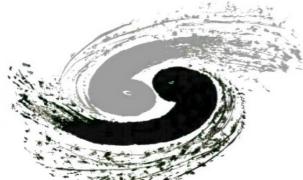


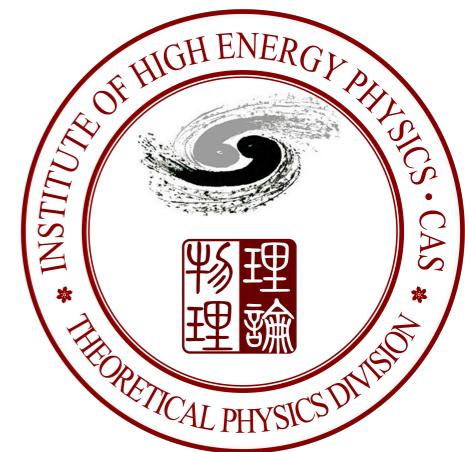
# Amplitude Reduction

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# **Amplitude/Tensor Reduction**

**Multi-loop amplitude calculation needs tensor reduction:**

- Projection Method (needs analytical inverse of base projection matrix, multi-base, multi-scale)
- Tarasov's Method (needs analytical inverse of dimension-shift matrix, multi-MI, multi-scale)
- IBP reduction (not systematic enough => slow & unstable)

# Difficulties in amplitude

$$\mathcal{M} = \int \mathbb{D}^L q \frac{N(\{q_j\}_{j=1}^L, \{k_e\}_{e=1}^E)}{\prod_{i=1}^n \mathcal{D}_i^{\nu_i}},$$

**Unlike SP between loop momentum and external momentum**

$$q \cdot k$$

**Numerator may contain ISP**

$$\bar{u} \not k \not q v \quad \text{or} \quad q \cdot \varepsilon$$

# Series Representation?

$$\frac{1}{\mathcal{D}_i} \equiv \frac{1}{P_i^2 - m_i^2} \rightarrow \frac{1}{\tilde{\mathcal{D}}_i} \equiv \frac{1}{P_i^2 - m_i^2 + i\eta},$$

$$\widetilde{\mathcal{M}}(\eta) = \int \mathbb{D}^L q \frac{\eta^{LD/2} N(\{\eta^{1/2} q_j\}_{j=1}^L, \{k_e\}_{e=1}^E)}{\prod_{i=1}^n [(\eta^{1/2} Q_i + K_i)^2 - m_i^2 + i\eta]^{\nu_i}}.$$

$$\widetilde{\mathcal{M}}(\eta) = \int \mathbb{D}^L q \eta^{LD/2-N_\nu} N(\{\eta^{1/2} q_j\}_{j=1}^L, \{k_e\}_{e=1}^E) \prod_{i=1}^n \left[ \frac{1}{Q_i^2 + i} \sum_{r_i=0}^{\infty} \left( -\frac{2\eta^{1/2} Q_i \cdot K_i + K_i^2 - m_i^2}{Q_i^2 + i} \right)^{r_i} \frac{1}{\eta^{r_i}} \right]^{\nu_i}$$

**Collect loop momenta from all numerators (2-loop as example):**

$$\frac{q_{i_1}^{\mu_1} \dots q_{i_n}^{\mu_n}}{[q_1^2 + i]^{\nu_1} [q_2^2 + i]^{\nu_2} [(q_1 + q_2)^2 + i]^{\nu_3}}$$

**Conventional reduction can give vacuum master integrals.**

# Find and collect form factors

**After reduction on vacuum integrals,  
all fermion chains and contraction with polarization vectors have no loop momentum.  
These can be used to generate form factors.**

$$\begin{aligned}\widetilde{\mathcal{M}}(\eta) &= \sum_i \mathcal{C}_i \mathcal{F}_i, \\ \mathcal{C}_i &= \eta^{L\mathcal{D}/2 - N_\nu + \lfloor r_i^{\max}/2 \rfloor} \sum_j \sum_{p=0}^{\infty} \mathcal{A}_{ijp} I_{L,j}^{\text{bub}} \eta^{-p},\end{aligned}$$

**By choosing certain set of scalar (master) integrals, amplitude can be reduced.**

$$\widetilde{\mathcal{M}}(\eta) = \sum_i C_i(\eta) \widetilde{I}_i(\eta),$$

# Improvement on series expansion for generic tensor integral

$$\tilde{G}_{\ell_1 \dots \ell_R}^{\mu_1 \dots \mu_R} = \int \mathbb{D}^L q \frac{\eta^{(LD+R)/2} q_{\ell_1}^{\mu_1} \dots q_{\ell_R}^{\mu_R}}{\prod_{i=1}^n [(\eta^{1/2} Q_i + K_i)^2 - m_i^2 + i\eta]^{\nu_i}}.$$

## Feynman parameterization

$$\begin{aligned} \tilde{G}_{\ell_1 \dots \ell_R}^{\mu_1 \dots \mu_R} &= \frac{(-1)^{N_\nu}}{\prod_{j=1}^N \Gamma(\nu_j)} \int \prod_{j=1}^N dx_j \ x_j^{\nu_j-1} \delta(1 - \sum_{l=1}^N x_l) \\ &\times \sum_{m=0}^{[R/2]} \frac{\Gamma(N_\nu^{(m)})}{(-2)^m} \left[ (\tilde{M}^{-1} \otimes g)^{(m)} \tilde{\ell}^{(R-2m)} \right]^{\Gamma_1, \dots, \Gamma_R} \\ &\times U^{-D/2+m-R} \left( \frac{F}{U} - i\eta \right)^{-N_\nu^{(m)}}, \end{aligned} \quad (5)$$

where  $N_\nu^{(m)} = N_\nu - m - LD/2$ .

$$\left( \frac{F}{U} - i\eta \right)^{-N_\nu^{(m)}} = (-i\eta)^{-N_\nu^{(m)}} \sum_{n=0}^{\infty} \binom{-N_\nu^{(m)}}{n} \frac{F^n}{U^n (-i\eta)^n}.$$

$$\begin{aligned} \mathcal{U}(\vec{x}) &= \sum_{T \in \mathcal{T}_1} \left[ \prod_{j \in \mathcal{C}(T)} x_j \right], \\ \mathcal{F}_0(\vec{x}) &= \sum_{\hat{T} \in \mathcal{T}_2} \left[ \prod_{j \in \mathcal{C}(\hat{T})} x_j \right] (-s_{\hat{T}}), \\ \mathcal{F}(\vec{x}) &= \mathcal{F}_0(\vec{x}) + \mathcal{U}(\vec{x}) \sum_{j=1}^N x_j m_j^2. \end{aligned}$$

$$\tilde{F} = F - i\eta U$$

**By removing external momenta, any integral can shrink to vacuum integral.**

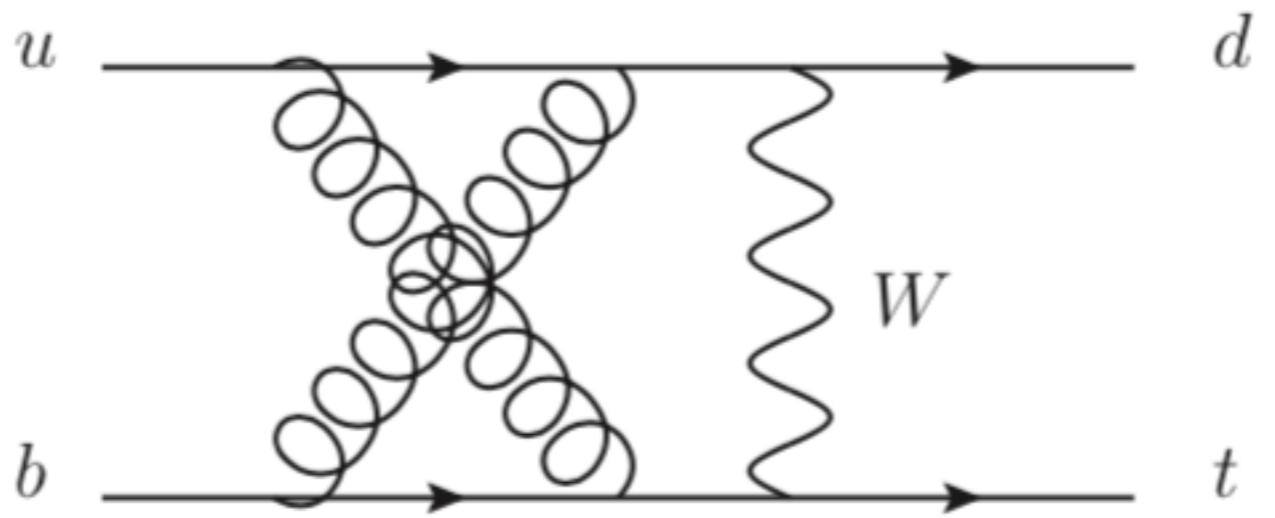
$$\int \prod_{j=1}^N dx_j \ x_j^{n_j-1} \delta(1 - \sum_{l=1}^N x_l) U^{-\tilde{D}/2}$$

**Distribute x's into y1, y2 and y3.**

$$U = y_1 y_2 + y_1 y_3 + y_2 y_3.$$

$$\begin{aligned} & \int dx_1 \cdots dx_m \ x_1^{n_1} \cdots x_m^{n_m} \int dy_1 \delta(y_1 - x_1 - \cdots - x_m) \\ &= \int dy_1 \ y_1^{n_1+\cdots+n_m+m-1} \frac{\Gamma(n_1+1) \cdots \Gamma(n_m+1)}{\Gamma(n_1 + \cdots + n_m + m)}. \end{aligned}$$

$$\begin{aligned} I_{\nu_1 \nu_2 \nu_3}^{(vac), D} &= \int \frac{d^D q_1 d^D q_2}{[q_1^2 + i]^{\nu_1} [q_2^2 + i]^{\nu_2} [(q_1 + q_2)^2 + i]^{\nu_3}} \\ &= (-i)^{D+N_\nu} \int \frac{y_1^{\nu_1-1} dy_1}{\Gamma(\nu_1)} \frac{y_2^{\nu_2-1} dy_2}{\Gamma(\nu_2)} \frac{y_3^{\nu_3-1} dy_3}{\Gamma(\nu_3)} \\ &\quad \times \delta(1 - y_1 - y_2 - y_3) \Gamma(N_\nu - D) U^{-D/2}. \end{aligned}$$



$$\begin{aligned}
 \mathcal{F}_1 &= (\bar{t}P_R\gamma^{\mu_1\mu_2\mu_3\mu_4\mu_5}b) (\bar{d}P_R\gamma^{\mu_1\mu_2\mu_3\mu_4\mu_5}u), \\
 \mathcal{F}_2 &= (\bar{t}P_L\gamma^{\mu_1\mu_2\mu_3\mu_4}b) (\bar{d}P_R\gamma^{\mu_1\mu_2\mu_3\mu_4}k_2u), \\
 \mathcal{F}_3 &= (\bar{t}P_R\gamma^{\mu_1\mu_2\mu_3\mu_4}k_1b) (\bar{d}P_R\gamma^{\mu_1\mu_2\mu_3\mu_4}k_2u), \\
 \mathcal{F}_4 &= (\bar{t}P_L\gamma^{\mu_1\mu_2\mu_3}k_1b) (\bar{d}P_R\gamma^{\mu_1\mu_2\mu_3}u), \\
 \mathcal{F}_5 &= (\bar{t}P_R\gamma^{\mu_1\mu_2\mu_3}b) (\bar{d}P_R\gamma^{\mu_1\mu_2\mu_3}u), \\
 \mathcal{F}_6 &= (\bar{t}P_L\gamma^{\mu_1\mu_2}b) (\bar{d}P_R\gamma^{\mu_1\mu_2}k_2u), \\
 \mathcal{F}_7 &= (\bar{t}P_R\gamma^{\mu_1\mu_2}k_1b) (\bar{d}P_R\gamma^{\mu_1\mu_2}k_2u), \\
 \mathcal{F}_8 &= (\bar{t}P_L\gamma^{\mu_1}k_1b) (\bar{d}P_R\gamma^{\mu_1}u), \\
 \mathcal{F}_9 &= (\bar{t}P_R\gamma^{\mu_1}b) (\bar{d}P_R\gamma^{\mu_1}u), \\
 \mathcal{F}_{10} &= (\bar{t}P_Lb) (\bar{d}P_Rk_2u), \\
 \mathcal{F}_{11} &= (\bar{t}P_Rk_1b) (\bar{d}P_Rk_2u),
 \end{aligned}$$

# Progress & Prospect

- Alternative approach to solving reduction equations.
- Single-top calculation (ambitious ttH?)
- B-physics  $k_T$  factorization calculation
- Gluon amplitudes

# Reduction Problems

- Many target scalar (master) integrals lead to many x's.
- Number of equations relies on number of scales.
- Higher series gives more equations
- Can we have additional equations/relations???