Minkowski-space solutions of the Schwinger-Dyson equation for the fermion propagator with the rainbow-ladder truncation

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The Lagrangian of QED with massive photons

\[ \mathcal{L} = \bar{\psi}(i\mathcal{D} - m_B)\psi + \frac{1}{2} A_\mu \left[ g^{\mu\nu} \left( \partial^2 + m_A^2 \right) - \left( 1 - \frac{1}{\xi} \right) \partial^\mu \partial^\nu \right] A_\nu, \]

(1)

with the covariant derivative defined as

\[ D^\mu = \partial^\mu - ieA^\mu. \]

(2)

Here \( \psi \) is the fermion field. \( A^\mu \) are the vector boson fields. \( m_B \) is the bare mass of the fermion. \( m_A \) is the mass of the gauge boson. \( e \) is the elementary charge.

The bare propagators are given by

\[ S^0_F(p) = \frac{1}{p - m_B + i\varepsilon}, \quad D^0_{\mu\nu}(q) = \frac{g_{\mu\nu} + (\xi - 1)q_\mu q_\nu/(q^2 - \xi m_A^2)}{q^2 - m_A^2 + i\varepsilon}. \]

(3)

The dressed fermion propagator \( S_F(p) \) is to be solved from its Schwinger-Dyson equation.
The Schwinger-Dyson equation for $S_F(p)$

\[ S_F^{-1}(p) = p - m_B - \Sigma(p). \]  \hspace{1cm} (4)

\[ \Sigma(p) = -ie^2 \int d_k \gamma^\nu S_F(k) \Gamma^\mu(k, p) D_{\mu\nu}(q) \]
\[ \equiv p\Sigma_v(p^2) + \Sigma_s(p^2). \] \hspace{1cm} (5)

Figure: The Schwinger-Dyson equation for the fermion propagator in QED.

Here $\Gamma^\mu(k, p)$ is the one-particle-irreducible fermion-photon vertex. In the rainbow-ladder truncation, we take $\Gamma^\mu(k, p) = \gamma^\mu$. 
The Källén-Lehmann spectral representation

The spectral representation of the fermion propagator is given by

$$S_F(p) = \int_{|W|=m}^{+\infty} dW \frac{\rho(W)}{p - W + i \text{sign}(W) \varepsilon},$$  \hspace{1cm} (6)

where the support of the spectral integral is $W \in (-\infty, -m] \cup [m, +\infty)$. The spectral function $\rho(W)$ is related to two scalar spectral functions by

$$\rho(W) = \text{sign}(W)[W \rho_1(W^2) + \rho_2(W^2)],$$ \hspace{1cm} (7)

such that $S_F(p) = \rho S_v(p^2) + S_s(p^2)$ with

$$\begin{cases}
S_v(p^2) = \int_{m^2}^{+\infty} ds \frac{\rho_1(s)}{p^2 - s + i \varepsilon} \\
S_s(p^2) = \int_{m^2}^{+\infty} ds \frac{\rho_2(s)}{p^2 - s + i \varepsilon}
\end{cases}.$$ \hspace{1cm} (8)

The mass-shell pole and the branch-cuts

\[ \rho_j(s) = Z_j \delta(s - m^2) + r_j(s), \quad \text{for } j \in \{1, 2\}. \]

The renormalized mass is to be solved from

\[ m \in \left\{ \sqrt{p^2} \mid \forall \sqrt{p^2} \in \mathbb{R} \text{ such that } \sqrt{p^2}[1 - \Sigma_v(p^2)] = m_B + \Sigma_s(p^2) \right\}. \]

Meanwhile, the mass-pole residue is given by

\[ Z = \left[ 1 - \Sigma_v(m^2) \right]^{-1}. \quad (9) \]

To calculate the branch-cuts of the spectral functions, define

\[ A(p^2) = 1 - \Sigma_v(p^2) \quad \text{and} \quad B(p^2) = -m_B - \Sigma_s(p^2), \quad (10) \]

such that the functions \( r_{s,v}(s) \) are given by

\[
\begin{aligned}
    r_v(s) &= -\frac{1}{\pi} \text{Im} \left\{ \frac{A(s)}{A^2(s)p^2 - B^2(s)} \right\} \bigg|_{s \geq p_{th}^2} \\
    r_s(s) &= -\frac{1}{\pi} \text{Im} \left\{ \frac{-B(s)}{A^2(s)p^2 - B^2(s)} \right\} \bigg|_{s \geq p_{th}^2}.
\end{aligned}
\quad (11)
\]

Here \( p_{th}^2 \) is the threshold above which the imaginary parts of the fermion self-energy become nonzero.
The quenched approximation in the Feynman gauge

Within the quenched approximation, the fermion self-energy in the Feynman gauge ($\xi = 1$) is given by

$$\frac{\delta \Sigma_F(p)}{\delta \rho(W)} = -ie^2 \int dl \int dF^2 \frac{(2 - d)y \rho + dW}{(l^2 - \Delta + i\varepsilon)^2}, \quad (12)$$

with $\int dF^2 = \int_0^1 dx \int_0^1 dy \delta(1 - x - y)$ and

$$\Delta(x, y, m_\sigma, k^2) = x s + y m_A^2 - xy p^2 - i\varepsilon. \quad (13)$$

The imaginary part of the fermion self-energy is subsequently given by

$$-\frac{1}{\pi} \text{Im}\{\Sigma_{Fv}(p^2)\}$$

$$= \frac{\alpha}{4\pi} \int_{m_A^2} (\sqrt{p^2 - m_A^2})^2 ds \frac{p^2 - m_A^2 + s}{p^4} \sqrt{(p^2 - m_A^2 + s)^2 - 4p^2s \rho_1(s)}, \quad (14a)$$

$$-\frac{1}{\pi} \text{Im}\{\Sigma_{Fs}(p^2)\}$$

$$= \frac{\alpha}{4\pi} \int_{m_A^2} (\sqrt{p^2 - m_A^2})^2 ds \left(-\frac{4}{p^2}\right) \sqrt{(p^2 - m_A^2 + s)^2 - 4p^2s \rho_2(s)}. \quad (14b)$$
Pauli-Villars regularization

- In the Pauli-Villars regularization scheme, divergences in the fermion self-energy are removed by:

\[ \Sigma_{PV}(p) = \Sigma_{bare}(p) - (m_A \to \Lambda), \quad (15) \]

where \( \Lambda > m_A \) is the mass of the regulator.

- The direct implementation of the spectral representation for the self-energy is implied in the Pauli-Villars regularization scheme due to the asymptotic behavior of \( \Sigma_{PV}(p) \).

- With dimensional regularization, subtracted spectral representation applies to the fermion-self energy.

- In other \( R_\xi \) gauges, the partial-fraction decomposition could be used to simplify the loop integral.
Numerical solution for the fermion spectral functions

Figure: $m_B = 0.5$, $\frac{e^2}{4\pi} = 0.3$, $m_A = 1$, $\Lambda = 10$. $Z = 0.9096$ and $m_R = 0.6502$. 
Numerical solution for the fermion self-energy

Solutions in the spacelike region

\begin{align*}
M(p^2) &\quad F(p^2) \\
S_v &\quad S_s
\end{align*}
The Nakanishi representation of the Bethe-Salpeter vertex and amplitude for scalar bound states

\[
\begin{align*}
\frac{P}{2-k} & \quad \frac{P}{2+k} \\
\frac{P}{2-k} & \quad \frac{P}{2+k}
\end{align*}
\]

Define the Bethe-Salpeter amplitude

\[
\chi(k, k \cdot P) = \psi(k, k \cdot P) D \left( \frac{P}{2} + k \right) D \left( \frac{P}{2} - k \right), \quad (16)
\]

with \( \psi(k, k \cdot P) \) being the Bethe-Salpeter vertex.
Spectral representation for the scalar propagator

\[ D(p) = \int ds \frac{\rho(s)}{p^2 - s + i\varepsilon} \]

and the Nakanishi representations for the BS vertex and the amplitude

\[ \psi(k, k \cdot P) = \int d\gamma \int_{-1}^{+1} dz \frac{\phi(\gamma, z)}{(k^2 + zk \cdot P - \gamma + i\varepsilon)^n} \quad (17) \]

\[ \chi(k, k \cdot P) = \int d\gamma \int_{-1}^{+1} dz \frac{\Phi(\gamma, z)}{(k^2 + zk \cdot P - \gamma + i\varepsilon)^{n+2}} \quad (18) \]

indicate the functional relation of \( \Phi \) in terms of \( \phi \) and \( \rho \).

\[ \Phi(\gamma, z) = \frac{\Gamma(n+2)}{\Gamma(n)} \int ds \int dt \rho(s) \rho(t) \left[ s + t + (s - t)z - 2 \left( \gamma + \frac{P^2}{4} \right) \right]^{n-1} \]

\[ \times \left\{ \theta \left( [s + t + (s - t)z] - 2 \left( \gamma + \frac{P^2}{4} \right) \right) \left[ \int_{-1}^{z} dz' \int_{\gamma_{\text{th}}(-z, \gamma, s, -z')}^{\gamma_{\text{th}}(z, \gamma, t, z')} d\gamma' + \int_{z}^{1} dz' \int_{0}^{\gamma_{\text{th}}(z, \gamma, t, z')} d\gamma' \right] \right. \]

\[ \left. - \theta \left( 2 \left( \gamma + \frac{P^2}{4} \right) - [s + t + (s - t)z] \right) \left[ \int_{-1}^{z} dz' \int_{\gamma_{\text{th}}(-z, \gamma, s, -z')}^{+\infty} d\gamma' + \int_{z}^{1} dz' \int_{0}^{\gamma_{\text{th}}(z, \gamma, t, z')} d\gamma' \right] \right\} \]

\[ \times \left[ s + t + (s - t)z' - 2 \left( \gamma' + \frac{P^2}{4} \right) \right]^{-n} \phi(\gamma', z'). \quad (19) \]

\[ \gamma_{\text{th}}(z, \gamma, u, z'; P^2) \equiv \frac{(1 + z')\gamma - (z' - z) \left( u - \frac{P^2}{4} \right)}{1 + z}. \]
Specifically when $n = 1$, the result is further reduced to

$$
\Phi(\gamma, z) = \int ds \int dt \rho(s) \rho(t) \left\{ \theta \left( [s + t + (s - t)z] - 2 \left( \gamma + \frac{P^2}{4} \right) \right) \times \left[ \int_{-1}^{1} dz' \int_{0}^{+\infty} d\gamma' + \int_{z}^{1} dz' \int_{0}^{+\infty} d\gamma' \right] \right\}
$$

$$
- \theta \left( 2 \left( \gamma + \frac{P^2}{4} \right) - [s + t + (s - t)z] \right) \times \left[ \int_{-1}^{z} dz' \int_{0}^{+\infty} d\gamma' + \int_{z}^{1} dz' \int_{0}^{+\infty} d\gamma' \right] \right\}
$$

$$
\times \frac{2 \phi(\gamma', z')}{s + t + (s - t)z' - 2 \left( \gamma' + \frac{P^2}{4} \right)}, \quad (20)
$$

with $\gamma_{th}(z, \gamma, u, z'; P^2) \equiv [(1 + z')\gamma - (z' - z) (u - P^2 2/4)] /(1+z)$. With the bare propagator, $\Phi(\gamma, z)$ starts from $\Gamma_{th} = m^2 - P^2/4$. 
The Bethe-Salpeter equation in the ladder truncation

\[ \psi(k, k \cdot P) = ig^2 \int dq \chi(k + q, (k + q) \cdot P) D(q). \]  

Figure: The diagrammatic representation of the Bethe-Salpeter equation for scalar bound states in the rainbow-ladder truncation.
The propagator of the exchange particle is given by

$$D(q) = \frac{1}{q^2 - \mu^2 - i\epsilon}.$$ 

With the Nakanishi representation for both the Bethe-Salpeter amplitude and the vertex, the BSE becomes

$$\int d\gamma \int dz \frac{\phi(\gamma, z)}{(k^2 + zk \cdot P - \gamma + i\epsilon)^n} = ig^2 \int dq \int d\gamma \int dz \Phi(\gamma, z) \frac{\phi(\gamma, z)}{(k + q)^2 + z(k + q) \cdot P - \gamma + i\epsilon}^{n+2} (q^2 - \mu^2 + i\epsilon)$$

which indicates

$$\phi(\gamma, z) = -\frac{g^2}{(4\pi)^2} \frac{1}{n(n + 1)} \int d\gamma' \int dz' \int_0^1 dx \Phi(\gamma', z') \frac{1}{(1 - x)^{n+1}}$$

$$\times \delta(z - z') \delta \left( \gamma - \left[ \frac{\gamma'}{1 - x} + \frac{\mu^2}{x} + \frac{xz'^2 P^2}{4(1 - x)} \right] \right) \frac{\partial}{\partial \gamma}.$$
Within the condition that $\gamma + z^2 \frac{P^2}{4} \geq \mu^2$, we have

$$
\phi(\gamma, z) = -\frac{g^2}{(4\pi)^2} \frac{1}{n(n+1)} \int_{\mu^2-\gamma-z^2P^2/2}^{\gamma+\mu^2-2\mu\sqrt{\gamma+z^2P^2/4}} \Phi(\gamma', z) \sqrt{(\gamma + \mu^2 - \gamma')^2 - 4\mu^2(\gamma + z^2 P^2/4)} \\
\times \left\{ \frac{x_-(\gamma', z, \gamma)}{[1 - x_-(\gamma', z, \gamma)]^n} + \frac{x_+(\gamma', z, \gamma)}{[1 - x_+(\gamma', z, \gamma)]^n} \right\} \frac{\partial}{\partial \gamma} \\
\equiv -\Theta(\gamma, z) \frac{\partial}{\partial \gamma} \tag{23}
$$

with $x_\pm$ given by

$$
x_\pm(\gamma', z', \gamma) = \frac{(\gamma + \mu^2 - \gamma') \pm \sqrt{(\gamma + \mu^2 - \gamma')^2 - 4\mu^2(\gamma + z'^2P^2/4)}}{2(\gamma + z'^2P^2/4)}.
$$
Specifically in the case of \( n = 1 \), \( \Theta(\gamma, z) \) is reduced to

\[
\Theta(\gamma, z) = \frac{g^2}{(4\pi)^2} \int_{\mu^2 - \gamma - z^2P^2/2}^{\gamma + \mu^2 - 2\mu \sqrt{\gamma + z^2P^2/4}} d\gamma' \frac{\gamma - \mu^2 - \gamma'}{2(\gamma' + z^2P^2/4)} \times \frac{\Phi(\gamma', z)}{\sqrt{(\gamma + \mu^2 - \gamma')^2 - 4\mu^2(\gamma + z^2P^2/4)}}.
\]

(24)

The square-root singularity is integratable with

\[
x = \text{arccosh} \frac{\gamma + \mu^2 - \gamma'}{2\mu \sqrt{\gamma + z^2P^2/4}},
\]

(25)

such that \( \gamma' = \gamma + \mu^2 - 2\mu \sqrt{\gamma + z^2P^2/4} \) \( \text{cosh} \ x \), and

\[
\Theta(\gamma, z) = \frac{g^2}{(4\pi)^2} \int_{\text{arccosh} \sqrt{\frac{\gamma + z^2P^2/4}{\mu^2}}}^{0} dx \frac{\mu \sqrt{\gamma + z^2P^2/4} \text{cosh} \ x - \mu^2}{\gamma + \mu^2 + z^2P^2/4 - 2\mu \sqrt{\gamma + z^2P^2/4} \text{cosh} \ x} \times \Phi \left( \gamma + \mu^2 - 2\mu \sqrt{\gamma + z^2P^2/4} \text{cosh} \ x, z \right).
\]

(26)

\( \Theta(\gamma, z) \) is nonvanishing when \( \gamma > \mu^2 - z^2P^2/4 \) and \( \gamma > \Gamma_{\text{th}}(z) + \mu^2 + 2\mu \sqrt{\Gamma_{\text{th}}(z) + z^2P^2/4} \).
Figure: Solution of the Nakanishi spectral functions for a scalar bound state from the Bethe-Salpeter equation in the rainbow-ladder truncation with the bare propagator, $m^2 = 1$, $P^2 = 3$, and $\mu^2 = 2$. 