Three-body System in a Box

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Collaborate with M. Döring, J. J. Wu
1 Introduction

2 Lattice QCD and 3-body Effective Field Theory

3 Application on 3-body System

4 Summary and Outlook
**Hadron Physics**

- **Quantum ChromoDynamics**
- **Hadron Spectroscopy and Strong Decay Process**, e.g., \(N^*(1440) \to N\pi\pi\)

**Effective Field Theory**

- **QCD**
- **LQCD**
- **EFT**
- **EXP.**

- symmetry
- \(\text{ab initial}\)
- non-perturbative
- LEC
- unphysical setting
Lattice QCD in Hadron Physics

Lattice Quantum Chromodynamics

\[ \mathcal{L}_{\text{QCD}} = \sum_i \bar{\psi}_i (i\not\!D - m_i) \psi_i - \frac{1}{4} G_{\mu\nu}^a G^{a,\mu\nu} \]

where quarks' flavor \( i = u, d, s, c, b \) and gluon field

\[ G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c \]

**Input:** lattice spacing \( a \), lattice size \( L \)

mesons, e.g., \( \pi \), \( K \), \( D_s \), \( B_s \)

Build operators for specific system

Output: Finite volume spectrum

Relation to experiment?

\( \rho (\pi\pi) \), Roper Resonance (\( N\pi \) and \( N\pi\pi \)), …
From QCD to Experiment (Extract 3-body Physical Observable from LQCD)

- Efimov State in a Box and 3-body Force [arXiv:1706.07700]
- Transparent Theoretical Framework for LQCD 3-body Simulation [arXiv:1707.02176]
- 3-body State in Cubic Representation, i.e., $A_1^+, E^+, T_1^-$, etc. [arXiv:1802.03362]
- Energy Shift of 3-body State and 3-body Physical Observable [arXiv:1902.01111]
- Triton in a Box (including Higher Partial Wave, e.g., $S-D$ mixing in Deuteron)
- Twisted Boundary Condition, Relativistic Kinematics
- Roper Resonance, $N^*(1440) \rightarrow N\pi$ and $N\pi\pi$
From QCD to Experiment

3-body Finite Volume Spectrum

3-body Force in EFT

3-body Observables
3-body EFT and 3-body Observables

- **Effective Theory of 3-body System**

\[ \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I^{(2\text{-body})} + \mathcal{L}_I^{(3\text{-body})} \]

- **3-body Scattering Amplitude**

![Dalitz Plot for 3-body Decay](image URL)
### Effective Field Theory for 3-body System

<table>
<thead>
<tr>
<th>EFT</th>
<th>2-body Physics</th>
<th>3-body Physics</th>
</tr>
</thead>
<tbody>
<tr>
<td>2-body Operators</td>
<td>2-body Potential</td>
<td>3-body Potential</td>
</tr>
<tr>
<td>with 2-body LEC</td>
<td>Unitarity</td>
<td>2-body Amplitude</td>
</tr>
</tbody>
</table>

#### QCD

- **2-body Potential**
- **3-body Potential**

- **2-body Amplitude**
- **3-body Scattering Amplitude**

- **3-body Scattering Equation**
  - **Faddeev Equation**

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### Particle-Dimer Formalism

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<tr>
<td>2-body Operators</td>
<td>Dimer Self-Energy</td>
<td>Dimer-Spectator Propagation</td>
</tr>
<tr>
<td>with 2-body LEC</td>
<td>Unitarity</td>
<td>Particle-Dimer Potential</td>
</tr>
<tr>
<td></td>
<td>Dimer Propagation</td>
<td></td>
</tr>
<tr>
<td>3-body Operators</td>
<td></td>
<td></td>
</tr>
<tr>
<td>with 3-body LEC (3-body Force)</td>
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</tr>
</tbody>
</table>

- **Dimer Self-Energy**
- **Unitarity**
- **Dimer Propagation**
- **Particle-Dimer Scattering Equation**
- **Particle-Dimer Scattering Amplitude**
3-body EFT and 3-body Observables

3-body Force in EFT

Efimov States: 3-body Bound States

- **Particle-Dimer Scattering Equation**

  \[ M = \begin{array}{c}
  \hline
  \hline
  (\text{Relative Momentum between Dimer and Spectator})
  \\
  \hline
  \hline
  \end{array} \]

- **Projection and Cut-off Regularization** *(Relative Momentum between Dimer and Spectator)*
From QCD to Experiment

QCD → EFT → EXP.

LEC

finite volume effect

ab initial

QCD

EFT

LEC

finte volume effect

ab initial

LQCD

3-body Finite Volume Spectrum

3-body Force in EFT

3-body Observables
3-body EFT and 3-body Finite Volume Spectrum

- **Scattering Amplitude in a Box**

- **Quantization Condition**
<table>
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<td>Dimer Propagation</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{p \cot \delta - S_F}$</td>
<td>Dimer-Spectator</td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{p_* \cot \delta - S_F(p_*)}$</td>
<td>Potential</td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{p^2 + q^2 + pq - mE}$</td>
<td></td>
</tr>
</tbody>
</table>

3-body Operators

with 3-body LEC

(3-body Force)

Particle-Dimer Scattering Equation in a Box

Particle-Dimer Scattering Amplitude in a Box
 Finite Volume Correction  \((\text{Discrete Momentum in a Box})\)

 Particle-Dimer Scattering Equation in a Box

- **Dimer Propagation in a Box**

\[
\mathcal{M}_L = \frac{1}{2} \mathcal{M}_L + \mathcal{M}_L
\]

- **Scattering Equation in a Box**

- **Projection by Cubic Symmetry**

Rotaional Symmetry \(\rightarrow\) Partial Wave Expansion

Cubic Symmetry \(\rightarrow\) Cubic Irreps. Expansion (Shells and Irreps \(A_1, A_2, E, T_1, T_2\))
3-body EFT and 3-body Finite Volume Spectrum

- Projected Quantization Condition

\[
\det \left( \tau_{L}^{-1}(r, E) \delta_{sr} \delta_{ij} - \frac{8\pi}{L^3} Z_{ij}^{(\Gamma)}(s, r, E) \right) = 0
\]
Lattice QCD and Effective Field Theory

3-body Finite Volume Spectrum

3-body Force in EFT

3-body Observables

- **Quantization Condition**

\[
\det \left( \tau^{-1}_L (r; E) \delta_{sr} \delta_{ij} - \frac{8 \pi}{L^3} Z_{ij}^{(\Gamma)} (s, r; E) \right) = 0
\]

- **Particle-Dimer Scattering Equation in the Infinite Volume**

  \[\mathcal{M} = \mathcal{M} + \mathcal{M} + \mathcal{M} + \mathcal{M}\]
Avoided Level Crossing (3-body Threshold and Particle-Dimer Threshold) \([\text{arXiv:1809.07350}]\)

Energy Shifts \([\text{arXiv:1902.01111}]\)

Ground State: \(\kappa^2 = \frac{g_0}{L} \left(1 + \frac{g_1}{L} + \frac{g_2}{L^2} + \frac{g_3}{L^3} \ln \frac{mL}{2\pi} + \frac{g_4}{L^3} + \cdots\right)\)

Excited State: \(\kappa^2 - 1 = \frac{h_0}{L} \left(1 + \frac{h_1}{L} + \frac{h_2}{L^2} + \frac{h_3}{L^3} \ln \frac{mL}{2\pi} + \frac{h_4}{L^3} + \cdots\right)\),

where \(g_4 = \left(-5.159159617 + 6\pi \left(\frac{r}{a}\right) - 8\pi \left(\frac{\hat{M}}{a^2}\right)\right) a^3\)

\(h_4 = \left(-172.001650 + 83.745841 \left(\frac{r}{a}\right) - \frac{27}{5} \times 8\pi \left(\frac{\hat{M}}{a^2}\right)\right) a^3\).
- Build 3-body Operators and Obtain 3-body Finite Volume Spectrum from LQCD Simulation
- Quantization Condition: 3-body Finite Volume Spectrum $\rightarrow$ 3-body Force in Effective Theory
- Particle-Dimer Scattering Equation: 3-body Force $\rightarrow$ 3-body Physical Observables
Triton in a Box (including Higher Partial Wave, e.g., $S - D$ mixing in Deuteron)

Twisted Boundary Condition, Relativistic Kinematics

Roper Resonance, $N^*(1440) \rightarrow N\pi$ and $N\pi\pi$
Thank you for your attention!
**Shell Structure** Shell is a set of momenta with the same $|\mathbf{p}|$, which can be obtained from reference momentum $\mathbf{p}_0$, $\mathbf{p} = g \mathbf{p}_0$, $g \in O_h$. The momenta unrelated by the $O_h$, but having $|\mathbf{p}| = |\mathbf{p}'|$, belong to the different shells.

**Cubic Irreps. Expansion**

\[
f(\mathbf{p}) = f(g\mathbf{p}_0(s)) = \sum_{\Gamma, ij} f_{ij}^{(\Gamma)}(s) T_{ij}^{(\Gamma)}(g).
\]  

(1)

\[
\sum_{\mathbf{p}} f(\mathbf{p}) = \sum_s \frac{d_s}{G} \sum_g f(g\mathbf{p}_0(s)).
\]  

(2)
Appendix: Energy Shifts of Bound States

\[ \Delta E = 8\pi \int \frac{d^3 q}{(2\pi)^3} \phi^\dagger(q) \sum_{n \neq 0} e^{inqL} \tau(q) \phi(q) + \ldots \]  

\[ \tau(q; E) = \frac{1}{-a^{-1} + \sqrt{\frac{3}{4} q^2 - mE - i\epsilon}} \]

- Regular Wave Function \( \phi \sim \text{const.} \)
- Cut and Pole of \( \tau(q; E) \)
The energy of the scattering states vanishes in the infinite volume limit. We quote their finite volume energy $E$ in terms of the quantity $\kappa^2 = L^2 mE/(2\pi)^2$.

The energy shift of the ground state (which resides in the $A_1^+$ irrep) is:

$$\kappa^2 = \frac{g_0}{L} \left( 1 + \frac{g_1}{L} + \frac{g_2}{L^2} + \frac{g_3}{L^3} \ln \frac{mL}{2\pi} + \frac{g_4}{L^3} + \cdots \right),$$

with

$$g_0 = \frac{3}{\pi} a,$$
$$g_1 = 2.837297480 \, a,$$
$$g_2 = 9.725330808 \, a^2,$$
$$g_3 = 8\pi \left( \frac{2\sqrt{3}}{\pi} - \frac{8}{3} \right) a^3,$$
$$g_4 = \left(-5.159159617 + 6\pi \left( \frac{r}{a} \right) - 8\pi \left( \frac{M}{a^2} \right) \right) a^3.$$
The energy shift of the 1st excited state in the $A_1^+$ irrep is

$$\kappa^2 - 1 = \frac{h_0}{L} \left(1 + \frac{h_1}{L} + \frac{h_2}{L^2} + \frac{h_3}{L^3} \ln \frac{mL}{2\pi} + \frac{h_4}{L^3} + \cdots\right),$$

with

$$h_0 = \frac{10}{\pi} a,$$

$$h_1 = 0.279070 \ a,$$

$$h_2 = \left(8.494802 + \frac{7\pi^2}{5} \left(\frac{r}{a}\right)\right) a^2,$$

$$h_3 = \frac{27}{5} \times 8\pi \left(\frac{2\sqrt{3}}{\pi} - \frac{8}{3}\right) a^3,$$

$$h_4 = \left(-172.001650 + 83.745841 \left(\frac{r}{a}\right) - \frac{27}{5} \times 8\pi \left(\frac{\mathcal{M}}{a^2}\right)\right) a^3.$$