Considerations on Schmid theorem

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Triangle singularities

\[ t_T = i \int \frac{d^4q}{(2\pi)^4} \bar{\epsilon}_{f_1} \cdot \bar{\epsilon}_{K^*} \cdot \bar{\epsilon}_{K^*} \cdot (2\vec{K} + \vec{q}) \frac{1}{q^2 - m_K^2 + i\epsilon} \frac{1}{(P - q)^2 - m_{K^*}^2 + i\epsilon} \frac{1}{(P - q - k)^2 - m_K^2 + i\epsilon} \]

\[ \tilde{t}_T = \int \frac{d^3q}{(2\pi)^3} \left( 2 + \frac{\vec{k} \cdot \vec{q}}{k^2} \right) \frac{1}{8 \omega(q)\omega'(q)\omega^*(q)} \frac{1}{k^0 - \omega'(q) - \omega^*(q) + i\epsilon} \frac{1}{P^0 - \omega^*(q) - \omega(q) + i\epsilon} \times \frac{2P^0\omega(q) + 2k^0\omega'(q) - 2(\omega(q) + \omega'(q))(\omega(q) + \omega'(q) + \omega^*(q))}{(P^0 - \omega(q) - \omega'(q) - k^0 + i\epsilon)(P^0 + \omega(q) + \omega'(q) - k^0 - i\epsilon)} , \]

\[ \omega(q) = \sqrt{q^2 + m_K^2}, \quad \omega'(q) = \sqrt{(q + \vec{k})^2 + m_{K^*}^2}, \quad \omega^*(q) = \sqrt{q^2 + m_K^2} \]

**Poles in the integration**

\[ P^0 - \omega^*(q) - \omega(q) + i\epsilon = 0, \quad q_{on+} = q_{on} + i\epsilon \text{ with } q_{on} = \frac{1}{2M} \sqrt{\lambda(M^2, m_1^2, m_2^2)} \]

\[ P^0 - \omega(q) - \omega'(q) - k^0 + i\epsilon = 0 \]
If we fix \( \cos(\theta) = \pm 1 \) and we make this expression zero, then in the integral of \( \cos(\theta) \) one cannot cancel the divergence with the principal value, and the divergence remains. \( \Theta \) is the angle between \( \vec{k} \) and \( \vec{q} \).

For \( \cos(\theta) = -1 \)

\[
q_{a+} = \gamma \left( v E_2^* + p_2^* \right) + i \epsilon, \quad q_{a-} = \gamma \left( v E_2^* - p_2^* \right) - i \epsilon
\]

\[
v = \frac{k}{E_{23}}, \quad E_2^* = \frac{1}{2m_{23}} \left( m_{23}^2 + m_2^2 - m_3^2 \right), \quad p_2^* = \frac{1}{2m_{23}} \sqrt{\lambda(m_{23}^2, m_2^2, m_3^2)}
\]

Calculate solution when \( p_{23} \) is at rest (\( p_{23}^* \)) and make a boost.
For \( \cos(\theta) = 1 \)

\[
q_{0+} = \gamma (-\nu E_2^* + p_2^*) + i \epsilon, \quad q_{0-} = -\gamma (\nu E_2^* + p_2^*) - i \epsilon
\]

Threshold singularity

Very simple expression to see where the TS appears, and to explain the Coleman-Norton theorem, Nuovo Cim. 1965, (TS appears when the decays in the loop can occur at the classical level).
Considerations on the Schmid theorem for triangle singularities


\( t_{t}^{(0)} \) is the \( L=0 \) partial wave of the tree level amplitude

\[ t_{t}^{(0)} + t_{L} = t_{t}^{(0)} e^{2i\delta} \cdot t_{L} \]

is the loop amplitude

\[ \left| t_{t}^{(0)} + t_{L} \right|^2 = \left| t_{t}^{(0)} e^{2i\delta} \right|^2 = \left| t_{t}^{(0)} \right|^2 \]


The rescattering mechanism does not change the width given by the tree level
\[
\Gamma_A = \frac{1}{2M_A} \int \frac{d^3 p_3}{(2\pi)^3} \frac{1}{2\omega_3} \frac{1}{8\pi} \int_{-1}^{1} d\cos\theta \tilde{p}_1 \frac{1}{M_{\text{inv}}(12)} |t|^2
\]

\[
t_L = \frac{g_A g_R}{2\omega_R(p_3 - \vec{q})} \frac{1}{\tilde{E}_A - \omega_1(q) - \omega_R(p_3 - \vec{q}) + i\epsilon}.
\]

If \(\epsilon\) becomes \(i\Gamma/2\) the theorem is no longer exact.
\[ t_L = -i \frac{1}{4\pi} \frac{1}{|q_{on}|} \frac{1}{M^{\text{inv}}(12)} t_{12,12}^{(0)} \]
\[ f = -\frac{1}{8\pi} \frac{1}{M^{\text{inv}}} t \]
\[ t_L = 2i |q_{on}| f t^{(0)}_t \]
\[ f = \frac{\eta e^{2i\delta} - 1}{2i |q_{on}|} \]
\[ t^{(0)}_t + t_L = t^{(0)}_t (1 + 2i |q_{on}| f) = \eta e^{2i\delta} t^{(0)}_t \]

\[ \int_{-1}^{1} d\cos \theta |t_t + t_L|^2 = \int_{-1}^{1} d\cos \theta \left\{ |t_t^{(\ell \neq 0)}|^2 + \eta^2 |t_t^{(0)}|^2 \right\} \]
\[ = \int_{-1}^{1} d\cos \theta \left\{ |t_t^{(\ell \neq 0)}|^2 + |t_t^{(0)}|^2 - (1 - \eta^2) |t_t^{(0)}|^2 \right\} \]
\[ = \int_{-1}^{1} d\cos \theta |t_t|^2 - (1 - \eta^2) \int_{-1}^{1} d\cos \theta |t_t^{(0)}|^2 \]
\[ (46) \]

and since \( t_t^{(0)} \) contains the same singularity as \( t_L \), the singularity of the triangle diagram will show up with a strength of \( 1 - \eta^2 \).

1) First consequence: in the limit where Schmid theorem is exact, $\Gamma = 0$, it is also irrelevant because the tree level is infinitely larger than the triangle singularity.

2) The theorem does not hold when the $1+2 \rightarrow 1+2$ amplitude has inelasticities.

3) What happens when $\Gamma$ is not small?

\[
\frac{d\Gamma^{(L)}_A}{dM_{\text{inv}}(12)} \propto \int_{-1}^{1} d\cos \theta \left| \ell_t^{(0)} \right|^2 \\
\sim 2 \left( \frac{g_A g_R}{4 p_3 q} \right)^2 \left| \ln \left( \frac{\Gamma_R}{\omega_R(p_3 - q) - \omega_R(p_3 + q)} \right) \right|^2 .
\]
Example of realistic calculation

\[
\begin{align*}
M_A &= 2154 \text{ MeV} \\
M_R &= 1600 \text{ MeV}, \quad \Gamma_R = 30 \text{ MeV} \\
M_1 &= 500 \text{ MeV} \\
M_2 &= 200 \text{ MeV} \\
M_3 &= 900 \text{ MeV}.
\end{align*}
\]

\[
t_{12, 12} = \frac{g^2}{M_{\text{inv}}^2(12) - M_{\text{BW}}^2 + i \Gamma_{\text{BW}}(M_{\text{inv}}(12)) M_{\text{inv}}(12)}
\]

\[
M_A = 2200 \text{ MeV}, \quad \Gamma_R = 0.5 \text{ MeV}.
\]

\[
M_A = 2200 \text{ MeV}, \quad \Gamma_R = 0.1 \text{ MeV}.
\]

\[
M_A = 2154 \text{ MeV}, \quad \Gamma_R = 0.1 \text{ MeV}.
\]
Conclusions:

1) First consequence: in the limit where Schmid theorem is exact, $\Gamma=0$, it is also irrelevant because the tree level is infinitely larger than the triangle singularity.

2) The theorem does not hold when the $1+2 \rightarrow 1+2$ amplitude has inelasticities.

3) For realistic widths do not rely upon Schmid theorem and evaluate the rescattering diagram.

4) Even then, the coherent sum of tree level and TS rescattering changes the cross section, or differential width, less than an incoherent sum of the processes, and much less than a coherent sum of terms with the same phase. THE PROCESS STILL HAS SOME MEMORY OF SCHMID THEOREM.