Effective Light Front QCD Hamiltonian and spectral equation for quark-antiquark states

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Outline of the talk

• QCD as a field theory describing hadrons in terms of quarks and gluons. Unlike to QED this problem in QCD is nonperturbative in coupling.

• LF formulation advantages: simple vacuum and physical Fock space for quarks and gluons.

• LF zero mode, its relation to nonperturbative effects like condensates. Difficalties with taking this mode into account on the LF.

• Approaching to LF formulation from standard space-like formulations close to the LF as a way to investigate zero mode dynamics. This limit transition should be different for zero and nonzero modes.

• The expression for m^2_{eff} including the contribution of zero mode.

• Construction of effective LF Hamiltonian with zero mode as independent dynamical variable.

• The description of hadron states in terms of constituent quarks and gluons, related to nonzero modes.

• From effective LF Hamiltonian to spectral equation for quark-antiquark bound states.

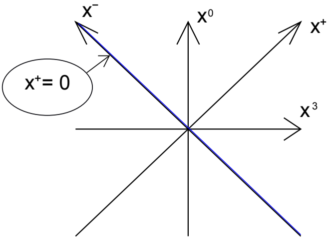
• Large quark mass limit and analitical solution of the spectral equation.

Relativistic physics of elementary particles is related to QFT. In QCD hadrons are to be described as bound states of quark and gluon fields. However this is difficult due to strong coupling. Nonperturbative methods like calculations on the space-time lattice are required.

The proposed by Dirac LF formulation of field theory allows to simplify the description of quantum vacuum state, identifying it with the state having zero LF longitudinal momentum

$$p_- = \frac{p_0 - p_3}{\sqrt{2}} \geq 0$$

(LF coordinates: $x^{\pm} = (x^0 \pm x^3)/\sqrt{2}$, $x^{\perp} = (x^1, x^2)$, x^+ plays the role of time x^0 and fields are quantized on the LF: $x^+ = 0$).



The component

$$P_+ = \frac{P_0 + P_3}{\sqrt{2}}$$

plays the role of Hamiltonian and depends on interaction, while P_- is kinematical quantity, independing on the coupling.
Moreover, on the LF not only free fields, but also interacting ones, can be represented in terms of creation and anihilation operators in LF Fock space:

$$\varphi(x) = \int\limits_0^\infty \frac{dp_- dp_\perp}{\sqrt{2p_-}} \left(a(p_-, x^\perp; x^+) e^{-ip_- x^-} + H.c. \right),$$

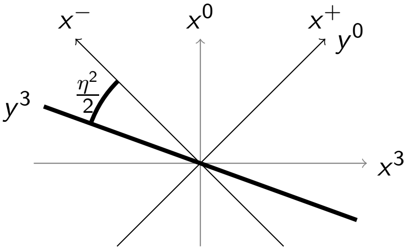
$$P_- = \int_0^\infty dp_- \int dp_\perp p_- a^+(p_-, x^\perp; x^+) a(p_-, x^\perp; x^+),$$

$$a(p_-, x^\perp; x^+)|0\rangle = 0.$$

Therefore LF formalism could be used as an alternative to other nonperturbative methods.

The difficulties of LF formulation are related to the singularity at $p_- \to 0$, so that zero ($p_- = 0$) mode of field is not defined. However this mode is responsible for vacuum effects (e.g. condensate description). It is not possible to ignore this mode in nonperturbative domain of QCD. The regularization $|x^-| \leq L$ and introduction of p. b. c. makes $p_- = \pi n/L$, $n \in \mathbb{Z}$, so that zero mode, $n = 0$, is present, but in canonical formalism it has no independent dynamics. It should be expressed through others, nonzero, modes via canonical constraints (note, that there the momentum $\Pi_{(0)}$ canonically conjugated to zero mode field $\varphi_{(0)}$ is zero: $\Pi_{(0)} = 0$). The constraints are so complicated that they can not be practically solved for zero mode. To overcome this difficulty with zero mode we start to investigate how it arises when one goes from usual formulation of field theory on space-like hyperplanes to LF.

With this aim we introduce approximating LF coordinates, $y^0 = x^+ + (\eta^2/2)x^-$, $y^3 = x^-$, $y^\perp = x^\perp$ with small parameter $\eta > 0$, components of momentum in these coordinates: $q_0 = p_+$, $q_3 = p_- - (\eta^2/2)p_+$, $q_\perp = p_\perp$.



These coordinates are closely related to Lorentz coordinates in the Lorentz frame fastly moving w. r. to x^μ-frame: $x'^\pm = (\eta/\sqrt{2})^{\mp 1}x^\pm$, $x'^\perp = x^\perp$.
We use the Hamiltonian on the plane $y^0 = 0$ and extract the term $H_{(0)}$ with only zero ($q_3 = 0$) modes of fields.

Metric tensor $g_{\mu\nu}$: $g_{00} = 0$, $g_{03} = 1$, $g_{33} = -\eta^2/2$ ($g^{00} = \eta^2/2$, $g^{03} = 1$, $g^{33} = 0$), $g_{kk'} = -\delta_{kk'}$.

$$\mathcal{L}_{gluon}(y) = -\frac{1}{2} Tr \left(F_{\mu\nu} F_{\rho\lambda} g^{\rho\mu} g^{\lambda\nu} \right) =$$

$$= Tr \left(F_{03}^2(y) + 2F_{0k}(y) F_{3k}(y) + \eta^2 F_{0k}^2(y) - F_{12}^2(y) \right),$$

$$\mathcal{L}_{quark}(y) = i\sqrt{2}\, \psi_+^+(y)\, D_0\, \psi_+(y) + \frac{i\eta^2}{\sqrt{2}}\, \psi_-^+(y)\, D_0\, \psi_-(y) +$$

$$+ i\sqrt{2}\, \psi_-^+(y)\, D_3\, \psi_-(y) + i\, \psi_-^+(y)\, (D_\perp - M)\, \psi_+(y) +$$

$$+ i\, \psi_+^+(y)\, (D_\perp + M)\, \psi_-(y),$$

where $A_0(y)$, $A_3(y)$, $F_{03}(y)$, $F_{0k}(y)$, $F_{3k}(y)$, D_0, D_3 are related to "y" coordinate system, and $\psi_\pm(y) \equiv \psi_\pm(x(y))$.

Canonical formulation on $y^0 = 0$:

$$\Pi_3^a(y) = F_{03}^a(y), \quad \Pi_k^a(y) = F_{3k}^a(y) + \eta^2 F_{0k}^a(y), \quad F_{0k}^a = \frac{\Pi_k^a - F_{3k}^a}{\eta^2},$$

$$\int d^4y \mathcal{L}_{gluon}(y) = \int d^4y \left(\Pi_3^a \partial_0 A_3^a + \Pi_k^a \partial_0 A_k^a - \mathcal{H}_g(y)\right),$$

$$\mathcal{H}_g(y) = \frac{1}{2}(\Pi_3^a)^2 + \frac{\eta^2}{2}(F_{0k}^a)^2 + \frac{1}{2}(F_{12}^a)^2 - A_0^a(D_3\Pi_3 + D_k\Pi_k)^a =$$

$$= \frac{1}{2}(\Pi_3^a)^2 + \frac{(\Pi_k^a - F_{3k}^a)^2}{2\eta^2} + \frac{1}{2}(F_{12}^a)^2 - A_0^a(D_3\Pi_3 + D_k\Pi_k)^a,$$

$\mathcal{H}_{quark}(y)$ contains $i\sqrt{2}\,\psi_+^+(y)\,D_0\,\psi_+(y)$ which gives

$\sqrt{2}gA_0^a\psi_+^+(y)\,\dfrac{\lambda^a}{2}\,\psi_+(y)$ term, so that the constraint at A_0^a has the form

$(D_3\Pi_3 + D_k\Pi_k)^a + g\sqrt{2}\psi_+^+\,\dfrac{\lambda^a}{2}\,\psi_+ = 0$ this gives in $A_3^a = 0$ gauge:

$$\Pi_3^a = -\partial_3^{-1}\left((D_k\Pi_k)^a + g\sqrt{2}\psi_+^+\frac{\lambda^a}{2}\psi_+\right).$$

We can separate zero mode $A_{k(0)}$ $\quad (A_k = A_{k(0)} + \widetilde{A_k})$,

$$\partial_3 A_{k(0)} = 0, \quad \int_{-L}^{L} dy^3 \widetilde{A_k} = 0.$$

$$A_{k(0)}(y^\perp, y^0 = 0) \quad \text{has conjugated momentum} \quad 2L\Pi_{k(0)} :$$

$$\int d^4y \left(\Pi_{k(0)} \partial_0 A_{k(0)}\right) = \int dy^0 dy^\perp \left(2L\Pi_{k(0)} \partial_0 A_{k(0)}\right).$$

The momentum conjugated to $A_{k(0)}(y)$ will be denoted by $\pi_k^a(y^\perp = x^\perp, y^0 = 0) \equiv 2L\Pi_{k(0)}(y)$. It enters into the Hamiltonian

$$\text{as} \quad \frac{1}{4L\eta^2} \int d^2y^\perp \, (\pi_k^a)^2 \,.$$

As for usual QFT Hamiltonians, which are similar to oscillator Hamiltonians, the canonical momentum $\Pi(y)$ enters quadratically into the Hamiltonian. If we denote by $\pi_k^a(x^\perp)$ the canonical momentum conjugated to zero mode of gluon field $A^a_{k(0)}(x^\perp)$ ($a = 1, \ldots, N^2 - 1$ for SU(N) color group symmetry, $k = 1, 2$ – transverse Lorentz indeces) we can describe this term $H_{(0)}$ as follows:

$$H_{(0)} = \frac{1}{4L\eta^2} \int d^2x^\perp \pi_k^a \pi_k^a.$$

This expression explicitly shows why in $\eta \to 0$ limit, i.e. on the LF, we need to have $\pi_k^a = 0$ as a condition for finiteness of LF "energy". However, with the more simple models, like QED(1+1) we can find that vacuum effects may be taken into account only if we change the form of limit transition to the LF for zero mode terms in the Hamiltonian, namely, one should take $\eta \to 0$ and $L \to \infty$ simultaneously at $L_0 = \eta L$ fixed.

So we try to consider zero mode terms of QCD Hamiltonian in "η"-coordinates in the same way: we require the finite contribution of $H_{(0)}$ to mass squared

$$m^2 = 2q_0 q_3 + \eta^2 q_0^2 - q_\perp^2$$

in $\eta \to 0$ limit: for terms with only zero modes the m^2 gets the contribution from $\eta^2 H_{(0)}$ term, which is equal to

$$\left(\frac{1}{4L\eta} \int d^2 x^\perp \pi_k^a \pi_k^a \right)^2 .$$

Therefore it remains finite at the limit $\eta \to 0$, $L \to \infty$, $L_0 = \eta L$ fixed. We keep the usual LF limit, $\eta \to 0$ at fixed L, for terms with nonzero modes.

This leads us to propose the following effective form for mass squared operator on the LF:

$$m_{eff}^2 = \left(\frac{1}{4L_0} \int d^2x^{\perp} \pi_k^a \pi_k^a \right)^2 + 2P_+ P_- - P_{\perp}^2,$$

where P_+, P_-, $P_\perp$ are canonical expressions for momentum operators in terms of fields on the LF.

"η"-coordinates: $y^0 = x^+ + (\eta^2/2)x^-$, $y^3 = x^-$, $y^\perp = x^\perp$.
Components of momentum in η-coordinates:
$q_0 = p_+$, $q_3 = p_- - (\eta^2/2)p_+$, $q_\perp = p_\perp$.
The quantization on $y^0 = 0$ is closely related to quantization in Lorentz coordinates $x'^\pm = (\eta/\sqrt{2})^{\mp 1}x^\pm$, $x'^\perp = x^\perp$, in Lorentz frame, fastly moving w. r. to x^μ-frame. Indeed, one has $x'^0 = \eta^{-1}(x^+ + (\eta^2/2)x^-) = \eta^{-1}y^0$, so that the plane $y^0 = 0$ coincide with the plane $x'^0 = 0$. Also
$p'_3 = -\eta^{-1}(p_- - (\eta^2/2)p_+) = -\eta^{-1}q_3$, so at $\eta \to 0 \quad p'_3 \to \infty$ and only $q_3 = 0$ corresponds to $p'_3 = 0$.
In QCD, formulated in η-coordinates, we have zero modes as canonically independent variables, i.e. they have nonzero canonically conjugated momentum. However when we approach to the LF ($\eta \to 0$), we see that the term containing this canonically conjugated momentum becomes singular and to avoid the infinite energy we need to set this momentum equal to zero in the limit.

Indeed, let us consider gluon fields, $A_\mu(y)$, and quark fields, $\psi(y)$, in η-coordinates with p. b. c. in y^3 for $A_\mu(y)$ and antiperiodic boundary conditions for $\psi(y)$. We have $A_0(y) = A_+(x)$, $A_3(y) = A_-(x) - (\eta^2/2)A_+(x)$, $A_\perp(y) = A_\perp(x)$ and take $A_3(y) = 0$. The $A_0(y)$ plays the role of Lagrange multiplier so that gluons are described by $A_\perp(y)$. Let us denote by $\pi^a_\perp(y^0 = 0, y^\perp)$ the canonical momentum conjugated to zero mode field $A_{\perp(0)}(y^0 = 0, y^\perp)$, i.e. $\left[A^a_{k(0)}(y^\perp), \pi^b_{k'}(y'^\perp)\right]_{y^0=y'^0=0} = i\delta_{kk'}\delta^{ab}\delta^{(2)}(y^\perp - y'^\perp)$. The $\pi^a_k(y^0 = 0, y^\perp)$ enters quadratically into the Hamiltonian. Let us extract this term and denote it as $H_{(0)}$,

$$H_{(0)} = \frac{1}{4L\eta^2}\int d^2y^\perp \pi^a_k \pi^a_k.$$

In the limit $\eta \to 0$ at fixed L this term is bounded only if $\pi^a_k \to 0$ in accordance with the LF canonical formalism.

However for QED(1+1) we noticed that the correct description of nonperturbative vacuum effects by zero modes corresponds to the limit $\eta \to 0$, $L \to \infty$ at fixed $L_0 \equiv \eta L$. So we can propose to consider different limit transitions for zero and nonzero modes as a possibility to take into account nonperturbative effects semiphenomenologically.

Let us estimate the contribution of zero mode ($q_3 = 0$) to the expression of mass squared operator

$$m^2 = 2Q_3 Q_0 + \eta^2 Q_0^2 - Q_\perp^2.$$

If we neglect the contribution of nonzero modes, then $Q_0 = H_{(0)}$ and we get

$$m_{(0)}^2 = \eta^2 H_{(0)}^2 - Q_\perp^2 = \left(\frac{1}{4L\eta} \int d^2 y^\perp \pi_k^a \pi_k^a \right)^2 - Q_\perp^2.$$

This contribution can be finite at finite $Q_\perp$ and L_0, and $\pi_k^a \neq 0$.

So our proposition in the LF limit:

$$m_{eff}^2 = \left(\frac{1}{4L_0} \int d^2x^\perp \pi_k^a(x)\pi_k^a(x) \right)^2 + 2P_+P_- - P_\perp^2,$$

where $\pi_k^a(x)$ is the momentum, conjugated to zero mode of gluon field on the LF and P_+ is the canonical expression for the LF Hamiltonian including zero mode of gluon field as independent canonical variable. We can generate this expression for m_{eff}^2 in usual form in LF coordinates if we introduce formally the expression for the effective LF Hamiltonian P_+^{eff}:

$$P_+^{eff} = \frac{1}{2P_-} \left(\frac{1}{4L_0} \int d^2x^\perp \pi_k^a \pi_k^a \right)^2 + P_+^{can}.$$

Here we define the vacuum state $|0\rangle$ as $P_-|0\rangle = \pi_k^a(x^\perp)|0\rangle = 0$. Besides we exclude the term in the P_+ containing only zero modes to avoid states with $p_- = 0$ but $\pi_k^a(x^\perp) \neq 0$.

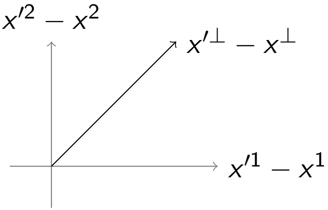
This model can be applied to the description of "constituent" quarks and gluons in hadrons describing this particles by nonzero modes of corresponding fields. Zero mode of gluon field is used for the construction of the state which is invariant w. r. to the gauge symmetry transformations remaining after fixing the gauge $A^a_- = 0$. We use gluon zero mode for the "string" connecting "constituent" particles separated in transverse coordinates. We take this "string" in the form of "path ordered" exponent

$$U_{x^\perp, x'^\perp} = Pexp\left(-ig\int_{x^\perp}^{x'^\perp}\sum_{k=1,2} dz^k A_{k(0)}(z^\perp)\right),$$

$A_{k(0)}(z^\perp) = A^a_{k(0)}(z^\perp)(\lambda^a/2)$, where $\lambda^a/2$ are Gell-Mann like matrices for SU(N), g is coupling constant, $U_{x^\perp, x'^\perp} = U^{-1}_{x'^\perp, x^\perp}$. We choose the path as the straight line in transverse plane, from $x^\perp$ to $x'^\perp$.

Let us consider the action of new term in the P_+^{eff} on the state containing only quark and antiquark connected by such a string. Let us describe the path of this string between $x^\perp$ and $x'^\perp$ as follows: $x'^1 - x^1 = \rho\cos\varphi, \quad x'^2 - x^2 = \rho\sin\varphi,$ then

$$U_{x^\perp, x'^\perp} = \lim_{\Delta\to 0} \prod_{i=1}^{i=\rho/\Delta} \exp\{-ig\Delta\left(A_{1(0)}(z_i)\cos\varphi + A_{2(0)}(z_i)\sin\varphi\right)\}.$$



Let us introduce the fermion field as a bispinor $\psi(x) = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$, where on the LF

$$\psi_- = -\frac{1}{\sqrt{2}} \partial_-^{-1} (D_\perp - M) \psi_+,$$

$D_\perp \equiv \sum_{k=1,2} D_k \sigma_k$, $D_k = \partial_k - igA_k$, σ_k are Pauli matrices and M is fermion mass. We can write:

$$\psi_+(x) = \frac{2^{-1/4}}{\sqrt{2L}} \sum_{n>0} \left(b_n(x^\perp) e^{-ip_n x^-} + d_n^+(x^\perp) e^{ip_n x^-} \right), \; n \in \mathbb{Z}+1/2,$$

$p_n = \pi n/L$, where b_n^+ and d_n^+ correspond to quarks and antiquarks creation operators respectively.

Introducing the ortonormal basis:

$$\frac{1}{\sqrt{N}} b_m^+(x^\perp) U_{x^\perp, x'^\perp} d_{m'}^+(x'^\perp)|0\rangle \equiv |mx^\perp, m'x'^\perp\rangle,$$

we project our effective Hamiltonian onto this basis. Let us denote the new term in the P_+^{eff} as $P_{+(0)}^{eff}$, then

$$P_{+(0)}^{eff}|mx^\perp, m'x'^\perp\rangle = \lim_{\Delta \to 0} \frac{g^4 \left(N - \frac{1}{N}\right)^2 (x^\perp - x'^\perp)^2}{4(4L_0\Delta)^2 2p_{m+m'}} |mx^\perp, m'x'^\perp\rangle.$$

So we obtain confinement in transverse direction.

Canonical quantization on the Light Front (LF) of QCD in $A_- = 0$ gauge.
Lagrangian density:

$$\mathcal{L} = -\frac{1}{2} Tr(F_{\mu\nu} F^{\mu\nu}) + \overline{\psi}(i\gamma^\mu D_\mu - M)\psi,$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu],$$

$$D_\mu = \partial_\mu - igA_\mu, \qquad A_\mu \equiv \{A_\mu^{ij}\}, \quad i, j = 1, ..., N \quad \text{for SU(N)},$$

$$\psi(x) = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}.$$

We define γ^μ as follows:

$$\gamma^0 = i \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \quad \gamma^3 = i \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

$$\gamma^k = i \begin{pmatrix} -\sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}, \quad k = 1, 2,$$

where σ_k is Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Lagrangian density:

$$\mathcal{L} = Tr\left(F_{+-}^2 + 2F_{+k}F_{-k} - F_{12}^2\right) + i\sqrt{2}\,\psi_+^+ D_+\,\psi_+ + i\sqrt{2}\,\psi_-^+ D_-\,\psi_- + \\ + \left(i\,\psi_-^+\,(D_\perp - M)\,\psi_+ + H.c.\right), \qquad D_\perp \equiv D_k\sigma_k,$$

Under gauge transformation $\Omega(x)$

$$\psi(x) \to \Omega(x)\psi(x), \quad A_\mu(x) \to \Omega(x)A_\mu(x)\,\Omega^+(x) + \frac{i}{g}\Omega(x)\partial_\mu\Omega^+(x),$$

$\mathcal{L}$ is invariant.

The adjoint representation $A^a_\mu(x)$ can be obtained with Gell-Mann-like matrices $\lambda^a = (\lambda^a)^+$, $a = 1, ..., N^2 - 1$,

$$A_\mu = \frac{\lambda^a}{2} A^a_\mu, \quad Tr(\lambda^a \lambda^b) = 2\delta^{ab}, \quad \left[\frac{\lambda^a}{2}, \frac{\lambda^b}{2}\right] = i f^{abc} \frac{\lambda^c}{2},$$

$$\sum_a \left(\frac{\lambda^a}{2}\frac{\lambda^a}{2}\right) = \frac{1}{2}\left(N - \frac{1}{N}\right), \quad F_{\mu\nu} = \frac{\lambda^a}{2} F^a_{\mu\nu}.$$

Canonical formulation on the LF, $x^+ = 0$:

$$\Pi^a_-(x) = \frac{\partial \mathcal{L}}{\partial(\partial_+ A^a_-(x))} = F^a_{+-}(x), \quad \Pi^a_k(x) = \frac{\partial \mathcal{L}}{\partial(\partial_+ A^a_k(x))} = F^a_{-k}(x),$$

$$\int d^4x\, \mathcal{L}_{gluon}(x) = \int d^4x \left(\Pi^a_- \partial_+ A^a_- + \Pi^a_k \partial_+ A^a_k - \mathcal{H}_{gluon}(x)\right),$$

$$\int d^4x\, \mathcal{H}_{gluon}(x) =$$

$$= \int d^4x \left(\frac{1}{2}\left(\Pi^a_- \Pi^a_- + F^a_{12} F^a_{12}\right) - A^a_+ \left(D_- \Pi_- + D_k \Pi_k\right)^a\right),$$

$$D_\mu \Pi_\nu \equiv \partial_\mu \Pi_\nu - ig[A_\mu, \Pi_\nu].$$

$$\frac{\partial \mathcal{L}}{\partial \psi_-^+(x)} = i\left(\sqrt{2} D_- \psi_- + (D_\perp - M)\psi_+\right) = 0$$

is the 2nd class constraint.

$$\psi_- = -\frac{1}{\sqrt{2}} D_-^{-1}(D_\perp - M)\psi_+ : \qquad \int d^4x\, \mathcal{L}_{quark} =$$

$$= \int d^4x \left(i\sqrt{2}\, \psi_+^+\, D_+\, \psi_+ - \frac{i}{\sqrt{2}} \psi_+^+ (D_\perp + M) D_-^{-1} (D_\perp - M)\psi_+ \right).$$

So we get $\Pi_{\psi_+} = i\sqrt{2}\, \psi_+^+$ and the Hamiltonian:

$$H = \int dx^- \int d^2x^\perp \left[\frac{1}{2} \left(\Pi_-^a \Pi_-^a + F_{12}^a F_{12}^a \right) - A_+^a \left(D_- \Pi_- + D_k \Pi_k \right)^a + \right.$$

$$\left. + \frac{i}{\sqrt{2}} \psi_+^+ (D_\perp + M) D_-^{-1} (D_\perp - M)\psi_+ - g\sqrt{2} A_+^a\, \psi_+^+\, \frac{\lambda^a}{2}\, \psi_+ \right],$$

here the A_+ is the Lagrangian multiplier, and the 1st class constraint

$$(D_-\Pi_- + D_k\Pi_k)^a + g\sqrt{2}\,\psi_+^+\,\frac{\lambda^a}{2}\,\psi_+ = 0 \quad \text{gives}$$

$$\Pi_-^a = -D_-^{-1}\left((D_k\Pi_k)^a + g\sqrt{2}\,\psi_+^+\,\frac{\lambda^a}{2}\,\psi_+\right).$$

Thus we get for Hamiltonian:

$$H = \int dx^- \int d^2x^\perp \Bigg[\frac{1}{2}\left(D_-^{-1}\left((D_k\Pi_k)^a + g\sqrt{2}\,\psi_+^+\,\frac{\lambda^a}{2}\,\psi_+\right)\right)^2 +$$

$$+\frac{1}{2}\,(F_{12}^a)^2 + \frac{i}{\sqrt{2}}\psi_+^+(D_\perp + M)D_-^{-1}(D_\perp - M)\psi_+\Bigg] \equiv H_{LF}.$$

The operator $D_- = \partial_-$ in $A_-^a = 0$ gauge.

Let us consider the action of the other terms in our effective Hamiltonian on the chosen basis states. Actually we remain with only that part of the Hamiltonian which contains fermion modes and gluon zero modes. This part of the Hamiltonian has the following form:

$$\int_{-L}^{L} dx^- \int d^2x^\perp \left[\frac{i}{\sqrt{2}} \psi_+^+ (D_\perp + M) \partial_-^{-1} (D_\perp - M) \psi_+ + \right.$$
$$\left. + g^2 \partial_-^{-1} \left(\psi_+^+ \frac{\lambda^a}{2} \psi_+ \right) \partial_-^{-1} \left(\psi_+^+ \frac{\lambda^a}{2} \psi_+ \right) \right].$$

It contains the four-fermion term, which can be rewritten as follows:

$$\int_{-L}^{L} dx^- \int_{-L}^{L} dx'^- \int d^2x^\perp \left[\left(\psi_+^+ \frac{\lambda^a}{2} \psi_+ \right)_{x^-} |x^- - x'^-| \left(\psi_+^+ \frac{\lambda^a}{2} \psi_+ \right)_{x'^-} \right].$$

This form of the interaction can be responsible for confinement in x^- direction, as was demonstrated by t' Hooft in the example of QCD(1+1).

Due to locality of 4-fermion term in $x^\perp$ it gives zero on our basis states, except for the case when quark and antiquark are not separated in $x^\perp$.
To restore the interaction between separated quark-antiquark we introduce nonlocal modification of this 4-fermion term in gauge invariant way:

$$\frac{g^2}{\pi R^2}\int_{-L}^{L} dx^- \int d^2x^\perp \int\limits_{|x'^\perp - x^\perp| \le R} d^2x'^\perp \partial_-^{-1}\left(\psi_+^+ \frac{\lambda^a}{2}\psi_+\right)_{x^\perp} \times$$

$$\times \partial_-^{-1}\left(\psi_+^+(x^-, x'^\perp) U_{x'^\perp, x^\perp} \frac{\lambda^a}{2} U_{x^\perp, x'^\perp} \psi_+(x^-, x'^\perp)\right),$$

where $1/(\pi R^2)$ appears due to the averaging over the circle in transverse plane of finite radius R, when we integrate over $(x'^\perp - x^\perp)$ at $|x'^\perp - x^\perp| \le R$. To obtain spectral equation we consider the action of this term on the basis states.

The spectral equation has the following form: $2P_{-}P_{+}^{eff}|f\rangle = m^2|f\rangle$ at $P_{\perp}|f\rangle = 0$, where $|f\rangle$ is the superposition of basis states:

$$|f\rangle = \sum_{m,m'>0} \delta_{n,m+m'} \int d^2x^{\perp} d^2x'^{\perp} f_{m,m'}(x'^{\perp} - x^{\perp})|mx^{\perp}, m'x'^{\perp}\rangle,$$

$$2P_{-}|f\rangle = 2p_n|f\rangle, \quad \langle mx^{\perp}, m'x'^{\perp}|2P_{-}P_{+}^{eff}|f\rangle = m^2\langle mx^{\perp}, m'x'^{\perp}|f\rangle.$$

Using ortonormality property of basis states we obtain the eigenvalue equation for wave functions $f_{m,m'}(x'^{\perp} - x^{\perp})$:

$$m^2 f_{m,m'}(x^\perp) =$$

$$= \left(\frac{g^4 \left(N - \frac{1}{N}\right)^2 \rho^2}{4(L_0 \Delta)^2} + p_n \left(\frac{1}{p_m} + \frac{1}{p'_m} \right) \left(M^2 - \bigtriangledown_\perp^2 \right) \right) f_{m,m'}(x^\perp) -$$

$$- \frac{g^2 \left(N - \frac{1}{N}\right)}{2L\pi R^2} \sum_{m_1, m_2 > 0} \delta_{n, m_1 + m_2} \frac{p_n}{(p_m - p_{m_1})^2} f_{m_1, m_2}(x^\perp) -$$

$$- \frac{g^2 \left(N - \frac{1}{N}\right)}{4L\pi R^2} \sum_{m_1 > 0} p_n \left(\frac{1}{(p_m + p_{m_1})^2} + \frac{1}{(p_{m'} + p_{m_1})^2} \right) f_{m,m'}(x^\perp),$$

$$\bigtriangledown_\perp^2 = \partial_1^2 + \partial_2^2, \quad \rho^2 = (x^1)^2 + (x^2)^2, \quad p_n = p_m + p_{m'}.$$

To write this equation in $L \to \infty$ limit let us introduce new variables:

$$\xi = \frac{p_m}{p_n}, \quad 1-\xi = \frac{p_{m'}}{p_n}, \quad \xi' = \frac{p_{m_1}}{p_n}, \quad dp_{m_1} \simeq \frac{\pi}{L}, \quad d\xi' = \frac{dp_{m_1}}{p_n} \simeq \frac{\pi}{Lp_n},$$

$$\frac{\pi}{Lp_n} \sum_{m_1>0} p_n^2 \left(\frac{1}{(p_m + p_{m_1})^2} + \frac{1}{(p_{m'} + p_{m_1})^2} \right) \simeq$$

$$\simeq \int_0^\infty d\xi' \left(\frac{1}{(\xi + \xi')^2} + \frac{1}{(1 - \xi + \xi')^2} \right) = -\frac{1}{\xi} - \frac{1}{1-\xi},$$

$$\frac{\pi}{Lp_n} \sum_{m_1>0} p_n^2 \left(\frac{1}{(p_m - p_{m_1})^2} \right) \simeq \int_0^1 \frac{d\xi'}{(\xi' - \xi)^2}.$$

Thus we obtain the spectral equation in the following form:

$$\left(\frac{M^2 - \nabla_\perp^2}{\xi(1-\xi)} + \frac{g^4\left(N - \frac{1}{N}\right)^2 \rho^2}{4(4L_0\Delta)^2} - \right.$$

$$\left. - \frac{g^2\left(N - \frac{1}{N}\right)}{4\pi^2 R^2}\left(\frac{1}{\xi(1-\xi)}\right) - C\right) f(\xi, x^\perp) -$$

$$- \frac{g^2\left(N - \frac{1}{N}\right)}{2\pi^2 R^2} P \int_0^1 \frac{d\xi'\ f(\xi', x^\perp)}{(\xi' - \xi)^2} = m^2 f(\xi, x^\perp),$$

where we renormalize the singularity at $\xi' = \xi$ in the integral by introducing into equation the principal value symbol,

$$P\frac{1}{x^2} = \frac{1}{2}\left(\frac{1}{(x+i\varepsilon)^2} + \frac{1}{(x-i\varepsilon)^2}\right),$$

and arbitrary renormalization constant C as the new parameter. One can meet similar equation in papers S. J. Brodsky, G. F. de Teramond, J. P. Vary, X. Zhao, J. R. Hiller, ...

To solve this equation we start with the investigation of the large quark mass limit. Let us rewrite the spectral equation in terms of dimensionless variables: $\overline{x} = x^{\perp}/R$, $\overline{m} = mR$, $\overline{M} = MR$,

$$\beta = \frac{g^2\left(N - \frac{1}{N}\right)}{2L_0 a}R^2, \quad \gamma = \frac{g^2\left(N - \frac{1}{N}\right)}{2\pi^2}, \quad \overline{\nabla}_{\perp}^2 = \nabla_{\perp}^2 R^2$$

and change the variable ξ to $\omega = 2\xi - 1$, $-1 \le \omega \le 1$,

$$\left(\frac{\overline{M}^2 - \overline{\nabla}_{\perp}^2 - \frac{\gamma}{2}}{1-\omega^2} + \frac{\beta^2}{4}\overline{x}^2\right)\overline{f}(\omega,\overline{x}) - \frac{\gamma}{2}P\int_{-1}^{1}\frac{d\omega'\,\overline{f}(\omega',\overline{x})}{(\omega'-\omega)^2} =$$

$$= \frac{\overline{m}^2 + C}{4}\,\overline{f}(\omega',\overline{x}).$$

Large quark mass limit corresponds to $\omega^2 \ll 1$. It is convinient to introduce new variable $s = \overline{M}\omega$, $-\overline{M} \le s \le \overline{M}$, and expand the equation in $\overline{M}^{-1}$. Assume also that $\gamma\overline{M} \equiv \gamma_1$ is finite in this limit then we get the following result:

$$\left(-\overline{\nabla}_\perp^2 + s^2 + \frac{\beta^2}{4}\overline{x}^2\right)\widetilde{f}(s,\overline{x}) - \frac{\gamma_1}{2} P \int_{-\infty}^{\infty} \frac{ds'\ \widetilde{f}(s',\overline{x})}{(s'-s)^2} = \frac{\overline{m}^2 + C}{4}\ \widetilde{f}(s',\overline{x}),$$

where we fix the arbitrary parameter C:

$$C = \overline{M}^2 + \frac{\gamma}{2}.$$

Let us make Fourier transformation in s:

$$\varphi(z,\overline{x}) = \int ds \exp(isz)\widetilde{f}(s,\overline{x}),$$

$$P \int \frac{ds\, ds' \exp\{i(s-s')z\}}{(s-s')^2} \exp(is'z)\ \widetilde{f}(s',\overline{x}) = -\pi|z|\varphi(z,\overline{x}).$$

Thus we obtain the following equation:

$$\left(-\overline{\nabla}_{\perp}^{2} - \partial_z^2 + \frac{\beta^2}{4}\overline{x}^2 + \frac{\pi}{2}\gamma_1|z|\right)\varphi(z,\overline{x}) = \frac{\overline{m}^2}{4}\varphi(z,\overline{x}).$$

We can write $\varphi(z,\overline{x}) = \varphi_1(\overline{x})\varphi_2(z)$, where $\varphi_1(\overline{x})$ and $\varphi_2(z)$ satisfy the following equations:

$$\left(-\overline{\nabla}_{\perp}^{2} + \frac{\beta^2}{4}\overline{x}^2\right)\varphi_1(\overline{x}) = \frac{\overline{m}_1^2}{4}\varphi_1(\overline{x}),$$

$$\left(-\partial_z^2 + \frac{\pi}{2}\gamma_1|z|\right)\varphi_2(z) = \frac{\overline{m}_2^2}{4}\varphi_2(z), \quad \overline{m}^2 = \overline{m}_1^2 + \overline{m}_2^2.$$

The first equation is the equation of 2-dimensional quantum harmonic oscillator and the second one is the Airy equation. Therefore we know the spectrum of $\overline{m}_1^2$ and $\overline{m}_2^2$:

$$\overline{m}_1^2(n_1,n_2) = 4\beta\sum_{k=1,2}\left(n_k + \frac{1}{2}\right), \quad \overline{m}_2^2(n_3) = 4\left(\frac{\pi\gamma_1}{2}\right)^{\frac{2}{3}}|\zeta_{n_3}|,$$

where ζ_n are zeros of corresponding Airy eigenfunctions.

For this roots we have the following: $|\zeta_0| \approx 1$ and for $n > 0$

$$|\zeta_n| \approx \left(\frac{3\pi}{4}\left(n+\frac{1}{2}\right)\right)^{\frac{2}{3}}.$$

Thus $\quad \overline{m}_2^2(n_3=0) \approx 4\left(\frac{\pi\gamma_1}{2}\right)^{\frac{2}{3}}, \quad \overline{m}_2^2(n_3>0) \approx 4\left(\frac{\pi^2\gamma_1}{8}\left(n+\frac{1}{2}\right)\right)^{\frac{2}{3}}.$

In order to introduce the classification of the states in orbital momentum l let us take

$$\overline{m}_1^2(n_1=n_2=0) = \overline{m}_2^2(n_3=0), \quad \text{i.e.} \quad \left(\frac{\pi\gamma_1}{2}\right)^{\frac{2}{3}} = \frac{\beta}{2}$$

and try to approximate the Airy equation for lowest eigenstates by oscillator equation. This gives a possibility to restore 3-dim spherical symmetry because we get the 3-dim oscillator equation with this spectrum:

$$\overline{m}^2 = 8\beta\left(n+\frac{l}{2}+\frac{3}{4}\right).$$

$$\overline{m}^2 = 8\beta\left(n + \frac{l}{2} + \frac{3}{4}\right).$$

The experimental meson spectrum can be approximated using the following formula by Sergey Afonin

$$m^2 = [1.1(n + l) + 0.7] GeV^2,$$

where the lowest state corresponds to ρ-meson. We can compare these spectra. We suppose that the discrepancy in orbital momentum l is due to rather simple description of the gluon "string" so that we assume a possibility to add $l/2$ to the spectral equation by hand. In this case we can identify our parameter 8β with the $1.1 GeV^2 R^2$ and again fit the parameter C so that the lowest state of our spectrum corresponds to $0.7 GeV^2$.

Thank you for your attention!

1. *Dirac P. A. M.* Forms of relativistic dynamics // Rev. Mod. Phys. 1949. V. 21. P. 392.

2. *Bakker B. L. G. at al.* Light-Front Quantum Chromodynamics: A framework for the analysis of hadron physics // Nucl. Phys. B (Proc. Suppl.). 2014. V. 251. P. 165, arXiv:1309.6333 [hep-ph].

3. *M. Burkardt, A. Langnau* // Phis. Rev. D, 1991. V. 44. P. 1187, 3857.

4. *S. A. Paston, V. A. Franke* // Theoretical and mathematical physics. 1997. V. 112, Issue 3, pp. 1117-1130, hep-th/9901110.

5. *Prokhvatilov E. V., Franke V. A.* Limit Transition to Light Front Coordinates in Field Theory and QCD-Hamiltonian // Sov. J. Nucl. Phys. 1989. V. 49. P. 688.