Generalized Parton Distributions from light-front wave functions

Universidad de Huelva

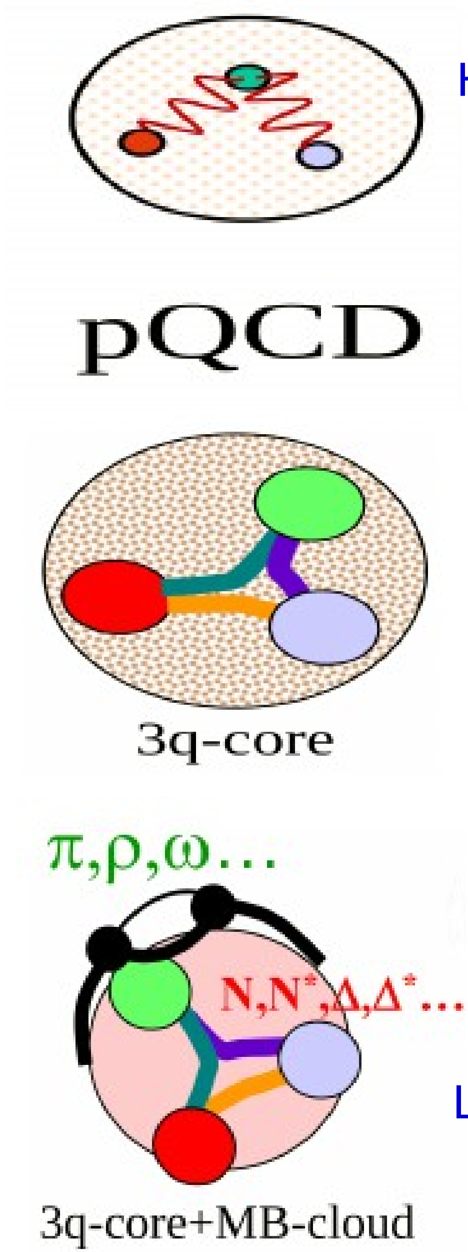
Chang Lei
Khépani Raya
Craig D. Roberts,
José Rodríguez-Quintero,

...

11th Workshop on Hadron Physics in China...; Nankai U. August 22nd- 28th

Hadron Physics. General Motivation.

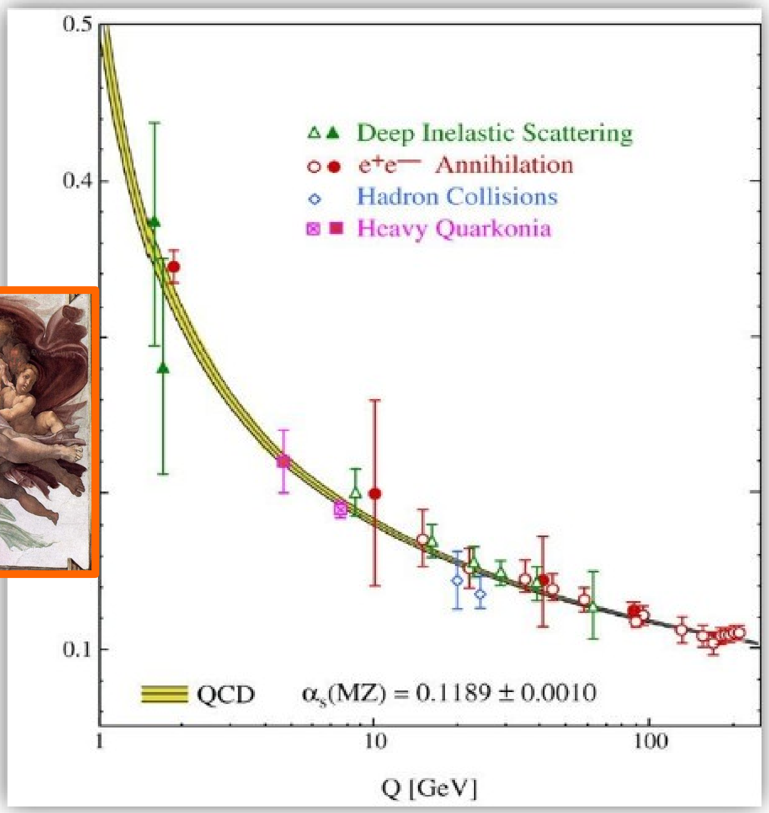
The QCD Holy Grail: the understanding of hadrons in terms of its elementary excitations; namely, quarks and gluons!



High

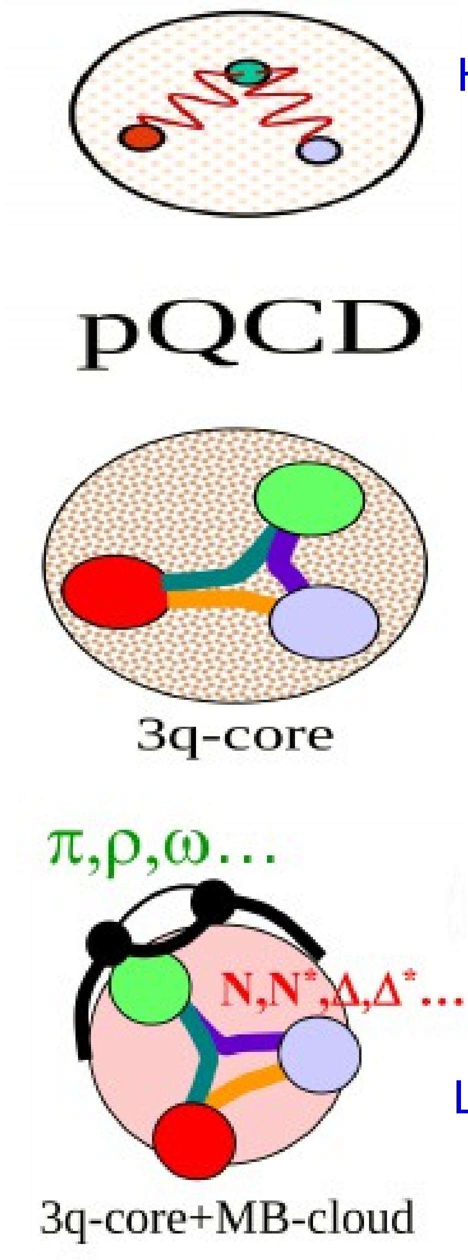
Q^2

Low



Hadron Physics. General Motivation.

The QCD Holy Grail: the understanding of hadrons in terms of its elementary excitations; namely, quarks and gluons!



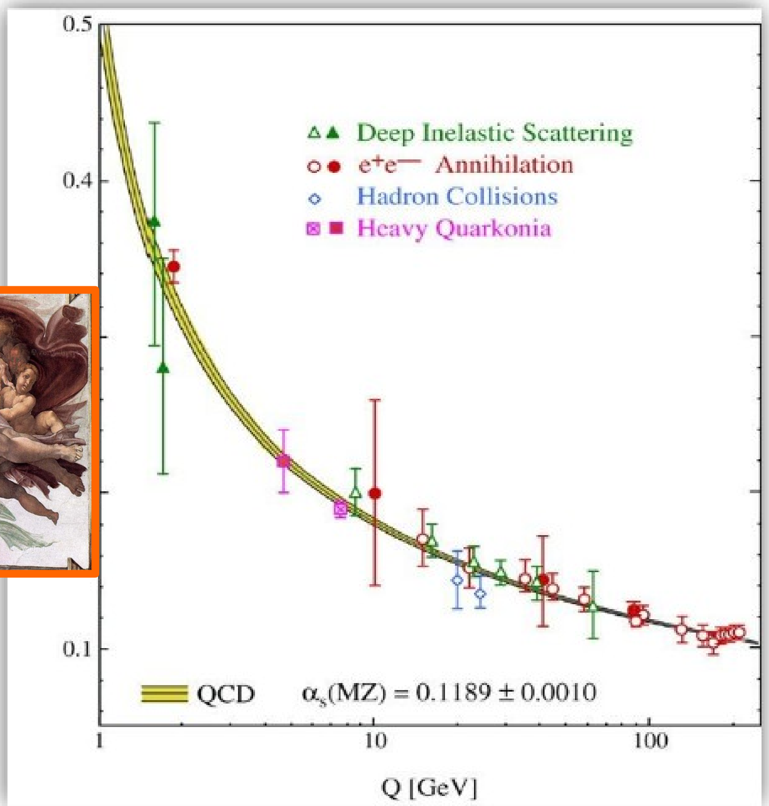
High

Q^2

Low

Confinement

Colored bound states have never been seen to exist as particles in nature



DCSB

Chiral symmetry appears dynamically violated in the Hadron spectrum

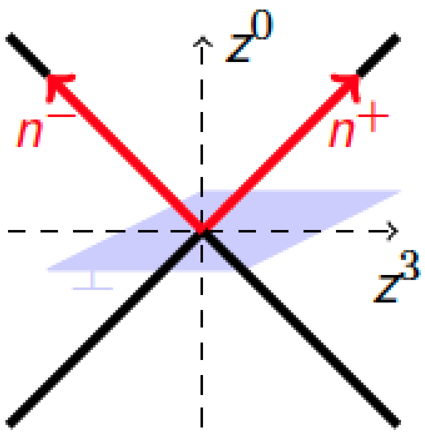
Emergent phenomena playing a dominant role in the real world dominated by the IR dynamics of QCD.

Antecedents:

GPD definition:

$$H_\pi^q(x,\xi,t) = \frac{1}{2}\int \frac{dz^-}{2\pi} e^{ixP^+z^-} \left\langle \pi, P+\frac{\Delta}{2}\right| \bar{q}\left(-\frac{z}{2}\right)\gamma^+ q\left(\frac{z}{2}\right)\left|\pi, P-\frac{\Delta}{2}\right\rangle_{\substack{z^+=0\\ z_\perp=0}}$$

with $t=\Delta^2$ and $\xi = -\Delta^+/(2P^+)$.



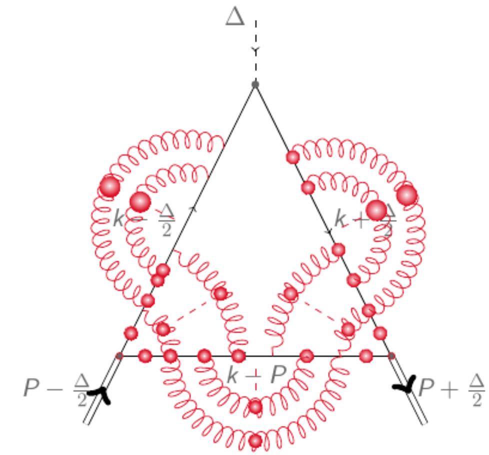
References

Muller et al., Fortchr. Phys. **42**, 101 (1994)
Radyushkin, Phys. Lett. **B380**, 417 (1996)
Ji, Phys. Rev. Lett. **78**, 610 (1997)

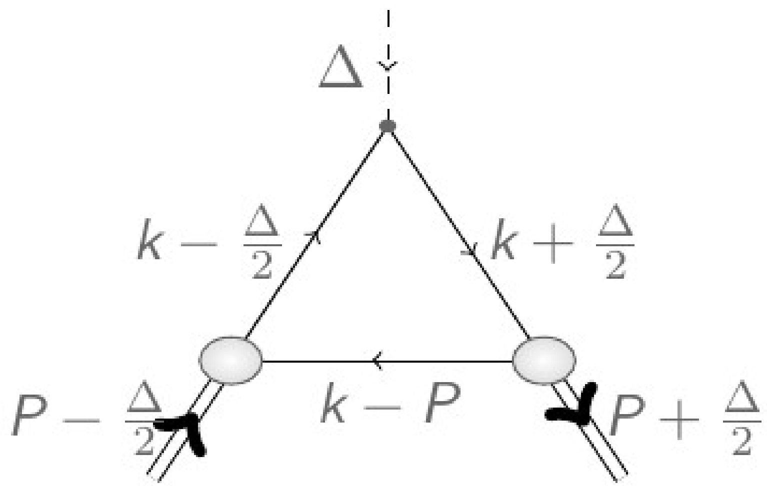
- From **isospin symmetry**, all the information about pion GPD is encoded in $H^u_{\pi^+}$ and $H^d_{\pi^+}$.
- Further constraint from **charge conjugation**: $H^u_{\pi^+}(x,\xi,t) = -H^d_{\pi^+}(-x,\xi,t)$.

Antecedents:

GPDs in the Schwinger-Dyson and Bethe-Salpeter approach

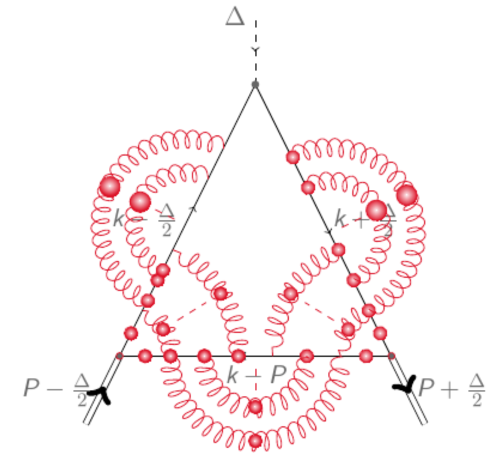


$$\langle x^m \rangle^q = \frac{1}{2(P^+)^{n+1}} \left\langle \pi, P+\frac{\Delta}{2} \middle| \bar{q}(0)\gamma^+(i\overleftrightarrow{D}^+)^m q(0) \middle| \pi, P-\frac{\Delta}{2} \right\rangle$$



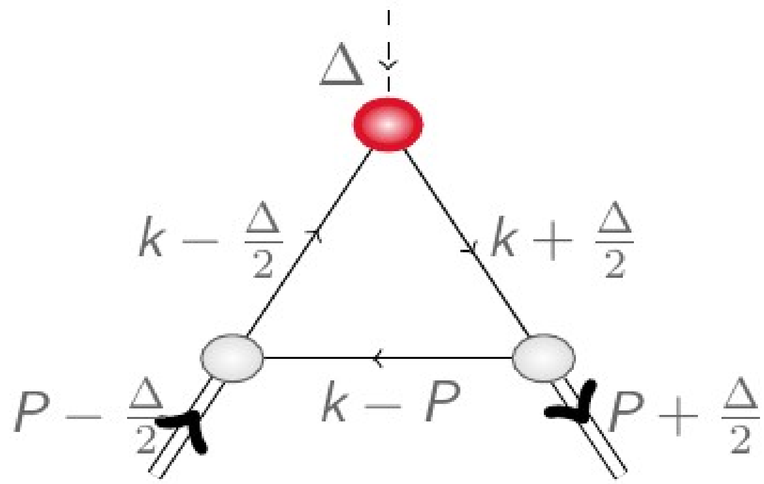
- Compute **Mellin moments** of the pion GPD H.

Antecedents:



GPDs in the Schwinger-Dyson and Bethe-Salpeter approach

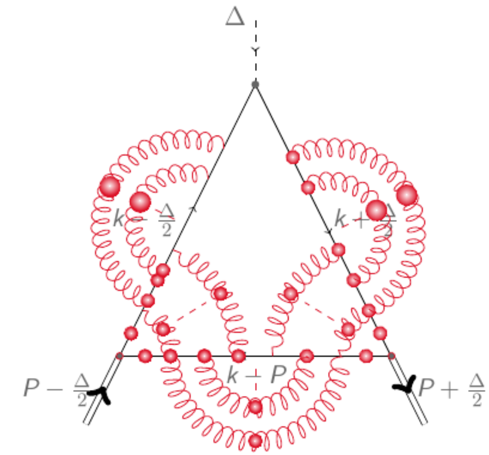
$$\langle x^m\rangle^q = \frac{1}{2(P^+)^{n+1}} \left\langle \pi, P+\frac{\Delta}{2} \left| \bar{q}(0)\gamma^+(i\overleftrightarrow{D}^+)^m q(0)\right| \pi, P-\frac{\Delta}{2}\right\rangle$$



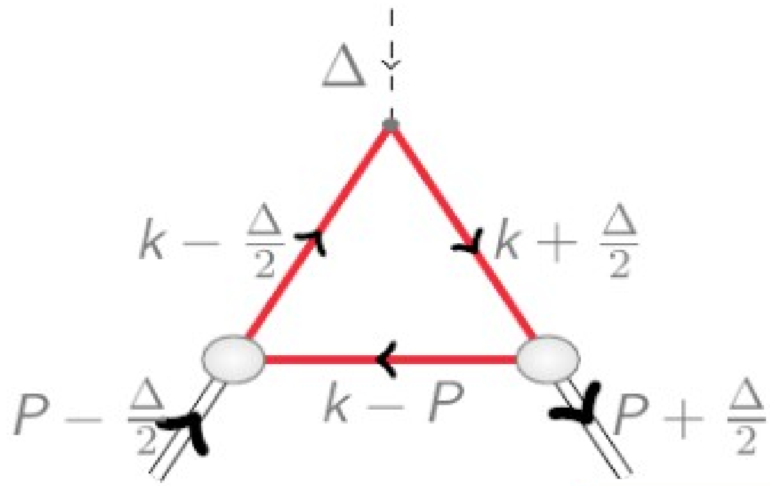
- Compute **Mellin moments** of the pion GPD H.
- Triangle diagram approx.

Antecedents:

GPDs in the Schwinger-Dyson and Bethe-Salpeter approach



$$\langle x^m\rangle^q = \frac{1}{2(P^+)^{n+1}}\left\langle \pi, P+\frac{\Delta}{2}\right| \bar{q}(0)\gamma^+(i\overleftrightarrow{D}^+)^m q(0)\left|\pi, P-\frac{\Delta}{2}\right\rangle$$

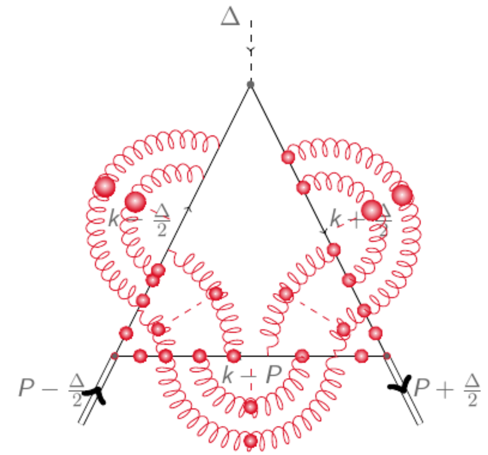


- Compute **Mellin moments** of the pion GPD H.
- Triangle diagram approx.
- Resum **infinitely many** contributions.

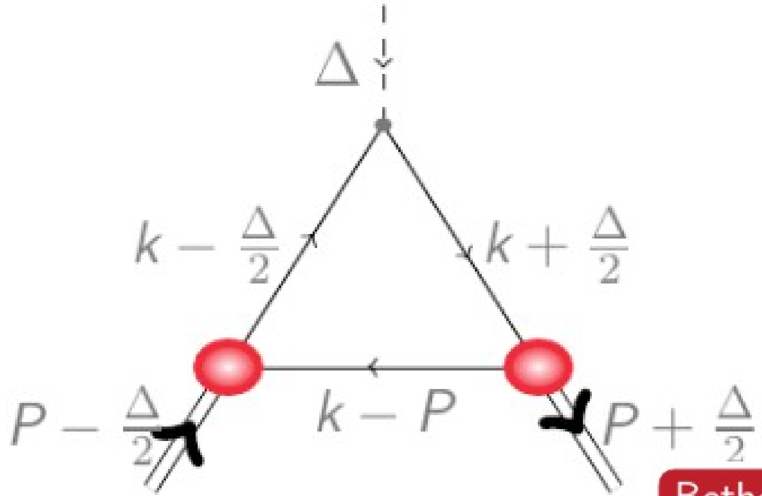
Dyson - Schwinger equation

Antecedents:

GPDs in the Schwinger-Dyson and Bethe-Salpeter approach

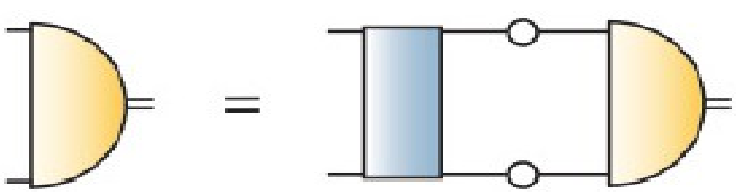


$$\langle x^m \rangle^q = \frac{1}{2(P^+)^{n+1}} \left\langle \pi, P + \frac{\Delta}{2} \middle| \bar{q}(0)\gamma^+ (i\overleftrightarrow{D}^+)^m q(0) \middle| \pi, P - \frac{\Delta}{2} \right\rangle$$



- Compute **Mellin moments** of the pion GPD H.
- Triangle diagram approx.
- Resum **infinitely many** contributions.

Bethe - Salpeter equation



Antecedents:

GPD asymptotic algebraic model:

- Expressions for vertices and propagators:

$$
\begin{aligned}
S(p) &= \left[-i\gamma \cdot p + M \right] \Delta_M(p^2) \\
\Delta_M(s) &= \frac{1}{s + M^2} \\
\Gamma_\pi(k,p) &= i\gamma_5 \frac{M}{f_\pi} M^{2\nu} \int_{-1}^{+1} \mathrm{d}z\, \rho_\nu(z) \, \left[\Delta_M(k_{+z}^2) \right]^\nu \\
\rho_\nu(z) &= R_\nu (1 - z^2)^\nu
\end{aligned}
$$

  with R_ν a normalization factor and $k_{+z} = k - p(1-z)/2$.

  Chang *et al.*, Phys. Rev. Lett. **110**, 132001 (2013)

- Only two parameters:
  - Dimensionful parameter M.
  - Dimensionless parameter ν. **Fixed to 1** to recover asymptotic pion DA.

Antecedents:

GPD asymptotic algebraic model:

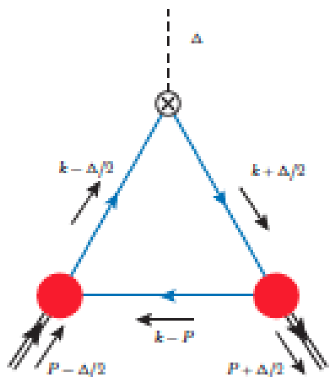
- **Analytic expression** in the DGLAP region.

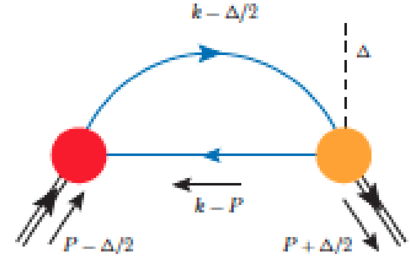
$$
\begin{aligned}
H^{u}_{x\geq\xi}(x,\xi,0) \quad = \quad & \frac{48}{5}\Bigg\{ \frac{3\left(-2(x-1)^4\left(2x^2-5\xi^2+3\right)\log(1-x)\right)}{20\left(\xi^2-1\right)^3} \\
& \frac{3\left(+4\xi\left(15x^2(x+3)+(19x+29)\xi^4+5(x(x(x+11)+21)+3)\xi^2\right)\tanh^{-1}\left(\frac{(x-1)}{x-\xi^2}\right.\right.}{20\left(\xi^2-1\right)^3} \\
& +\frac{3\left(x^3(x(2(x-4)x+15)-30)-15(2x(x+5)+5)\xi^4\right)\log\left(x^2-\xi^2\right)}{20\left(\xi^2-1\right)^3} \\
& +\frac{3\left(-5x(x(x(x+2)+36)+18)\xi^2-15\xi^6\right)\log\left(x^2-\xi^2\right)}{20\left(\xi^2-1\right)^3} \\
& +\frac{3\left(2(x-1)\left((23x+58)\xi^4+(x(x(x+67)+112)+6)\xi^2+x(x((5-2x)x+15)+\right.\right.}{20\left(\xi^2-1\right)^3} \\
& +\frac{3\left(\left(15(2x(x+5)+5)\xi^4+10x(3x(x+5)+11)\xi^2\right)\log\left(1-\xi^2\right)\right)}{20\left(\xi^2-1\right)^3} \\
& +\frac{3\left(2x(5x(x+2)-6)+15\xi^6-5\xi^2+3\right)\log\left(1-\xi^2\right)}{20\left(\xi^2-1\right)^3}\Bigg\}
\end{aligned}
$$

Antecedents:

GPD asymptotic algebraic model (completion):

The full model:



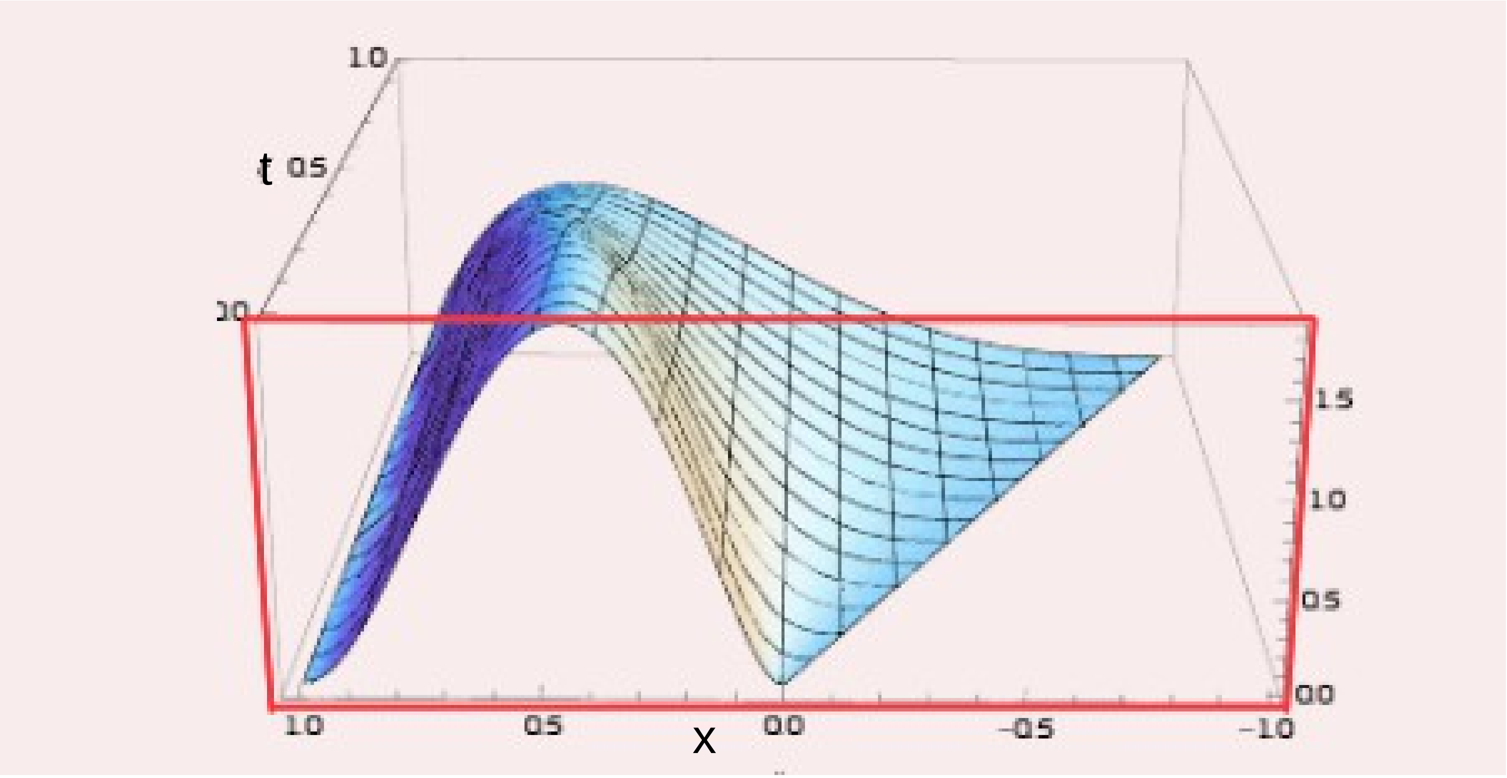


$$
\begin{aligned}
2(P\cdot n)^{m+1}\langle x^m\rangle^u \;=\;& \mathrm{tr}_{CFD}\int \frac{\mathrm{d}^4k}{(2\pi)^4}(k\cdot n)^m\,\tau_+ i\Gamma_\pi\left(\eta(k-P)+(1-\eta)\left(k-\frac{\Delta}{2}\right),P-\frac{\Delta}{2}\right)\\
& S(k-\frac{\Delta}{2})\; i\gamma\cdot n\; S(k+\frac{\Delta}{2})\\
& \tau_- i\bar{\Gamma}_\pi\left((1-\eta)\left(k+\frac{\Delta}{2}\right)+\eta(k-P),P+\frac{\Delta}{2}\right)S(k-P),
\end{aligned}
$$

$$
\begin{aligned}
2(P\cdot n)^{m+1}\langle x^m\rangle^u \;=\;& \mathrm{tr}_{CFD}\int \frac{\mathrm{d}^4k}{(2\pi)^4}(k\cdot n)^m\,\tau_+ i\Gamma_\pi\left(\eta(k-P)+(1-\eta)\left(k-\frac{\Delta}{2}\right),P-\frac{\Delta}{2}\right)\\
& S(k-\frac{\Delta}{2})\tau_-\;\frac{\partial}{\partial k}\bar{\Gamma}_\pi\left((1-\eta)\left(k+\frac{\Delta}{2}\right)+\eta(k-P),P+\frac{\Delta}{2}\right)S(k-P)
\end{aligned}
$$

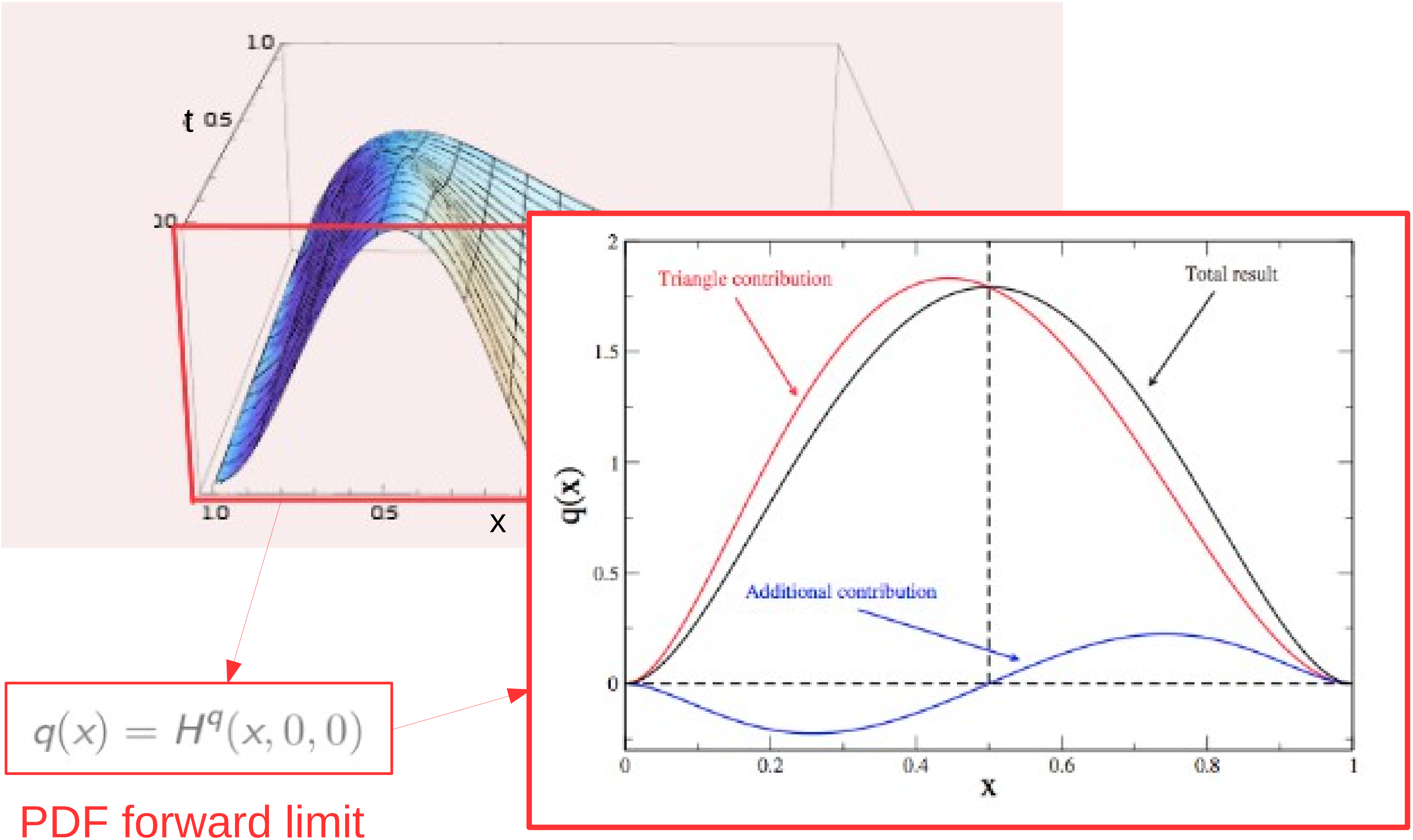
Antecedents:

GPD asymptotic algebraic model (completion):



Antecedents:

GPD asymptotic algebraic model (completion):



$q(x) = H^q(x, 0, 0)$

PDF forward limit

Antecedents:

GPD overlap approach: The pion light front wave function

$$|H;P,\lambda\rangle = \sum_{N,\beta}\int [\mathrm{d}x]_N[\mathrm{d}^2\mathbf{k}_\perp]_N \Psi^\lambda_{N,\beta}(\Omega)\,|N,\beta,k_1\cdots k_N\rangle$$

$$\Omega = (x_1,\mathbf{k}_{\perp 1},\cdots,x_N,\mathbf{k}_{\perp N})$$

$$[\mathrm{d}x]_N = \prod_{i=1}^N \mathrm{d}x_i\,\delta\left(1-\sum_{i=1}^N x_i\right),$$

$$[\mathrm{d}^2\mathbf{k}_\perp]_N = \frac{1}{(16\pi^3)^{N-1}}\prod_{i=1}^N \mathrm{d}^2\mathbf{k}_{\perp i}\,\delta^2\left(\sum_{i=1}^N \mathbf{k}_{\perp i}-\mathbf{P}_\perp\right)$$

N-partons LCWF for the hadron H

$$\sum_{N,\beta}\int [\mathrm{d}x]_N[\mathrm{d}^2\mathbf{k}_\perp]_N|\Psi^\lambda_{N,\beta}(\Omega)|^2 = 1.$$

Let's consider the two-body pion LCWF:

$$|\pi^+,P\rangle|^{2\text{-body}}_{\uparrow\downarrow} = \int \frac{\mathrm{d}^2\mathbf{k}_\perp}{(2\pi)^3}\frac{\mathrm{d}x}{\sqrt{x(1-x)}}\Psi_{\uparrow\downarrow}(k^+,\mathbf{k}_\perp)\Big[b^\dagger_{u\uparrow}(x,\mathbf{k}_\perp)d^\dagger_{d\downarrow}(1-x,-\mathbf{k}_\perp) + b^\dagger_{u\downarrow}(x,\mathbf{k}_\perp)d^\dagger_{d\uparrow}(1-x,-\mathbf{k}_\perp)\Big]|0\rangle\,,$$

$$\Gamma_\pi(k,P) = S^{-1}(-k_2)\,\chi(k,P)\,S^{-1}(k_1).$$

BS wave function

$$2P^+\Psi_{\uparrow\downarrow}(k^+,\mathbf{k}_\perp) = \int\frac{\mathrm{d}k^-}{2\pi}\mathrm{Tr}\left[\gamma^+\gamma_5\chi(k,P)\right]$$

Antecedents:

GPD overlap approach: The pion light front wave function

$$2P^+\Psi_{\uparrow\downarrow}(k^+,\mathbf{k}_\perp) = \int \frac{\mathrm{d}k^-}{2\pi}\mathrm{Tr}\left[\gamma^+\gamma_5\chi(k,P)\right]$$

BS wave function

$$\Gamma_\pi(k,P) = S^{-1}(-k_2)\,\chi(k,P)\,S^{-1}(k_1).$$

- Expressions for vertices and propagators:

$$S(p) = \left[-i\gamma\cdot p + M\right]\Delta_M(p^2)$$

$$\Delta_M(s) = \frac{1}{s+M^2}$$

$$\Gamma_\pi(k,p) = i\gamma_5\frac{M}{f_\pi}M^{2\nu}\int_{-1}^{+1}\mathrm{d}z\,\rho_\nu(z)\,\left[\Delta_M(k_{+z}^2)\right]^\nu$$

$$\rho_\nu(z) = R_\nu(1-z^2)^\nu$$

with R_ν a normalization factor and $k_{+z} = k - p(1-z)/2$.

Chang *et al.*, Phys. Rev. Lett. **110**, 132001 (2013)

Keeping so contact with the previous “covariant” approach” based on DSE and BSE.

$$\Psi_{\uparrow\downarrow}(x,\mathbf{k}_\perp) = -\frac{\Gamma(\nu+1)}{\Gamma(\nu+2)}\frac{M^{2\nu+1}4^\nu R_\nu}{\left[\mathbf{k}_\perp^2+M^2\right]^{\nu+1}}x^\nu(1-x)^\nu.$$

Antecedents:

GPD overlap approach:

Helicity-0 two-body pion LCWF:

$$\Psi_{\uparrow\downarrow}(x,\mathbf{k}_\perp) = -\frac{\Gamma(\nu+1)}{\Gamma(\nu+2)}\frac{M^{2\nu+1}4^\nu R_\nu}{[\mathbf{k}_\perp^2+M^2]^{\nu+1}}x^\nu(1-x)^\nu.$$

GPD in the overlap approach:

$$H(x,\xi,t) = \sqrt{2}\sum_{N,N'}\sum_{\beta,\beta'}\int [\mathrm{d}\hat{x}']_{N'}[\mathrm{d}^2\hat{\mathbf{k}}'_\perp]_{N'}[\mathrm{d}\tilde{x}]_N[\mathrm{d}^2\tilde{\mathbf{k}}_\perp]_N \Psi^*_{N',\beta'}(\hat{\Omega}')\Psi_{N,\beta}(\tilde{\Omega})$$

$$\times \int \frac{\mathrm{d}z^-}{2\pi}e^{iP^+z^-}\langle N',\beta,k'_1\cdots k'_N|\,\phi^{q\dagger}\left(-\frac{z}{2}\right)\phi^q\left(\frac{z}{2}\right)|N,\beta,k_1\cdots k_N\rangle$$

$$= \sum_N \sqrt{1-\xi}^{2-N}\sqrt{1+\xi}^{2-N}\sum_{\beta=\beta'}\sum_j \delta_{s_j q}$$

$$\times \int [\mathrm{d}\bar{x}]_N[\mathrm{d}^2\bar{\mathbf{k}}_\perp]_N \delta(x-\bar{x}_j)\Psi^*_{N,\beta'}(\hat{\Omega}')\Psi_{N,\beta}(\tilde{\Omega})$$

In DGLAP kinematics: $\zeta \leqslant x \leqslant 1$

$$= \int [\mathrm{d}\bar{x}]_2[\mathrm{d}^2\bar{\mathbf{k}}_\perp]_2\delta(x-\bar{x}_j)\Psi^*_{\uparrow\downarrow}(\hat{\Omega}')\Psi_{\uparrow\downarrow}(\tilde{\Omega})$$

In the pion 2-body case

$+$ Helicity-1 component

$$\frac{x-\zeta}{1-\zeta} \qquad \frac{x+\zeta}{1+\zeta}$$

$$= \frac{\Gamma(2\nu+2)}{\Gamma(\nu+2)^2}\int \mathrm{d}u\mathrm{d}v\; u^\nu v^\nu \delta(1-u-v)\frac{\left(2M^{2\nu}4^\nu R_\nu\right)^2\hat{x}^\nu(1-\hat{x})^\nu\tilde{x}^\nu(1-\tilde{x})^\nu}{\left(t\,uv\frac{(1-x)^2}{1-\xi^2}+M^2\right)^{2\nu+1}},$$

Antecedents:

GPD overlap approach:

Helicity-0 two-body pion LCWF:

$$\Psi_{\uparrow\downarrow}(x,\mathbf{k}_\perp) = -\frac{\Gamma(\nu+1)}{\Gamma(\nu+2)}\frac{M^{2\nu+1}4^\nu R_\nu}{\left[\mathbf{k}_\perp^2+M^2\right]^{\nu+1}}x^\nu(1-x)^\nu.$$

GPD in the overlap approach:

$$H(x,\xi,t) = \frac{\Gamma(2\nu+2)}{\Gamma(\nu+2)^2}\int \mathrm{d}u\mathrm{d}v\, u^\nu v^\nu \delta(1-u-v)\frac{\left(2M^{2\nu}4^\nu R_\nu\right)^2 \hat{x}^\nu(1-\hat{x})^\nu \tilde{x}^\nu(1-\tilde{x})^\nu}{\left(t\,uv\frac{(1-x)^2}{1-\xi^2}+M^2\right)^{2\nu+1}}, \qquad \xi \leqslant x \leqslant 1$$

$$\hat{x} = \frac{x-\xi}{1-\xi}, \qquad \tilde{x} = \frac{x+\xi}{1+\xi}$$

$$= 30\frac{(1-x)^2(x^2-\xi^2)}{(1-\xi^2)^2}\frac{1}{(1+z)^2}\left(\frac{3}{4}+\frac{1}{4}\frac{1-2z}{1+z}\frac{\operatorname{arctanh}\sqrt{\frac{z}{1+z}}}{\sqrt{\frac{z}{1+z}}}\right)$$

$$z = \frac{t}{4M^2}\frac{(1-x)^2}{1-\xi^2}$$

Encoding the correlations of kinematical variables

Antecedents:

GPD overlap approach:

Helicity-0 two-body pion LCWF:

$$\Psi_{\uparrow\downarrow}(x,\mathbf{k}_\perp) = -\frac{\Gamma(\nu+1)}{\Gamma(\nu+2)}\frac{M^{2\nu+1}4^\nu R_\nu}{[\mathbf{k}_\perp^2+M^2]^{\nu+1}}x^\nu(1-x)^\nu.$$

GPD in the overlap approach:

$$H(x,\xi,t) = 30\frac{(1-x)^2(x^2-\xi^2)}{(1-\xi^2)^2}\frac{1}{(1+z)^2}\left(\frac{3}{4}+\frac{1}{4}\frac{1-2z}{1+z}\ \frac{\text{arct}\ldots\sqrt{\frac{z}{\ldots}}}{\sqrt{1+z}}\right)$$

0 1

$$0 \leqslant x \leqslant 1$$

Forward limit

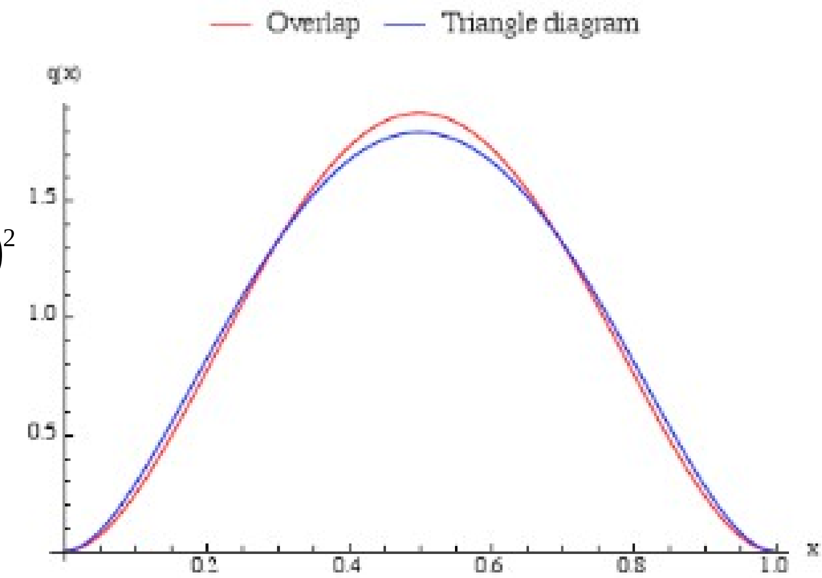
$$z = \frac{t}{4M^2}\frac{(1-x)^2}{1-\xi^2}$$

Encoding the correlations of kinematical variables

PDF:

$$H(x,0,0) = q(x) = 30\,x^2(1-x)^2$$

Compares numerically very well with the results obtained from the Triangle diagram!!!



Consistent descriptions from both approaches!!! (tested with a simple model)

Pion (kaon maybe) realistic picture:

- The pseudoscalar LFWF can be written:

$$f_K \psi_K^{\uparrow\downarrow}(x,k_\perp^2) = \mathrm{tr}_{CD} \int_{dk_{||}} \delta(n\cdot k - x n\cdot P_K)\gamma_5 \gamma\cdot n \chi_K^{(2)}(k_-^K;P_K) \ .$$

- The moments of the distribution are given by:

$$< x^m >_{\psi_K^{\uparrow\downarrow}} = \int_0^1 dx x^m \psi_K^{\uparrow\downarrow}(x,k_\perp^2) = \frac{1}{f_K n\cdot P}\int_{dk_{||}} \left[\frac{n\cdot k}{n\cdot P}\right]^m \gamma_5\gamma\cdot n \chi_K^{(2)}(k_-^K;P_K)$$

$$\int_0^1 d\alpha \alpha^m \left[\frac{12}{f_K}\mathcal{Y}_K(\alpha;\sigma^2)\right] \ , \quad \mathcal{Y}_K(\alpha;\sigma^2) = [M_u(1-\alpha)+M_s\alpha]\mathcal{X}(\alpha;\sigma_\perp^2) \ .$$

Uniqueness of Mellin moments ⟶

$$\psi_K^{\uparrow\downarrow}(x,k_\perp^2) = \frac{12}{f_K}\mathcal{Y}_K(x;\sigma_\perp^2)$$

$$\chi_K(\alpha;\sigma^3) = \left[\int_{-1}^{1-2\alpha} d\omega \int_{1+\frac{2\alpha}{\omega-1}}^{1} dv + \int_{1-2\alpha}^{1} d\omega \int_{\frac{\omega-1+2\alpha}{\omega+1}}^{1} dv\right] \frac{\rho_K(\omega)}{n_K}\frac{\Lambda_K^2}{\sigma^3} \ .$$

The spectral density $\rho_K(z)$ can be modelled...
...Or taken with BSE solutions as an input!

$$\Rightarrow \psi_K^{\uparrow\downarrow}(x,k_\perp^2) \sim \int d\omega \ \cdots \rho_K(\omega) \cdots$$

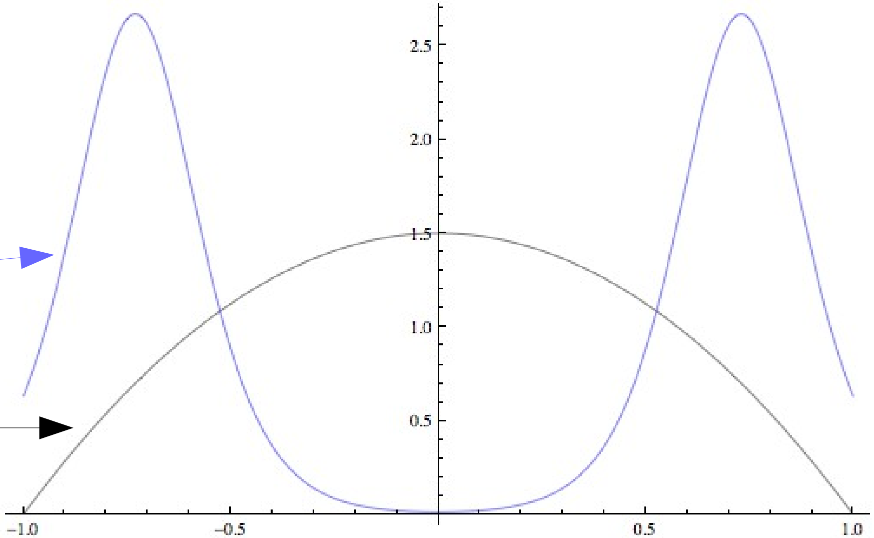
Pion realistic picture:

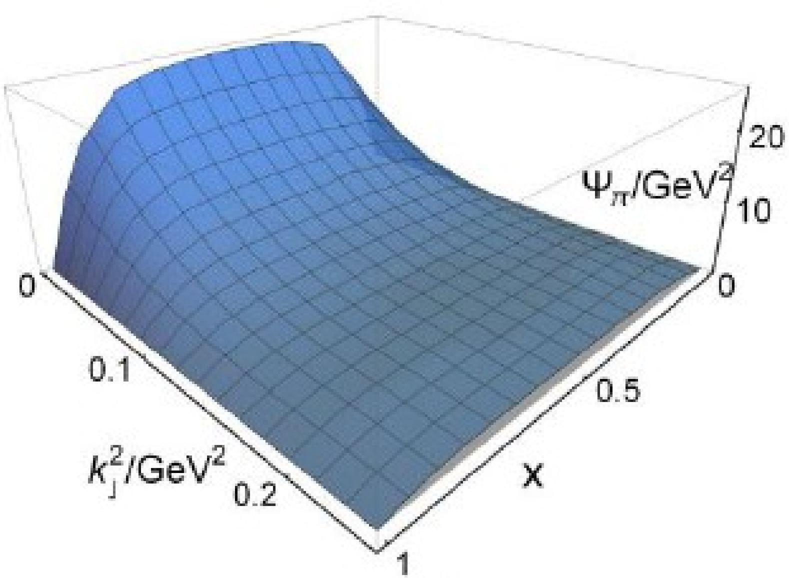
- Spectral density is chosen as:

$$u_G \rho_G(\omega) = \frac{1}{2b_0^G}\left[\mathrm{sech}^2\left(\frac{\omega - \omega_0^G}{2b_0^G}\right) + \mathrm{sech}^2\left(\frac{\omega + \omega_0^G}{2b_0^G}\right)\right]$$

Phenomelogical model: $b_0^\pi = 0.1, b_0^\pi = 0.73;$

Asymptotic case: $\rho(\omega;\nu) \sim (1-\omega^2)^\nu$



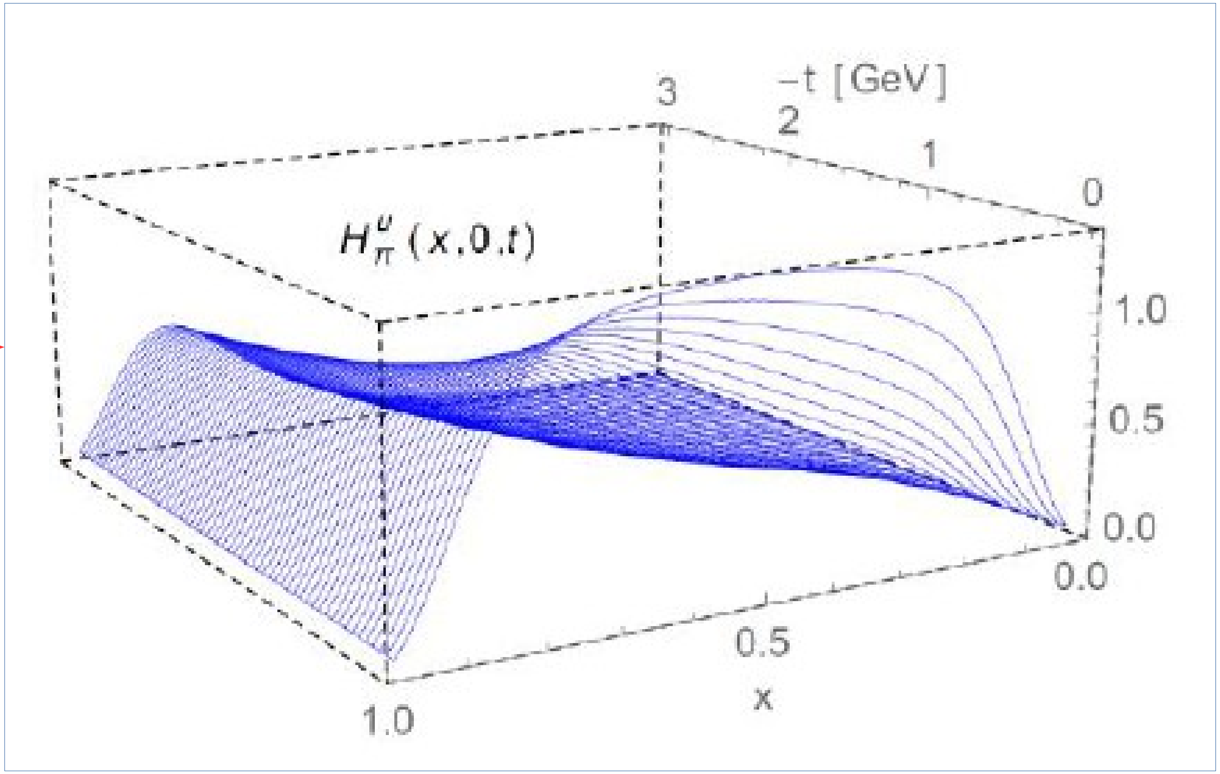


Pion realistic picture:

GPD overlap representation:

$$H_M^q(x,\xi,t) = \int \frac{d^2\mathbf{k}_\perp}{16\,\pi^3} \Psi^*_{u\bar{f}}\left(\frac{x-\xi}{1-\xi}, \mathbf{k}_\perp + \frac{1-x}{1-\xi}\frac{\Delta_\perp}{2}\right) \Psi_{u\bar{f}}\left(\frac{x+\xi}{1+\xi}, \mathbf{k}_\perp - \frac{1-x}{1+\xi}\frac{\Delta_\perp}{2}\right)$$

Phenomenological model

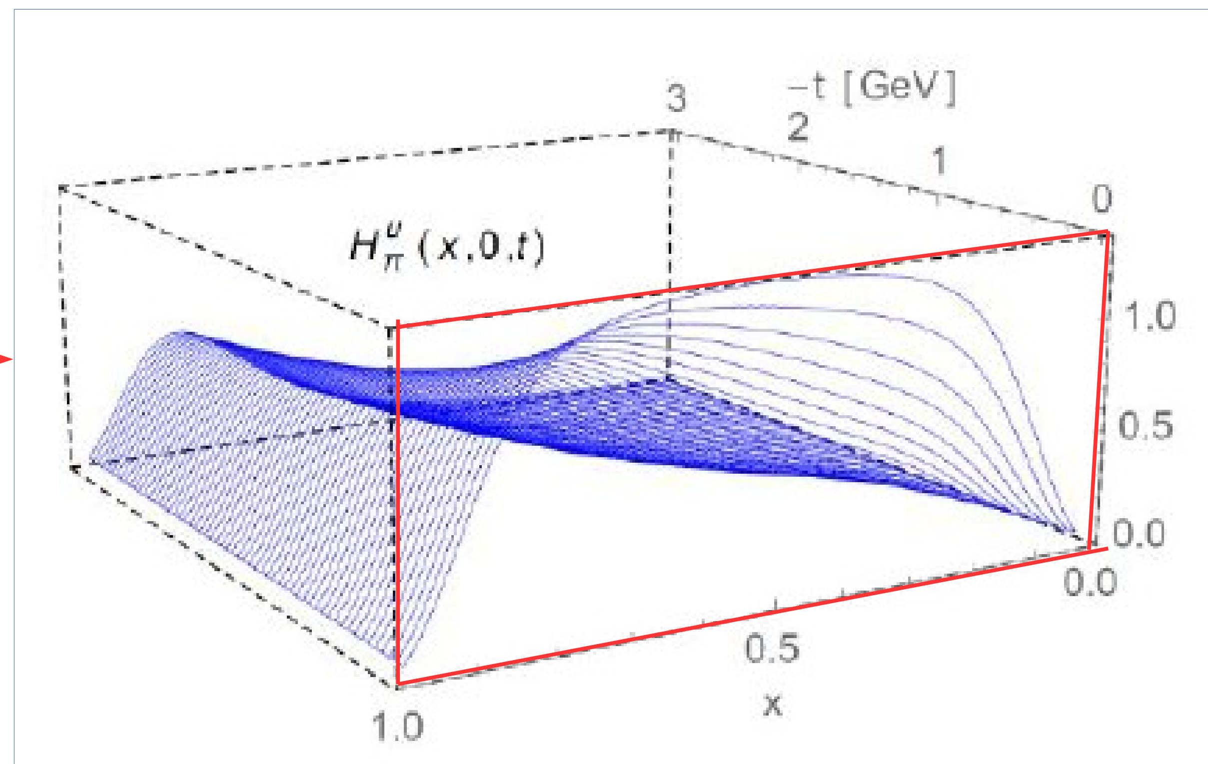


Pion realistic picture: PDF as benchmark

GPD overlap representation: forward limit

$$H_M^q(x,\xi,t) = \int \frac{\mathrm{d}^2\mathbf{k}_\perp}{16\,\pi^3} \Psi^*_{u\bar{f}}\left(\frac{x-\xi}{1-\xi}, \mathbf{k}_\perp + \frac{1-x}{1-\xi}\frac{\Delta_\perp}{2}\right) \Psi_{u\bar{f}}\left(\frac{x+\xi}{1+\xi}, \mathbf{k}_\perp - \frac{1-x}{1+\xi}\frac{\Delta_\perp}{2}\right)$$

Phenomenological model

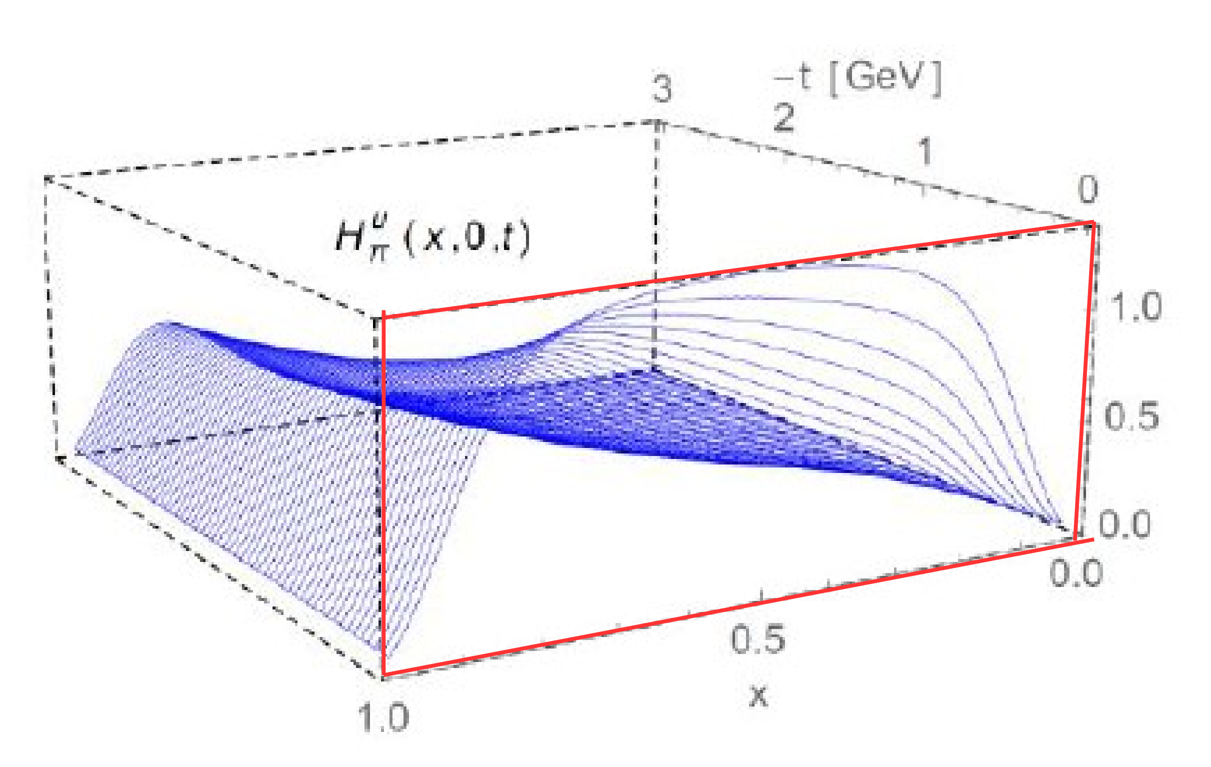


Pion realistic picture: PDF as benchmark

The pion PDF can be computed as the lightfront projection of the hadronic matrix element of a bilocal operator that, in the overlap representation at low Fock states, can be expressed in terms of 2-body LFWFs at a given hadronic scale

$$q^\pi(x;\zeta_H) = \frac{1}{2}\int \frac{dz^-}{2\pi} e^{ixP^+z^-} \left\langle P \middle| \overline{\psi}^q(-z)\gamma^+\psi^q(z) \middle| P \right\rangle \Big|_{z^+=0, z_\perp=0} = \int \frac{d^2k_\perp}{16\pi^3} \Psi^*_{u\bar{f}}(x,\mathbf{k}_\perp)\,\Psi_{u\bar{f}}(x,\mathbf{k}_\perp)$$

Phenomenological model



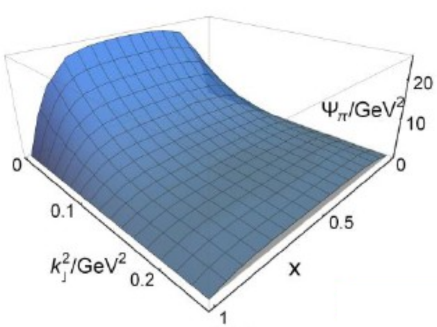
Pion realistic picture: PDF as benchmark

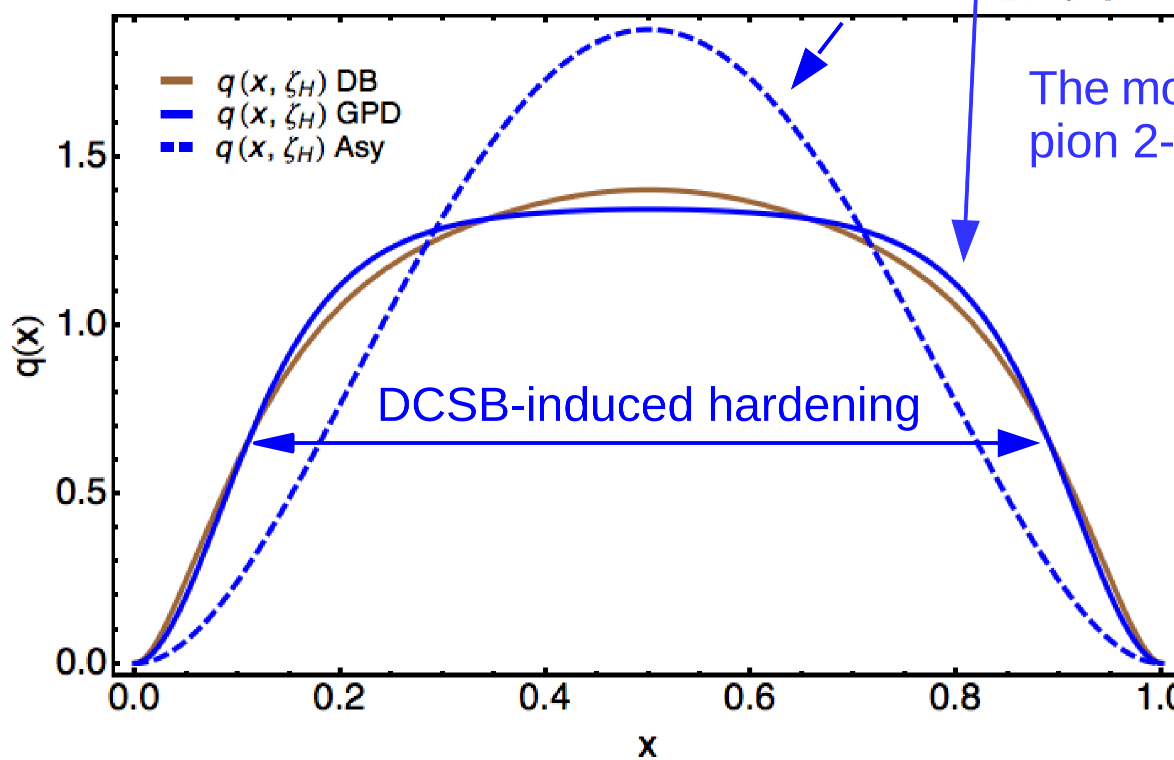
The pion PDF can be computed as the lightfront projection of the hadronic matrix element of a bilocal operator and, in the overlap representation at low Fock states, can be expressed in terms of 2-body LFWFs at a given hadronic scale

$$q^\pi(x;\zeta_H) = \frac{1}{2}\int \frac{dz^-}{2\pi} e^{ixP^+z^-} \left\langle P \middle| \overline{\psi}^q(-z)\gamma^+\psi^q(z) \middle| P \right\rangle \Big|_{z^+=0,z_\perp=0} = \int \frac{d^2k_\perp}{16\pi^3} \Psi^*_{u\bar{f}}(x,\mathbf{k}_\perp)\,\Psi_{u\bar{f}}(x,\mathbf{k}_\perp)$$

LFWF leading to asymptotic PDAs

$$q_{\rm sf}(x) \approx 30\,x^2(1-x)^2$$





The more realistic pion 2-body LFWF

Pion realistic picture: PDF as benchmark

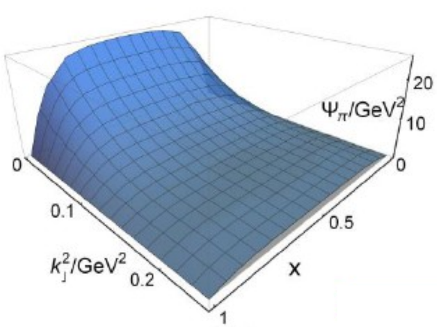
The pion PDF can be computed as the lightfront projection of the hadronic matrix element of a bilocal operator and, in the overlap representation at low Fock states, can be expressed in terms of 2-body LFWFs at a given hadronic scale

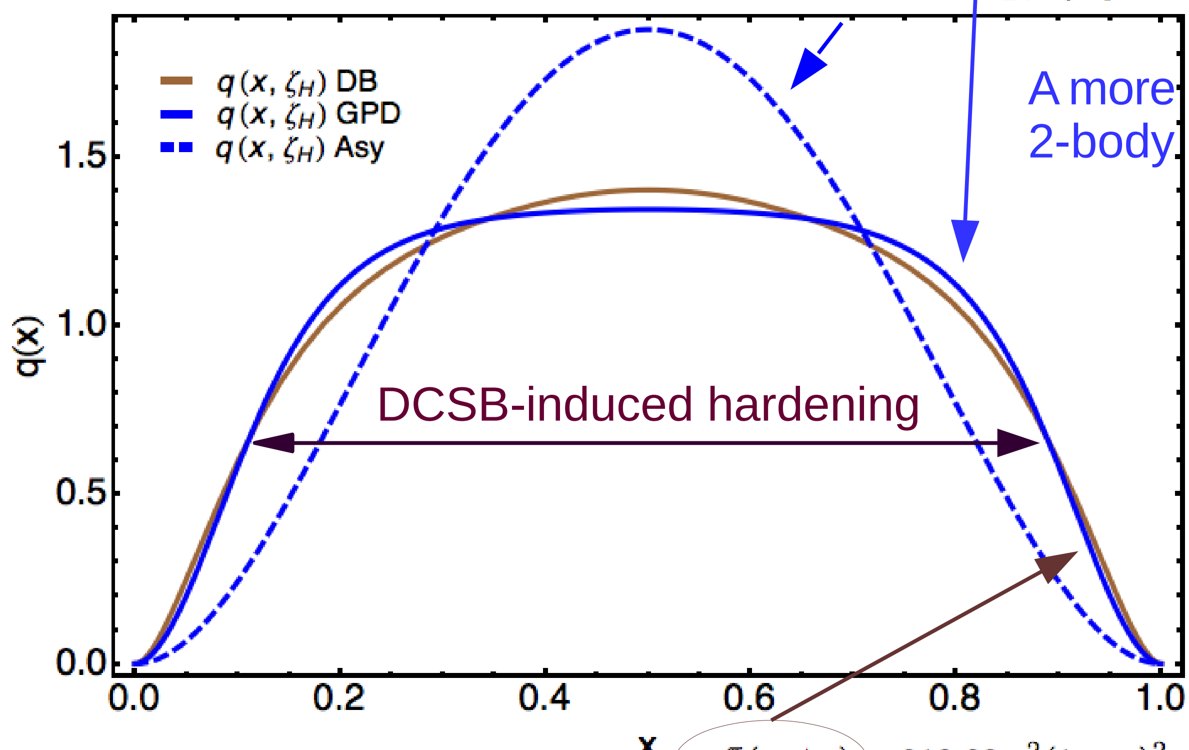
$$q^\pi(x;\zeta_H) = \frac{1}{2}\int \frac{dz^-}{2\pi} e^{ixP^+z^-} \left\langle P \middle| \bar{\psi}^q(-z)\gamma^+\psi^q(z) \middle| P \right\rangle \Big|_{z^+=0,z_\perp=0} = \int \frac{d^2k_\perp}{16\pi^3} \Psi^*_{u\bar{f}}(x,\mathbf{k}_\perp)\, \Psi_{u\bar{f}}(x,\mathbf{k}_\perp)$$

LFWF leading to asymptotic PDAs

$$q_{\rm sf}(x) \approx 30\,x^2(1-x)^2$$

A more realistic pion 2-body LFWF





Direct computation of Mellin moments:

$$\langle x^m\rangle^\pi_{\zeta_H} = \int_0^1 dx\, x^m q^\pi(x;\zeta_H)$$

$$= \frac{N_c}{n\cdot P}\mathrm{tr}\int_{dk}\left[\frac{n\cdot k_\eta}{n\cdot P}\right]^m \Gamma_\pi(k_{\bar{\eta}},P)\, S(k_{\bar{\eta}})\, n\cdot\partial_{k_\eta}\left[\Gamma_\pi(k_\eta,-P)S(k_\eta)\right]$$

$$q^\pi(x;\zeta_H) = 213.32\,x^2(1-x)^2 \times \left[1 - 2.9342\sqrt{x(1-x)} + 2.2911\,x(1-x)\right]$$

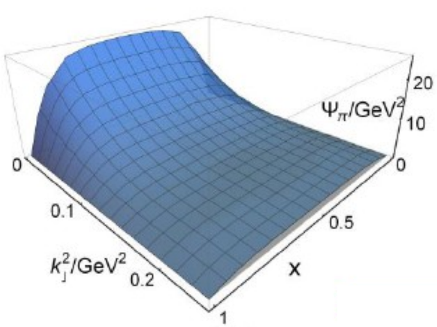
Pion realistic picture: PDF as benchmark

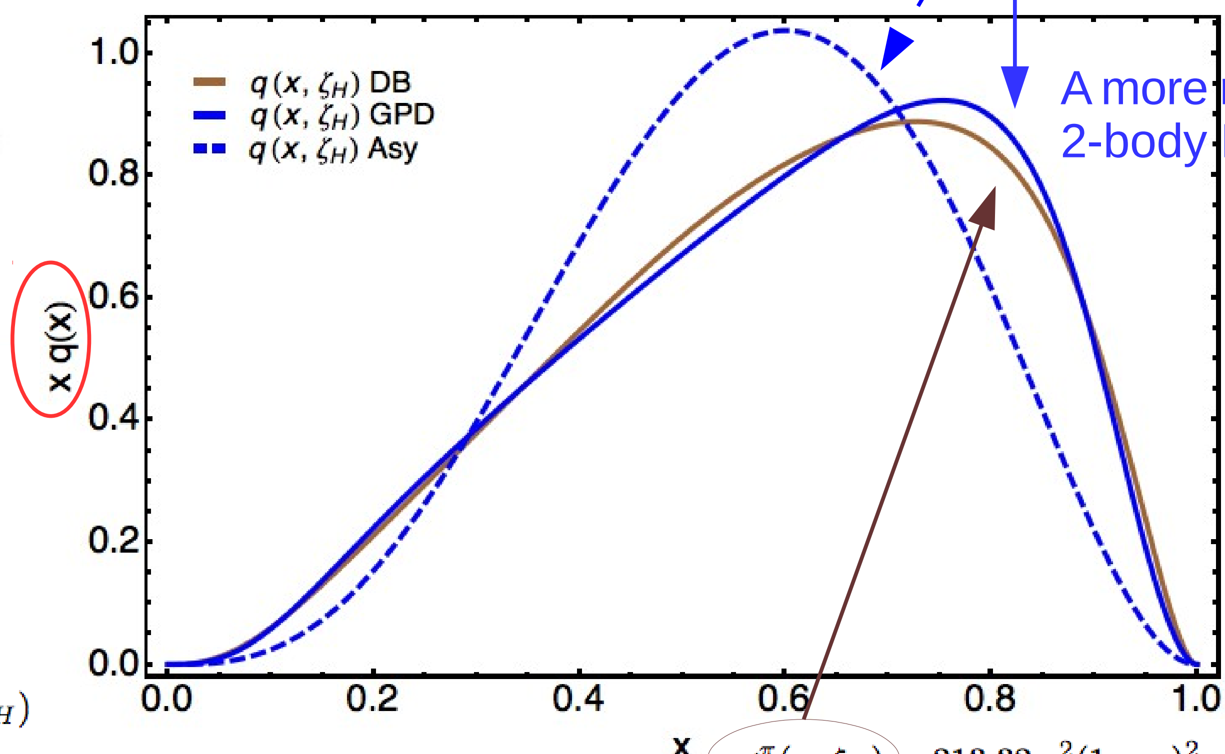
The pion PDF can be computed as the lightfront projection of the hadronic matrix element of a bilocal operator and, in the overlap representation at low Fock states, can be expressed in terms of 2-body LFWFs at a given hadronic scale

$$q^\pi(x;\zeta_H) = \frac{1}{2}\int \frac{dz^-}{2\pi} e^{ixP^+z^-} \left\langle P \middle| \overline{\psi}^q(-z)\gamma^+\psi^q(z) \middle| P \right\rangle \Big|_{z^+=0, z_\perp=0} = \int \frac{d^2k_\perp}{16\pi^3} \Psi^*_{u\bar{f}}(x,\mathbf{k}_\perp)\, \Psi_{u\bar{f}}(x,\mathbf{k}_\perp)$$

LFWF leading to asymptotic PDAs

$$q_{\rm sf}(x) \approx 30\, x^2(1-x)^2$$





A more realistic pion 2-body LFWF

Direct computation of Mellin moments:

$$\langle x^m \rangle^\pi_{\zeta_H} = \int_0^1 dx\, x^m q^\pi(x;\zeta_H)$$

$$= \frac{N_c}{n\cdot P} \mathrm{tr} \int_{dk} \left[\frac{n\cdot k_\eta}{n\cdot P}\right]^m \Gamma_\pi(k_{\bar{\eta}},P)\, S(k_{\bar{\eta}})\, n\cdot\partial_{k_\eta} \left[\Gamma_\pi(k_\eta,-P) S(k_\eta)\right]$$

$$q^\pi(x;\zeta_H) = 213.32\, x^2(1-x)^2 \times \left[1 - 2.9342\sqrt{x(1-x)} + 2.2911\, x(1-x)\right]$$

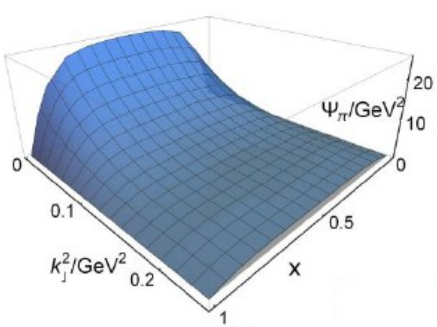
Pion realistic picture: PDF as benchmark

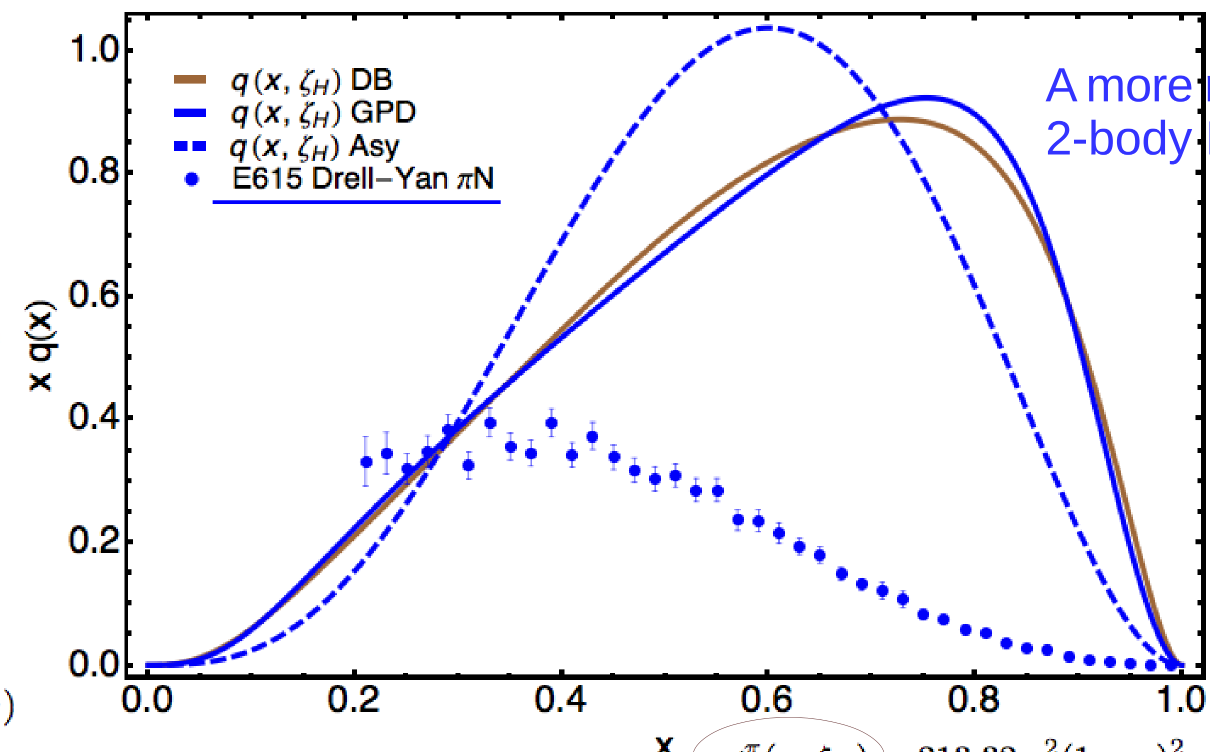
The pion PDF can be computed as the lightfront projection of the hadronic matrix element of a bilocal operator and, in the overlap representation at low Fock states, can be expressed in terms of 2-body LFWFs at a given hadronic scale

$$q^\pi(x;\zeta_H) = \frac{1}{2}\int \frac{dz^-}{2\pi} e^{ixP^+z^-} \left\langle P \middle| \overline{\psi}^q(-z)\gamma^+\psi^q(z) \middle| P \right\rangle \Big|_{z^+=0,z_\perp=0} = \int \frac{d^2k_\perp}{16\pi^3} \Psi^*_{u\bar{f}}(x,\mathbf{k}_\perp)\,\Psi_{u\bar{f}}(x,\mathbf{k}_\perp)$$

LFWF leading to asymptotic PDAs

$$q_{\rm sf}(x) \approx 30\,x^2(1-x)^2$$





A more realistic pion 2-body LFWF

$$q^\pi(x;\zeta_H) = 213.32\,x^2(1-x)^2 \times \left[1 - 2.9342\sqrt{x(1-x)} + 2.2911\,x(1-x)\right]$$

Direct computation of Mellin moments:

$$\langle x^m \rangle^\pi_{\zeta_H} = \int_0^1 dx\, x^m q^\pi(x;\zeta_H)$$

$$= \frac{N_c}{n\cdot P}\mathrm{tr}\int_{dk}\left[\frac{n\cdot k_\eta}{n\cdot P}\right]^m \Gamma_\pi(k_{\bar{\eta}},P)\,S(k_{\bar{\eta}})\,n\cdot\partial_{k_\eta}\left[\Gamma_\pi(k_\eta,-P)S(k_\eta)\right]$$

Pion realistic picture: PDF as benchmark

The pion PDF can be computed as the lightfront projection of the hadronic matrix element of a bilocal operator and, in the overlap representation at low Fock states, can be expressed in terms of 2-body LFWFs at a given hadronic scale

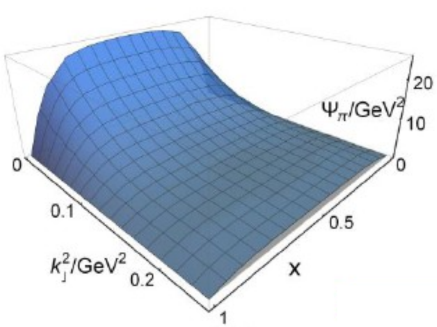
$$q^\pi(x;\zeta_H) = \frac{1}{2}\int \frac{dz^-}{2\pi} e^{ixP^+z^-} \left\langle P \left| \overline{\psi}^q(-z)\gamma^+\psi^q(z) \right| P \right\rangle \Big|_{z^+=0, z_\perp=0} = \int \frac{d^2k_\perp}{16\pi^3} \Psi^*_{u\bar{f}}(x,\mathbf{k}_\perp)\, \Psi_{u\bar{f}}(x,\mathbf{k}_\perp)$$

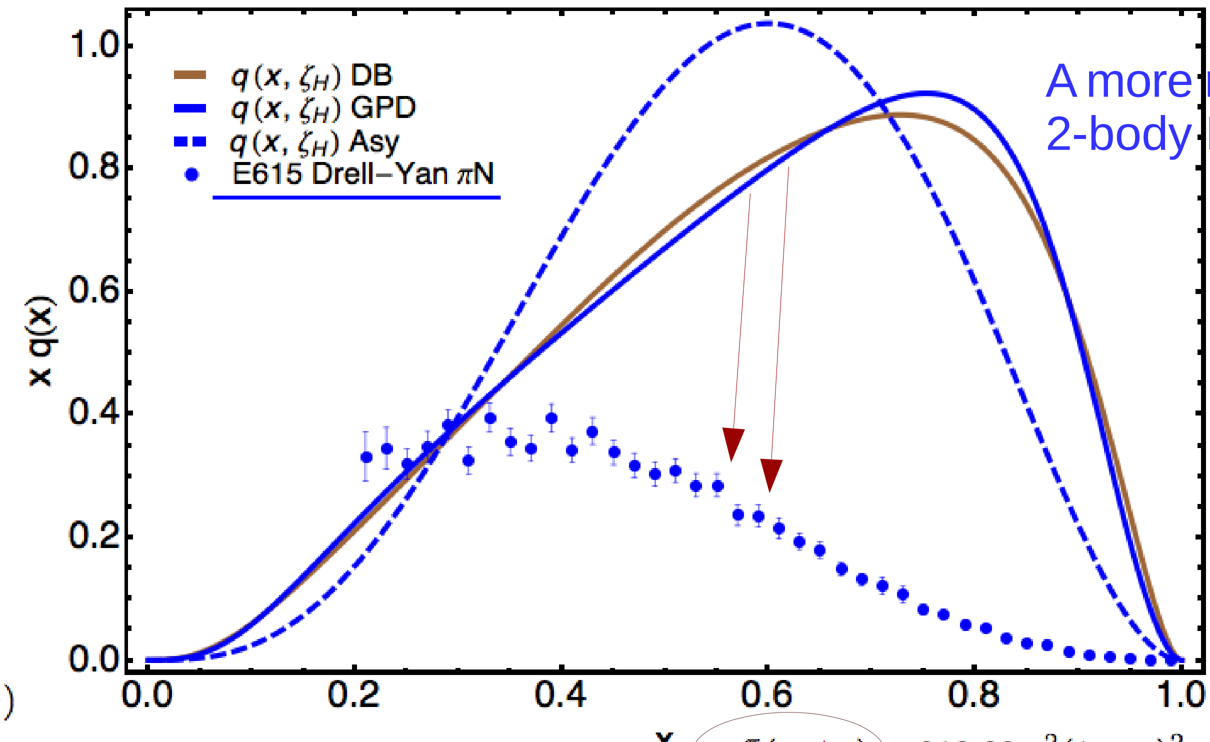
c.f. Craig Roberts' talk!

$\zeta_H \rightarrow \zeta_2 = 5.2$ GeV

LFWF leading to asymptotic PDAs

$$q_{\rm sf}(x) \approx 30\, x^2(1-x)^2$$





A more realistic pion 2-body LFWF

Direct computation of Mellin moments:

$$\langle x^m \rangle^\pi_{\zeta_H} = \int_0^1 dx\, x^m q^\pi(x;\zeta_H)$$

$$= \frac{N_c}{n\cdot P} \mathrm{tr} \int_{dk} \left[\frac{n\cdot k_\eta}{n\cdot P}\right]^m \Gamma_\pi(k_{\bar\eta},P)\, S(k_{\bar\eta})\, n\cdot\partial_{k_\eta} \left[\Gamma_\pi(k_\eta,-P) S(k_\eta)\right]$$

$$q^\pi(x;\zeta_H) = 213.32\, x^2(1-x)^2 \times \left[1 - 2.9342\sqrt{x(1-x)} + 2.2911\, x(1-x)\right]$$

Pion realistic picture: DGLAP evolution

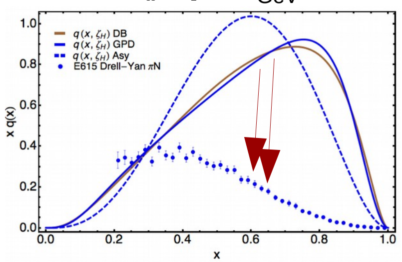
$$M_n(t) = \int_0^1 dx\, x^n q(x,t)$$

$$t = \ln\left(\frac{\zeta^2}{\zeta_0^2}\right)$$

Moments' evolution (1-loop):

$$\frac{d}{dt} M_n(t) = -\frac{\alpha(t)}{4\pi} \gamma_0^n M_n(t) + \ldots$$

$\zeta_H \rightarrow \zeta_2 = 5.2$ GeV



Pion realistic picture: DGLAP evolution

$$M_n(t) = \int_0^1 dx\, x^n q(x,t)$$

$$t = \ln\left(\frac{\zeta^2}{\zeta_0^2}\right)$$

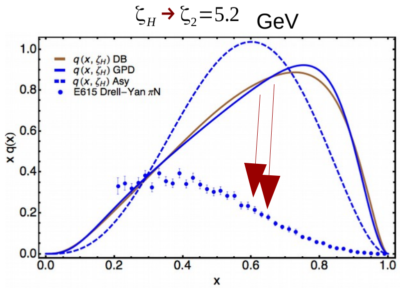
A master equation for the (1-loop) moments' evolution:

$$\frac{d}{dt} q(x,t) = -\frac{\alpha(t)}{4\pi} \int_x^1 \frac{dy}{y} q(y,t) P\left(\frac{x}{y}\right) + \ldots$$

$$\int_0^1 dx\, P(x) = \gamma_0^n$$

Moments' evolution (1-loop):

$$\frac{d}{dt} M_n(t) = -\frac{\alpha(t)}{4\pi} \gamma_0^n M_n(t) + \ldots$$



Pion realistic picture: DGLAP evolution

$$M_n(t)=\int_0^1 dx\, x^n q(x,t)$$

$$t=\ln\left(\frac{\zeta^2}{\zeta_0^2}\right)$$

A master equation for the (1-loop) moments' evolution:

$$\frac{d}{dt} q(x,t) = -\frac{\alpha(t)}{4\pi}\int_x^1 \frac{dy}{y} q(y,t) P\left(\frac{x}{y}\right) + \ldots$$

$$\int_0^1 dx\, P(x) = \gamma_0^n$$

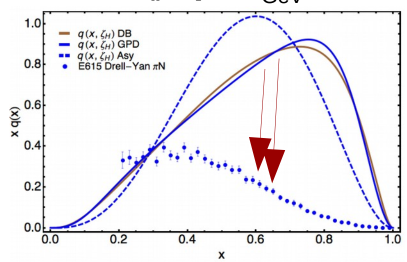
Moments' evolution (1-loop):

$$\frac{d}{dt} M_n(t) = -\frac{\alpha(t)}{4\pi}\gamma_0^n M_n(t) + \ldots$$

$$P(x) = \frac{8}{3}\left(\frac{1+z^2}{(1-x)_+} + \frac{3}{2}\delta(x-1)\right)$$

$$\gamma_n = -\frac{4}{3}\left(3 + \frac{2}{(n+2)(n+3)} - 4\sum_{i=1}^{n+1}\frac{1}{i}\right)$$

$\zeta_H \rightarrow \zeta_2 = 5.2$ GeV



Pion realistic picture: DGLAP evolution

$$M_n(t)=\int_0^1 dx\, x^n q(x,t)$$

$$t=\ln\left(\frac{\zeta^2}{\zeta_0^2}\right)$$

A master equation for the (1-loop) moments' evolution:

$$\frac{d}{dt}q(x,t) = -\frac{\alpha(t)}{4\pi}\int_x^1 \frac{dy}{y} q(y,t) P\left(\frac{x}{y}\right)+\dots$$

$$\int_0^1 dx\, P(x) = \gamma_0^n$$

Moments' evolution (1-loop):

$$\frac{d}{dt}M_n(t) = -\frac{\alpha(t)}{4\pi}\gamma_0^n M_n(t)+\dots$$

$$P(x) = \frac{8}{3}\left(\frac{1+z^2}{(1-x)_+}+\frac{3}{2}\delta(x-1)\right)$$

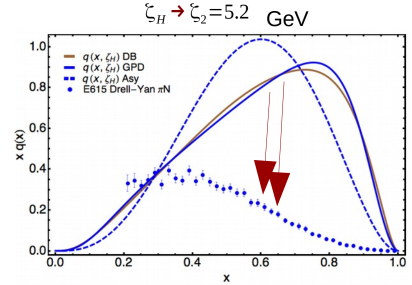
$$\frac{d}{dt}\alpha(t) = -\frac{\alpha^2(t)}{4\pi}\beta_0+\dots$$

$$\gamma_0^n = -\frac{4}{3}\left(3+\frac{2}{(n+2)(n+3)}-4\sum_{i=1}^{n+1}\frac{1}{i}\right)$$

$$\alpha(t) = \frac{4\pi}{\beta_0(t-t_\Lambda)}+\dots$$

$$t_\Lambda=\ln\left(\frac{\Lambda^2}{\zeta_0^2}\right)$$

$\zeta_H \rightarrow \zeta_2=5.2$ GeV



Pion realistic picture: DGLAP evolution

$$M_n(t) = \int_0^1 dx\, x^n q(x,t)$$

$$t = \ln\left(\frac{\zeta^2}{\zeta_0^2}\right)$$

A master equation for the (1-loop) moments' evolution:

$$\frac{d}{dt} q(x,t) = -\frac{\alpha(t)}{4\pi} \int_x^1 \frac{dy}{y} q(y,t) P\left(\frac{x}{y}\right) + \ldots$$

$$\int_0^1 dx\, P(x) = \gamma_0^n$$

Moments' evolution (1-loop):

$$\frac{d}{dt} M_n(t) = -\frac{\alpha(t)}{4\pi} \gamma_0^n M_n(t) + \ldots$$

$$P(x) = \frac{8}{3}\left(\frac{1+z^2}{(1-x)_+} + \frac{3}{2}\delta(x-1)\right)$$

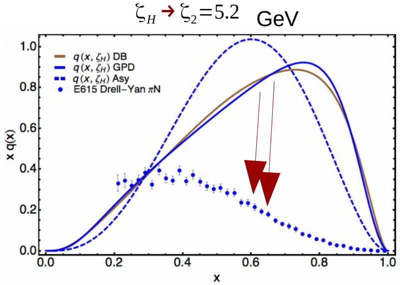
$$\frac{d}{dt} \alpha(t) = -\frac{\alpha^2(t)}{4\pi} \beta_0 + \ldots$$

$$\gamma_0^n = -\frac{4}{3}\left(3 + \frac{2}{(n+2)(n+3)} - 4\sum_{i=1}^{n+1} \frac{1}{i}\right)$$

$$\alpha(t) = \frac{4\pi}{\beta_0 (t - t_\Lambda)} + \ldots$$

$$t_\Lambda = \ln\left(\frac{\Lambda^2}{\zeta_0^2}\right)$$

$$M_n(t) = M_n(t_0)\left(\frac{\alpha(t)}{\alpha(t_0)}\right)^{\gamma_0^n/\beta_0}$$



Pion realistic picture: DGLAP evolution

Which value of Lambda?

$$\alpha(t) = \frac{4\pi}{\beta_0 (t - t_\Lambda)} + \ldots = \frac{4\pi}{\beta_0 \ln\left(\frac{\zeta^2}{\Lambda^2}\right)} + \ldots$$

Pion realistic picture: DGLAP evolution

Which value of Lambda? It depends on the scheme... Indeed, at the one-loop level, its value defines by itself the scheme!!!

$$\alpha(t) = \frac{4\pi}{\beta_0 (t - t_\Lambda)} + \ldots = \frac{4\pi}{\beta_0 \ln\left(\frac{\zeta^2}{\Lambda^2}\right)} + \ldots$$

$$\alpha(t) = \overline{\alpha}(t)\left(1 + c\,\overline{\alpha}(t) + \ldots\right)$$

$$\ln\left(\frac{\Lambda^2}{\overline{\Lambda}^2}\right) = \frac{4\pi}{\beta_0}\left(\frac{1}{\alpha(t)} - \frac{1}{\overline{\alpha}(t)}\right) + \ldots = \frac{4\pi c}{\beta_0}$$

Pion realistic picture: DGLAP evolution

Which value of Lambda? It depends on the scheme... Indeed, at the one-loop level, its value defines by itself the scheme!!!

$$\alpha(t) = \frac{4\pi}{\beta_0 (t - t_\Lambda)} + \ldots = \frac{4\pi}{\beta_0 \ln\left(\frac{\zeta^2}{\Lambda^2}\right)} + \ldots$$

$$\alpha(t) = \overline{\alpha}(t)\left(1 + c\ \overline{\alpha}(t) + \ldots\right)$$

$$\ln\left(\frac{\Lambda^2}{\overline{\Lambda}^2}\right) = \frac{4\pi}{\beta_0}\left(\frac{1}{\alpha(t)} - \frac{1}{\overline{\alpha}(t)}\right) + \ldots = \frac{4\pi c}{\beta_0}$$

$$\frac{d}{dt} M_n(t) = -\frac{\alpha(t)}{4\pi} \gamma_0^n M_n(t) + \ldots$$

$$\frac{d}{dt} \alpha(t) = -\frac{\alpha^2(t)}{4\pi} \beta_0 + \ldots$$

The evolution will thus depend on the scheme *via* the perturbative truncation

Pion realistic picture: DGLAP evolution

Which value of Lambda? It depends on the scheme... Indeed, at the one-loop level, its value defines by itself the scheme!!!

$$\alpha(t) = \frac{4\pi}{\beta_0 (t - t_\Lambda)} + \dots = \frac{4\pi}{\beta_0 \ln\left(\frac{\zeta^2}{\Lambda^2}\right)} + \dots$$

$$\alpha(t) = \overline{\alpha}(t)\left(1 + c\,\overline{\alpha}(t) + \dots\right)$$

$$\ln\left(\frac{\Lambda^2}{\overline{\Lambda}^2}\right) = \frac{4\pi}{\beta_0}\left(\frac{1}{\alpha(t)} - \frac{1}{\overline{\alpha}(t)}\right) + \dots = \frac{4\pi c}{\beta_0}$$

$$\frac{d}{dt} M_n(t) = -\frac{\overline{\alpha}(t)}{4\pi} \gamma_0^n M_n(t) + \dots$$

$$\frac{d}{dt} \overline{\alpha}(t) = -\frac{\overline{\alpha}^2(t)}{4\pi} \beta_0 + \dots$$

The evolution will thus depend on the scheme *via* the perturbative truncation and the usual prejudice is that truncation errors are optimally small in MS scheme.

PDG2018: [PRD98(2018)030001]

$$\Lambda^{(5)}_{\overline{MS}} = (210 \pm 14)\ \text{MeV}, \qquad (9.24b)$$

$$\Lambda^{(4)}_{\overline{MS}} = (292 \pm 16)\ \text{MeV}, \qquad (9.24c)$$

$$\Lambda^{(3)}_{\overline{MS}} = (332 \pm 17)\ \text{MeV}, \qquad (9.24d)$$

Pion realistic picture: DGLAP evolution

Then, one can evolve the pion PDF, e.g. the one obtained by direct computation of Mellin moments, by using DGLAP evolution from one unknown hadronic scale up to the relevant one for the E615 experiment:

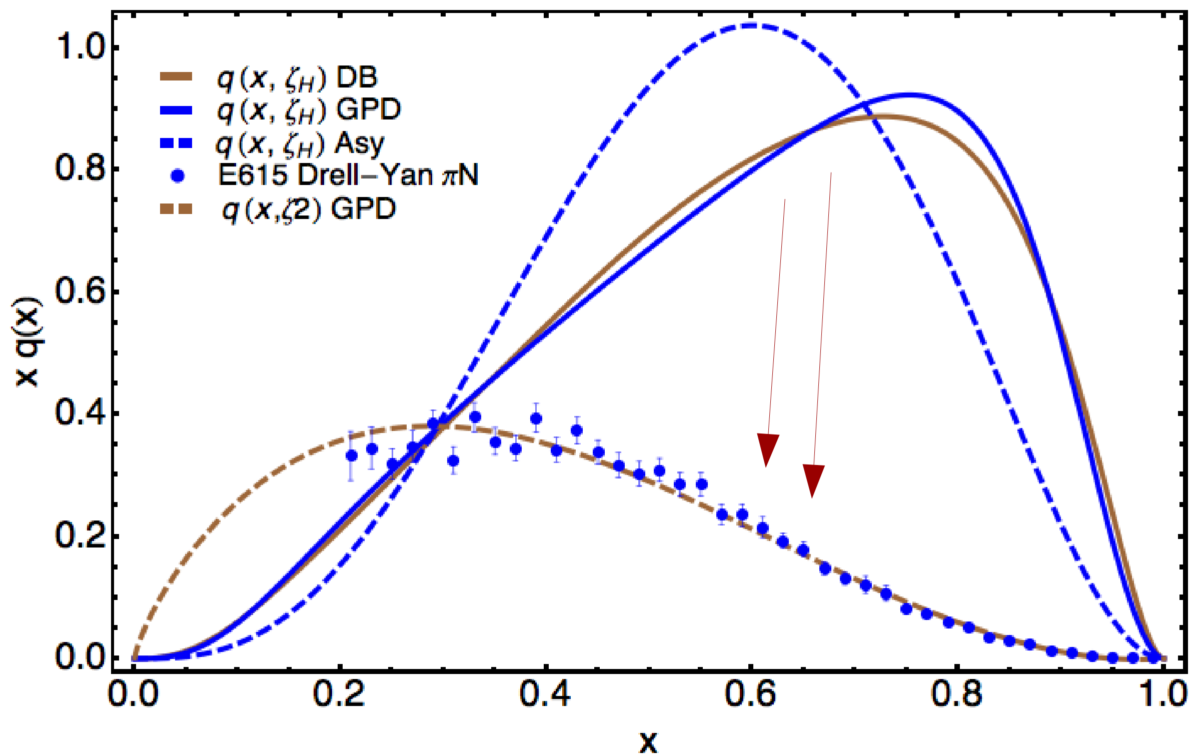
$$\langle x^m \rangle^\pi_{\zeta_H} = \int_0^1 dx\, x^m q^\pi(x;\zeta_H)$$
$$= \frac{N_c}{n\cdot P} \mathrm{tr} \int_{dk} \left[\frac{n\cdot k_\eta}{n\cdot P}\right]^m \Gamma_\pi(k_{\bar\eta},P)\, S(k_{\bar\eta})\, n\cdot\partial_{k_\eta}\left[\Gamma_\pi(k_\eta,-P)S(k_\eta)\right]$$

$$q^\pi(x;\zeta_H) = 213.32\, x^2(1-x)^2 \times \left[1 - 2.9342\sqrt{x(1-x)} + 2.2911\, x(1-x)\right]$$

$\zeta_H \rightarrow \zeta_2 = 5.2$ GeV

Optimal best-fitting parameters:

$\Lambda_{QCD} = 0.234$ GeV ;
$\zeta_H = 0.349$ GeV .



Pion realistic picture: DGLAP evolution

Then, one can evolve the pion PDF, e.g. the one obtained by direct computation of Mellin moments, by using DGLAP evolution from one unknown hadronic scale up to the relevant one for the E615 experiment:

$$\langle x^m\rangle^{\pi}_{\zeta_H} = \int_0^1 dx\, x^m q^{\pi}(x;\zeta_H)$$

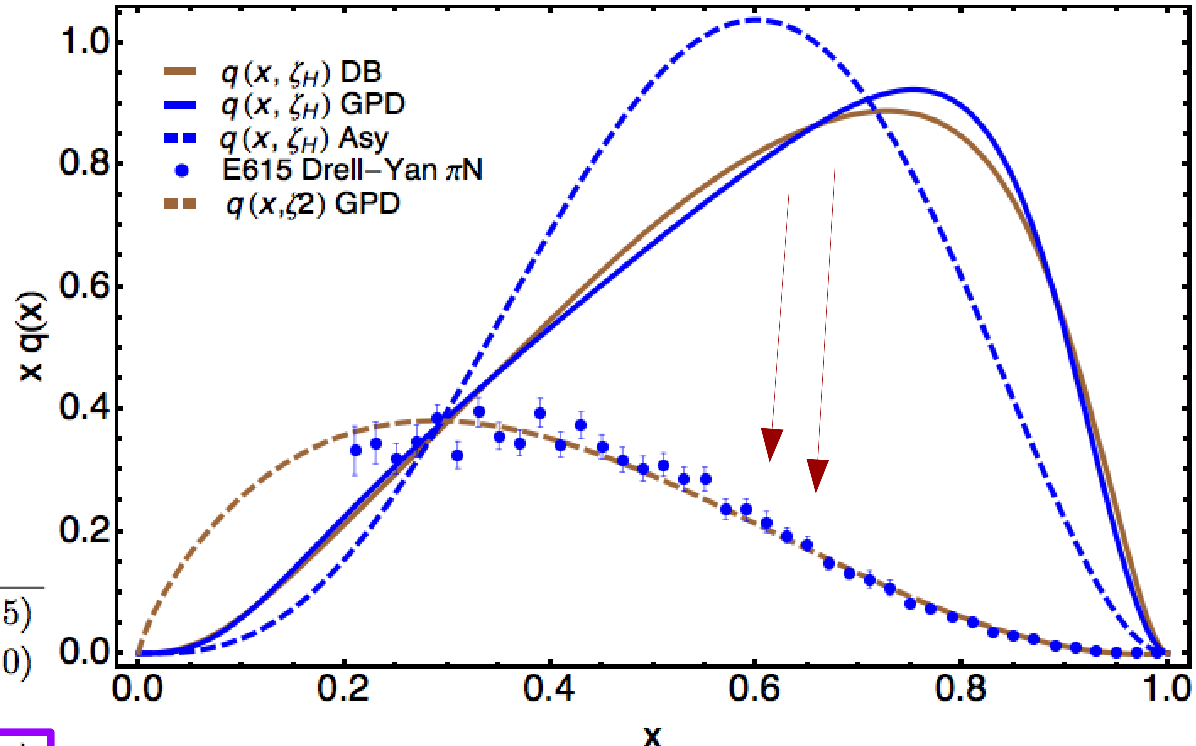
$$= \frac{N_c}{n\cdot P}\mathrm{tr}\int_{dk}\left[\frac{n\cdot k_\eta}{n\cdot P}\right]^m \Gamma_\pi(k_{\bar\eta},P)\,S(k_{\bar\eta})\,n\cdot\partial_{k_\eta}\left[\Gamma_\pi(k_\eta,-P)S(k_\eta)\right]$$

$$q^{\pi}(x;\zeta_H) = 213.32\,x^2(1-x)^2 \times\left[1-2.9342\sqrt{x(1-x)}+2.2911\,x(1-x)\right]$$

$$\zeta_H \rightarrow \zeta_2 = 5.2\ \text{GeV}$$

Optimal best-fitting parameters:

$$\Lambda_{QCD} = 0.234\ \text{GeV}\ ;\quad \zeta_H = 0.349\ \text{GeV}\ .$$



ζ_2	$\langle x\rangle^\pi_u$	$\langle x^2\rangle^\pi_u$	$\langle x^3\rangle^\pi_u$
Ref. [33]	0.24(2)	0.09(3)	0.053(15)
Ref. [34]	0.27(1)	0.13(1)	0.074(10)
Ref. [35]	0.21(1)	0.16(3)	
average	0.24(2)	0.13(4)	0.064(18)
Herein	0.24(2)	0.098(10)	0.049(07)

Comparison with the three first moments obtained from lQCD

Pion realistic picture: DGLAP evolution

Then, one can evolve the pion PDF, e.g. the one obtained by direct computation of Mellin moments, by using DGLAP evolution from one unknown hadronic scale up to the relevant one for the E615 experiment:

$$\langle x^m \rangle^\pi_{\zeta_H} = \int_0^1 dx\, x^m q^\pi(x;\zeta_H)$$
$$= \frac{N_c}{n\cdot P} \mathrm{tr} \int_{dk} \left[\frac{n\cdot k_\eta}{n\cdot P}\right]^m \Gamma_\pi(k_{\bar\eta},P)\, S(k_{\bar\eta})\, n\cdot\partial_{k_\eta} \left[\Gamma_\pi(k_\eta,-P) S(k_\eta)\right]$$

$$q^\pi(x;\zeta_H) = 213.32\, x^2(1-x)^2 \times \left[1 - 2.9342\sqrt{x(1-x)} + 2.2911\, x(1-x)\right]$$

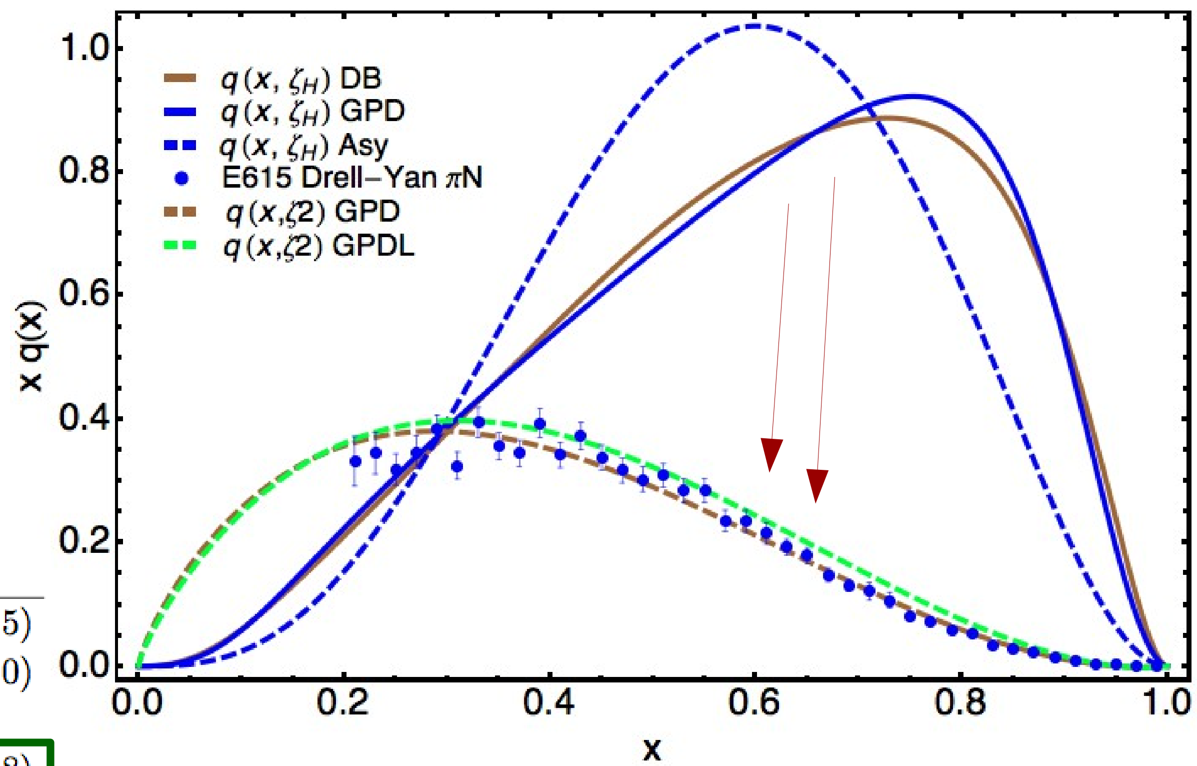
$$\zeta_H \rightarrow \zeta_L = 2\ \text{GeV} \rightarrow \zeta_2 = 5.2\ \text{GeV}$$

Optimal best-fitting parameters:

$\Lambda_{QCD} = 0.234$ GeV ; $\zeta_H = 0.349$ GeV .

$\Lambda_{QCD} = 0.234$ GeV; $\zeta_H = 0.374$ GeV .

ζ_2	$\langle x \rangle^\pi_u$	$\langle x^2 \rangle^\pi_u$	$\langle x^3 \rangle^\pi_u$
Ref. [33]	0.24(2)	0.09(3)	0.053(15)
Ref. [34]	0.27(1)	0.13(1)	0.074(10)
Ref. [35]	0.21(1)	0.16(3)	
average	0.24(2)	0.13(4)	0.064(18)



Matching the three first moments obtained from lQCD

Pion realistic picture: DGLAP evolution

Which value of Lambda? It depends on the scheme... Indeed, at the one-loop level, its value defines by itself the scheme!!!

$$\alpha(t) = \frac{4\pi}{\beta_0 (t - t_\Lambda)} + \ldots = \frac{4\pi}{\beta_0 \ln\left(\frac{\zeta^2}{\Lambda^2}\right)} + \ldots$$

$$\alpha(t) = \overline{\alpha}(t)\left(1 + c\,\overline{\alpha}(t) + \ldots\right)$$

$$\ln\left(\frac{\Lambda^2}{\overline{\Lambda}^2}\right) = \frac{4\pi}{\beta_0}\left(\frac{1}{\alpha(t)} - \frac{1}{\overline{\alpha}(t)}\right) + \ldots = \frac{4\pi c}{\beta_0}$$

$$\frac{d}{dt} M_n(t) = -\frac{\alpha(t)}{4\pi} \gamma_0^n M_n(t) + \ldots$$

$$\frac{d}{dt} \alpha(t) = -\frac{\alpha^2(t)}{4\pi} \beta_0 + \ldots$$

The evolution will thus depend on the scheme *via* the perturbative truncation

The use of $\Lambda = 0.234$ GeV can be thus interpreted as the choice of particular scheme, differing from MS.

Pion realistic picture: DGLAP evolution

Which value of Lambda? It depends on the scheme... Indeed, at the one-loop level, its value defines by itself the scheme!!!

$$\alpha(t) = \frac{4\pi}{\beta_0 (t - t_\Lambda)} + \ldots = \frac{4\pi}{\beta_0 \ln\left(\frac{\zeta^2}{\Lambda^2}\right)} + \ldots$$

$$\alpha(t) = \overline{\alpha}(t)\left(1 + c\,\overline{\alpha}(t) + \ldots\right)$$

$$\ln\left(\frac{\Lambda^2}{\overline{\Lambda}^2}\right) = \frac{4\pi}{\beta_0}\left(\frac{1}{\alpha(t)} - \frac{1}{\overline{\alpha}(t)}\right) + \ldots = \frac{4\pi c}{\beta_0}$$

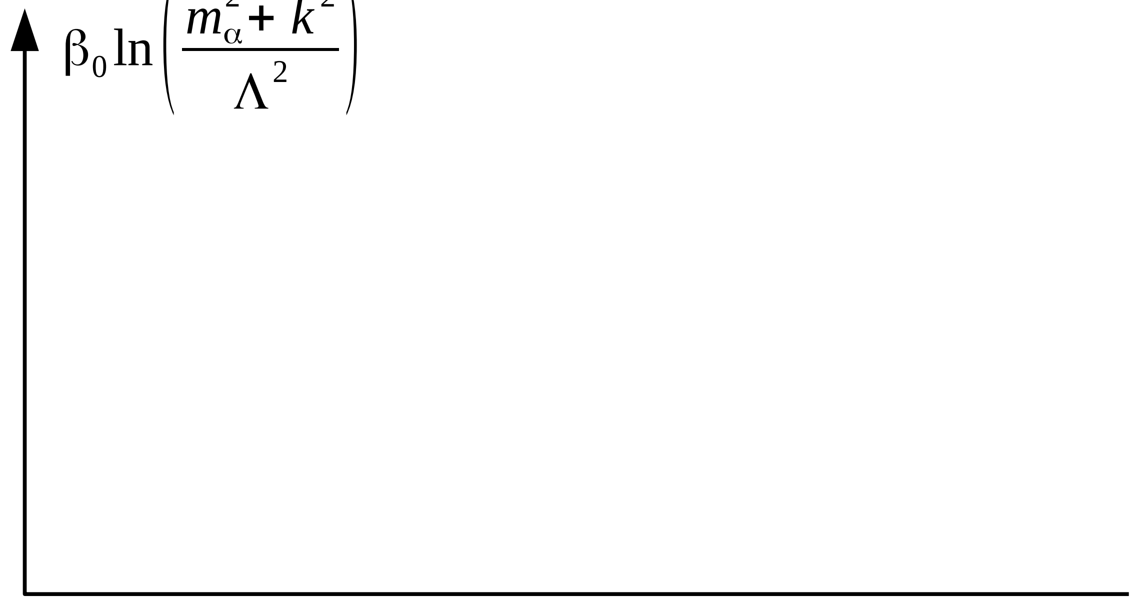
$$\frac{d}{dt} M_n(t) = -\frac{\alpha(t)}{4\pi} \gamma_0^n M_n(t)$$

The evolution will thus depend on the scheme *via* the perturbative truncation

The use of $\Lambda = 0.234$ GeV can be thus interpreted as the choice of particular scheme, differing from MS. Beyond this, the scheme can be defined in such a way that one-loop DGLAP is exact at all orders (Grunberg's effective charge).

Pion realistic picture: DGLAP evolution

$$\alpha(t) = \frac{4\pi}{\beta_0 \ln\left(\frac{m_\alpha^2 + \zeta_0^2 \exp(t)}{\Lambda^2}\right)} = \frac{4\pi}{\beta_0 \ln\left(\frac{m_\alpha^2 + k^2}{\Lambda^2}\right)}$$



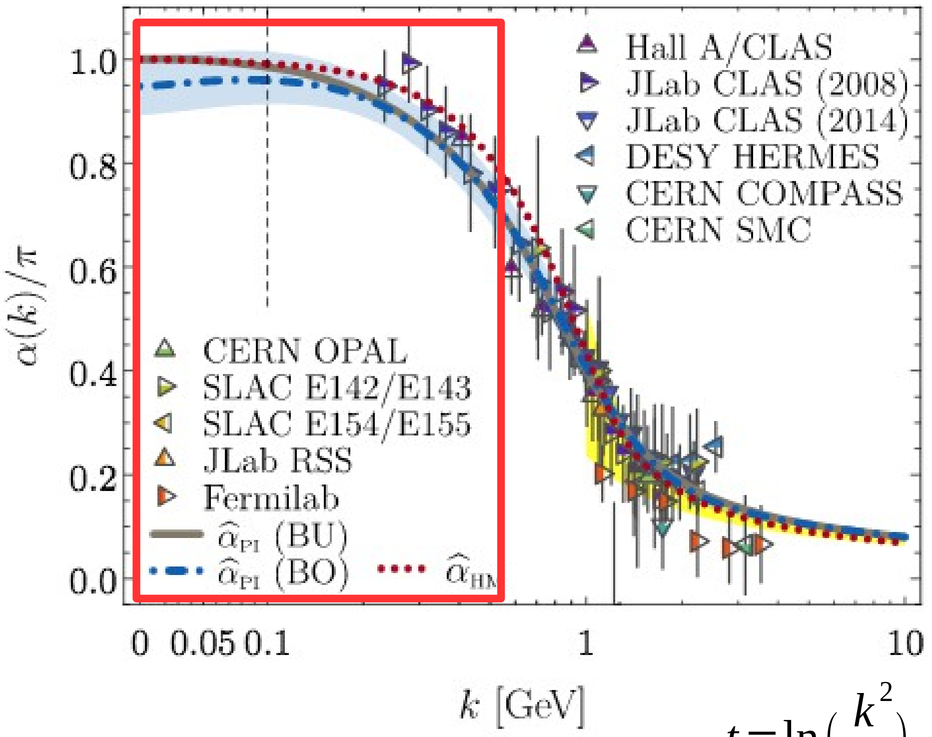
$$t = \ln\left(\frac{k^2}{\zeta_0^2}\right)$$

Pion realistic picture: DGLAP evolution

$$\alpha(t) = \frac{4\pi}{\beta_0 \ln\left(\frac{m_\alpha^2 + \zeta_0^2 \exp(t)}{\Lambda^2}\right)} = \frac{4\pi}{\beta_0 \ln\left(\frac{m_\alpha^2 + k^2}{\Lambda^2}\right)}$$

$$\alpha(0) = \alpha_{PI}(0) \rightarrow m_\alpha = 0.300 \text{ GeV}$$

c.f. Craig Roberts' talk!

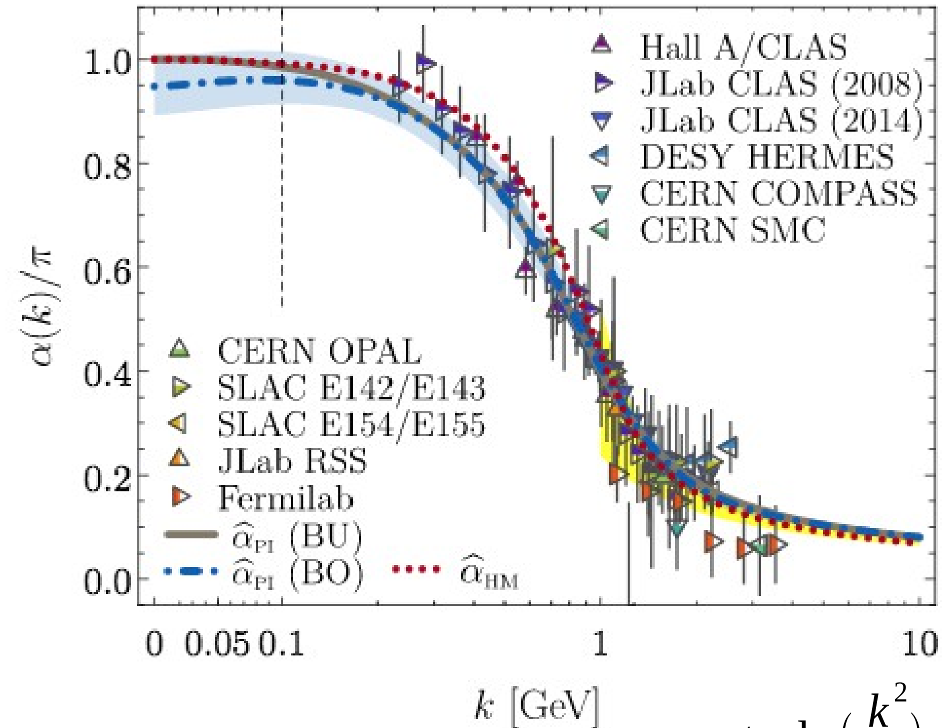


$$t = \ln\left(\frac{k^2}{\zeta_0^2}\right)$$

Pion realistic picture: DGLAP evolution

$$\alpha(t) = \frac{4\pi}{\beta_0 \ln\left(\frac{m_\alpha^2 + \zeta_0^2 \exp(t)}{\Lambda^2}\right)} = \frac{4\pi}{\beta_0 \ln\left(\frac{m_\alpha^2 + k^2}{\Lambda^2}\right)}$$

$$\alpha(0) = \alpha_{PI}(0) \rightarrow m_\alpha = 0.300 \text{ GeV}$$



$$t = \ln\left(\frac{k^2}{\zeta_0^2}\right)$$

$$\frac{d}{dt} M_n(t) = -\frac{\alpha(t)}{4\pi} \gamma_0^n M_n(t)$$

$$M_n(t) = \int_0^1 dx\, x^n q(x,t)$$

Numerical integration with the effective charge

$$M_n(t) = M_n(t_0) \exp\left(-\frac{\gamma_0^n}{4\pi} \int_{t_0}^{t} dz\, \alpha(z)\right)$$

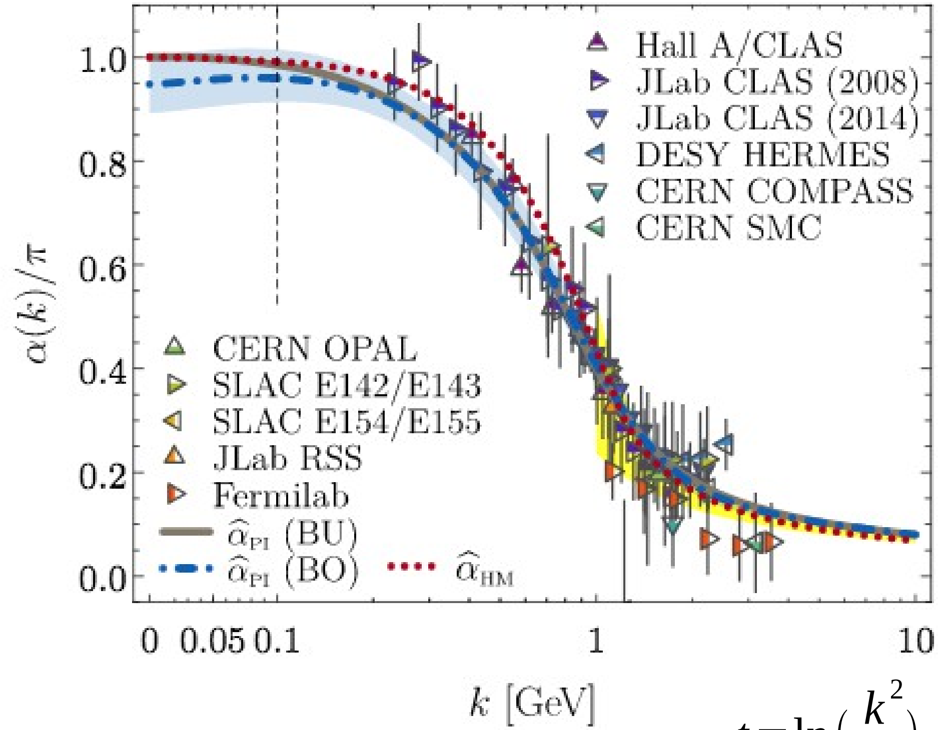
$$\gamma_0^n = -\frac{4}{3}\left(3 + \frac{2}{(n+2)(n+3)} - \sum_{i=1}^{n+1} \frac{1}{i}\right)$$

Pion realistic picture: DGLAP evolution

$$\alpha(t) = \frac{4\pi}{\beta_0 \ln\left(\frac{m_\alpha^2 + \zeta_0^2 \exp(t)}{\Lambda^2}\right)} = \frac{4\pi}{\beta_0 \ln\left(\frac{m_\alpha^2 + k^2}{\Lambda^2}\right)}$$

$$\alpha(0) = \alpha_{PI}(0) \;\rightarrow\; m_\alpha = 0.300 \text{ GeV}$$

$$\frac{d}{dt} M_n(t) = -\frac{\alpha(t)}{4\pi} \gamma_0^n M_n(t)$$



Numerical integration with the effective charge

$$t = \ln\left(\frac{k^2}{\zeta_0^2}\right)$$

$$M_n(t) = M_n(t_0) \exp\left(-\frac{\gamma_0^n}{4\pi} \int_{t_0}^{t} dz\, \alpha(z)\right)$$

$$M_n(t) = \int_0^1 dx\, x^n q(x,t)$$

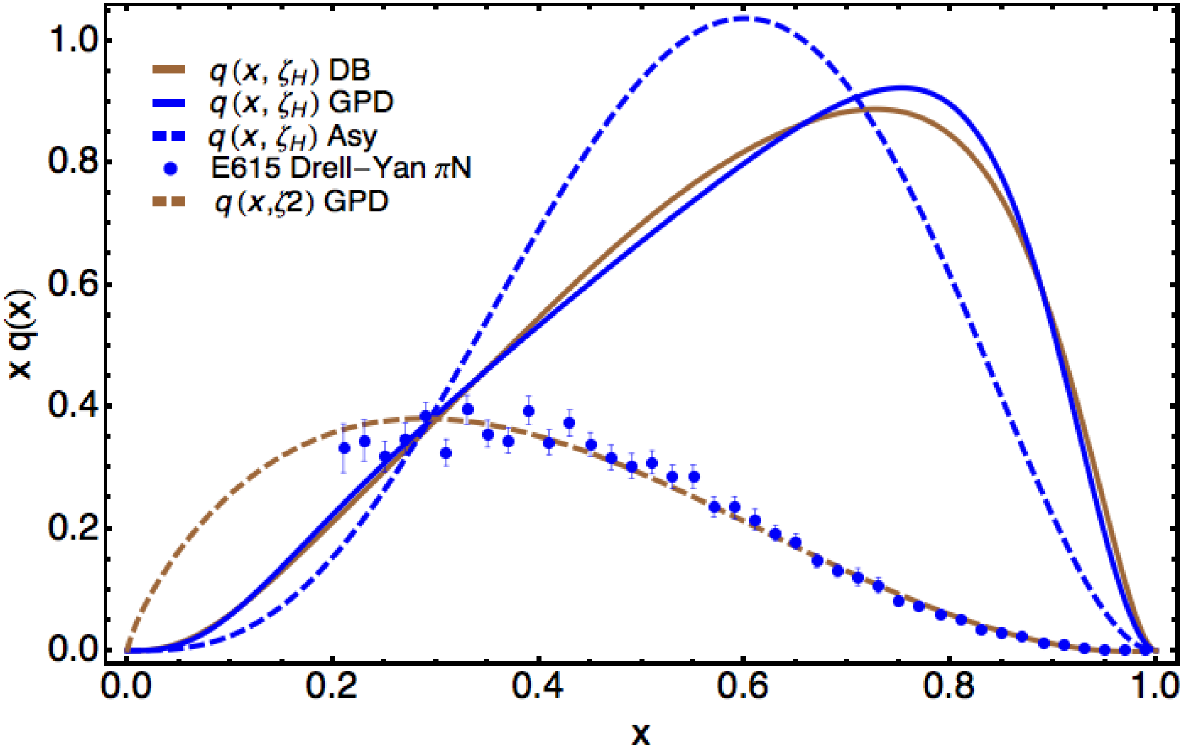
$$\gamma_0^n = -\frac{4}{3}\left(3 + \frac{2}{(n+2)(n+3)} - \sum_{i=1}^{n+1} \frac{1}{i}\right)$$

If one identifies: $m_\alpha \equiv \zeta_H$, all the scales (and the evolution between them) appear thus fixed, apart from Λ_{QCD} (fixed by the scheme).

Pion realistic picture: DGLAP evolution

Then, one can evolve the pion PDF, e.g. the one obtained by direct computation of Mellin moments, by using DGLAP evolution from one unknown hadronic scale up to the relevant one for the E615 experiment:

$\Lambda_{QCD} = 0.234$ GeV; $\zeta_H \equiv m_\alpha \rightarrow \zeta_2 = 5.2$ GeV

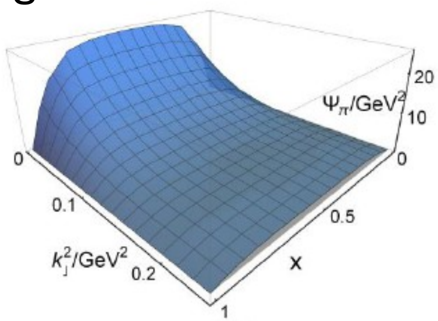


If one identifies: $m_\alpha \equiv \zeta_H$, all the scales (and the evolution between them) appear thus fixed, apart from Λ_{QCD} (fixed by the scheme). And the agreement with E615 data is perfect!!!

Pion realistic picture: DGLAP evolution

Then, one can evolve the pion PDF, e.g. the one obtained by direct computation of Mellin moments, by using DGLAP evolution from one unknown hadronic scale up to the relevant one for the E615 experiment:

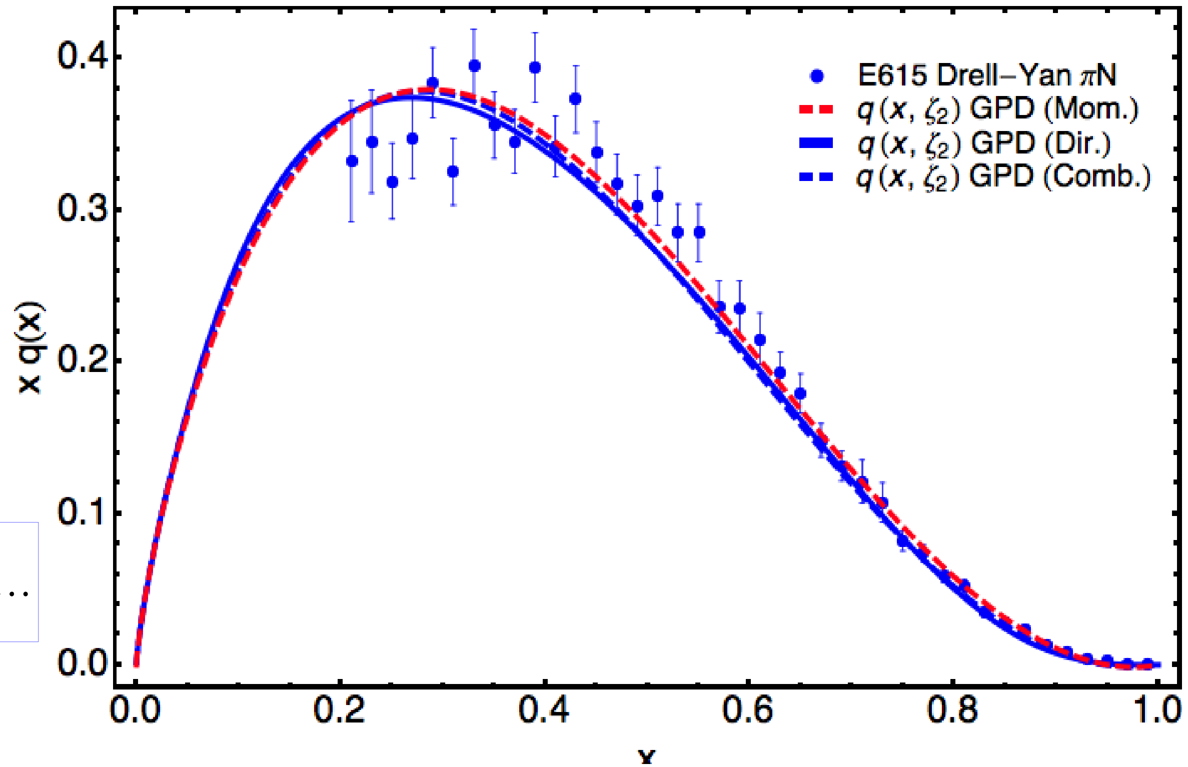
The same is obtained from the overlap of realistic pion 2-body LFWFs



and after integration of the DGLAP master equation

$$\frac{d}{dt}q(x,t) = -\frac{\alpha(t)}{4\pi}\int_x^1 \frac{dy}{y} q(y,t) P\left(\frac{x}{y}\right) + \dots$$

$\Lambda_{QCD} = 0.234$ GeV; $\zeta_H \equiv m_\alpha \rightarrow \zeta_2 = 5.2$ GeV



If one identifies: $m_\alpha \equiv \zeta_H$, all the scales (and the evolution between them) appear thus fixed, apart from Λ_{QCD} (fixed by the scheme). And the agreement with E615 data is perfect!!!

Pion realistic picture: PDF as benchmark

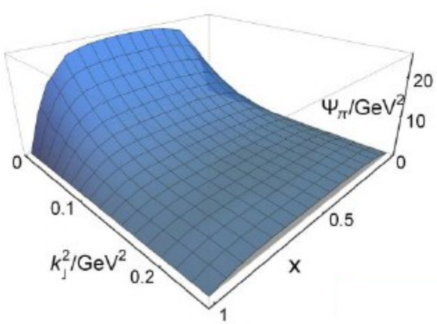
The pion PDF can be computed as the lightfront projection of the hadronic matrix element of a bilocal operator and, in the overlap representation at low Fock states, can be expressed in terms of 2-body LFWFs at a given hadronic scale

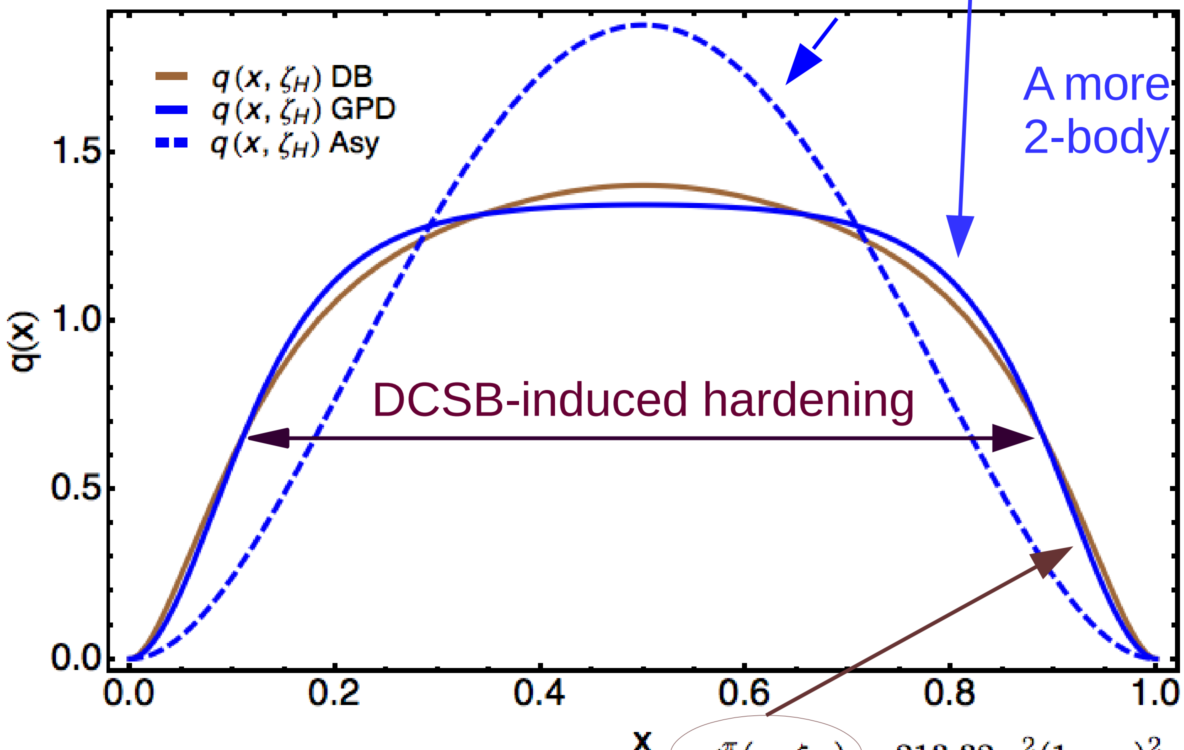
$$q^\pi(x;\zeta_H) = \frac{1}{2}\int \frac{dz^-}{2\pi} e^{ixP^+z^-} \left\langle P \middle| \overline{\psi}^q(-z)\gamma^+\psi^q(z) \middle| P \right\rangle \Big|_{z^+=0, z_\perp=0} = \int \frac{d^2k_\perp}{16\pi^3} \Psi^*_{u\bar{f}}(x,\mathbf{k}_\perp)\,\Psi_{u\bar{f}}(x,\mathbf{k}_\perp)$$

LFWF leading to asymptotic PDAs

$$q_{\rm sf}(x) \approx 30\, x^2(1-x)^2$$

A more realistic pion 2-body LFWF





Direct computation of Mellin moments:

$$\langle x^m\rangle^\pi_{\zeta_H} = \int_0^1 dx\, x^m q^\pi(x;\zeta_H)$$

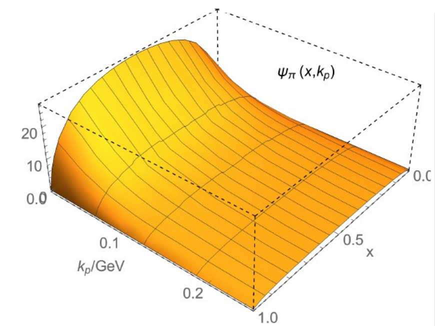
$$= \frac{N_c}{n\cdot P}\mathrm{tr}\int_{dk}\left[\frac{n\cdot k_\eta}{n\cdot P}\right]^m \Gamma_\pi(k_{\bar\eta},P)\,S(k_{\bar\eta})\,n\cdot\partial_{k_\eta}\left[\Gamma_\pi(k_\eta,-P)S(k_\eta)\right]$$

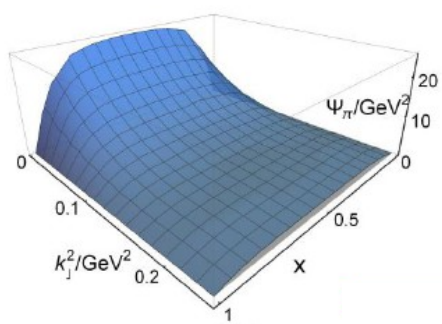
$$q^\pi(x;\zeta_H) = 213.32\, x^2(1-x)^2 \times \left[1 - 2.9342\sqrt{x(1-x)} + 2.2911\, x(1-x)\right]$$

Pion (more) realistic picture: PDF as benchmark

The pion PDF can be computed as the lightfront projection of the hadronic matrix element of a bilocal operator and, in the overlap representation at low Fock states, can be expressed in terms of 2-body LFWFs at a given hadronic scale

$$q^\pi(x;\zeta_H) = \frac{1}{2}\int \frac{dz^-}{2\pi} e^{ixP^+z^-} \left\langle P \middle| \overline{\psi}^q(-z)\gamma^+\psi^q(z) \middle| P \right\rangle \Big|_{z^+=0, z_\perp=0} = \int \frac{d^2k_\perp}{16\pi^3} \Psi^*_{u\bar{f}}(x,\mathbf{k}_\perp)\,\Psi_{u\bar{f}}(x,\mathbf{k}_\perp)$$

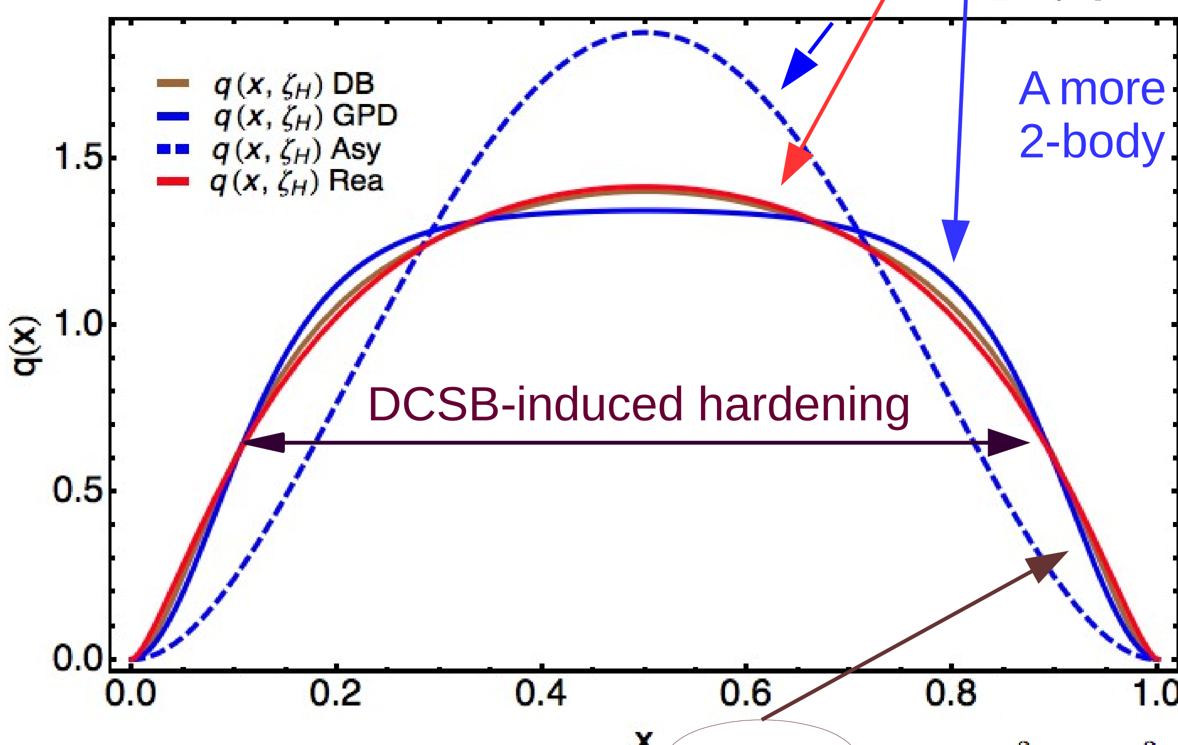




LFWF leading to asymptotic PDAs

$$q_{\rm sf}(x) \approx 30\,x^2(1-x)^2$$

A more realistic pion 2-body LFWF



Direct computation of Mellin moments:

$$\langle x^m\rangle^\pi_{\zeta_H} = \int_0^1 dx\, x^m q^\pi(x;\zeta_H)$$

$$= \frac{N_c}{n\cdot P}\mathrm{tr}\int_{dk}\left[\frac{n\cdot k_\eta}{n\cdot P}\right]^m \Gamma_\pi(k_{\bar\eta},P)\,S(k_{\bar\eta})\,n\cdot\partial_{k_\eta}\left[\Gamma_\pi(k_\eta,-P)S(k_\eta)\right]$$

$$q^\pi(x;\zeta_H) = 213.32\,x^2(1-x)^2 \times\left[1 - 2.9342\sqrt{x(1-x)} + 2.2911\,x(1-x)\right]$$

Pion (more) realistic picture: PDF as benchmark

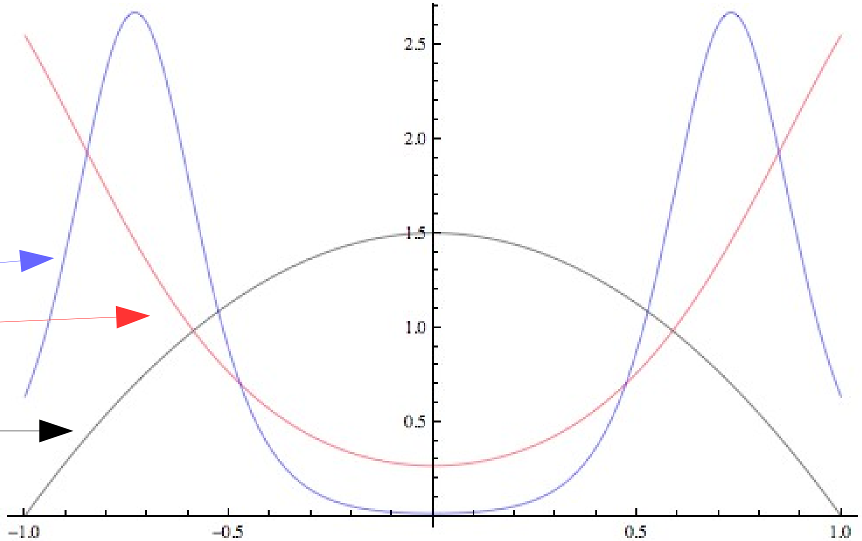
- Spectral density is chosen as:

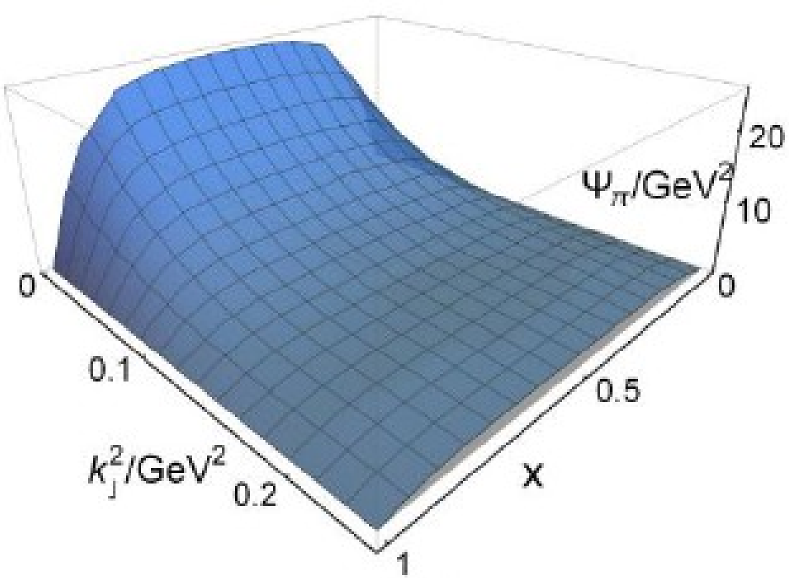
$$u_G \rho_G(\omega) = \frac{1}{2b_0^G}\left[\mathrm{sech}^2\left(\frac{\omega - \omega_0^G}{2b_0^G}\right) + \mathrm{sech}^2\left(\frac{\omega + \omega_0^G}{2b_0^G}\right)\right]$$

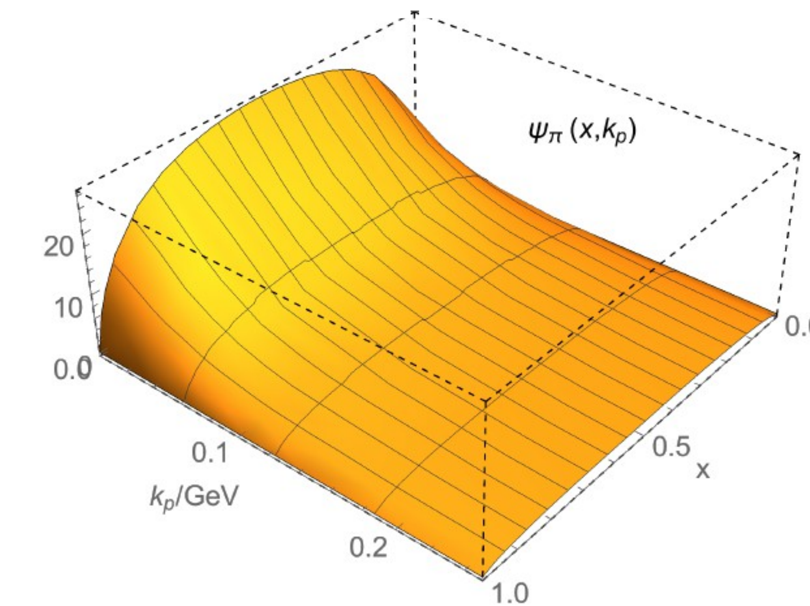
Phenomelogical model: $b_0^\pi = 0.1, w_0^\pi = 0.73;$

Realistic case: $b_0^\pi = 0.275, b_0^\pi = 1.23;$

Asymptotic case: $\rho(\omega;\nu) \sim (1-\omega^2)^\nu$



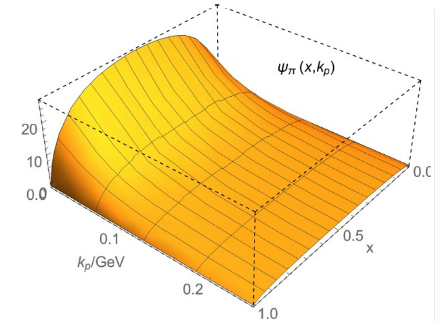


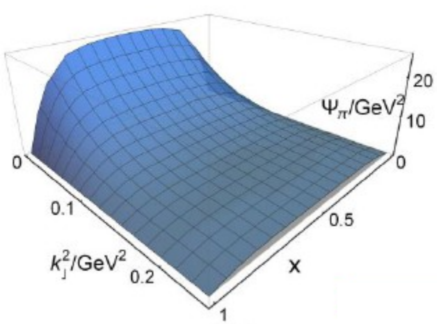


Pion (more) realistic picture: PDF as benchmark

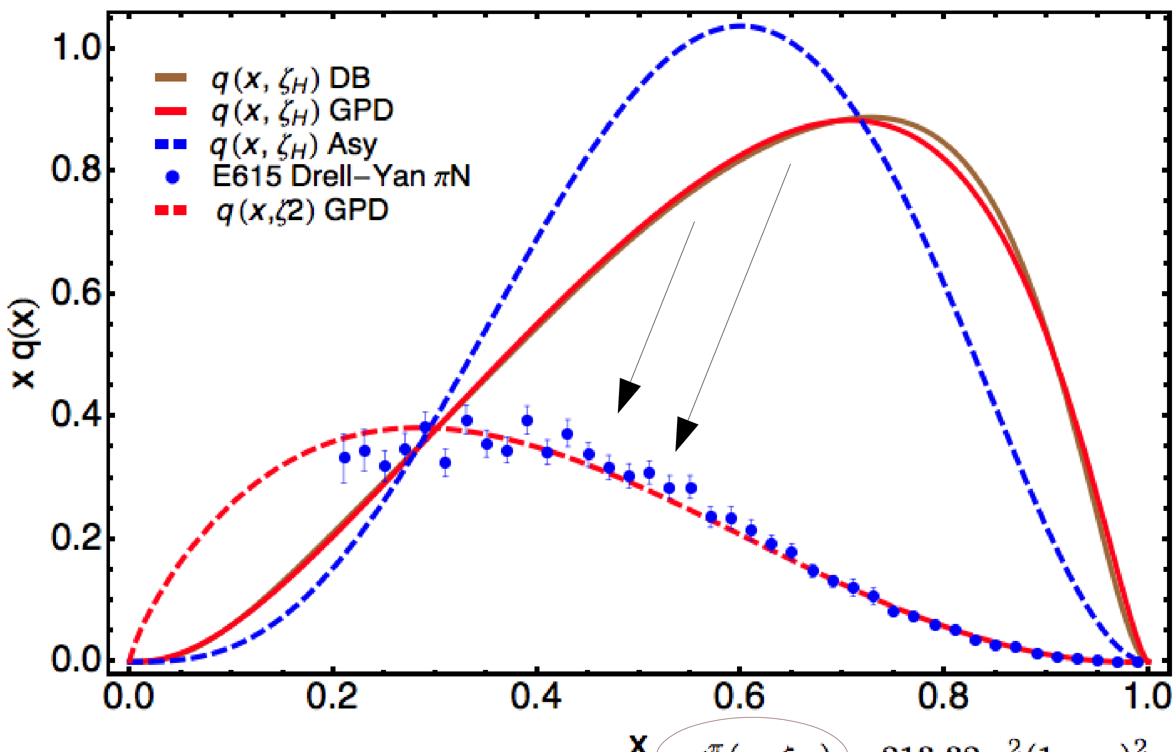
The pion PDF can be computed as the lightfront projection of the hadronic matrix element of a bilocal operator and, in the overlap representation at low Fock states, can be expressed in terms of 2-body LFWFs at a given hadronic scale

$$q^\pi(x;\zeta_H) = \frac{1}{2}\int \frac{dz^-}{2\pi} e^{ixP^+z^-} \left\langle P \middle| \overline{\psi}^q(-z)\gamma^+\psi^q(z) \middle| P \right\rangle \Big|_{z^+=0,z_\perp=0} = \int \frac{d^2k_\perp}{16\pi^3} \Psi^*_{u\bar{f}}(x,\mathbf{k}_\perp)\,\Psi_{u\bar{f}}(x,\mathbf{k}_\perp)$$





$\zeta_H \equiv m_\alpha \rightarrow \zeta_2 = 5.2$ GeV



Direct computation of Mellin moments:

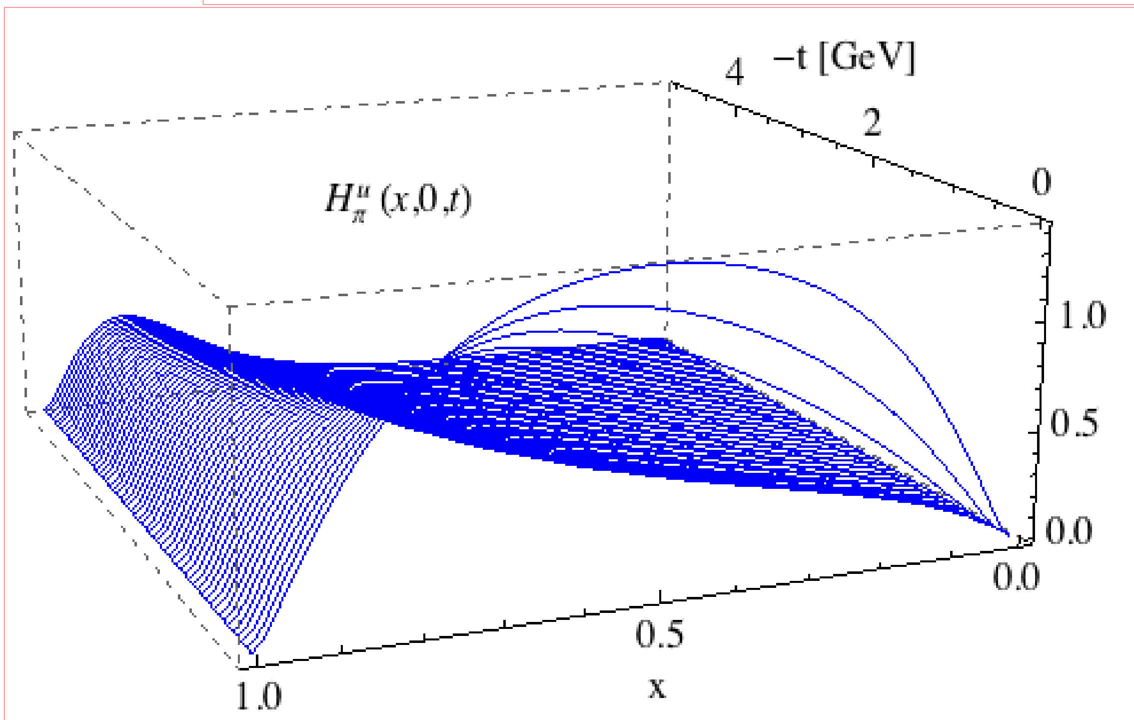
$$\langle x^m\rangle^\pi_{\zeta_H} = \int_0^1 dx\, x^m q^\pi(x;\zeta_H)$$

$$= \frac{N_c}{n\cdot P}\mathrm{tr}\int_{dk}\left[\frac{n\cdot k_\eta}{n\cdot P}\right]^m \Gamma_\pi(k_{\bar{\eta}},P)\,S(k_{\bar{\eta}})\,n\cdot\partial_{k_\eta}\left[\Gamma_\pi(k_\eta,-P)S(k_\eta)\right]$$

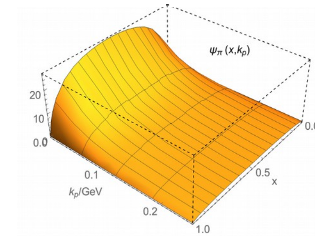
$$q^\pi(x;\zeta_H) = 213.32\,x^2(1-x)^2 \times \left[1 - 2.9342\sqrt{x(1-x)} + 2.2911\,x(1-x)\right]$$

Pion (more) realistic picture: GPD

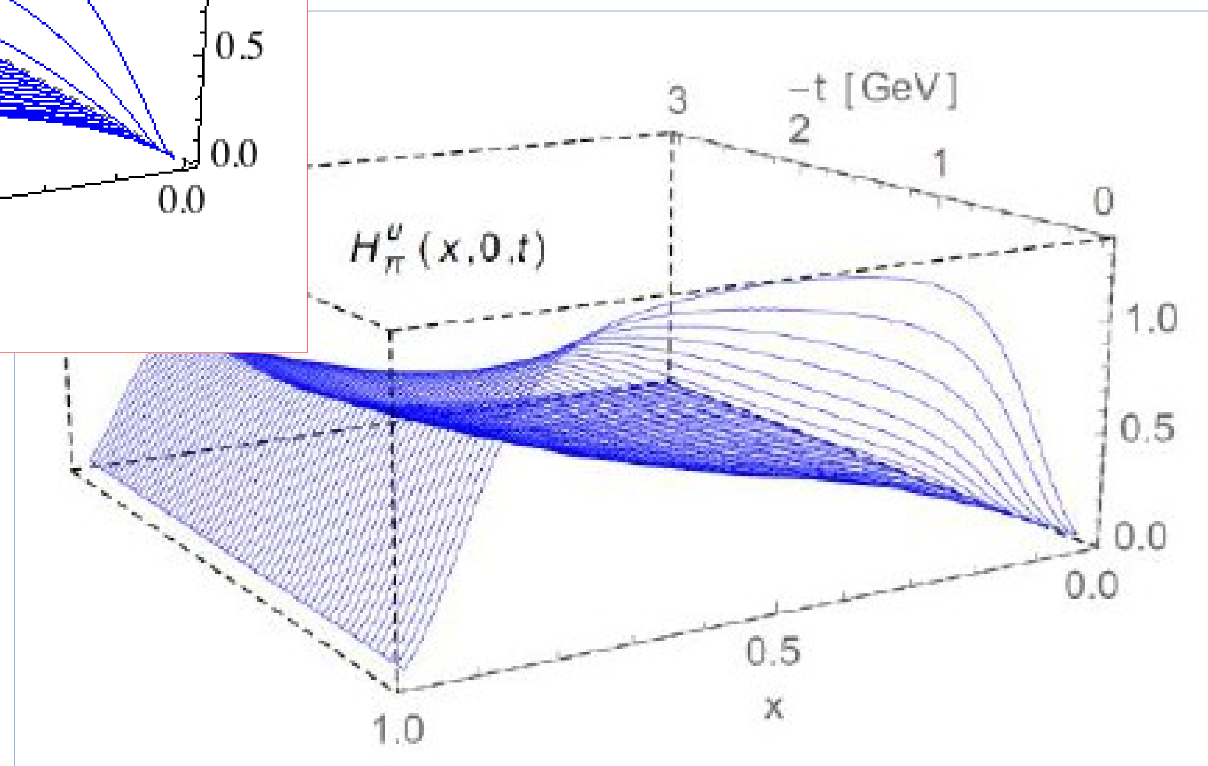
$$H_M^q(x,\xi,t) = \int \frac{\mathrm{d}^2\mathbf{k}_\perp}{16\,\pi^3} \Psi^*_{u\bar{f}}\left(\frac{x-\xi}{1-\xi}, \mathbf{k}_\perp + \frac{1-x}{1-\xi}\frac{\Delta_\perp}{2}\right) \Psi_{u\bar{f}}\left(\frac{x+\xi}{1+\xi}, \mathbf{k}_\perp - \frac{1-x}{1+\xi}\frac{\Delta_\perp}{2}\right)$$



Realistic case



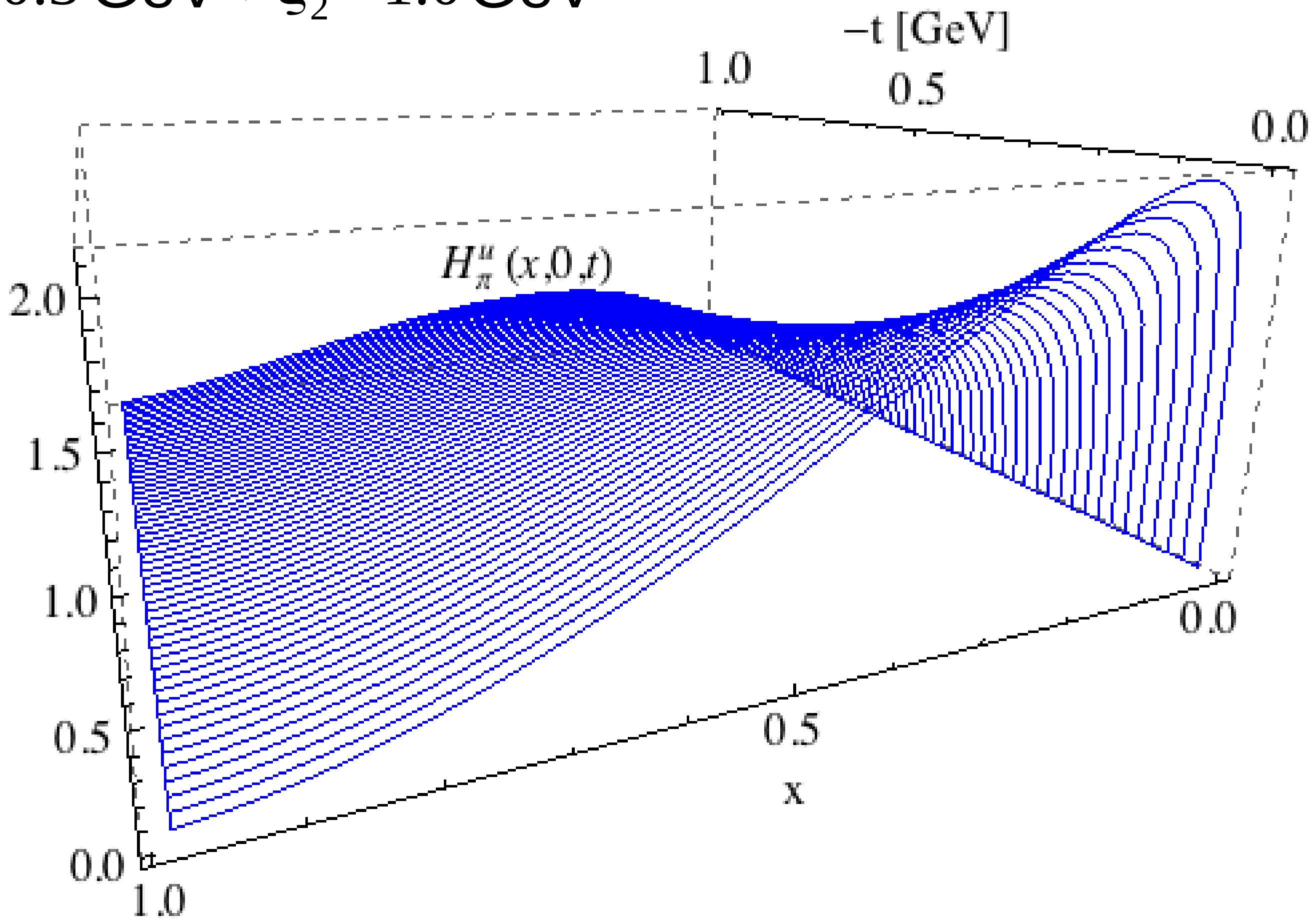
Phenomenological model



Pion (more) realistic picture: DGLAP evolution

$$H_M^q(x,\xi,t) = \int \frac{d^2\mathbf{k}_\perp}{16\,\pi^3} \Psi^*_{u\bar{f}}\left(\frac{x-\xi}{1-\xi}, \mathbf{k}_\perp + \frac{1-x}{1-\xi}\frac{\Delta_\perp}{2}\right) \Psi_{u\bar{f}}\left(\frac{x+\xi}{1+\xi}, \mathbf{k}_\perp - \frac{1-x}{1+\xi}\frac{\Delta_\perp}{2}\right)$$

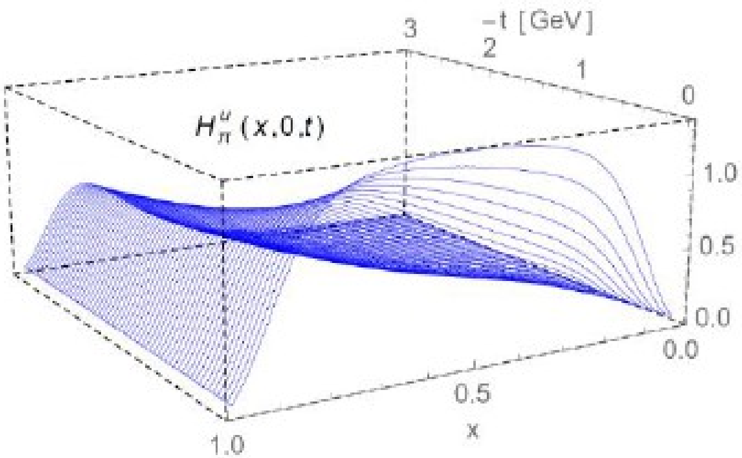
$\zeta_0 = \zeta_H = 0.3\,\mathrm{GeV} \rightarrow \zeta_2 = 1.0\,\mathrm{GeV}$

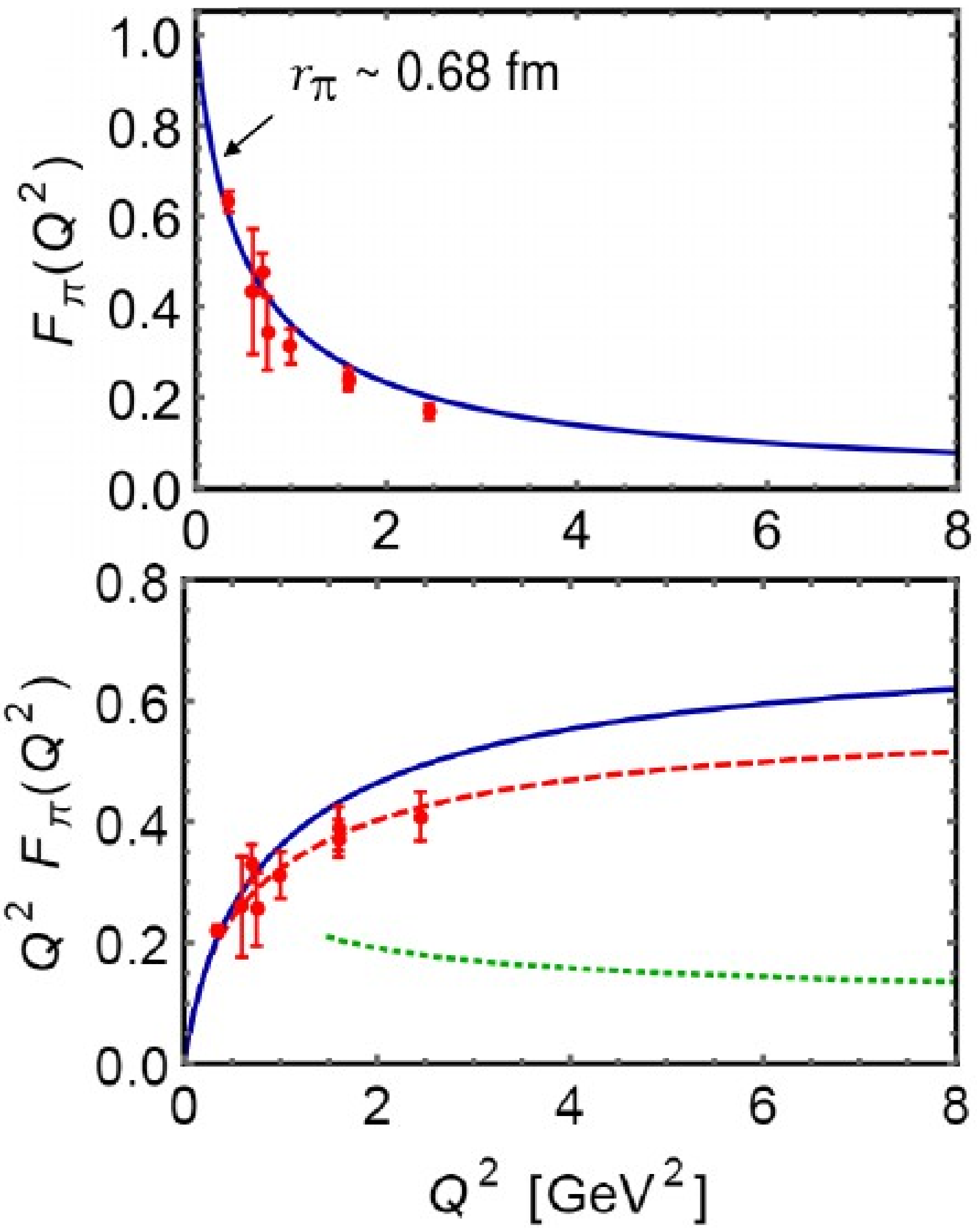


Pion (more) realistic picture: Elect. Form Factor

$$F_M(\Delta^2) = e_u F_M^u(\Delta^2) + e_f F_M^f(\Delta^2)\ ,\ F_M^q(-t=\Delta^2) = \int_{-1}^{1} dx\ H_M^q(x,\xi,t)$$

Electric charges



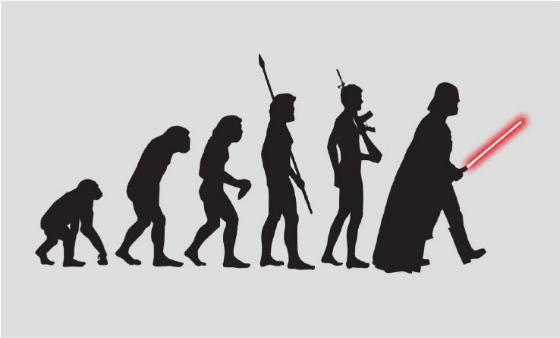


Blue: Computed from GPD

Green: Computed from HS formula

Red: '*Evolved*' form factor

PDA and LFWF evolution



LFWF evolution:

$$\phi(x) = \frac{1}{16\pi^3} \int d^2\vec{k}_\perp \psi^{\uparrow\downarrow}(x, k_\perp^2)$$

- We look for a way to evolve the LFWF.
- First, let's assume that the LFWF admits a similar Gegenbauer expansion. That is:

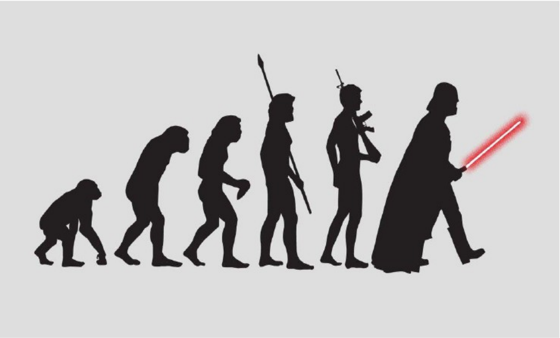
$$\psi(x, k_\perp^2; \zeta) = 6x(1-x) \left[\sum_{n=0} b_n(k_\perp^2; \zeta)\, C_n^{3/2}(2x-1) \right] ,$$

$$a_n(\zeta) \; = \; \frac{1}{16\pi^3} \int d^2\vec{k}_\perp\, b_n(k_\perp^2; \zeta) \text{ (for } n \geq 1) \,, \; \frac{1}{16\pi^3} \int d^2\vec{k}_\perp\, b_0(k_\perp^2; \zeta) = 1 \,.$$

- 1-loop ERBL evolution of $a_n(\zeta)$ implies:

$$\frac{1}{a_n(\zeta)} \frac{d}{d\ln\zeta^2} a_n(\zeta) \; = \; \frac{\displaystyle\int d^2\vec{k}_\perp \frac{d}{d\ln\zeta^2} b_n(k_\perp^2; \zeta)}{\displaystyle\int d^2\vec{k}_\perp\, b_n(k_\perp^2; \zeta)} \,,$$

PDA and LFWF evolution



Standard PDA evolution:

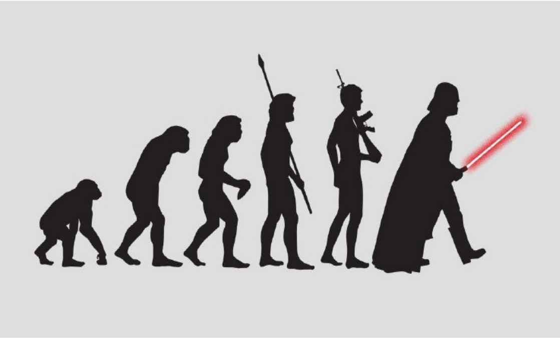
- We project **PDA** onto a 3/2-Gegenbauer polynomial basis. Such that it **evolves**, from an initial scale ζ_0 to a final scale ζ, **according to** the corresponding **ERBL equations**:

$$\phi(x;\zeta) = 6x(1-x)\left[1 + \sum_{n=1} a_n(\zeta) C_n^{3/2}(2x-1)\right] ,$$

$$a_n(\zeta) = a_n(\zeta_0)\left[\frac{\alpha(\zeta^2)}{\alpha(\zeta_0^2)}\right]^{\gamma_0^n/\beta_0} , \quad \gamma_0^n = -\frac{4}{3}\left[3 + \frac{2}{(n+1)(n+2)} - 4\sum_{k=1}^{n+1}\frac{1}{k}\right] .$$

- Thus, any PDA at hadronic scale evolves logarithmically towards its conformal distribution, $\phi(x)=6x(1-x)$.
    - Quark mass and flavor become irrelevant. Broad PDA becomes narrower, skewed PDA becomes symmetric.

PDA and LFWF evolution



LFWF evolution:

$$\phi(x) = \frac{1}{16\pi^3}\int d^2\vec{k}_\perp \psi^{\uparrow\downarrow}(x, k_\perp^2)$$

- Now, if we take a factorization assumtion, we arrive at:

$$\frac{b_n(k_\perp^2;\zeta)}{b_n(k_\perp^2;\zeta_0)} = \frac{\widehat{b}_n(\zeta)}{\widehat{b}_n(\zeta_0)} = \left[\frac{\alpha(\zeta^2)}{\alpha(\zeta_0^2)}\right]^{\gamma_0^n/\beta_0}, \quad b_n(k_\perp^2;\zeta) \equiv \widehat{b}_n(\zeta)\chi_n(k_\perp^2) \ .$$

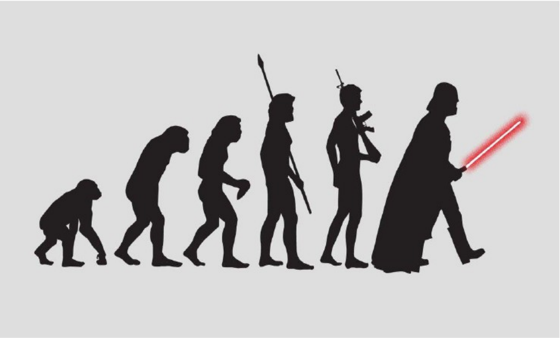
- Suplemented by the condition $\chi_n(k_\perp^2) \equiv \chi(k_\perp^2)$, one gets $\widehat{b}_n(\zeta) \equiv a_n(\zeta)$.
- Such that, the followiong factorised form is obtained:

$$\psi(x, k_\perp^2;\zeta) \equiv \phi(x;\zeta)\,\chi(k_\perp^2)$$

→ **LFWF Evolves like PDA**

- Which is far from being a general result, but an useful approximation instead.

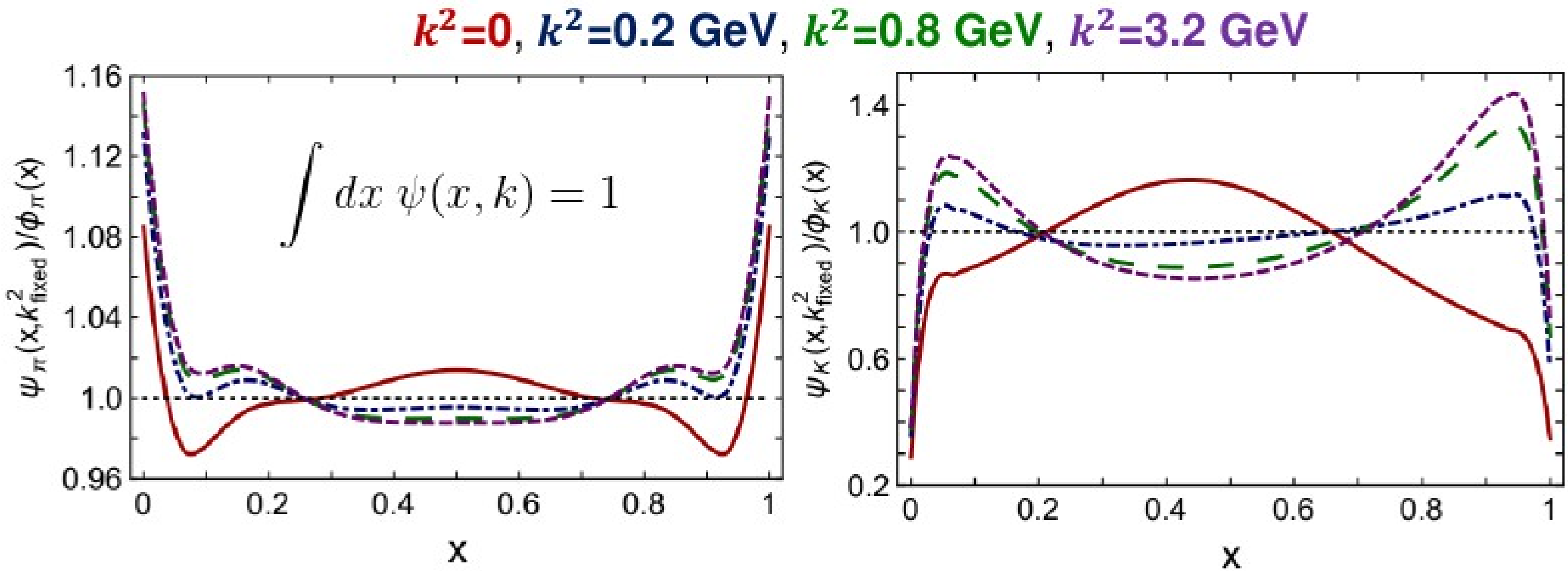
PDA and LFWF evolution



Testing the factorization ansatz:

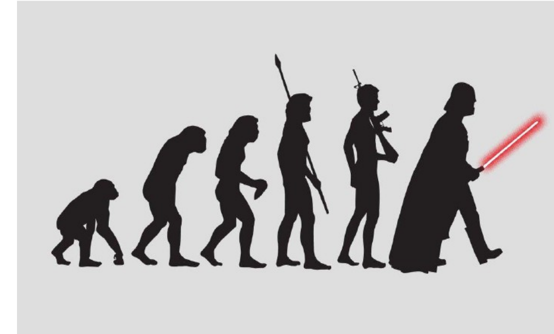
$$\psi(x, k_\perp^2; \zeta) \equiv \phi(x; \zeta)\, \chi(k_\perp^2)$$

- A first validation of the factorized ansätz is addressed in **Phys.Rev. D97 (2018) no.9, 094014**:

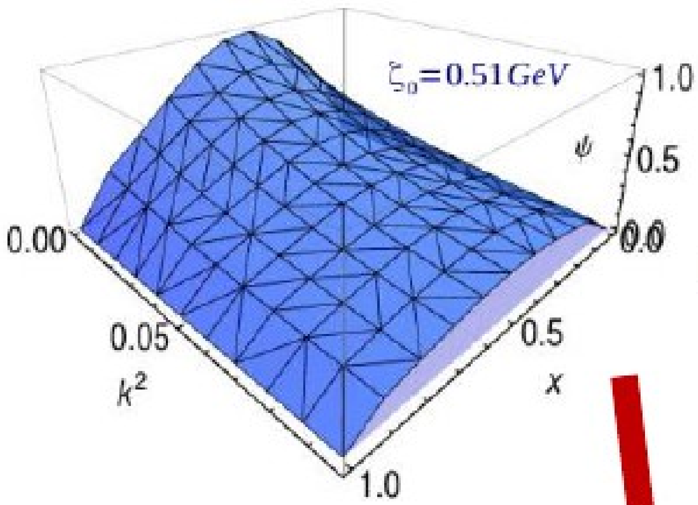


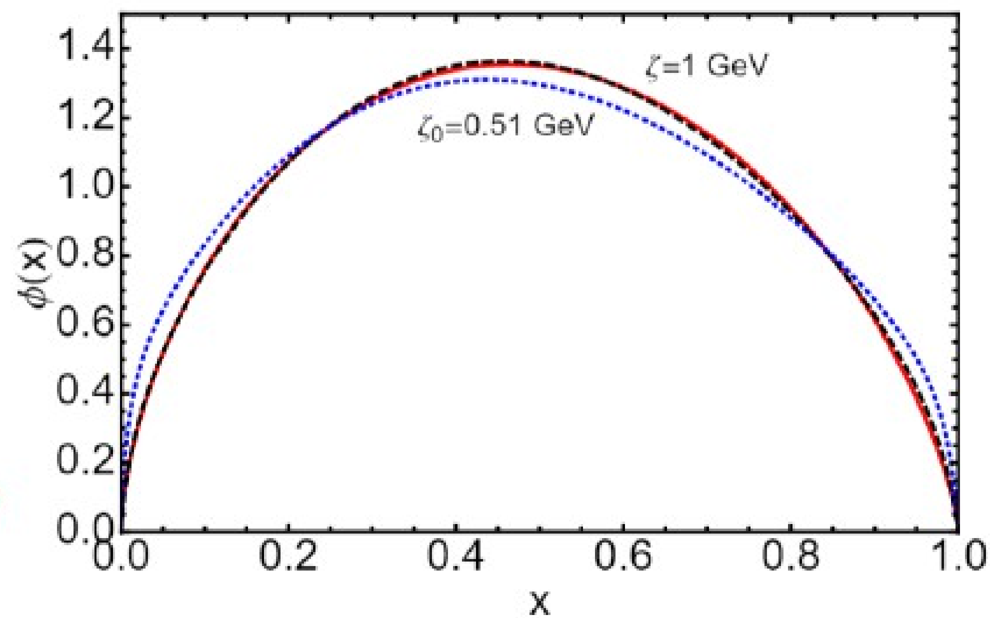
- If the factorized ansatz is a good approximation, then the plotted ratio must be 1. For the pion, it slightly deviates from 1; for the kaon, the deviation is much larger.

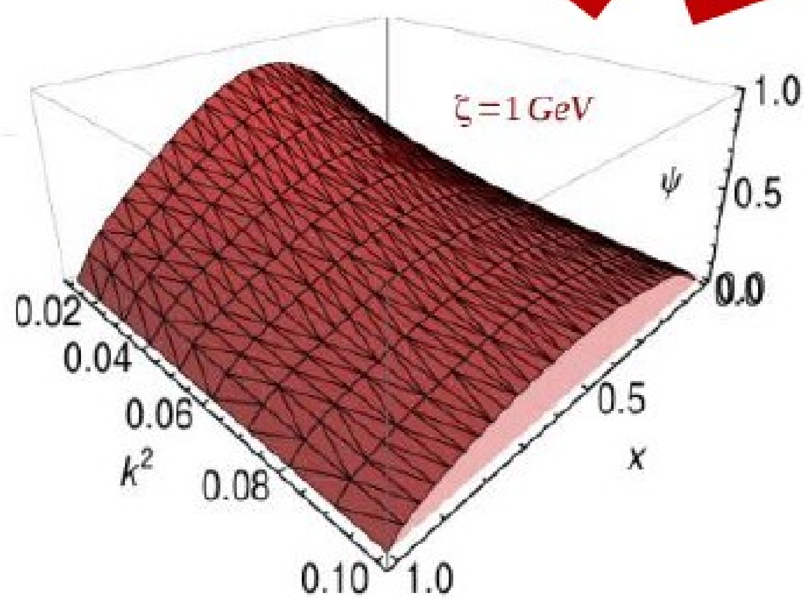
PDA and LFWF evolution



Testing the factorization ansatz:







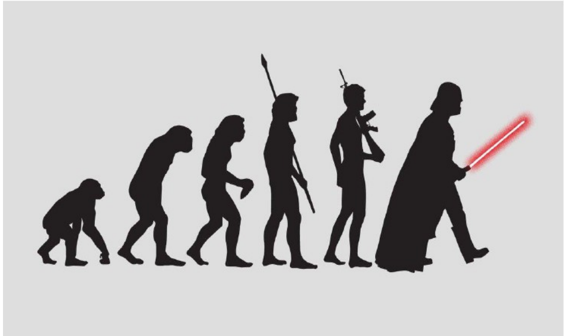
1) Compute LFWF and ERBL running of PDA

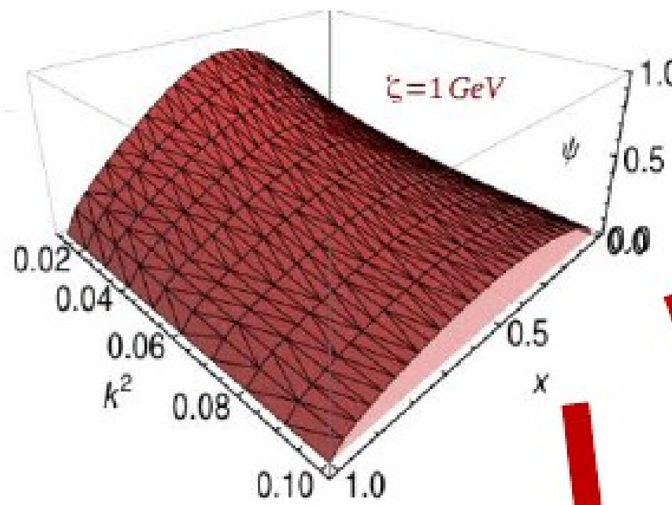
2) ERBL running of LFWF and compute PDA

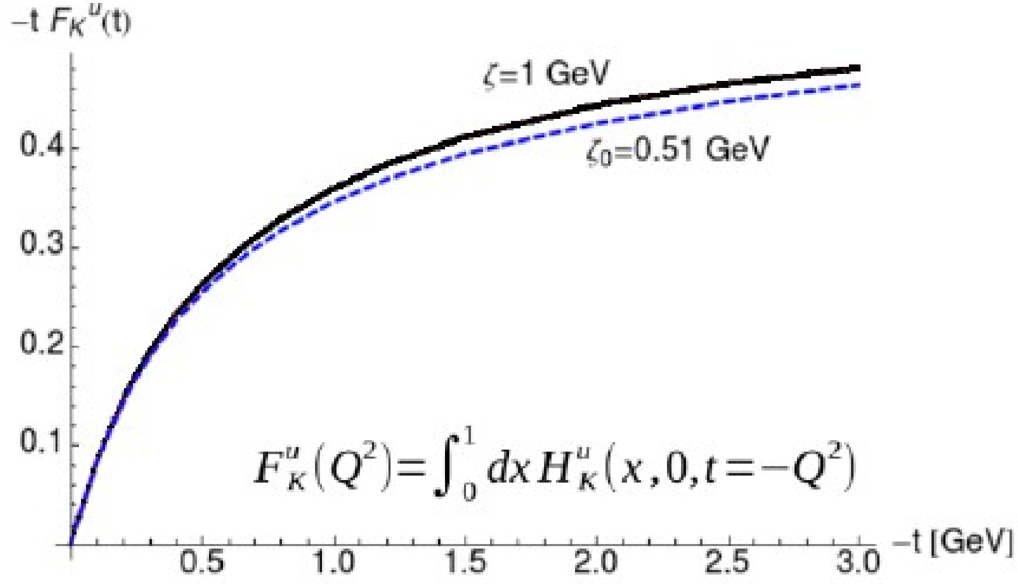
Notably, **1)** and **2)** are **equivalent**. Factorization assumption and evolution seem reasonable.

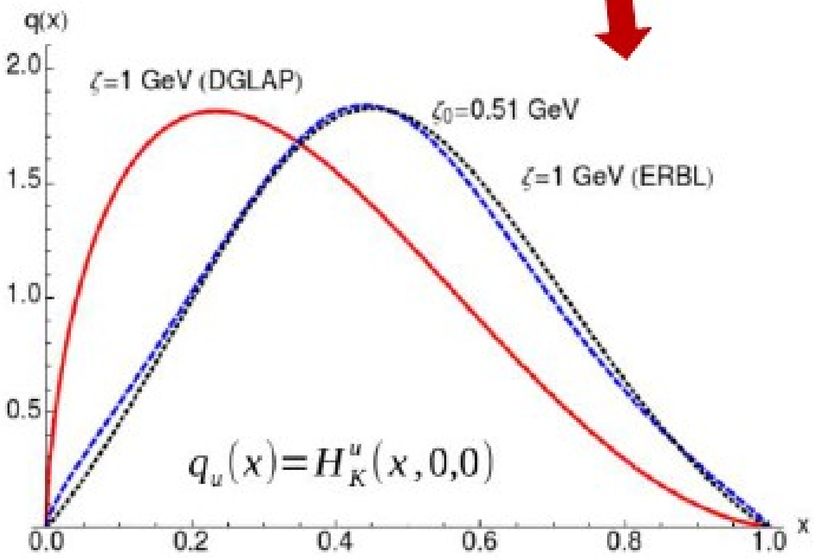
PDA and LFWF evolution

How ERBL and DGLAP evolutions make contact:









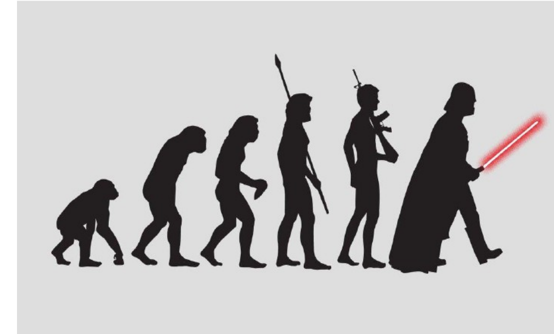
1) Obtained from ERBL evolution of LFWF

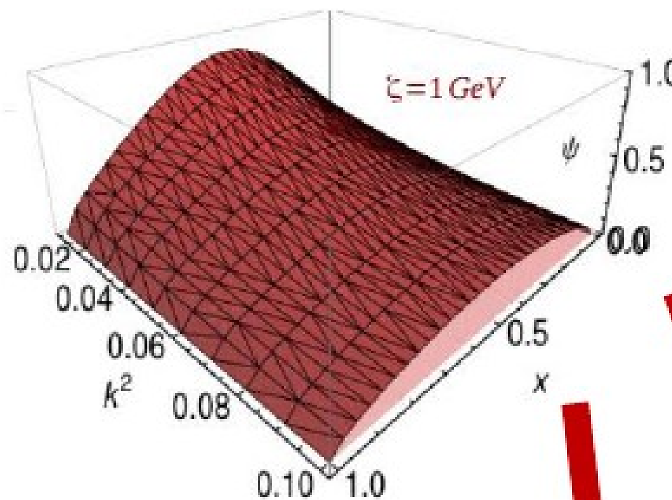
2) Obtained from DGLAP evolution of GPD

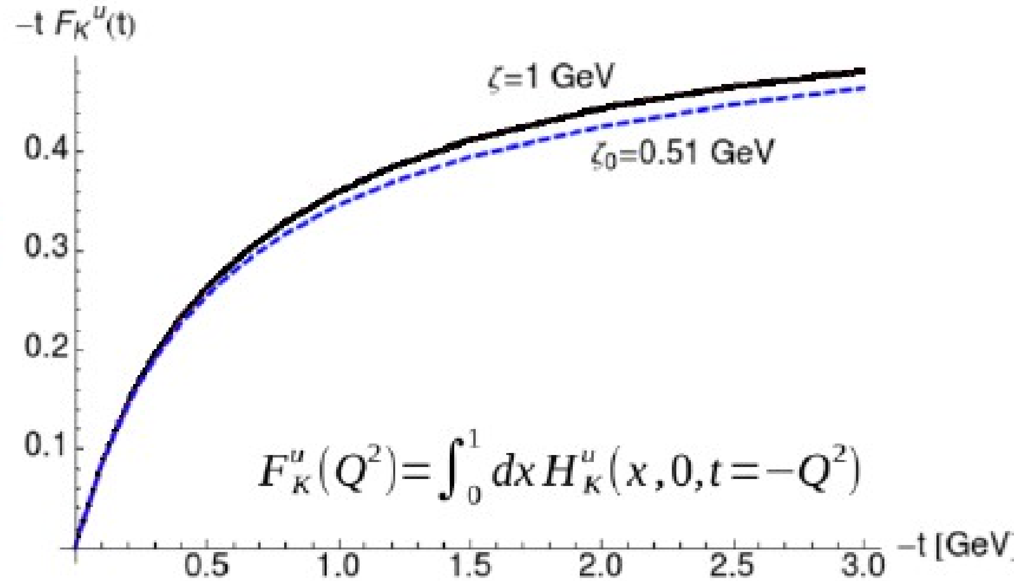
Clearly, **1)** and **2)** are **not equivalent**.

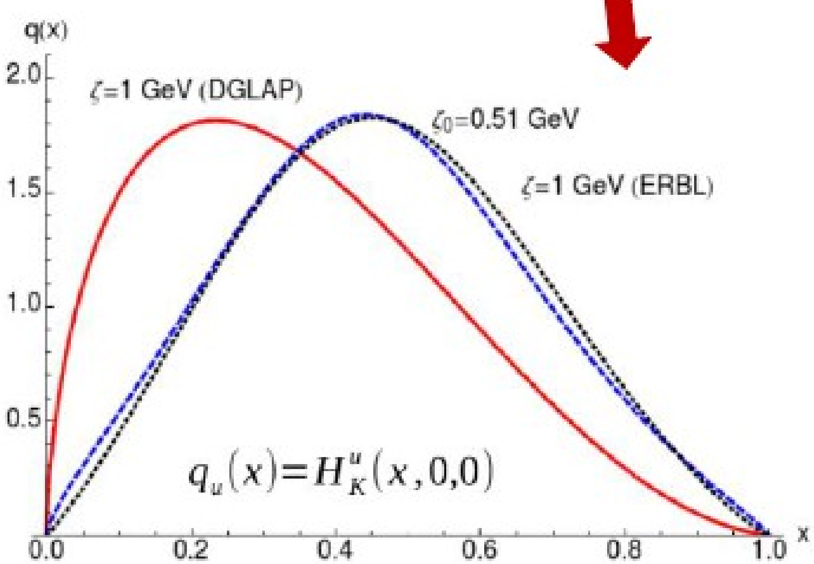
PDA and LFWF evolution

How ERBL and DGLAP evolutions make contact:









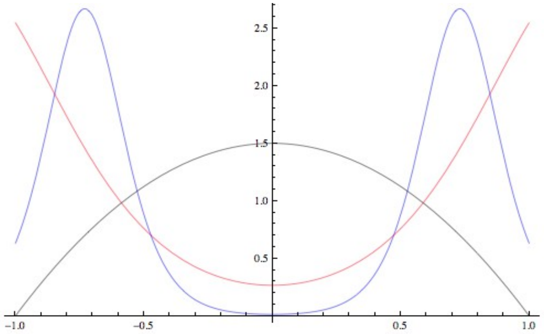
1) Obtained from ERBL evolution of LFWF

2) Obtained from DGLAP evolution of GPD

Clearly, **1)** and **2)** are **not equivalent**.

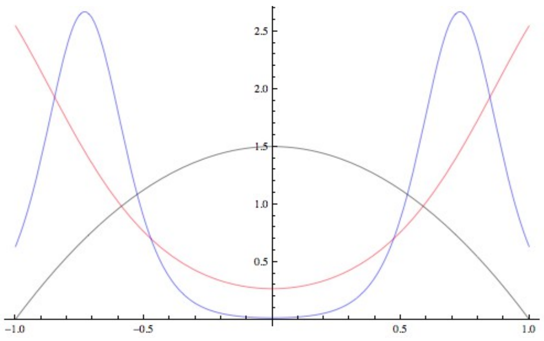
Sea-quark and gluon content incorporated to the parton distribution by DGLAP are obviously not present in the valence-quark PDF from LFWFs!!!

Conclusions

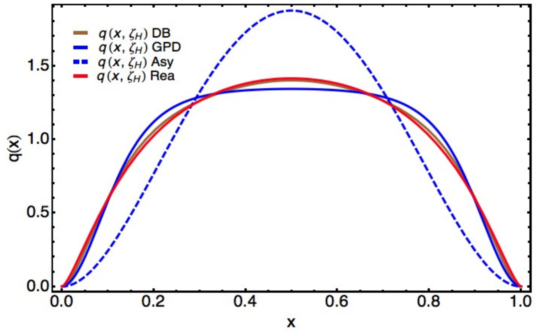


Owing to a sensible parametrisation of the BSA grounded on the so-called Nakanishi representation, one is left with a flexible algebraic model for the LFWF in terms of a spectral density.

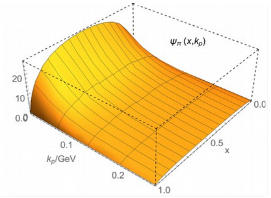
Conclusions



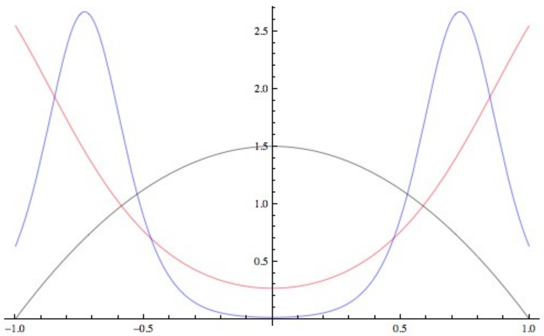
Owing to a sensible parametrisation of the BSA grounded on the so-called Nakanishi representation, one is left with a flexible algebraic model for the LFWF in terms of a spectral density.



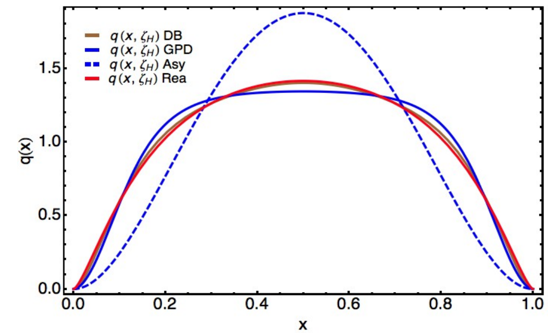
A direct calculation of the PDF from realistic quark gap and Bethe-Salpeter equations' solutions (in the forward kinematical limit) delivers a benchmark result to identify the spectral density which corresponds to the realistic LFWF.



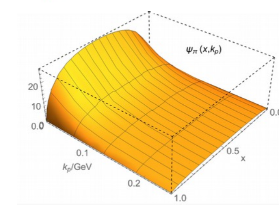
Conclusions

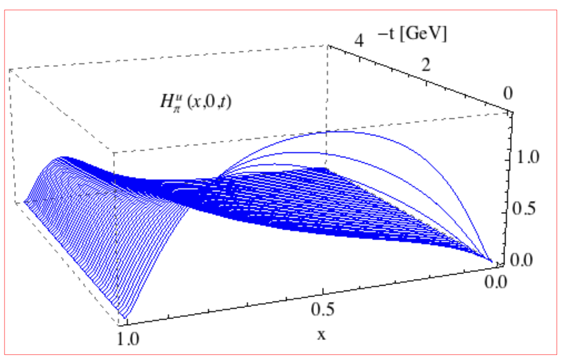


Owing to a sensible parametrisation of the BSA grounded on the so-called Nakanishi representation, one is left with a flexible algebraic model for the LFWF in terms of a spectral density.



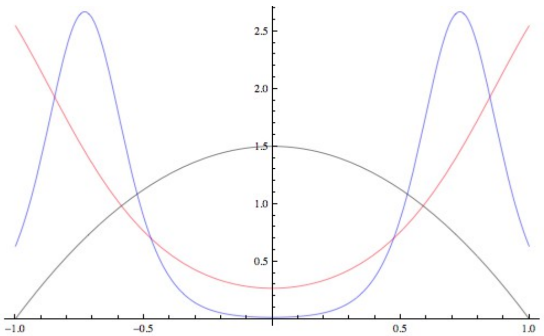
A direct calculation of the PDF from realistic quark gap and Bethe-Salpeter equations' solutions (in the forward kinematical limit) delivers a benchmark result to identify the spectral density which corresponds to the realistic LFWF.



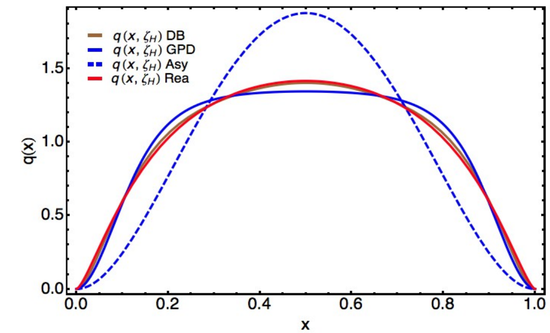


The overlap representation provides with a simple way to calculate beyond the forward kinematic limit, and thus obtain the GPD, although only in the DGLAP region.

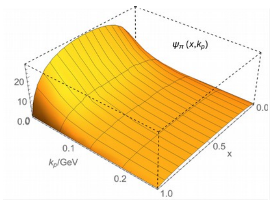
Conclusions

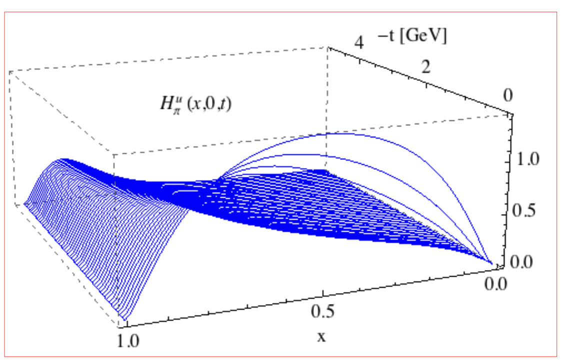


Owing to a sensible parametrisation of the BSA grounded on the so-called Nakanishi representation, one is left with a flexible algebraic model for the LFWF in terms of a spectral density.

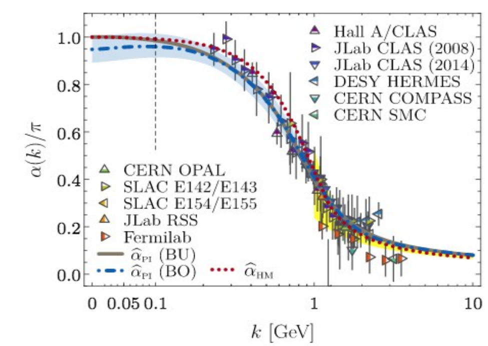


A direct calculation of the PDF from realistic quark gap and Bethe-Salpeter equations' solutions (in the forward kinematical limit) delivers a benchmark result to identify the spectral density which corresponds to the realistic LFWF.



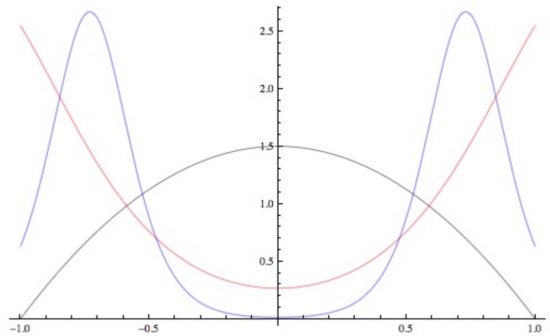


The overlap representation provides with a simple way to calculate beyond the forward kinematic limit, and thus obtain the GPD, although only in the DGLAP region.

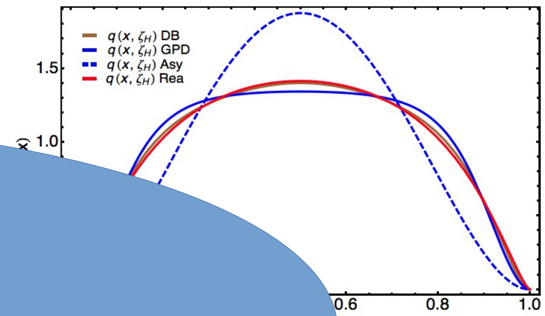


A recently proposed PI effective charge can be used to make the DGLAP GPD evolve from the hadronic scale (where quasi-particle DSE's solutions are the correct degrees-of-freedom) up to any other relevant scale.

Conclusions

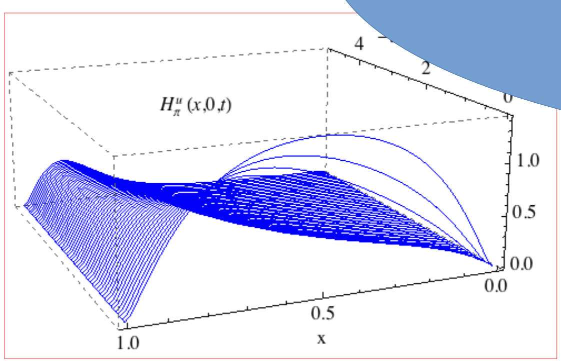


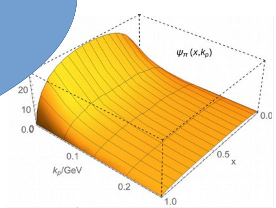
Owing to a sensible parametrisation of the BSA grounded on the so-called Nakanishi representation, one is left with a flexible algebraic model for the LFWF in terms of a spectral density.



A direct calculation of the PDF f
Bethe-Salpeter equ
kinematical limit)
spectral density

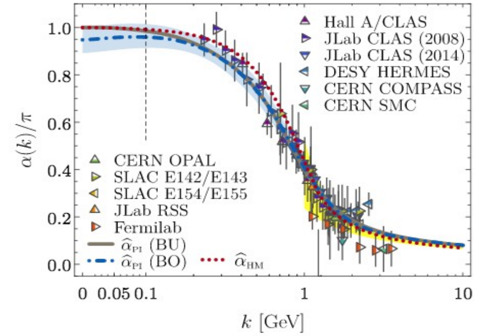
Thank you!!





The overlap representation provides with a simple way to calculate beyond the forward kinematic limit, and thus obtain the GPD, although only in the DGLAP region.

A recently proposed PI effective charge can be used to make the DGLAP GPD evolve from the hadronic scale (where quasi-particle DSE's solutions are the correct degrees-of-freedom) up to any other relevant scale.



Backslides

A word about bout gravitational Form Factors

A word about GPD polinomiality first:

- Express Mellin moments of GPDs as **matrix elements**:

$$\int_{-1}^{+1} \mathrm{d}x\, x^m H^q(x,\xi,t) = \frac{1}{2(P^+)^{m+1}} \left\langle P+\frac{\Delta}{2} \right| \bar{q}(0)\gamma^+ (i\overleftrightarrow{D}^+)^m q(0) \left| P-\frac{\Delta}{2} \right\rangle$$

- Identify the **Lorentz structure** of the matrix element:

linear combination of $(P^+)^{m+1-k}(\Delta^+)^k$ for $0 \leq k \leq m+1$

- Remember definition of **skewness** $\Delta^+ = -2\xi P^+$.
- Select **even powers** to implement time reversal.
- Obtain **polynomiality condition**:

$$\int_{-1}^{1} \mathrm{d}x\, x^m H^q(x,\xi,t) = \sum_{\substack{i=0 \\ \text{even}}}^{m} (2\xi)^i C^q_{mi}(t) + (2\xi)^{m+1} C^q_{mm+1}(t) \; .$$

A word about bout gravitational Form Factors

Definition and evaluation:

- Pion gravitational form factors are defined through*: Polinomiality!

$$J_{\pi^+}(-t,\xi) \equiv \int_{-1}^{1} dx\, x H_{\pi^+}(x,\xi,t) = \Theta_2(t) - \Theta_1(t)\xi^2 \,.$$

- Taking ξ=0 + isospin symmetric limit, one can readily compute:

$$\Theta_2(t) = \int_0^1 dx\, x[H^u_{\pi^+}(x,0,t) + H^d_{\pi^+}(x,0,t)] = \int_0^1 dx\, 2x H^u_{\pi^+}(x,0,t)\,.$$

- To obtain $\Theta_1(t)$, we need to take a non zero value of ξ; hence requiring the knowledge of the GPD in the ERBL region.

- Nevertheless, one can approximate $\Theta_1(t)$, by estimating the derivative of $J_{\pi^+}(-t,\xi)$ with respect to ξ^2 as:

$$D(\xi + \Delta/2) \equiv \frac{J(\xi+\Delta) - J(\xi)}{2(\xi+\Delta/2)\Delta}\,,\ \Delta \to 0\,.$$

*Phys.Rev. D78 (2008) 094011.

A word about bout gravitational Form Factors

Definition and evaluation:

- Pion gravitational form factors are defined through*:

Polynomiality!

$$J_{\pi^+}(-t,\xi) \equiv \int_{-1}^{1} dx\; x H_{\pi^+}(x,\xi,t) = \Theta_2(t) - \Theta_1(t)\xi^2 \;.$$

- Taking ξ=0 + isospin symmetric limit, one can readily compute:

$$\Theta_2(t) = \int_0^1 dx\; x[H^u_{\pi^+}(x,0,t) + H^d_{\pi^+}(x,0,t)] = \int_0^1 dx\; 2x H^u_{\pi^+}(x,0,t) \;.$$

- To obtain $\Theta_1(t)$, we need to take a non zero value of ξ; hence requiring the knowledge of the GPD in the ERBL region.

- Nevertheless, one can approximate $\Theta_1(t)$, by estimating the derivative of $J_{\pi^+}(-t,\xi)$ with respect to ξ^2 as:

$$D(\xi + \Delta/2) \equiv \frac{J(\xi+\Delta) - J(\xi)}{2(\xi+\Delta/2)\Delta} \;,\; \Delta \to 0 \;.$$

*Phys.Rev. D78 (2008) 094011.

Polinomiality tells us that it is enough to evaluate in the vicinity of zero!

A word about bout gravitational Form Factors

Definition and evaluation:

- Pion gravitational form factors are defined through*:

Polynomiality!

$$J_{\pi^+}(-t,\xi) \equiv \int_{-1}^{1} dx\; x H_{\pi^+}(x,\xi,t) = \Theta_2(t) - \Theta_1(t)\xi^2 \; .$$

- Taking ξ=0 + isospin symmetric limit, one can readily compute:

$$\Theta_2(t) = \int_0^1 dx\; x[H^u_{\pi^+}(x,0,t) + H^d_{\pi^+}(x,0,t)] = \int_0^1 dx\; 2x H^u_{\pi^+}(x,0,t) \; .$$

- To obtain $\Theta_1(t)$, we need to take a non zero value of ξ; hence requiring the knowledge of the GPD in the ERBL region.

- Nevertheless, one can approximate $\Theta_1(t)$, by estimating the derivative of $J_{\pi^+}(-t,\xi)$ with respect to ξ^2 as:

$$D(\Delta/2) \equiv \frac{J(\Delta) - J(0)}{\Delta^2} \; , \; \Delta \to 0$$

***Phys.Rev. D78 (2008) 094011.**

Polinomiality tells us that it is enough to evaluate in the vicinity of zero!

A word about bout gravitational Form Factors

Definition and evaluation:

- To get a clearer picture, let's split $J(-t,\xi)$ as follows:

$$J(-t,\xi) = \int_{-\xi}^{1} dx\ 2xH(x,\xi,t) = \left[\int_{-\xi}^{\xi} dx + \int_{\xi}^{1} dx\right] 2xH(x,\xi,t)$$

$$\Rightarrow J(-t,\xi) = J^{\mathrm{ERBL}}(-t,\xi) + J^{\mathrm{DGLAP}}(-t,\xi)\ ,$$

- Notice that, because of the polinomiality of the *complete* GPD:

$$J^{\mathrm{DGLAP}}(-t,\xi) = \Theta_2(t) - \xi^2\Theta_1(t)^{\mathrm{DGLAP}} + \sum_{i=1}^{\infty} c_i(t)\xi^{2+i}\ ,$$

$$J^{\mathrm{ERBL}}(-t,\xi) = -\xi^2\Theta_1(t)^{\mathrm{ERBL}} - \sum_{i=1}^{\infty} c_i(t)\xi^{2+i}$$

- Thus, since so far we can only access DGLAP region: (overlap approximation)

$$J^{\mathrm{DGLAP}}(-t,\xi) = \Theta_2(t) - \xi^2\Theta_1(t)^{\mathrm{DGLAP}} + \sum_{i=1}^{\infty} c_i(t)\xi^{2+i},$$

A word about bout gravitational Form Factors

Definition and evaluation:

- To get a clearer picture, let's split $J(-t,\xi)$ as follows:

$$J(-t,\xi) = \int_{-\xi}^{1} dx\; 2xH(x,\xi,t) = \left[\int_{-\xi}^{\xi} dx + \int_{\xi}^{1} dx\right] 2xH(x,\xi,t)$$

$$\Rightarrow J(-t,\xi) = J^{\mathrm{ERBL}}(-t,\xi) + J^{\mathrm{DGLAP}}(-t,\xi)\ ,$$

- Notice that, because of the polinomiality of the *complete* GPD:

$$J^{\mathrm{DGLAP}}(-t,\xi) = \Theta_2(t) - \xi^2\Theta_1(t)^{\mathrm{DGLAP}} + \sum_{i=1}^{\infty} c_i(t)\xi^{2+i}\ ,$$

$$J^{\mathrm{ERBL}}(-t,\xi) = -\xi^2\Theta_1(t)^{\mathrm{ERBL}} - \sum_{i=1}^{\infty} c_i(t)\xi^{2+i}$$

?

- Thus, since so far we can only access DGLAP region: (overlap approximation)

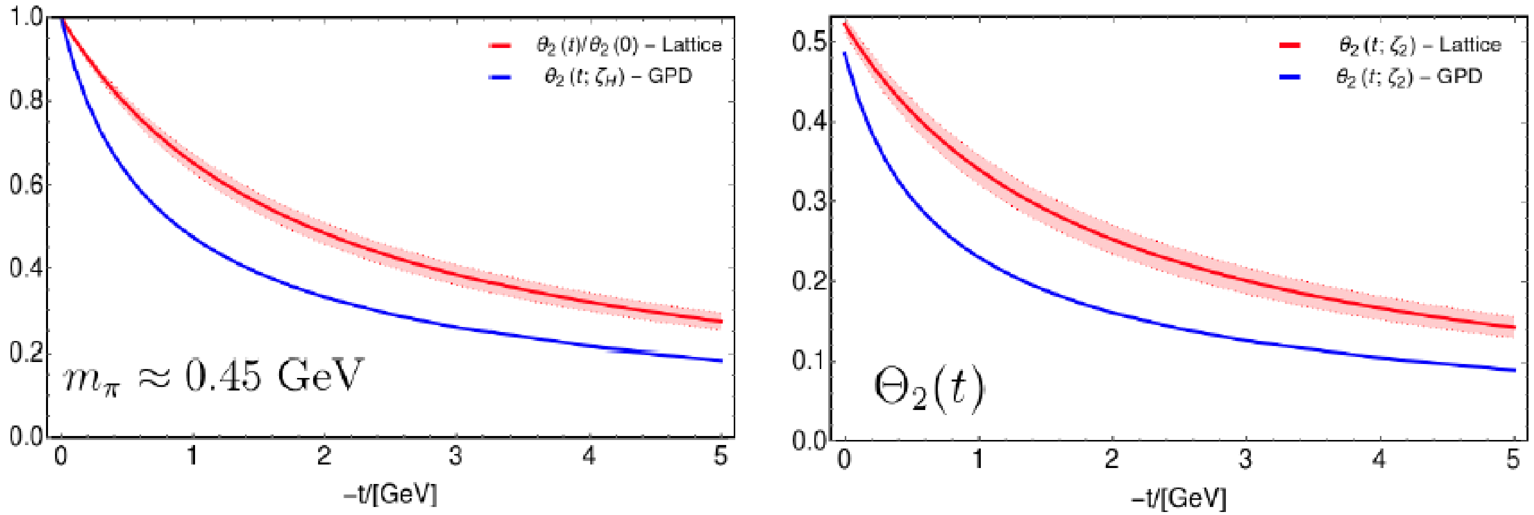
$$J^{\mathrm{DGLAP}}(-t,\xi) = \Theta_2(t) - \xi^2\Theta_1(t)^{\mathrm{DGLAP}} + \sum_{i=1}^{\infty} c_i(t)\xi^{2+i},$$

A word about bout gravitational Form Factors

Definition and evaluation:

- The extensión to **ERBL region** is then **needed**. Taking advantage of the soft-pion theorem, one can conect PDA with $J(-t,\xi)^{ERBL}$ and thus with $\Theta_1(t)^{ERBL}$.

- Nonetheless, polinomiality of GPD is not fulfilled without the ERBL región. Such extension is necessary to provide a more reliable computation of Θ_1.



Lattice: (2007) Brömmel's dissertation. $\Theta_2(0)/2 = \langle x \rangle = 0.261(5)$

GPD + Ding et al. $\Theta_2(0)/2 = \langle x \rangle = 0.242(20)$

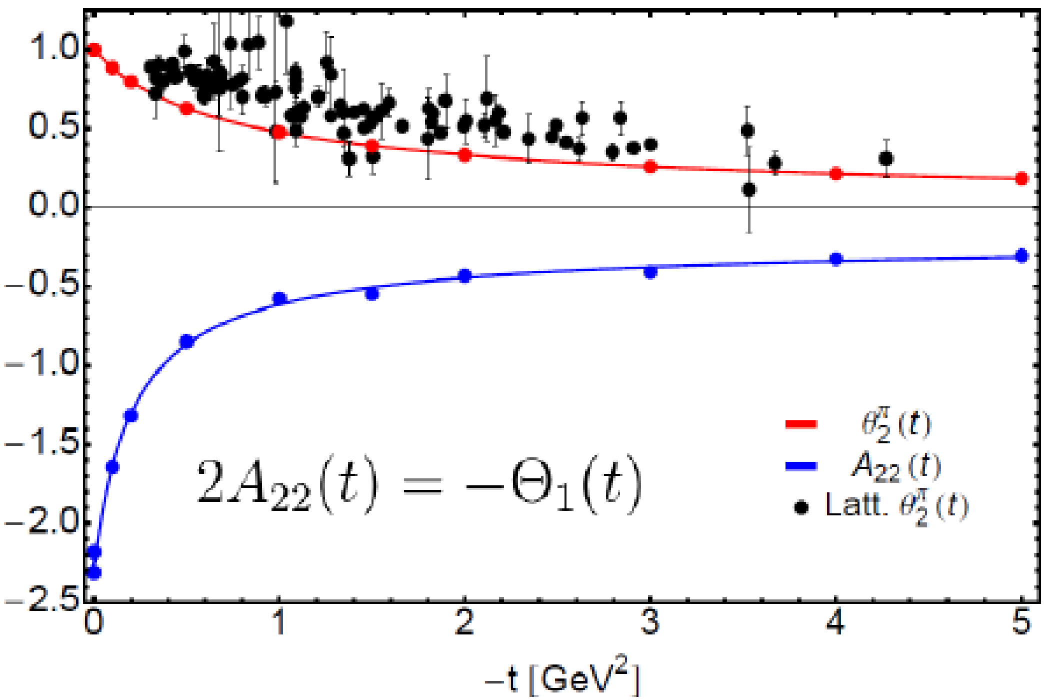
Latt.: D. Brommel, Ph.D. thesis, University of Regensburg, Regensburg, Germany (2007), DESY-THESIS-2007-023

A word about bout gravitational Form Factors

Definition and evaluation:

- The extensión to **ERBL region** is then **needed**. Taking advantage of the soft-pion theorem, one can conect PDA with $J(-t,\xi)^{ERBL}$ and thus with $\Theta_1(t)^{ERBL}$.

- Nonetheless, polinomiality of GPD is not fulfilled without the ERBL región. Such extension is necessary to provide a more reliable computation of Θ_1.



Latt.: D. Brommel, Ph.D. thesis, University of Regensburg, Regensburg, Germany (2007), DESY-THESIS-2007-023