Minkowski-space solutions of the Schwinger-Dyson equation for the fermion propagator with the rainbow-ladder truncation

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The Lagrangian of QED with massive photons

$$\mathcal{L} = \overline{\psi}(i\not{D} - m_{\rm B})\psi + \frac{1}{2}A_\mu \left[g^{\mu\nu}\left(\partial^2 + m_{\rm A}^2\right) - \left(1 - \frac{1}{\xi}\right)\partial^\mu\partial^\nu\right] A_\nu, \tag{1}$$

with the covariant derivative defined as

$$D^\mu = \partial^\mu - ieA^\mu. \tag{2}$$

Here ψ is the fermion field. A^μ are the vector boson fields. $m_{\rm B}$ is the bare mass of the fermion. $m_{\rm A}$ is the mass of the gauge boson. e is the elementary charge.
The bare propagators are given by

$$S_{\rm F}^0(p) = \frac{1}{\not{p} - m_{\rm B} + i\varepsilon}, \quad D_{\mu\nu}^0(q) = \frac{g_{\mu\nu} + (\xi - 1)q_\mu q_\nu/(q^2 - \xi m_{\rm A}^2)}{q^2 - m_{\rm A}^2 + i\varepsilon}. \tag{3}$$

The dressed fermion propagator $S_{\rm F}(p)$ is to be solved from its Schwinger-Dyson equation.

The Schwinger-Dyson equation for $S_{\mathrm{F}}(p)$

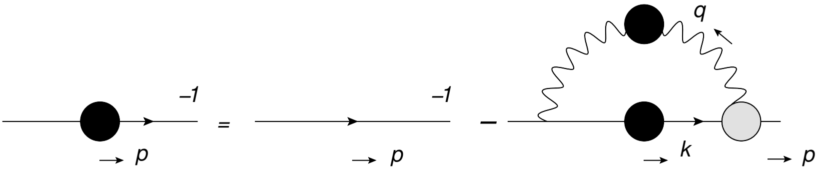


Figure: The Schwinger-Dyson equation for the fermion propagator in QED.

$$S_{\mathrm{F}}^{-1}(p) = \not{p} - m_{\mathrm{B}} - \Sigma(p). \tag{4}$$

$$\begin{aligned}\Sigma(p) &= -ie^2 \int d\underline{k}\, \gamma^\nu\, S_{\mathrm{F}}(k)\Gamma^\mu(k,p)D_{\mu\nu}(q) \\ &\equiv \not{p}\Sigma_{\mathrm{v}}(p^2) + \Sigma_{\mathrm{s}}(p^2).\end{aligned} \tag{5}$$

Here $\Gamma^\mu(k,p)$ is the one-particle-irreducible fermion-photon vertex. In the rainbow-ladder truncation, we take $\Gamma^\mu(k,p) = \gamma^\mu$.

The Källén-Lehmann spectral representation

The spectral representation of the fermion propagator is given by

$$S_{\mathrm{F}}(p) = \int_{|W|=m}^{+\infty} dW \frac{\rho(W)}{\not{p} - W + i\,\mathrm{sign}(W)\,\varepsilon}, \tag{6}$$

where the support of the spectral integral is $W \in (-\infty, -m] \cup [m, +\infty)$. The spectral function $\rho(W)$ is related to two scalar spectral functions by

$$\rho(W) = \mathrm{sign}(W)[W\,\rho_1(W^2) + \rho_2(W^2)], \tag{7}$$

such that $S_{\mathrm{F}}(p) = \not{p} S_{\mathrm{v}}(p^2) + S_{\mathrm{s}}(p^2)$ with

$$\begin{cases} S_{\mathrm{v}}(p^2) = \int_{m^2}^{+\infty} ds \dfrac{\rho_1(s)}{p^2 - s + i\varepsilon} \\ S_{\mathrm{s}}(p^2) = \int_{m^2}^{+\infty} ds \dfrac{\rho_2(s)}{p^2 - s + i\varepsilon} \end{cases} . \tag{8}$$

[J.Phys. A10 1049 Delbourgo:1977jc]

The mass-shell pole and the branch-cuts

$$\rho_j(s) = Z_j\,\delta(s - m^2) + r_j(s), \quad \text{for}\, j \in \{1, 2\}.$$

The renormalized mass is to be solved from

$$m \in \left\{ \sqrt{p^2} \,|\, \forall \sqrt{p^2} \in \mathbb{R}\, \text{such that}\, \sqrt{p^2}[1 - \Sigma_v(p^2)] = m_{\rm B} + \Sigma_{\rm s}(p^2) \right\}.$$

Meanwhile, the mass-pole residue is given by

$$Z = \left[1 - \Sigma_{\rm v}(m^2)\right]^{-1}. \tag{9}$$

To calculate the branch-cuts of the spectral functions, define

$$A(p^2) = 1 - \Sigma{\rm v}(p^2) \quad \text{and} \quad B(p^2) = -m_{\rm B} - \Sigma_{\rm s}(p^2), \tag{10}$$

such that the functions $r_{\rm s,v}(s)$ are given by

$$\begin{cases} r_{\rm v}(s) = -\frac{1}{\pi}{\rm Im}\left\{ \frac{A(s)}{A^2(s)p^2 - B^2(s)} \right\}\Big|_{s \geq p^2_{\rm th}} \\ r_{\rm s}(s) = -\frac{1}{\pi}{\rm Im}\left\{ \frac{-B(s)}{A^2(s)p^2 - B^2(s)} \right\}\Big|_{s \geq p^2_{\rm th}} \end{cases}. \tag{11}$$

Here $p^2_{\rm th}$ is the threshold above which the imaginary parts of the fermion self-energy become nonzero.

The quenched approximation in the Feynman gauge

Within the quenched approximation, the fermion self-energy in the Feynman gauge ($\xi = 1$) is given by

$$\frac{\delta\,\Sigma_{\rm F}(p)}{\delta\,\rho(W)} = -ie^2 \int d\underline{l} \int dF^2 \, \frac{(2-d)y\not{p} + dW}{(l^2 - \Delta + i\varepsilon)^2}\,, \qquad (12)$$

with $\int dF^2 = \int_0^1 dx \int_0^1 dy\, \delta(1-x-y)$ and

$$\Delta(x, y, m_\sigma, k^2) = x\,s + y\,m_{\rm A}^2 - xyp^2 - i\varepsilon. \qquad (13)$$

The imaginary part of the fermion self-energy is subsequently given by

$$\begin{aligned}
&-\frac{1}{\pi}{\rm Im}\{\Sigma_{\rm Fv}(p^2)\} \\
&= \frac{\alpha}{4\pi}\int_{m^2}^{(\sqrt{p^2}-\sqrt{m_{\rm A}^2})^2} ds\, \frac{p^2 - m_{\rm A}^2 + s}{p^4}\sqrt{(p^2 - m_{\rm A}^2 + s)^2 - 4p^2 s}\,\rho_1(s), \qquad (14a) \\
&-\frac{1}{\pi}{\rm Im}\{\Sigma_{\rm Fs}(p^2)\} \\
&= \frac{\alpha}{4\pi}\int_{m^2}^{(\sqrt{p^2}-\sqrt{m_{\rm A}^2})^2} ds\, \left(-\frac{4}{p^2}\right)\sqrt{(p^2 - m_{\rm A}^2 + s)^2 - 4p^2 s}\,\rho_2(s). \qquad (14b)
\end{aligned}$$

Pauli-Villars regularization

- In the Pauli-Villars regularization scheme, divergences in the fermion self-energy are removed by:

$$\Sigma_{\mathrm{PV}}(p) = \Sigma_{\mathrm{bare}}(p) - (m_{\mathrm{A}} \to \Lambda), \tag{15}$$

  where $\Lambda > m_{\mathrm{A}}$ is the mass of the regulator.
- The direct implementation of the spectral representation for the self-energy is implied in the Pauli-Villars regularization scheme due to the asymptotic behavior of $\Sigma_{\mathrm{PV}}(p)$.
- With dimensional regularization, subtracted spectral representation applies to the fermion-self energy.
- In other R_ξ gauges, the partial-fraction decomposition could be used to simplify the loop integral.

Numerical solution for the fermion spectral functions

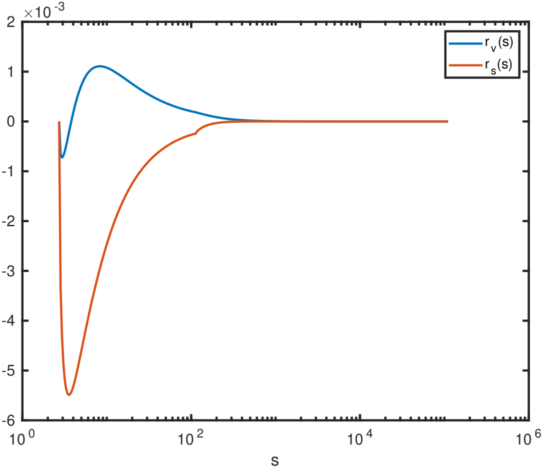
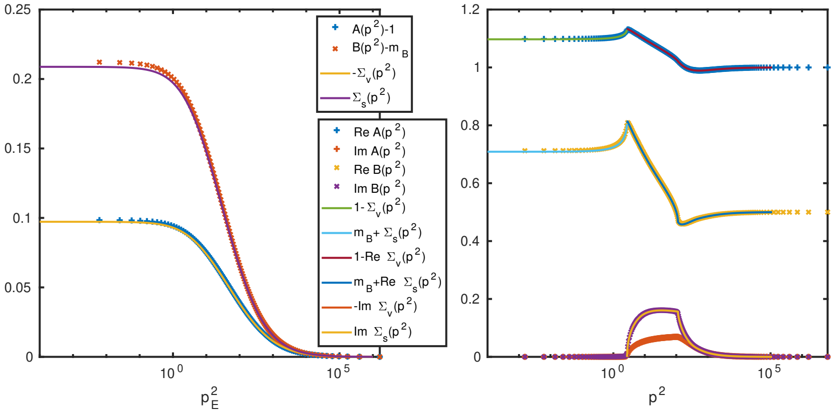


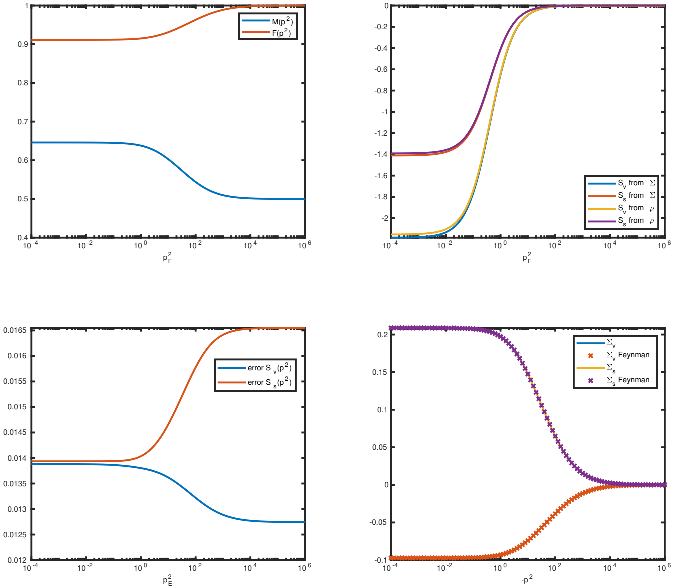
Figure: $m_{\mathrm{B}} = 0.5, \quad e^2/(4\pi) = 0.3, \quad m_{\mathrm{A}} = 1, \quad \Lambda = 10.$ $Z = 0.9096$ and $m_{\mathrm{R}} = 0.6502$.

Numerical solution for the fermion self-energy

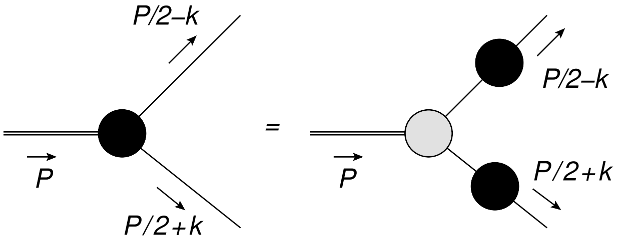


[arXiv:1905.00703 Frederico:2019noo]

Solutions in the spacelike region



The Nakanishi representation of the Bethe-Salpeter vertex and amplitude for scalar bound states



Define the Bethe-Salpeter amplitude

$$\chi(k, k \cdot P) = \psi(k, k \cdot P) D\left(P/2 + k\right) D\left(P/2 - k\right), \qquad (16)$$

with $\psi(k, k \cdot P)$ being the Bethe-Salpeter vertex.

- Spectral representation for the scalar propagator

$$D(p) = \int ds \frac{\rho(s)}{p^2 - s + i\varepsilon}$$

- and the Nakanishi representations for the BS vertex and the amplitude

$$\psi(k, k\cdot P) = \int d\gamma \int_{-1}^{+1} dz \, \frac{\phi(\gamma, z)}{(k^2 + zk\cdot P - \gamma + i\varepsilon)^n} \tag{17}$$

$$\chi(k, k\cdot P) = \int d\gamma \int_{-1}^{+1} dz \frac{\Phi(\gamma, z)}{(k^2 + zk\cdot P - \gamma + i\varepsilon)^{n+2}} \tag{18}$$

indicate the functional relation of Φ in terms of ϕ and ρ.

$$\begin{aligned}
\Phi(\gamma, z) &= \frac{\Gamma(n+2)}{\Gamma(n)} \int ds \int dt \, \rho(s)\, \rho(t) \left[s + t + (s-t)z - 2\left(\gamma + \frac{P^2}{4}\right)\right]^{n-1} \\
&\times \left\{ \theta\left([s + t + (s-t)z] - 2\left(\gamma + \frac{P^2}{4}\right)\right) \left[\int_{-1}^{z} dz' \int_0^{\gamma_{\rm th}(-z,\gamma,s,-z')} d\gamma' + \int_z^1 dz' \int_0^{\gamma_{\rm th}(z,\gamma,t,z')} d\gamma' \right] \right. \\
&- \left. \theta\left(2\left(\gamma + \frac{P^2}{4}\right) - [s + t + (s-t)z]\right) \left[\int_{-1}^{z} dz' \int_{\gamma_{\rm th}(-z,\gamma,s,-z')}^{+\infty} d\gamma' + \int_z^1 dz' \int_{\gamma_{\rm th}(z,\gamma,t,z')}^{+\infty} d\gamma' \right] \right\} \\
&\times \left[s + t + (s-t)z' - 2\left(\gamma' + \frac{P^2}{4}\right)\right]^{-n} \phi(\gamma', z').
\end{aligned} \tag{19}$$

$$\gamma_{\rm th}(z, \gamma, u, z'; P^2) \equiv \frac{(1+z')\gamma - (z'-z)\left(u - \frac{P^2}{4}\right)}{1+z}.$$

Specifically when $n = 1$, the result is further reduced to

$$\Phi(\gamma, z) = \int ds \int dt\, \rho(s)\, \rho(t) \Bigg\{ \theta\left([s + t + (s - t)z] - 2\left(\gamma + \frac{P^2}{4}\right) \right)$$
$$\times \left[\int_{-1}^{z} dz' \int_{0}^{\gamma_{\rm th}(-z,\gamma,s,-z')} d\gamma' + \int_{z}^{1} dz' \int_{0}^{\gamma_{\rm th}(z,\gamma,t,z')} d\gamma' \right]$$
$$- \theta\left(2\left(\gamma + \frac{P^2}{4}\right) - [s + t + (s - t)z] \right)$$
$$\times \left[\int_{-1}^{z} dz' \int_{\gamma_{\rm th}(-z,\gamma,s,-z')}^{+\infty} d\gamma' + \int_{z}^{1} dz' \int_{\gamma_{\rm th}(z,\gamma,t,z')}^{+\infty} d\gamma' \right] \Bigg\}$$
$$\times \frac{2\,\phi(\gamma', z')}{s + t + (s - t)z' - 2\left(\gamma' + \frac{P^2}{4}\right)}, \qquad (20)$$

with $\gamma_{\rm th}(z, \gamma, u, z'; P^2) \equiv \left[(1 + z')\gamma - (z' - z)\left(u - P^2 2/4\right) \right] / (1+z)$.

With the bare propagator, $\Phi(\gamma, z)$ starts from $\Gamma_{\rm th} = m^2 - P^2/4$.

The Bethe-Salpeter equation in the ladder truncation

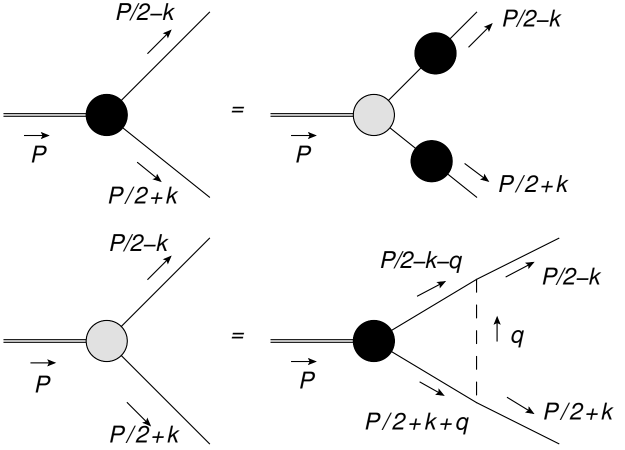


Figure: The diagrammatic representation of the Bethe-Salpeter equation for scalar bound states in the rainbow-ladder truncation.

$$\psi(k, k\cdot P) = ig^2 \int d\underline{q}\, \chi\left(k+q, (k+q)\cdot P\right) \mathcal{D}(q). \qquad (21)$$

The propagator of the exchange particle is given by

$$\mathcal{D}(q) = \frac{1}{q^2 - \mu^2 - i\varepsilon}.$$

With the Nakanishi representation for both the Bethe-Salpeter amplitude and the vertex, the BSE becomes

$$\int d\gamma \int dz \, \frac{\phi(\gamma, z)}{(k^2 + zk \cdot P - \gamma + i\varepsilon)^n} = ig^2 \int d\underline{q} \int d\gamma \int dz \times \frac{\Phi(\gamma, z)}{\left[(k+q)^2 + z\,(k+q) \cdot P - \gamma + i\varepsilon\right]^{n+2} (q^2 - \mu^2 + i\varepsilon)}, \quad (22)$$

which indicates

$$\phi(\gamma, z) = -\frac{g^2}{(4\pi)^2} \frac{1}{n(n+1)} \int d\gamma' \int dz' \int_0^1 dx \, \Phi(\gamma', z') \, \frac{1}{(1-x)^{n+1}} \times \delta(z - z') \, \delta\left(\gamma - \left[\frac{\gamma'}{1-x} + \frac{\mu^2}{x} + \frac{xz'^2 P^2}{4(1-x)}\right]\right) \frac{\partial}{\partial \gamma}.$$

Within the condition that $\gamma + z^2 \frac{P^2}{4} \geq \mu^2$, we have

$$\begin{aligned}
&\phi(\gamma, z) \\
&= -\frac{g^2}{(4\pi)^2} \frac{1}{n(n+1)} \int_{\mu^2-\gamma-z^2P^2/2}^{\gamma+\mu^2-2\mu\sqrt{\gamma+z^2P^2/4}} d\gamma' \\
&\quad \times \frac{\Phi(\gamma', z)}{\sqrt{(\gamma+\mu^2-\gamma')^2 - 4\mu^2(\gamma+z^2P^2/4)}} \\
&\quad \times \left\{ \frac{x_-(\gamma', z, \gamma)}{[1-x_-(\gamma', z, \gamma)]^n} + \frac{x_+(\gamma', z, \gamma)}{[1-x_+(\gamma', z, \gamma)]^n} \right\} \frac{\partial}{\partial\gamma} \\
&\equiv -\Theta(\gamma, z) \frac{\partial}{\partial\gamma} \qquad (23)
\end{aligned}$$

with $x_\pm$ given by

$$x_\pm(\gamma', z', \gamma) = \frac{(\gamma+\mu^2-\gamma') \pm \sqrt{(\gamma+\mu^2-\gamma')^2 - 4\mu^2(\gamma+z'^2P^2/4)}}{2(\gamma+z'^2P^2/4)}.$$

Specifically in the case of $n = 1$, $\Theta(\gamma, z)$ is reduced to

$$\Theta(\gamma, z) = \frac{g^2}{(4\pi)^2} \int_{\mu^2-\gamma-z^2P^2/2}^{\gamma+\mu^2-2\mu\sqrt{\gamma+z^2P^2/4}} d\gamma' \frac{\gamma-\mu^2-\gamma'}{2(\gamma'+z^2P^2/4)} \times \frac{\Phi(\gamma', z)}{\sqrt{(\gamma+\mu^2-\gamma')^2-4\mu^2(\gamma+z^2P^2/4)}}. \tag{24}$$

The square-root singularity is integratable with

$$x = \mathrm{arccosh} \frac{\gamma+\mu^2-\gamma'}{2\mu\sqrt{\gamma+z^2P^2/4}}, \tag{25}$$

such that $\gamma' = \gamma + \mu^2 - 2\mu\sqrt{\gamma + z^2\frac{P^2}{4}} \cosh x$, and

$$\Theta(\gamma, z) = \frac{g^2}{(4\pi)^2} \int_0^{\mathrm{arccosh}\sqrt{\frac{\gamma+z^2P^2/4}{\mu^2}}} dx \frac{\mu\sqrt{\gamma+z^2\frac{P^2}{4}} \cosh x - \mu^2}{\gamma+\mu^2+z^2\frac{P^2}{4}-2\mu\sqrt{\gamma+z^2\frac{P^2}{4}} \cosh x} \times \Phi\left(\gamma+\mu^2-2\mu\sqrt{\gamma+z^2\frac{P^2}{4}} \cosh x, z\right). \tag{26}$$

$\Theta(\gamma, z)$ is nonvanishing when $\gamma > \mu^2 - z^2P^2/4$ and $\gamma > \Gamma_{\mathrm{th}}(z) + \mu^2 + 2\mu\sqrt{\Gamma_{\mathrm{th}}(z) + z^2P^2/4}$.

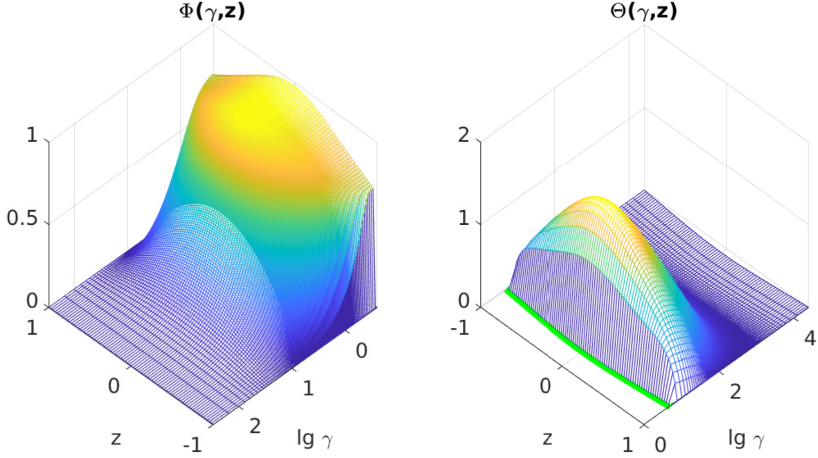


Figure: Solution of the Nakanishi spectral functions for a scalar bound state from the Bethe-Salpeter equation in the rainbow-ladder truncation with the bare propagator, $m^2 = 1$, $P^2 = 3$, and $\mu^2 = 2$.