Valence structures of light and strange mesons from the basis light-front quantization framework

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Light-front quantizaton conditions

Light-front coordinates $x^\pm = x^0 \pm x^3$ and momenta
$p^\pm = p^0 \pm p^3, \quad \vec{p}^\perp = (p^1, p^2), \quad k \cdot p = \frac{1}{2}(k^+p^- + k^-p^+) - \vec{k}^\perp \cdot \vec{p}^\perp$

	equal light-front time	equal time
quantization condition	$[\phi(x), \partial^+\phi(y)]_{x^+=y^+}$	$[\phi(x), \partial^0\phi(y)]_{x_0=y_0}$
kinetic energy	$(\vec{p}^{\perp 2} + m^2)/p^+$	$\sqrt{p_0^2 - \vec{p}^2 - m^2}$
dynamics	$\mathcal{H}_{\rm eff} = P^+\mathcal{P}^- - \vec{P}^{\perp 2}$	$\mathcal{L}_{\rm int}$

$$\phi(x)\big|_{x^+=0} = \int_0^{+\infty} \frac{dp^+}{4\pi p^+} \int \frac{d\vec{p}^\perp}{(2\pi)^2} \Big[a(p)\, e^{-ip\cdot x} + a^\dagger(p) e^{ip\cdot x}\Big]_{x^+=0}$$

Bound state structures are obtainable through the solutions of the light-front Schrödinger equation

$$\boxed{\mathcal{H}_{\rm eff}|\Psi\rangle = M^2|\Psi\rangle}, \qquad (1)$$

where $|\Psi\rangle$ is the light-front wavefunction (LFWF) for the bound state with mass M and conserved light-front 3-momentum $(P^+, \vec{P}^\perp)$.

LFWF for mesons

The LFWF can be expanded in terms of creation operators

$$|\Psi\rangle = 16\pi^3 P^+ \sum_{N=1}^{+\infty} \int d\underline{\mathbf{p}}\, \delta\left(\mathbf{P} - \sum_{j=1}^{N} \mathbf{p}_j^+\right) \psi_N(\mathbf{p}) \left[\prod_{k=1}^{N} a_k^\dagger(\mathbf{p})\right] \Big|0\Big\rangle, \quad (2)$$

or briefly $|\Psi\rangle = |q\bar{q}\rangle + |q\bar{q}q\bar{q}\rangle + \cdots + |q\bar{q}g\rangle + |q\bar{q}gg\rangle + \ldots$.
Truncated to the valence Fock sector of mesons:

$$\Big|\Psi(P^+, \vec{P}^\perp)\Big\rangle = \sum_{\mathrm{p,q},r,s} \int_0^1 \frac{dx}{4\pi x(1-x)} \int \frac{d\vec{\kappa}^\perp}{(2\pi)^2} \psi_{\mathrm{pq}rs}(x, \vec{\kappa}^\perp)$$
$$\times\, b^\dagger_{\mathrm{p}r}(xP^+, \vec{\kappa}^\perp + x\vec{P}^\perp)\, d^\dagger_{\mathrm{q}s}((1-x)P^+, -\vec{\kappa}^\perp + (1-x)\vec{P}^\perp)|0\rangle, \quad (3)$$

with P being the total momentum of the meson, $x = k^+/P^+$ the quark longitudinal momentum fraction, and $\vec{\kappa}^\perp = \vec{k}^\perp - x\vec{P}^\perp$ the relative transverse momentum. p and q are flavor labels.

Light-front kinetic energy and confinement

In the leading Fock sector of mesons, the kinetic energy, transverse confinement potential, and longitudinal confinement potential are given by

$$H_0 = \frac{\vec{\kappa}^{\perp 2} + \mathbf{m}^2}{x} + \frac{\vec{\kappa}^{\perp 2} + \overline{\mathbf{m}}^2}{1-x} + \kappa^4 x(1-x)\vec{r}^{\perp 2} - \frac{\kappa^4}{(\mathbf{m}+\overline{\mathbf{m}})^2}\partial_x x(1-x)\partial_x. \quad (4)$$

x	$\mathbf{m}$	$\overline{\mathbf{m}}$	κ	$\vec{r}^{\perp}$
quark longitudinal momentum fraction	quark mass	antiquark mass	confinement strength $\neq \lvert\vec{\kappa}\rvert$	$\frac{-i\partial}{\partial \vec{\kappa}^{\perp}}$

Basis function expansion of the valence LFWF

$$\psi_{rs}(x, \vec{\kappa}^{\perp}) = \sum_{n,m,l} \psi(n,m,l,r,s)\, \phi_{nm}\left(\frac{\vec{\kappa}^{\perp}}{\sqrt{x(1-x)}}\right) \chi_l(x),$$

such that $\boxed{H_0\, \phi_{nm}\, \chi_l = \Lambda_0(n,m,l)\, \phi_{nm}\, \chi_l}$ and

$$\Lambda_0(n,m,l) = (\mathbf{m}+\overline{\mathbf{m}})^2 + 2\kappa^2(2n+|m|+l+3/2) + \frac{\kappa^4}{(\mathbf{m}+\overline{\mathbf{m}})^2} l(l+1).$$

ϕ_{nm} is the 2-D harmonic oscillator (with $\vec{q}^{\perp} = \vec{\kappa}^{\perp}/\sqrt{x(1-x)}$)

$$\phi_{nm}(\vec{q}^{\perp}; b) = \frac{1}{b}\sqrt{\frac{4\pi n!}{(n+|m|)!}} \left(\frac{|\vec{q}^{\perp}|}{b}\right)^{|m|} \exp\left(-\frac{\vec{q}^{\perp 2}}{2b^2}\right) L_n^{|m|}\left(\frac{\vec{q}^{\perp 2}}{b^2}\right) e^{im\varphi},$$

with $\tan(\varphi) = q^2/q^1$ and $L_n^{|m|}$ being the associated Laguerre function. χ_l is the longitudinal basis function

$$\chi_l(x;\alpha,\beta) = \sqrt{4\pi(2l+\alpha+\beta+1)}\sqrt{\frac{\Gamma(l+1)\Gamma(l+\alpha+\beta+1)}{\Gamma(l+\alpha+1)\Gamma(l+\beta+1)}} x^{\beta/2}(1-x)^{\alpha/2} P_l^{(\alpha,\beta)}(2x-1),$$

with $P_l^{(\alpha,\beta)}(z)$ being the Jacobi polynomial and

$$\alpha = 2\overline{\mathbf{m}}(\mathbf{m}+\overline{\mathbf{m}})/\kappa^2, \quad \beta = 2\mathbf{m}(\mathbf{m}+\overline{\mathbf{m}})/\kappa^2.$$

With the basis diagonal Hamiltonian H_0 supplemented by an effective one-gluon exchange interaction, one could solve for the valence structures of heavy quakonia and B_c mesons.

[Li, Maris, and Vary Phys.Rev.,D96,016022 (2017)]

[Tang, Li, Maris, and Vary Phys.Rev.,D98,114038 (2018)]

Basis regularization

Define the basis cut-off

$$0 \le l \le L_{\max}, \quad -M_{\max} \le m \le M_{\max}, \quad \text{and} \quad -N_{\max} \le n \le N_{\max}.$$

Divergences encountered in the reduction of the Hamiltonian:

$$\mathcal{R}(\alpha, \beta, b, L_{\max}, N_{\max}) \equiv \int d\underline{p}_{12} \underline{\delta}(p_1 - p_2)\Big|_{\text{(basis-regularized)}}, \tag{5}$$

with

$$\int d\underline{p}_{123\cdots n} = \prod_{j=1}^{n} \int_0^{+\infty} \frac{dp_j^+}{4\pi p_j^+} \int \frac{d\vec{p}_j^{\perp}}{(2\pi)^2}. \tag{6}$$

We then have

$$\mathcal{R}(\alpha, \beta, b, L_{\max}, N_{\max}) \equiv \int_0^1 \frac{dx}{4\pi} \int_0^1 \frac{dx'}{4\pi} \int \frac{d\vec{k}^{\perp}}{(2\pi)^2} \int \frac{d\vec{p}^{\perp}}{(2\pi)^2} \frac{1}{xx'\sqrt{(1-x)(1-x')}}$$
$$\times \sum_{l=0}^{L_{\max}} \sum_{n=0}^{N_{\max}} \sum_{m=-M_{\max}}^{M_{\max}} \chi_l(x)\chi_l(x')\, \phi_{nm}\left(\frac{\vec{k}^{\perp}}{\sqrt{x(1-x)}}\right) \phi^*_{nm}\left(\frac{\vec{p}^{\perp}}{\sqrt{x'(1-x')}}\right)$$

$$\lim_{L_{\max}\to+\infty} \mathcal{R}(\alpha, \beta, b, L_{\max}, N_{\max}) = \frac{b^2}{8\pi^2}(N_{\max}+1) \tag{7}$$

Consider color-singlet four-fermion interactions in the three-flavor Nambu–Jona-Lasinio model. The Lagrangian is given by

$$
\begin{aligned}
\mathcal{L}^{(4)}_{\mathrm{NJL,SU(3)}} &= \overline{\psi}(i\partial\!\!\!/ - m)\psi \\
&+ G_\pi \sum_{i=0}^{8} \left[\left(\overline{\psi}\lambda^i\psi\right)^2 + \left(\overline{\psi} i\gamma_5\lambda^i\psi\right)^2 \right] \\
&- G_\rho \sum_{i=0}^{8} \left[\left(\overline{\psi}\gamma_\mu\lambda^i\psi\right)^2 + \left(\overline{\psi}\gamma_\mu\gamma_5\lambda^i\psi\right)^2 \right] \\
&- G_{\mathrm{V}} \left(\overline{\psi}\gamma_\mu\psi\right)^2 - G_{\mathrm{A}} \left(\overline{\psi}\gamma_\mu\gamma_5\psi\right)^2 .
\end{aligned}
$$

[Klimt:1989pm Nucl.Phys. A516 429]

$\psi = (\mathrm{u}, \mathrm{d}, \mathrm{s})^{\mathrm{T}}$. And λ^i are the Gell-Mann matrices with $i \in \{0, 1, 2, \ldots, 8\}$.

$\mathrm{SU(3)_V} \otimes \mathrm{SU(3)_A} \otimes \mathrm{U(1)_V} \otimes \mathrm{U(1)_A}$
(isospin) ~~(chiral)~~ (baryonic) ~~(axial)~~

Chiral symmetry is broken by

- nonvanishing quark mass,
- and dynamics.

- In the NJL model, the $\mathrm{U(1)_A}$ symmetry breaking is accounted for by the Kobayashi-Maskawa-'t Hooft determinant terms:

$$
\mathcal{L}_{\mathrm{det}} = G_{\mathrm{D}} \left[\det \overline{\psi}(1+\gamma_5)\psi + \det \overline{\psi}(1-\gamma_5)\psi \right] . \qquad (8)
$$

Determinants are taken in the flavor space, resulting in six-fermion interactions.

Flavor-space expansions

- Four-fermion interactions expanded in the flavor space

$$(\overline{\psi}\,\lambda_a\gamma^?\,\psi)^2 = 2(\bar{\mathrm{u}}\gamma^?\mathrm{u})^2 + 2(\bar{\mathrm{d}}\gamma^?\mathrm{d})^2 + 2(\bar{\mathrm{s}}\gamma^?\mathrm{s})^2 + 4\,\bar{\mathrm{u}}\gamma^?\mathrm{d}\,\bar{\mathrm{d}}\gamma^?\mathrm{u} + 4\,\bar{\mathrm{u}}\gamma^?\mathrm{s}\,\bar{\mathrm{s}}\gamma^?\mathrm{u} + 4\,\bar{\mathrm{d}}\gamma^?\mathrm{s}\,\bar{\mathrm{s}}\gamma^?\mathrm{d}. \tag{9}$$

- Six-fermion determinant terms expanded in the flavor space

$$\det\,\overline{\psi}\gamma^?\psi = \bar{\mathrm{u}}\gamma^?\mathrm{u}\,\bar{\mathrm{d}}\gamma^?\mathrm{d}\,\bar{\mathrm{s}}\gamma^?\mathrm{s} - \bar{\mathrm{u}}\gamma^?\mathrm{u}\,\bar{\mathrm{d}}\gamma^?\mathrm{s}\,\bar{\mathrm{s}}\gamma^?\mathrm{d} + \bar{\mathrm{u}}\gamma^?\mathrm{d}\,\bar{\mathrm{d}}\gamma^?\mathrm{s}\,\bar{\mathrm{s}}\gamma^?\mathrm{u} - \bar{\mathrm{u}}\gamma^?\mathrm{d}\,\bar{\mathrm{d}}\gamma^?\mathrm{u}\,\bar{\mathrm{s}}\gamma^?\mathrm{s} + \bar{\mathrm{u}}\gamma^?\mathrm{s}\,\bar{\mathrm{d}}\gamma^?\mathrm{u}\,\bar{\mathrm{s}}\gamma^?\mathrm{d} - \bar{\mathrm{u}}\gamma^?\mathrm{s}\,\bar{\mathrm{d}}\gamma^?\mathrm{d}\,\bar{\mathrm{s}}\gamma^?\mathrm{u}. \tag{10}$$

- After basis regularization, six-fermion interactions are reduced to

$$\det\overline{\psi}(1\pm\gamma^5)\psi \to -2\mathbf{m}_{\mathrm{a}}\mathcal{R}_{\mathrm{a}}\,\epsilon_{\mathrm{abc}}\epsilon_{\mathrm{agh}}\,\overline{\psi}_{\mathrm{b}}(1\pm\gamma^5)\psi_{\mathrm{g}}\,\overline{\psi}_{\mathrm{c}}(1\pm\gamma^5)\psi_{\mathrm{h}}. \tag{11}$$

- $\mathrm{SU}(2)$ isospin symmetry is exact when the up and the down quarks are of the same mass.

Decompose the valence light-front wave function into isospin symmetric and anti-symmetric terms:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\psi_{\rm l} \left(b^\dagger_{\rm u} d^\dagger_{\rm u} + b^\dagger_{\rm d} d^\dagger_{\rm d}\right)|0\rangle + \psi_{\rm s}\, b^\dagger_{\rm s} d^\dagger_{\rm s}|0\rangle + \frac{1}{\sqrt{2}}\psi_{\rm al} \left(b^\dagger_{\rm u} d^\dagger_{\rm u} - b^\dagger_{\rm d} d^\dagger_{\rm d}\right)|0\rangle. \tag{12}$$

The matrix elements in the flavor blocks are the given by

$$\langle\Psi'|H^{\rm eff}|\Psi\rangle = \begin{pmatrix}\psi_{\rm l} & \psi_{\rm s} & \psi_{\rm al}\end{pmatrix} \times \begin{pmatrix} H^{\rm eff}_{11} & \sqrt{2}\langle 0|d_{\rm l} b_{\rm l}\, H^{\rm eff}\, b^\dagger_{\rm s} d^\dagger_{\rm s}|0\rangle & 0 \\ \sqrt{2}\langle 0|d_{\rm s} b_{\rm s}\, H^{\rm eff}\, b^\dagger_{\rm l} d^\dagger_{\rm l}|0\rangle & \langle 0|d_{\rm s} b_{\rm s}\, H^{\rm eff}\, b^\dagger_{\rm s} d^\dagger_{\rm s}|0\rangle & 0 \\ 0 & 0 & H^{\rm eff}_{33}\end{pmatrix}\begin{pmatrix}\psi_{\rm l}\\ \psi_{\rm s}\\ \psi_{\rm al}\end{pmatrix}, \tag{13}$$

with

$$H^{\rm eff}_{11} = \frac{1}{2}\Big\{\langle 0|d_{\rm u} b_{\rm u}\, H^{\rm eff}\, b^\dagger_{\rm u} d^\dagger_{\rm u}|0\rangle + \langle 0|d_{\rm u} b_{\rm u}\, H^{\rm eff}\, b^\dagger_{\rm d} d^\dagger_{\rm d}|0\rangle + \langle 0|d_{\rm d} b_{\rm d}\, H^{\rm eff}\, b^\dagger_{\rm u} d^\dagger_{\rm u}|0\rangle + \langle 0|d_{\rm d} b_{\rm d}\, H^{\rm eff}\, b^\dagger_{\rm d} d^\dagger_{\rm d}|0\rangle\Big\} \tag{14}$$

$$H^{\rm eff}_{33} = \frac{1}{2}\Big\{\langle 0|d_{\rm u} b_{\rm u}\, H^{\rm eff}\, b^\dagger_{\rm u} d^\dagger_{\rm u}|0\rangle - \langle 0|d_{\rm u} b_{\rm u}\, H^{\rm eff}\, b^\dagger_{\rm d} d^\dagger_{\rm d}|0\rangle - \langle 0|d_{\rm d} b_{\rm d}\, H^{\rm eff}\, b^\dagger_{\rm u} d^\dagger_{\rm u}|0\rangle + \langle 0|d_{\rm d} b_{\rm d}\, H^{\rm eff}\, b^\dagger_{\rm d} d^\dagger_{\rm d}|0\rangle\Big\}. \tag{15}$$

We keep the G_π and G_D terms while ignoring instantaneous interactions such that $H^{\mathrm{eff}}_{\mathrm{NJL}} = -P^+ \mathcal{L}^{\mathrm{int}}_{\mathrm{NJL}}$.

$$H^{\mathrm{eff}}_{\mathrm{NJL}} = \int dx^2 \int d\vec{x}^{\perp} \Big\{ - G_\pi P^+ \sum_{i=0}^{8} \Big[\big(\overline{\psi}\gamma^i\psi\big)^2 - \big(\overline{\psi}\gamma^5\lambda^i\psi\big)^2 \Big] - G_\mathrm{D} P^+ \left[\det \overline{\psi}\left(1+\gamma^5\right)\psi + \det \overline{\psi}\left(1-\gamma^5\right)\psi\right] \Big\} \tag{16}$$

$$= \sum_{s1234} \int d\underline{k}_{1234}\, 4\pi P^+ \delta(k_1^+ + k_2^+ - k_3^+ - k_4^+)\,(2\pi)^2 \delta\left(\vec{k}_1^{\perp} + \vec{k}_2^{\perp} - \vec{k}_3^{\perp} - \vec{k}_4^{\perp}\right)$$
$$\times \Big\{ \sum_{\mathrm{q}\in\{\mathrm{u,d,s}\}} 4G_\pi \Big\{ -\overline{u}_{\mathrm{q}1} v_{\mathrm{q}2}\, \overline{v}_{\mathrm{q}3} u_{\mathrm{q}4} + \overline{u}_{\mathrm{q}1} u_{\mathrm{q}4}\, \overline{v}_{\mathrm{q}3} v_{\mathrm{q}2} + \overline{u}_{\mathrm{q}1}\gamma^5 v_{\mathrm{q}2}\, \overline{v}_{\mathrm{q}3}\gamma^5 u_{\mathrm{q}4}$$
$$- \overline{u}_{\mathrm{q}1}\gamma^5 u_{\mathrm{q}4}\, \overline{v}_{\mathrm{q}3}\gamma^5 v_{\mathrm{q}2} \Big\} b^\dagger_{\mathrm{q}1} d^\dagger_{\mathrm{q}2} d_{\mathrm{q}3} b_{\mathrm{q}4} + \sum_{\mathrm{q}\neq\mathrm{p}\in\{\mathrm{u,d,s}\}} \Big\{ \sum_{\mathrm{r}\in\{\mathrm{u,d,s}\}\backslash\{\mathrm{q,p}\}} 4G_\mathrm{D} \mathbf{m}_\mathrm{r} \mathcal{R}_\mathrm{r}$$
$$\times \left(\overline{u}_{\mathrm{q}1} v_{\mathrm{q}2}\, \overline{v}_{\mathrm{p}3} u_{\mathrm{p}4} + \overline{u}_{\mathrm{q}1} u_{\mathrm{p}4}\, \overline{v}_{\mathrm{p}3} v_{\mathrm{q}2} + \overline{u}_{\mathrm{q}1}\gamma^5 v_{\mathrm{q}2}\, \overline{v}_{\mathrm{p}3}\gamma^5 u_{\mathrm{p}4} + \overline{u}_{\mathrm{q}1}\gamma^5 u_{\mathrm{p}4}\, \overline{v}_{\mathrm{p}3}\gamma^5 v_{\mathrm{q}2}\right)$$
$$+ 4G_\pi \left(\overline{u}_{\mathrm{q}1} u_{\mathrm{p}4}\, \overline{v}_{\mathrm{p}3} v_{\mathrm{q}2} - \overline{u}_{\mathrm{q}1}\gamma^5 u_{\mathrm{p}4}\, \overline{v}_{\mathrm{p}3}\gamma^5 v_{\mathrm{q}2}\right) \Big\} b^\dagger_{\mathrm{q}1} d^\dagger_{\mathrm{q}2} d_{\mathrm{p}3} b_{\mathrm{p}4}. \tag{17}$$

When $\mathbf{m}_\mathrm{s}\mathcal{R}_\mathrm{s}G_\mathrm{D} = -G_\pi$ with the same m_l and κ_ll as in [Jia:2018ary Phys.Rev. C99, 035206],

m_l	κ_ll	G_π	N_max	M_max	L_max
337.01 MeV	227.00 MeV	18.5095 GeV^{-2}	8	2	8

the LFWF of the neutral pion is identical to that of the charged pion apart from the flavor wavefunction.

Mass	m_π	m_ρ
BLFQ-NJL	139.57 MeV	775.23 ± 0.04 MeV
PDG	139.57 MeV	775.26 ± 0.25 MeV
Decay constant	f_π	f_ρ
BLFQ-NJL	142.91 MeV	70.80 MeV
PDG	130.2 ± 1.7 MeV	221 ± 2 MeV

Table: Mass and the decay constant of ground state π^0 and ρ^0. The corresponding basis cutoff scales are $\Lambda_\mathrm{IR} = 55.06\,\mathrm{MeV}$ and $\Lambda_\mathrm{UV} = 935.9\,\mathrm{MeV}$. The PDG ref. is [PhysRevD.98.030001]. The ρ decay constant is further based on [Bhagwat:2006pu Phys.Rev. C77 025203].

η and η'

We expand the definition of the the decay constant for the pseudoscalar mesons $\langle 0|\overline{\psi}\gamma^\mu\gamma_5\psi|\mathrm{P}(p)\rangle = ip^\mu f_\mathrm{P}$ to include different flavor combinations of the axial-vector current such that

$$\langle 0|J^i_{\mu 5}|\mathrm{P}(p)\rangle = ip^\mu f^i_\mathrm{P}, \tag{18}$$

with

$$J^8_{\mu 5} = \frac{1}{\sqrt{6}}\left(\mathrm{u}\gamma_\mu\gamma_5\mathrm{u} + \mathrm{d}\gamma_\mu\gamma_5\mathrm{d} - 2\mathrm{s}\gamma_\mu\gamma_5\mathrm{s}\right) \tag{19}$$

$$J^0_{\mu 5} = \frac{1}{\sqrt{3}}\left(\mathrm{u}\gamma_\mu\gamma_5\mathrm{u} + \mathrm{d}\gamma_\mu\gamma_5\mathrm{d} + \mathrm{s}\gamma_\mu\gamma_5\mathrm{s}\right). \tag{20}$$

We then apply the following two-angle mixing scheme for the decay constants:

$$\begin{pmatrix} f^8_\eta & f^0_\eta \\ f^8_{\eta'} & f^0_{\eta'} \end{pmatrix} = \begin{pmatrix} \cos\theta_8 & -\sin\theta_0 \\ \sin\theta_8 & \cos\theta_0 \end{pmatrix} \begin{pmatrix} f_8 & \\ & f_0 \end{pmatrix}. \tag{21}$$

[Feldmann:2002kz Phys.Scripta T99 13] [Escribano:2005qq JHEP 06, 029 (2005)]

Preliminary results

m_l	κ_ll	m_s	κ_ss
300 MeV	400 MeV	450 MeV	400 MeV

G_π	G_D	N_max	M_max	L_max
1.7 GeV^{-2}	$-G_\pi m_\text{s} \mathcal{R}_\text{s}$	8	2	8

Table: Input parameters of the BLFQ-NJL model for the η and η' mesons

	m_η (MeV)	$m_{\eta'}$ (MeV)	θ_8	θ_0	f_8 (MeV)	f_0 (MeV)
BLFQ	554	939	4.23°	16.4°	211	89.4
Ref. [1]	547.86	957.8	-	-	-	-
Ref. [2]	-	-	-21.2°	-9.2°	164.1	152.3

Table: Masses, mixing angles, and decay constants of the η and η' mesons. Ref. [1] is [PDG 2018 PhysRevD.98.030001]. Ref. [2] is [Feldmann, Kroll, and Stech Phys. Rev. D 58, 114006 (1998)].

The Lagrangian of QCD before gauge fixing:

$$\mathcal{L}_{\mathrm{QCD}} = \overline{\psi}(i\not{D} - m)\psi - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a,$$

is a result of a local $\mathrm{SU}(3)_\mathrm{c}$ gauge symmetry:

$$\psi(x) \to \exp\left(-i \sum_{a=1}^{8} T_a \Theta_a(x)\right) \psi(x).$$

Consider quark field with three flavors:

$$\psi = (\mathrm{u},\, \mathrm{d},\, \mathrm{s})^{\mathrm{T}}, \quad m = \mathrm{diag}\,\{m_\mathrm{u},\, m_\mathrm{d},\, m_\mathrm{s}\}. \tag{22}$$

In the chiral limit, there exist global $\mathrm{U}(3)_\mathrm{L} \otimes \mathrm{U}(3)_\mathrm{R}$ chiral symmetries.

$$P_\mathrm{R} = (\mathbf{1} + \gamma_5)/2, \quad P_\mathrm{L} = (\mathbf{1} - \gamma_5)/2, \quad \psi_{\mathrm{L,R}} = P_{\mathrm{L,R}}\psi.$$

Symmetries of the strong interaction and corresponding dynamic theories

SYMMETRY	Local gauge	Global chiral	Local chiral
Theory	QCD	NJL	Chiral EFTs
D.o.f.	Quarks and gluons	Quarks	Mesons and baryons
Valid scale	0 to Λ_{GUT}	< 1 GeV	D.o.f. dependent

Reducing 6-fermion interactions into 4-fermion interactions

$$
\begin{aligned}
&\det \overline{\psi}\gamma^{?}\psi \\
&\to \frac{1}{6}\epsilon_{\rm abc}\epsilon_{\rm fgh}\Big\{\wick{\c{\overline{\psi}}_{\rm a}\gamma^{?}\c{\psi}_{\rm f}}\,\overline{\psi}_{\rm b}\gamma^{?}\psi_{\rm g}\,\overline{\psi}_{\rm c}\gamma^{?}\psi_{\rm h} + \wick{\c{\overline{\psi}}_{\rm a}\gamma^{?}\psi_{\rm f}\,\overline{\psi}_{\rm b}\gamma^{?}\c{\psi}_{\rm g}}\,\overline{\psi}_{\rm c}\gamma^{?}\psi_{\rm h} \\
&\qquad + \wick{\c{\overline{\psi}}_{\rm a}\gamma^{?}\psi_{\rm f}\,\overline{\psi}_{\rm b}\gamma^{?}\psi_{\rm g}\,\overline{\psi}_{\rm c}\gamma^{?}\c{\psi}_{\rm h}} + \overline{\psi}_{\rm a}\gamma^{?}\wick{\c{\psi}_{\rm f}\,\c{\overline{\psi}}_{\rm b}}\gamma^{?}\psi_{\rm g}\,\overline{\psi}_{\rm c}\gamma^{?}\psi_{\rm h} \\
&\qquad + \overline{\psi}_{\rm a}\gamma^{?}\wick{\c{\psi}_{\rm f}\,\overline{\psi}_{\rm b}\gamma^{?}\psi_{\rm g}\,\c{\overline{\psi}}_{\rm c}}\gamma^{?}\psi_{\rm h} + \overline{\psi}_{\rm a}\gamma^{?}\psi_{\rm f}\,\wick{\c{\overline{\psi}}_{\rm b}\gamma^{?}\c{\psi}_{\rm g}}\,\overline{\psi}_{\rm c}\gamma^{?}\psi_{\rm h} \\
&\qquad + \overline{\psi}_{\rm a}\gamma^{?}\psi_{\rm f}\,\wick{\c{\overline{\psi}}_{\rm b}\gamma^{?}\psi_{\rm g}\,\overline{\psi}_{\rm c}\gamma^{?}\c{\psi}_{\rm h}} + \overline{\psi}_{\rm a}\gamma^{?}\psi_{\rm f}\,\overline{\psi}_{\rm b}\gamma^{?}\wick{\c{\psi}_{\rm g}\,\c{\overline{\psi}}_{\rm c}}\gamma^{?}\psi_{\rm h} \\
&\qquad + \overline{\psi}_{\rm a}\gamma^{?}\psi_{\rm f}\,\overline{\psi}_{\rm b}\gamma^{?}\psi_{\rm g}\,\wick{\c{\overline{\psi}}_{\rm c}\gamma^{?}\c{\psi}_{\rm h}}\Big\} \\
&= \frac{1}{2}\epsilon_{\rm abc}\epsilon_{\rm fgh}\left[\wick{\c{\overline{\psi}}_{\rm a}\gamma^{?}\c{\psi}_{\rm f}}\,\overline{\psi}_{\rm b}\gamma^{?}\psi_{\rm g}\,\overline{\psi}_{\rm c}\gamma^{?}\psi_{\rm h} + 2\,\overline{\psi}_{\rm a}\gamma^{?}\wick{\c{\psi}_{\rm f}\,\c{\overline{\psi}}_{\rm b}}\gamma^{?}\psi_{\rm g}\,\overline{\psi}_{\rm c}\gamma^{?}\psi_{\rm h}\right].
\end{aligned}
\tag{23}
$$

Then with $\gamma^? = 1 \pm \gamma^5$, Eq. (23) becomes

$$\det \overline{\psi}(1 \pm \gamma^5)\psi \to -2\mathbf{m}_{\rm a}\mathcal{R}_{\rm a}\, \epsilon_{\rm abc}\epsilon_{\rm agh}\, \overline{\psi}_{\rm b}(1 \pm \gamma^5)\psi_{\rm g}\, \overline{\psi}_{\rm c}(1 \pm \gamma^5)\psi_{\rm h}. \quad (24)$$

The contribution to the effective Hamiltonian in the valence Fock block them becomes

$$\begin{aligned} H_{\rm det}^{\rm eff} = 4P^+ G_{\rm D}\Big\{ & \mathbf{m}_{\rm u}\mathcal{R}_{\rm u}\left(\bar{\rm d}{\rm d}\,\bar{\rm s}{\rm s} + \bar{\rm d}\gamma^5{\rm d}\,\bar{\rm s}\gamma^5{\rm s} - \bar{\rm d}{\rm s}\,\bar{\rm s}{\rm d} - \bar{\rm d}\gamma^5{\rm s}\,\bar{\rm s}\gamma^5{\rm d}\right) \\ & + \mathbf{m}_{\rm d}\mathcal{R}_{\rm d}\left(\bar{\rm s}{\rm s}\,\bar{\rm u}{\rm u} + \bar{\rm s}\gamma^5{\rm s}\,\bar{\rm u}\gamma^5{\rm u} - \bar{\rm s}{\rm u}\,\bar{\rm u}{\rm s} - \bar{\rm s}\gamma^5{\rm u}\,\bar{\rm u}\gamma^5{\rm s}\right) \\ & + \mathbf{m}_{\rm s}\mathcal{R}_{\rm s}\left(\bar{\rm u}{\rm u}\,\bar{\rm d}{\rm d} + \bar{\rm u}\gamma^5{\rm u}\,\bar{\rm d}\gamma^5{\rm d} - \bar{\rm u}{\rm d}\,\bar{\rm d}{\rm u} - \bar{\rm u}\gamma^5{\rm d}\,\bar{\rm d}\gamma^5{\rm u}\right) \Big\}, \end{aligned}$$

which is isospin symmetric in the isospin limit.

The light-front commutation relation of the fermion field is

$$\{\psi_+(x), \psi_+^\dagger(y)\}\big|_{x^+=y^+} = \Lambda^+ \delta(x^- - y^-)\delta(\overrightarrow{x}^\perp - \overrightarrow{y}^\perp), \qquad (25)$$

with the light-front projection of the fermion field given by $\psi_+ = \Lambda_+ \psi$. $\Lambda^\pm = \gamma^0\gamma^\pm/2$. With the following expansion of the fermion fields

$$\psi(x)|_{x^+=0} = \sum_{s=\pm 1/2} \int_0^{+\infty} \frac{dp^+}{4\pi p^+} \int \frac{d\overrightarrow{p}^\perp}{(2\pi)^2} \left[b_s(p) u_s(p) e^{-ip\cdot x} + d_s^\dagger(p) v_s(p) e^{ip\cdot x} \right] \Big|_{x^+=0}, \qquad (26)$$

Eq. (25) is satisfied by the following non-vanishing commutation relations for the creation and annihilation operators

$$\left\{ b_r(k), b_s^\dagger(p) \right\} = 2k^+\theta(k^+)(2\pi)\delta(k^+ - p^+)(2\pi)^2\delta(\overrightarrow{k}^\perp - \overrightarrow{p}^\perp)\delta_{rs}$$
$$\left\{ d_r(k), d_s^\dagger(p) \right\} = 2k^+\theta(k^+)(2\pi)\delta(k^+ - p^+)(2\pi)^2\delta(\overrightarrow{k}^\perp - \overrightarrow{p}^\perp)\delta_{rs}.$$

The Dirac spinors are then defined as

$$u_{\pm 1/2}(p) = \frac{1}{2\sqrt{p^+}}(\not{p} + m)\gamma^+ \chi_{\pm 1/2} \tag{27}$$

$$v_{\pm 1/2}(p) = \frac{1}{2\sqrt{p^+}}(\not{p} - m)\gamma^+ \chi_{\mp 1/2}. \tag{28}$$

With the spin bases satisfying

$$\Lambda^+ \chi_s = \chi_s, \quad \Lambda^- \chi_s = 0, \quad \chi_s^\dagger \chi_{s'} = \delta_{ss'}, \quad \text{and } \Sigma^3 \chi_{\pm 1/2} = \pm\frac{1}{2}\chi_{\pm 1/2}, \tag{29}$$

where the spin operators are defined as

$$S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j] = \frac{1}{2}\epsilon^{ijk} \begin{pmatrix} \sigma^k & \\ & \sigma^k \end{pmatrix} = \frac{1}{2}\epsilon^{ijk}\Sigma^k. \tag{30}$$