Partial Wave Mixing in Hamiltonian Effective Field Theory (HEFT)

Yan Li[1] Jia-jun Wu[1] Curtis D. Abell[2] Derek B. Leinweber[2] Anthony W. Thomas[2,3]

[1]School of Physical Sciences, University of Chinese Academy of Sciences (UCAS), Beijing 100049, China

[2]Special Research Center for the Subatomic Structure of Matter (CSSM), Department of Physics, University of Adelaide, Adelaide 5005, Australia

[3]ARC Centre of Excellence for Particle Physics at the Terascale (CoEPP), Department of Physics, University of Adelaide, Adelaide, South Australia 5005, Australia

Outline

Observables are different in experiments and lattice

- **Experiments**: Scattering in the infinite volume
  - Quantum states: Scattering states
  - Observables: Scattering observables like phase shifts $\delta_l(E)$
- **Lattice**: Scattering in a finite periodic box
  - Quantum states: Bound states
  - Observables: Finite volume spectra $E_n(L)$

Symmetry is different in experiments and lattice

- **Experiments**: Spherical symmetry → Rotation group: O(3)
  - Phase shifts $\delta_l(E)$ are classified by the irreps $l^{\pm}$ of O(3)
  - Irreps of O(3): $\mathbf{0}^{\pm}$, $\mathbf{1}^{\pm}$, $\mathbf{2}^{\pm}$, $\cdots$

  Irreducible representations

- **Lattice**: Cubic symmetry → Cubic group: $\mathrm{O_h}$
  - Spectra $E_n(\Gamma, L)$ are classified by the irreps Γ of $\mathrm{O_h}$
  - Irreps of $\mathrm{O_h}$: $\mathbf{A_1^{\pm}}$, $\mathbf{A_2^{\pm}}$, $\mathbf{E^{\pm}}$, $\mathbf{T_1^{\pm}}$, $\mathbf{T_2^{\pm}}$.

Symmetry is different in experiments and lattice

- O_h is a subgroup of O(3)
- The irreps of O(3) can be reduced into the irreps of O_h
- For each l, we have $m \to (\Gamma, f, \alpha)$
  - Γ denotes the irrep of O_h
  - f counts the number of Γ in the l (It's always 1 for $l \leq 4$)
  - α counts the dimension of Γ

$$\mathbf{0}^+ = \mathbf{A}_1^+$$
$$\mathbf{1}^- = \mathbf{T}_1^-$$
$$\mathbf{2}^+ = \mathbf{E}^+ \oplus \mathbf{T}_2^+$$
$$\mathbf{3}^- = \mathbf{A}_2^- \oplus \mathbf{T}_1^- \oplus \mathbf{T}_2^-$$
$$\mathbf{4}^+ = \mathbf{A}_1^+ \oplus \mathbf{E}^+ \oplus \mathbf{T}_1^+ \oplus \mathbf{T}_2^+$$

What is partial wave mixing

- Use l_{cut} to ignore higher partial wave ($l > l_{\text{cut}}$) contributions
- $l_{\text{cut}} = 0$: Only s-wave phase shifts are related to $\mathbf{A_1^+}$ spectra
- $l_{\text{cut}} = 4$: Both s- and g-wave phase shifts are related to $\mathbf{A_1^+}$ spectra
- **Partial wave mixing** between s- and g-waves.

$$\mathbf{0}^+ = \mathbf{A}_1^+$$
$$\mathbf{1}^- = \mathbf{T}_1^-$$
$$\mathbf{2}^+ = \mathbf{E}^+ \oplus \mathbf{T}_2^+$$
$$\mathbf{3}^- = \mathbf{A}_2^- \oplus \mathbf{T}_1^- \oplus \mathbf{T}_2^-$$
$$\mathbf{4}^+ = \mathbf{A}_1^+ \oplus \mathbf{E}^+ \oplus \mathbf{T}_1^+ \oplus \mathbf{T}_2^+$$

What is partial wave mixing

- Use l_{cut} to ignore higher partial wave ($l > l_{\text{cut}}$) contributions
- $l_{\text{cut}} = 0$: Only s-wave phase shifts are related to $\mathbf{A_1^+}$ spectra
- $l_{\text{cut}} = 4$: Both s- and g-wave phase shifts are related to $\mathbf{A_1^+}$ spectra
- **Partial wave mixing** between s- and g-waves.

$$
\begin{aligned}
\mathbf{0}^+ &= \mathbf{A}_1^+ \\
\mathbf{1}^- &= \mathbf{T}_1^- \\
\mathbf{2}^+ &= \mathbf{E}^+ \oplus \mathbf{T}_2^+ \\
\mathbf{3}^- &= \mathbf{A}_2^- \oplus \mathbf{T}_1^- \oplus \mathbf{T}_2^- \\
\mathbf{4}^+ &= \mathbf{A}_1^+ \oplus \mathbf{E}^+ \oplus \mathbf{T}_1^+ \oplus \mathbf{T}_2^+
\end{aligned}
$$

Model independent relation: Lüscher's formula

- $\delta_l(E)$ and $E_n(\Gamma, L)$ are related model independently (Up to exponentially small corrections)
- It's a one-to-one relation only in the simplest cases
  - $l_{\text{cut}} = 0$: $E_n^{\text{lat}}(\mathbf{A_1^+}, L) = 100\,\text{MeV}$
  - $\rightarrow$ equation of $\delta_0(100\,\text{MeV})$
  - $\rightarrow \delta_0(100\,\text{MeV})$

M. Lüscher:
Commun. Math. Phys., 105:153-188, 1986.
Commun. Math. Phys., 104:177, 1986.
Nucl. Phys., B354:531-578, 1991.

- It's not one-to-one in general cases
  - $l_{\text{cut}} = 4$: $E_n^{\text{lat}}(\mathbf{A_1^+}, L) = 100\,\text{MeV}$
  - $\rightarrow$ equation of $\delta_0(100\,\text{MeV})$ and $\delta_4(100\,\text{MeV})$
  - Need more than one energy levels with the same energy value
- Most data fail to find their partners to apply Lüscher's formula
- Fitting process is necessary (to relate quantities at different energy values)
- Normal approach: Parametrize the phase shifts

What is Hamiltonian effective field theory (HEFT)

- Build the infinite and finite volume Hamiltonians for a system

$$\text{Infinite: } \hat{H} = \hat{H}_0 + \hat{V}$$

$$\text{Finite: } \hat{H}_L = \hat{H}_{0L} + \hat{V}_L$$

- Parametrize the potentials of the Hamiltonians

$$\hat{V} \leftarrow \text{parameters} \rightarrow \hat{V}_L$$

- Fit the eigenvalues of $\hat{H}_L$ to $E_n^{\text{lat}}(\Gamma, L)$ to set the parameters
- Use $\hat{H}$ to calculate $\delta_l(E)$, which are potential model independent
- Use $\hat{H}$ and $\hat{H}_L$ to study the system

Outline

The infinite volume potential

- We start with the plane wave states: $|\mathbf{k}\rangle$
- We should deal with $|k; l, m\rangle := \int d\Omega_{\hat{\mathbf{k}}}\, Y_{lm}(\hat{\mathbf{k}})\ |\mathbf{k}\rangle$
- Rotational invariance only allows interaction like $|k'; l, m\rangle\, \langle k; l, m|$
- Wigner-Eckart theorem $\rightarrow$ different m correspond to the same v_l

$$\hat{V} = \int \frac{k'^2\, dk'}{(2\pi)^3} \int \frac{k^2\, dk}{(2\pi)^3} \sum_{l,m} v_l(k', k)\, |k'; l, m\rangle\, \langle k; l, m|$$

How does potential change from infinite to finite volume

$$\hat{V} = \int \frac{k'^2\, dk'}{(2\pi)^3} \int \frac{k^2\, dk}{(2\pi)^3} \sum_{l,m} v_l(k', k)\ |k'; l, m\rangle\, \langle k; l, m|$$

- $\mathbf{k} \to \frac{2\pi}{L}\mathbf{n}$ $\qquad \int_0^\infty \frac{k^2\, dk}{(2\pi)^3} \to \sum_{N=0}^{\infty}$ $\qquad N$ denotes $|\mathbf{n}|^2$
- $v_l(k', k) \to \tilde{v}_l(k_{N'}, k_N) := \frac{v_l(k_{N'}, k_N)}{4\pi L^3}$ $\qquad k_N$ denotes $\frac{2\pi}{L}\sqrt{N}$
- $|k; l, m\rangle := \int d\Omega_{\hat{\mathbf{k}}}\, Y_{lm}(\hat{\mathbf{k}})\ |\mathbf{k}\rangle$

$$\Longrightarrow \qquad |N; l, m\rangle := \sqrt{4\pi} \sum_{|\mathbf{n}|^2 = N} Y_{lm}(\hat{\mathbf{n}})\ |\mathbf{n}\rangle$$

$$\hat{V}_L = \sum_{N'} \sum_{N} \sum_{l,m} \tilde{v}_l(k_{N'}, k_N)\ |N'; l, m\rangle\, \langle N; l, m|$$

How does potential change from infinite to finite volume

$$\hat{V} = \int \frac{k'^2\,dk'}{(2\pi)^3} \int \frac{k^2\,dk}{(2\pi)^3} \sum_{l,m} v_l(k',k)\ |k';l,m\rangle\,\langle k;l,m|$$

- $\mathbf{k} \to \frac{2\pi}{L}\mathbf{n}$ $\qquad \int_0^\infty \frac{k^2\,dk}{(2\pi)^3} \to \sum_{N=0}^{\infty}$ $\qquad N$ denotes $|\mathbf{n}|^2$
- $v_l(k',k) \to \tilde{v}_l(k_{N'},k_N) := \frac{v_l(k_{N'},k_N)}{4\pi L^3}$ $\qquad k_N$ denotes $\frac{2\pi}{L}\sqrt{N}$
- $|k;l,m\rangle := \int d\Omega_{\hat{\mathbf{k}}}\, Y_{lm}(\hat{\mathbf{k}})\ |\mathbf{k}\rangle$

$$\Longrightarrow \qquad |N;l,m\rangle := \sqrt{4\pi} \sum_{|\mathbf{n}|^2=N} Y_{lm}(\hat{\mathbf{n}})\ |\mathbf{n}\rangle$$

$$\hat{V}_L = \sum_{N'} \sum_{N} \sum_{l,m} \tilde{v}_l(k_{N'},k_N)\ |N';l,m\rangle\,\langle N;l,m|$$

How does potential change from infinite to finite volume

$$\hat{V} = \int \frac{k'^2\, dk'}{(2\pi)^3} \int \frac{k^2\, dk}{(2\pi)^3} \sum_{l,m} v_l(k', k)\ |k'; l, m\rangle \langle k; l, m|$$

- $\mathbf{k} \to \frac{2\pi}{L}\mathbf{n}$ $\quad \int_0^\infty \frac{k^2\, dk}{(2\pi)^3} \to \sum_{N=0}^{\infty}$ $\quad N$ denotes $|\mathbf{n}|^2$
- $v_l(k', k) \to \tilde{v}_l(k_{N'}, k_N) := \frac{v_l(k_{N'}, k_N)}{4\pi L^3}$ $\quad k_N$ denotes $\frac{2\pi}{L}\sqrt{N}$
- $|k; l, m\rangle := \int d\Omega_{\hat{\mathbf{k}}}\, Y_{lm}(\hat{\mathbf{k}})\ |\mathbf{k}\rangle$

$$\Longrightarrow \qquad |N; l, m\rangle := \sqrt{4\pi} \sum_{|\mathbf{n}|^2 = N} Y_{lm}(\hat{\mathbf{n}})\ |\mathbf{n}\rangle$$

$$\hat{V}_L = \sum_{N'} \sum_{N} \sum_{l,m} \tilde{v}_l(k_{N'}, k_N)\ |N'; l, m\rangle \langle N; l, m|$$

How are partial waves mixed

$$\hat{V}_L = \sum_{N'} \sum_{N} \sum_{l,m} \tilde{v}_l(k_{N'}, k_N) \, |N'; l, m\rangle \langle N; l, m|$$

- (l, m) is no longer a good quantum number in the finite volume
- $|N; l, m\rangle$ belong to the reducible representations of O_h
- $|N; l, m\rangle$ have components of states belonging to the irreps of O_h

$$\begin{aligned}
\mathbf{0}^+ &= \mathbf{A}_1^+ \\
\mathbf{1}^- &= \mathbf{T}_1^- \\
\mathbf{2}^+ &= \mathbf{E}^+ \oplus \mathbf{T}_2^+ \\
\mathbf{3}^- &= \mathbf{A}_2^- \oplus \mathbf{T}_1^- \oplus \mathbf{T}_2^- \\
\mathbf{4}^+ &= \mathbf{A}_1^+ \oplus \mathbf{E}^+ \oplus \mathbf{T}_1^+ \oplus \mathbf{T}_2^+
\end{aligned}$$

- v_l are coupling to those states, hence coupling to the spectra of those irreps
- $\mathbf{0}^+$ and $\mathbf{4}^+$ can be mixed in $\mathbf{A}_1^+$

How are partial waves mixed

- The infinite volume states $|k;l,m\rangle$ are orthogonal to each other

$$\langle k';l',m'|k;l,m\rangle \propto \int d\Omega_{\hat{\mathbf{k}}} Y^*_{l'm'}(\hat{\mathbf{k}}) Y_{lm}(\hat{\mathbf{k}}) = \delta_{l',l}\delta_{m',m}$$

- The finite volume states $|N;l,m\rangle$ are not

$$\langle N';l',m'|N;l,m\rangle = 4\pi \sum_{|\mathbf{n}|^2=N} Y^*_{l'm'}(\hat{\mathbf{n}}) Y_{lm}(\hat{\mathbf{n}})$$

- $|N;l,m\rangle$ with different (l,m) can contain an overlap of their components $\Longrightarrow$ they are mixed

P-Matrix: Measure the degree of partial wave mixing

- $[P_N]_{l',m';l,m} := \langle N;l',m'|N;l,m\rangle = 4\pi \sum_{|\mathbf{n}|^2=N} Y^*_{l'm'}(\hat{\mathbf{n}})Y_{lm}(\hat{\mathbf{n}})$
- $[P_N]_{0,0;0,0} = \langle N;0,0|N;0,0\rangle = \sum_{|\mathbf{n}|^2=N} \mathbf{1} = C_3(N)$
- Recovery of spherical symmetry:

$$L \to \infty \text{ with } \frac{2\pi}{L}\sqrt{N} \text{ fixed} \quad \Longrightarrow \quad N \to \infty$$

$$\sum_{|\mathbf{n}|^2=N} \to \int d\Omega \quad \Longrightarrow \quad [P_N]_{l',m';l,m} \to C_3(N)\delta_{l',l}\delta_{m',m}$$

P-Matrix: Measure the degree of partial wave mixing

s-wave	p-wave			d-wave					f-wave							g-wave								
1.00	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1.05	0	0	0	1.75	0	0	0	1.05
0	1.00	0	0	0	0	0	0	0	0	0	−0.94	0	0	0	−1.21	0	0	0	0	0	0	0	0	0
0	0	1.00	0	0	0	0	0	0	0	0	0	1.53	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1.00	0	0	0	0	0	−1.21	0	0	0	−0.94	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1.25	0	0	0	1.25	0	0	0	0	0	0	0	0	0	−1.08	0	0	0	−1.08	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	2.50	0	0	0	0	0	0	0	0	0	−1.17	0	0	0	1.40	0	0	0	−1.17
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1.25	0	0	0	1.25	0	0	0	0	0	0	0	0	0	−1.08	0	0	0	−1.08	0	0
0	0	0	−1.21	0	0	0	0	0	1.46	0	0	0	1.13	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	−0.94	0	0	0	0	0	0	0	0	0	0.88	0	0	0	1.13	0	0	0	0	0	0	0	0	0
0	0	1.53	0	0	0	0	0	0	0	0	0	2.33	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	−0.94	0	0	0	0	0	1.13	0	0	0	0.88	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	−1.21	0	0	0	0	0	0	0	0	0	1.13	0	0	0	1.46	0	0	0	0	0	0	0	0	0
1.05	0	0	0	0	0	−1.17	0	0	0	0	0	0	0	0	0	1.64	0	0	0	1.18	0	0	0	1.64
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	−1.08	0	0	0	−1.08	0	0	0	0	0	0	0	0	0	0.94	0	0	0	0.94	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1.75	0	0	0	0	0	1.40	0	0	0	0	0	0	0	0	0	1.18	0	0	0	3.84	0	0	0	1.18
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	−1.08	0	0	0	−1.08	0	0	0	0	0	0	0	0	0	0.94	0	0	0	0.94	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1.05	0	0	0	0	0	−1.17	0	0	0	0	0	0	0	0	0	1.64	0	0	0	1.18	0	0	0	1.64

$$[P_{N=1}]\Big/C_3(1) \qquad\qquad C_3(1) = 6$$

$$[P_N]_{l',m';l,m} := \langle N; l', m' | N; l, m\rangle = 4\pi \sum_{|\mathbf{n}|^2 = N} Y^*_{l'm'}(\hat{\mathbf{n}}) Y_{lm}(\hat{\mathbf{n}})$$

25×25 matrix ordered as $(l, m) = (0,0),\ (1,-1),\ (1,0),\ (1,1),\ \cdots,\ (4,4)$

P-Matrix: Measure the degree of partial wave mixing

s-wave	p-wave			d-wave					f-wave							g-wave								
1.00	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.07	0	0	0	0.11	0	0	0	0.07
0	1.00	0	0	0	0	0	0	0	0	0	−0.06	0	0	0	−0.08	0	0	0	0	0	0	0	0	0
0	0	1.00	0	0	0	0	0	0	0	0	0	0.10	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1.00	0	0	0	0	0	−0.08	0	0	0	−0.06	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1.02	0	0	0	0.08	0	0	0	0	0	0	0	0	0	−0.07	0	0	0	−0.09	0	0
0	0	0	0	0	0.94	0	0	0	0	0	0	0	0	0	0	0	0	0	−0.01	0	0	0	0.03	0
0	0	0	0	0	0	1.10	0	0	0	0	0	0	0	0	0	−0.09	0	0	0	0.10	0	0	0	−0.09
0	0	0	0	0	0	0	0.94	0	0	0	0	0	0	0	0	0	0.03	0	0	0	−0.01	0	0	0
0	0	0	0	0.08	0	0	0	1.02	0	0	0	0	0	0	0	0	0	−0.09	0	0	0	−0.07	0	0
0	0	0	−0.08	0	0	0	0	0	1.03	0	0	0	0.09	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0.94	0	0	0	−0.03	0	0	0	0	0	0	0	0	0	0
0	−0.06	0	0	0	0	0	0	0	0	0	0.98	0	0	0	0.09	0	0	0	0	0	0	0	0	0
0	0	0.10	0	0	0	0	0	0	0	0	0	1.10	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	−0.06	0	0	0	0	0	0.09	0	0	0	0.98	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	−0.03	0	0	0	0.94	0	0	0	0	0	0	0	0	0	0
0	−0.08	0	0	0	0	0	0	0	0	0	0.09	0	0	0	1.03	0	0	0	0	0	0	0	0	0
0.07	0	0	0	0	0	−0.09	0	0	0	0	0	0	0	0	0	1.04	0	0	0	0.11	0	0	0	0.20
0	0	0	0	0	0	0	0.03	0	0	0	0	0	0	0	0	0	0.93	0	0	0	−0.04	0	0	0
0	0	0	0	−0.07	0	0	0	−0.09	0	0	0	0	0	0	0	0	0	1.03	0	0	0	0.08	0	0
0	0	0	0	0	−0.01	0	0	0	0	0	0	0	0	0	0	0	0	0	0.85	0	0	0	−0.04	0
0.11	0	0	0	0	0	0.10	0	0	0	0	0	0	0	0	0	0.11	0	0	0	1.30	0	0	0	0.11
0	0	0	0	0	0	0	−0.01	0	0	0	0	0	0	0	0	0	−0.04	0	0	0	0.85	0	0	0
0	0	0	0	−0.09	0	0	0	−0.07	0	0	0	0	0	0	0	0	0	0.08	0	0	0	1.03	0	0
0	0	0	0	0	0.03	0	0	0	0	0	0	0	0	0	0	0	0	0	−0.04	0	0	0	0.93	0
0.07	0	0	0	0	0	−0.09	0	0	0	0	0	0	0	0	0	0.20	0	0	0	0.11	0	0	0	1.04

$$[P_{N=581}] \Big/ C_3(581) \qquad\qquad C_3(581) = 336$$

$$[P_N]_{l',m';l,m} := \langle N; l', m' | N; l, m\rangle = 4\pi \sum_{|\mathbf{n}|^2=N} Y^*_{l'm'}(\hat{\mathbf{n}}) Y_{lm}(\hat{\mathbf{n}})$$

25×25 matrix ordered as $(l, m) = (0,0),\ (1,-1),\ (1,0),\ (1,1),\ \cdots,\ (4,4)$

P-Matrix: Measure the degree of partial wave mixing

s-wave	p-wave			d-wave					f-wave							g-wave								
1.00	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	−0.01	0	0	0	−0.02	0	0	0	−0.01
0	1.00	0	0	0	0	0	0	0	0	0	0.01	0	0	0	0.02	0	0	0	0	0	0	0	0	0
0	0	1.00	0	0	0	0	0	0	0	0	0	−0.02	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1.00	0	0	0	0	0	0.02	0	0	0	0.01	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1.00	0	0	0	−0.02	0	0	0	0	0	0	0	0	0	0.01	0	0	0	0.02	0	0
0	0	0	0	0	1.01	0	0	0	0	0	0	0	0	0	0	0	0	0	0.00	0	0	0	−0.00	0
0	0	0	0	0	0	0.98	0	0	0	0	0	0	0	0	0	0.02	0	0	0	−0.02	0	0	0	0.02
0	0	0	0	0	0	0	1.01	0	0	0	0	0	0	0	0	0	−0.00	0	0	0	0.00	0	0	0
0	0	0	0	−0.02	0	0	0	1.00	0	0	0	0	0	0	0	0	0	0.02	0	0	0	0.01	0	0
0	0	0	0.02	0	0	0	0	0	0.99	0	0	0	−0.02	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1.01	0	0	0	0.00	0	0	0	0	0	0	0	0	0	0
0	0.01	0	0	0	0	0	0	0	0	0	1.00	0	0	0	−0.02	0	0	0	0	0	0	0	0	0
0	0	−0.02	0	0	0	0	0	0	0	0	0	0.98	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0.01	0	0	0	0	0	−0.02	0	0	0	1.00	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0.00	0	0	0	1.01	0	0	0	0	0	0	0	0	0	0
0	0.02	0	0	0	0	0	0	0	0	0	−0.02	0	0	0	0.99	0	0	0	0	0	0	0	0	0
−0.01	0	0	0	0	0	0.02	0	0	0	0	0	0	0	0	0	0.99	0	0	0	−0.02	0	0	0	−0.03
0	0	0	0	0	0	0	−0.00	0	0	0	0	0	0	0	0	0	1.01	0	0	0	0.00	0	0	0
0	0	0	0	0.01	0	0	0	0.02	0	0	0	0	0	0	0	0	0	1.00	0	0	0	−0.01	0	0
0	0	0	0	0	0.00	0	0	0	0	0	0	0	0	0	0	0	0	0	1.02	0	0	0	0.00	0
−0.02	0	0	0	0	0	−0.02	0	0	0	0	0	0	0	0	0	−0.02	0	0	0	0.95	0	0	0	−0.02
0	0	0	0	0	0	0	0.00	0	0	0	0	0	0	0	0	0	0.00	0	0	0	1.02	0	0	0
0	0	0	0	0.02	0	0	0	0.01	0	0	0	0	0	0	0	0	0	−0.01	0	0	0	1.00	0	0
0	0	0	0	0	−0.00	0	0	0	0	0	0	0	0	0	0	0	0	0	0.00	0	0	0	1.01	0
−0.01	0	0	0	0	0	0.02	0	0	0	0	0	0	0	0	0	−0.03	0	0	0	−0.02	0	0	0	0.99

$$[P_{N=941}] \Big/ C_3(941) \qquad\qquad C_3(941) = 552$$

$$[P_N]_{l',m';l,m} := \langle N; l', m' | N; l, m \rangle = 4\pi \sum_{|\mathbf{n}|^2=N} Y^*_{l'm'}(\hat{\mathbf{n}}) Y_{lm}(\hat{\mathbf{n}})$$

25×25 matrix ordered as $(l, m) = (0,0), (1,-1), (1,0), (1,1), \cdots, (4,4)$

P-Matrix: Measure the degree of partial wave mixing

- P-Matrix also respects the cubic symmetry
- $m \to (\Gamma, f, \alpha) \Longrightarrow |N; l, m\rangle \to |N, l; \Gamma, f, \alpha\rangle$

$$|N, l; \Gamma, f, \alpha\rangle = \sum_m [C_l]_{\Gamma, f, \alpha; m} |N; l, m\rangle$$

- P-Matrix in new basis is obtained through a unitary transformation
  - The transformation matrix is given by $[C] = \text{diag}([C_0], [C_1], \cdots)$
  - P-Matrix will be block diagonal according to the irreps of O_h

$$[C][P_N][C]^\dagger \to \delta_{\Gamma', \Gamma}\delta_{\alpha', \alpha} \langle N, l'; \Gamma, f', \alpha | N, l; \Gamma, f, \alpha\rangle$$

How is the dimension of the Hamiltonian matrix reduced

- Original basis $|\mathbf{n}\rangle$: $\sum_{N=0}^{N_{\text{cut}}=600} C_3(N) \sim 60,000$
- Basis $|N; l, m\rangle$ with $l_{\text{cut}} = 4$: $\sum_{N=0}^{600} 25 \sim 600 \times 25$
- Basis $|N, l; \Gamma = \mathbf{A}_1^+, f, \alpha\rangle$: $\sum_{N=0}^{600} 2 \sim 600 \times 2$
- Orthonormalization needs the inner products——P-Matrix

Outline

The process

- $l_{\text{cut}} = 4$, only s-, d- and g-waves are present
- Separable potential model:

$$v_l(p,k) = f_l(p) G_l f_l(k)$$

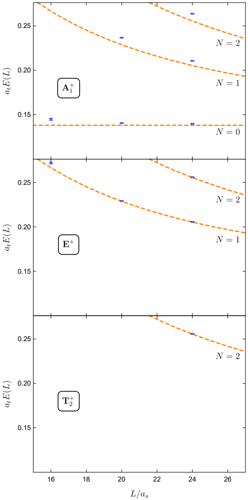
$$f_l(k) \sim \frac{(d_l \times k)^l}{(1 + (d_l \times k)^2)^{l/2+2}}$$

- 6 parameters: G_0, G_2, G_4, d_0, d_2, d_4
- Dimensions of Hamiltonians ($N_{\text{cut}} = 600$):

$$\mathbf{A}_1^+ : 923 \quad \mathbf{E}^+ : 965 \quad \mathbf{T}_2^+ : 963$$

- The fitted data: 11 energy levels

PRD 86, 034031 (2012)
Jozef J. Dudek et al.



PRELIMINARY

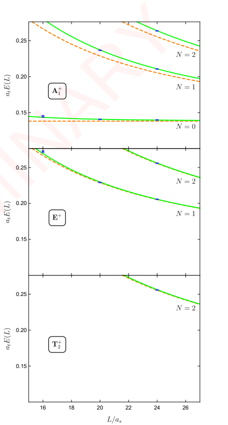
The results

Components of eigenstates

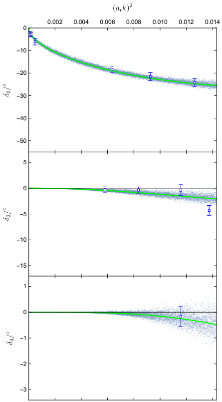
$\mathbf{A_1^+}$	$N=0$	$N=1$	$N=2$	$N=3$	$N=4$	$\cdots$
1st	99.7	0.2	0.0	0.0	0.0	$\cdots$
2nd	0.1	97.4	1.9	0.2	0.0	$\cdots$
3rd	0.0	1.5	94.5	2.8	0.3	$\cdots$

- Fitting $\rightarrow$ Parameters $\rightarrow$ $\hat{H}$ and $\hat{H}_L$
- $\hat{H}$ $\rightarrow$ $\delta_l(E)$
- $\hat{H}_L$ $\rightarrow$ $E_n(\Gamma, L)$
- $\hat{H}_L$ $\rightarrow$ Eigenstates

Volume dependent spectra



Phase shifts with errors



Outline

Summary

- Partial wave mixing is discussed in HEFT
  - How it happens in the viewpoint of HEFT
  - P-Matrix is defined to measure the degree of partial wave mixing
  - The dimension of the Hamiltonian can be highly reduced
- Example of isospin-2 $\pi\pi$ scattering is discussed

ACKNOWLEDGMENTS

Finite-volume energy levels taken from [PRD 86, 034031 (2012), Jozef J. Dudek et al.] were provided by the Hadron Spectrum Collaboration – no endorsement on their part of the analysis presented in the current paper should be assumed.

Thank you!

Backup

$$|\mathbf{n}\rangle \;\xrightarrow{\sum_{\hat{\mathbf{n}}} Y_{lm}(\hat{\mathbf{n}})\,|\mathbf{n}\rangle}\; |N; l, m\rangle$$

$N = \lvert\mathbf{n}\rvert^2$	0	1	2	3	$\cdots$
$\lvert\mathbf{n}\rangle$	1	6	12	8	$\cdots$
$\lvert N; l, m\rangle$	1	∞	∞	∞	$\cdots$

$$l = 0,\ 1,\ 2,\ \cdots$$
$$m = -l,\ \cdots,\ +l$$

$$N = 0: \quad (0,0,0)$$
$$N = 1: \quad (+1,0,0) \quad (0,+1,0) \quad (0,0,+1)$$
$$(-1,0,0) \quad (0,-1,0) \quad (0,0,-1)$$
$$\cdots\cdots$$

$$|\mathbf{n}\rangle \xrightarrow{\sum_{\hat{\mathbf{n}}} Y_{lm}(\hat{\mathbf{n}})\,|\mathbf{n}\rangle} |N;l,m\rangle$$

$N=\lvert\mathbf{n}\rvert^2$	0	1	2	3	$\cdots$
$\lvert\mathbf{n}\rangle$	1	6	12	8	$\cdots$
$\lvert N;l,m\rangle$	1	∞	∞	∞	$\cdots$
with l_{cut}	1	$(l_{\text{cut}}+1)^2$	$(l_{\text{cut}}+1)^2$	$(l_{\text{cut}}+1)^2$	$\cdots$

$$l = 0,\ 1,\ 2,\ \cdots,\ l_{\text{cut}}$$
$$m = -l,\ \cdots,\ +l$$

$$\sum_{l=0}^{l_{\text{cut}}}(2l+1) = (l_{\text{cut}}+1)^2 = 25 \text{ for } l_{\text{cut}} = 4$$

$$N = 0: \quad (0,0,0)$$
$$N = 1: \quad (+1,0,0) \quad (0,+1,0) \quad (0,0,+1)$$
$$(-1,0,0) \quad (0,-1,0) \quad (0,0,-1)$$
$$\cdots\cdots$$

Dimension: big number $\rightarrow (l_{\text{cut}}+1)^2 \times N_{\text{cut}}+1$

$\sim 25\times 600$ for $l_{\text{cut}} = 4,\ N_{\text{cut}} = 600$

$$|\mathbf{n}\rangle \xrightarrow{\sum_{\hat{\mathbf{n}}} Y_{lm}(\hat{\mathbf{n}})\,|\mathbf{n}\rangle} |N;l,m\rangle \xrightarrow{\sum_m [C_l]_{\Gamma,f,\alpha;m}\,|N;l,m\rangle} |N,l;\Gamma,f,\alpha\rangle$$

$N = \|\mathbf{n}\|^2$	0	1	2	3	$\cdots$
$\|\mathbf{n}\rangle$	1	6	12	8	$\cdots$
$\|N;l,m\rangle$	1	∞	∞	∞	$\cdots$
with l_{cut}	1	$(l_{\text{cut}}+1)^2$	$(l_{\text{cut}}+1)^2$	$(l_{\text{cut}}+1)^2$	$\cdots$
$\|N,l;\Gamma,f,\alpha\rangle$	1	F_{cut}	F_{cut}	F_{cut}	$\cdots$

$$\mathbf{0}^+ = \mathbf{A}_1^+ ,$$
$$\mathbf{1}^- = \mathbf{T}_1^- ,$$
$$\mathbf{2}^+ = \mathbf{E}^+ \oplus \mathbf{T}_2^+ ,$$
$$\mathbf{3}^- = \mathbf{A}_2^- \oplus \mathbf{T}_1^- \oplus \mathbf{T}_2^- ,$$
$$\mathbf{4}^+ = \mathbf{A}_1^+ \oplus \mathbf{E}^+ \oplus \mathbf{T}_1^+ \oplus \mathbf{T}_2^+ ,$$

$$F_{\text{cut}}(\Gamma = \mathbf{A}_1^+, l_{\text{cut}} = 4) = 2$$

$|\mathbf{n}\rangle$

$\downarrow \sum_{\hat{\mathbf{n}}} Y_{lm}(\hat{\mathbf{n}})\,|\mathbf{n}\rangle$

$|N;l,m\rangle$

$\downarrow \sum_{m} [C_l]_{\Gamma,f,\alpha;m}\,|N;l,m\rangle$

$|N,l;\Gamma,f,\alpha\rangle$

$\downarrow$ Orthonormalization

$|N,\Gamma,\alpha;F\rangle$

$N=\lvert\mathbf{n}\rvert^2$	0	1	2	3	$\cdots$
$\lvert\mathbf{n}\rangle$	1	6	12	8	$\cdots$
$\lvert N;l,m\rangle$	1	∞	∞	∞	$\cdots$
with l_{cut}	1	$(l_{\text{cut}}+1)^2$	$(l_{\text{cut}}+1)^2$	$(l_{\text{cut}}+1)^2$	$\cdots$
$\lvert N,l;\Gamma,f,\alpha\rangle$	1	F_{cut}	F_{cut}	F_{cut}	$\cdots$
$\lvert N,\Gamma,\alpha;F\rangle$	1	$\leq F_{\text{cut}}$	$\leq F_{\text{cut}}$	$\leq F_{\text{cut}}$	$\cdots$

$$\mathbf{0}^+ = \mathbf{A}_1^+\,, \qquad F_{\text{cut}}(\Gamma=\mathbf{A}_1^+, l_{\text{cut}}=4) = 2$$

$$\mathbf{1}^- = \mathbf{T}_1^-\,,$$

$$\mathbf{2}^+ = \mathbf{E}^+ \oplus \mathbf{T}_2^+\,,$$

$$\mathbf{3}^- = \mathbf{A}_2^- \oplus \mathbf{T}_1^- \oplus \mathbf{T}_2^-\,,$$

$$\mathbf{4}^+ = \mathbf{A}_1^+ \oplus \mathbf{E}^+ \oplus \mathbf{T}_1^+ \oplus \mathbf{T}_2^+\,,$$

Dimension: big number $\rightarrow$ at most $F_{\text{cut}}\times N_{\text{cut}}+1$

$\sim 2\times 600$ for $l_{\text{cut}}=4,\; N_{\text{cut}}=600$

$$|\mathbf{n}\rangle \qquad \hat{V}_L = \sum_{\mathbf{n}',\mathbf{n}\in\mathbb{Z}^3} V_L(\frac{2\pi\mathbf{n}'}{L}, \frac{2\pi\mathbf{n}}{L}) |\mathbf{n}'\rangle \langle\mathbf{n}| = \sum_{\mathbf{n}',\mathbf{n}\in\mathbb{Z}^3} L^{-3} \underbrace{V(\frac{2\pi\mathbf{n}'}{L}, \frac{2\pi\mathbf{n}}{L})}_{\sum_{l,m} v_l(k_{N'},k_N) Y_{lm}(\hat{\mathbf{n}}') Y^*_{lm}(\hat{\mathbf{n}})} |\mathbf{n}'\rangle \langle\mathbf{n}|$$

$$\downarrow \; \sum_{\hat{\mathbf{n}}} Y_{lm}(\hat{\mathbf{n}}) |\mathbf{n}\rangle$$

$$|N;l,m\rangle \qquad \hat{V}_L = \sum_{N',N} \sum_{l} L^{-3} v_l(k_{N'},k_N) \sum_{m} |N';l,m\rangle \langle N;l,m|$$

$$\downarrow \; \sum_{m} [C_l]_{\Gamma,f,\alpha;m} |N;l,m\rangle$$

$$|N,l;\Gamma,f,\alpha\rangle \qquad \hat{V}_L = \sum_{N',N} \sum_{l} L^{-3} v_l(k_{N'},k_N) \sum_{\Gamma,f,\alpha} |N',l;\Gamma,f,\alpha\rangle \langle N,l;\Gamma,f,\alpha|$$

↓ Orthonormalization

$$|N,\Gamma,\alpha;F\rangle \qquad \hat{V}_L = \sum_{N',N} \sum_{\Gamma,F',F} v_{\Gamma,F',F}(k_{N'},k_N) \sum_{\alpha} |N',\Gamma,\alpha;F'\rangle \langle N,\Gamma,\alpha;F|$$

$$|\mathbf{n}\rangle \qquad \hat{V}_L = \sum_{\mathbf{n}',\mathbf{n}\in\mathbb{Z}^3} V_L(\frac{2\pi\mathbf{n}'}{L}, \frac{2\pi\mathbf{n}}{L}) |\mathbf{n}'\rangle \langle\mathbf{n}| = \sum_{\mathbf{n}',\mathbf{n}\in\mathbb{Z}^3} L^{-3} \underbrace{V(\frac{2\pi\mathbf{n}'}{L}, \frac{2\pi\mathbf{n}}{L})}_{\sum_{l,m} v_l(k_{N'},k_N) Y_{lm}(\hat{\mathbf{n}}') Y^*_{lm}(\hat{\mathbf{n}})} |\mathbf{n}'\rangle \langle\mathbf{n}|$$

$$\downarrow \sum_{\hat{\mathbf{n}}} Y_{lm}(\hat{\mathbf{n}}) |\mathbf{n}\rangle$$

$$|N; l, m\rangle$$

$$v_{\Gamma,F',F}(k_{N'},k_N) = \sum_l L^{-3} v_l(k_{N'},k_N) [G_{l;\Gamma}]_{N',F';N,F}$$

$$\downarrow \sum_m [C_l]_{\Gamma,f,\alpha;m} |N; l, m\rangle$$

$$|N, l; \Gamma, f, \alpha\rangle$$

Need to know the inner products between $|N, l; \Gamma, f, \alpha\rangle$

$\downarrow$ Orthonormalization

$$|N, \Gamma, \alpha; F\rangle \qquad \hat{V}_L = \sum_{N',N} \sum_{\Gamma,F',F} v_{\Gamma,F',F}(k_{N'},k_N) \sum_\alpha |N', \Gamma, \alpha; F'\rangle \langle N, \Gamma, \alpha; F|$$

$|\mathbf{n}\rangle$

$\sum_{\hat{\mathbf{n}}} Y_{lm}(\hat{\mathbf{n}})\,|\mathbf{n}\rangle$

$|N; l, m\rangle$

$\sum_{m} [C_l]_{\Gamma,f,\alpha;m}\,|N; l, m\rangle$

$|N, l; \Gamma, f, \alpha\rangle$

Orthonormalization

$|N, \Gamma, \alpha; F\rangle$

$$[P_N]_{l',m';l,m} := \langle N; l', m' | N; l, m\rangle = \sum_{\hat{\mathbf{n}}} Y^*_{l'm'}(\hat{\mathbf{n}}) Y_{lm}(\hat{\mathbf{n}})$$

25×25

for $l_{\text{cut}} = 4$

$[C]^\dagger[P][C]$

2×2

$\mathbf{A}_1^+$

$$[P_{N;\Gamma,\alpha}]_{l',f';l,f} := \langle N, l'; \Gamma, f', \alpha | N, l; \Gamma, f, \alpha\rangle = \sum_{m',m} [C_{l'}]^*_{\Gamma,f',\alpha;m'} [P_N]_{l',m';l,m} [C_l]_{\Gamma,f,\alpha;m}$$

Combination Coefficients: $[G_{l;\Gamma}]_{N',F';N,F} = \sum_{f} \langle N', \Gamma, \alpha; F' | N'; l, \Gamma, f, \alpha\rangle \langle N; l, \Gamma, f, \alpha | N, \Gamma, \alpha; F\rangle$