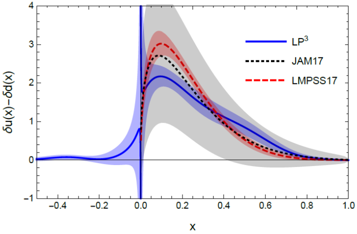

Figure: The proton transversity PDF compared to global fits by JAM17 and LMPSS17 , 1810.05043 (LP^3)

Conclusion and future work

In conclusion, LaMET allows first principle lattice calculation of light-cone quantities.

Conclusion and future work

In conclusion, LaMET allows first principle lattice calculation of light-cone quantities.

The method applies to many other functions, including work in progress:

- The meson DA(distribution-amplitude).
- The generalized parton distribution (GPD).
- TMD
- etc

- X. Ji, Phys. Rev. Lett. **110**, 262002 (2013) doi:10.1103/PhysRevLett.110.262002 [arXiv:1305.1539 [hep-ph]].
- X. Ji, Sci. China Phys. Mech. Astron. **57**, 1407 (2014) doi:10.1007/s11433-014-5492-3 [arXiv:1404.6680 [hep-ph]].
- Y. S. Liu, J. W. Chen, L. Jin, R. Li, H. W. Lin, Y. B. Yang, J. H. Zhang and Y. Zhao, arXiv:1810.05043 [hep-lat].
- C. Alexandrou, K. Cichy, M. Constantinou, K. Jansen, A. Scapellato and F. Steffens, Phys. Rev. Lett. **121**, no. 11, 112001 (2018) doi:10.1103/PhysRevLett.121.112001 [arXiv:1803.02685 [hep-lat]].
- T. Ishikawa, Y. Q. Ma, J. W. Qiu and S. Yoshida, Phys. Rev. D **96**, no. 9, 094019 (2017)

doi:10.1103/PhysRevD.96.094019 [arXiv:1707.03107 [hep-ph]].

T. Izubuchi, X. Ji, L. Jin, I. W. Stewart and Y. Zhao, Phys. Rev. D **98**, no. 5, 056004 (2018) doi:10.1103/PhysRevD.98.056004 [arXiv:1801.03917 [hep-ph]].

Y. Q. Ma and J. W. Qiu, arXiv:1404.6860 [hep-ph].