Novel Feynman Integral Reduction Method and its Application

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Based on: 1711.09572, 1801.10523
and work in preparation

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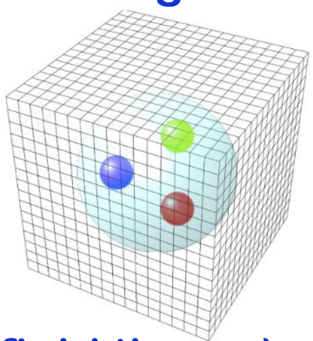
Outline

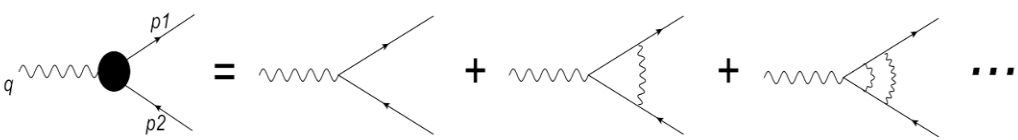
Quantum field theory

- **QFT: the underlying theory of modern physics**
  - Solving QFT is important for testing the SM and discovering NP



- **How to solve QFT:**

- Nonperturbatively (e.g. lattice field theory): discretize spacetime, numerical simulation complicated, application limited



- Perturbatively (small coupling constant): generate and calculate Feynman amplitudes, relatively simpler, the primary method

Super computer

Perturbative QFT

1. Generate Feynman amplitudes

- Feynman diagrams and Feynman rules
- New developments: unitarity, recurrence relation

2. Calculate Feynman loop integrals

3. Calculate phase-space integrals

- Monte Carlo simulation with IR subtractions
- Relating to loop integrals

$$\int \frac{d^D p}{(2\pi)^D}(2\pi)\delta_+(p^2) = \lim_{\eta\to 0^+}\int \frac{d^D p}{(2\pi)^D}\left(\frac{i}{p^2+i\eta}+\frac{-i}{p^2-i\eta}\right)$$

Feynman loop integrals

➢ **The key to apply pQFT**

$$\lim_{\eta\to 0^+}\int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \prod_{\alpha=1}^{N} \frac{1}{(q_\alpha^2 - m_\alpha^2 + \mathrm{i}\eta)^{\nu_\alpha}}$$

q_α: linear combination of loop momenta and external momenta

Multi-loop: a challenge for intelligence

- One-loop calculation: (up to 4 legs) satisfactory approaches existed as early as 1970s

  ’t Hooft, Veltman, NPB (1979); Passarino, Veltman, NPB (1979); Oldenborgh, Vermaseren (1990)

  Developments of unitarity-based method in the past decade made the calculation efficient for multi-leg problems

  Britto, Cachazo, Feng, 0412103; Ossola, Papadopoulos, Pittau, 0609007; Giele, Kunszt, Melnikov, 0801.2237

- About 40 years later, a satisfactory method for multi-loop calculation is still missing

Main strategy

1) Reduce loop integrals to basis (Master Integrals)

- Integration-by-parts (IBP) reduction: Chetyrkin, Tkachov, NPB (1981) Laporta, 0102033

  the only way (before our method), main bottleneck

  extremely time consuming for multi-scale problems

  unitarity-based reduction is efficient but cannot give complete reduction

2) Calculate MIs/original integrals

- Differential equations (depends on reduction and BCs) Kotikov, PLB (1991)
- Difference equations (depends on reduction and BCs) Laporta, 0102033
- Sector decomposition (extremely time-consuming) Binoth, Heinrich, 0004013
- Mellin-Barnes representation (nonplanar, time) Usyukina (1975) Smirnov, 9905323

IBP redution

➢ A result of dimensional regularization

Chetyrkin, Tkachov, NPB (1981)

$$\int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \frac{\partial}{\partial \ell_j^\mu} \left(v_k^\mu \prod_{\alpha=1}^{N} \frac{1}{(q_\alpha^2 - m_\alpha^2 + \mathrm{i}\eta)^{\nu_\alpha}} \right) = 0\,, \quad \forall j, k$$

$$\Downarrow$$

- **Linear equations:** $\sum_{i=1} Q_i(D, \vec{s}, \eta)\, \mathcal{M}_i(D, \vec{s}, \eta) = 0$
- M_i **scalar integrals,** Q_i **polynomials in** $D, \vec{s}, \eta$

➢ For each problem, the number of MIs is FINITE

Smirnov, Petukhov, 1004.4199

- **Feynman integrals form a finite dimensional linear space**
- **Reduce thousands of loop integrals to much less MIs**

Difficulty of IBP reduction

➢ Solve IBP equations

$$\sum_{i=1} Q_i(D,\vec{s},\eta)\,\mathcal{M}_i(D,\vec{s},\eta) = 0$$

- Very large scale of linear equations (even millions of)
- Fully coupled
- Hard to do Gaussian elimination for many variables $D,\vec{s},\eta$
- Too slow if solve it numerically for each phase space point

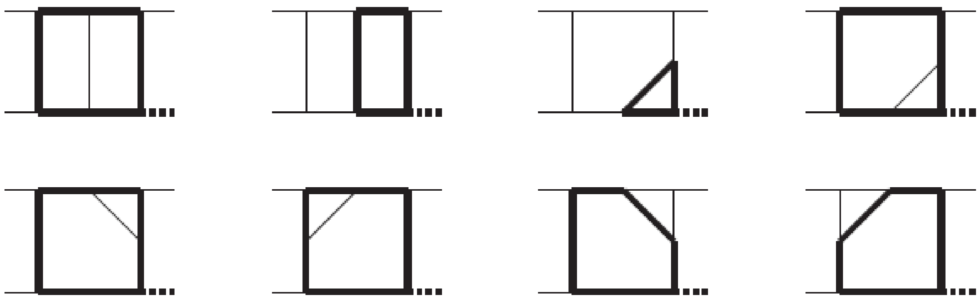
➢ Cutting-edge problems

- Hundreds GB RAM
- Months of runtime using super computer

Difficulty of MIs calculation

- **Analytical: Higgs → 3 partons (Euclidean Region)**

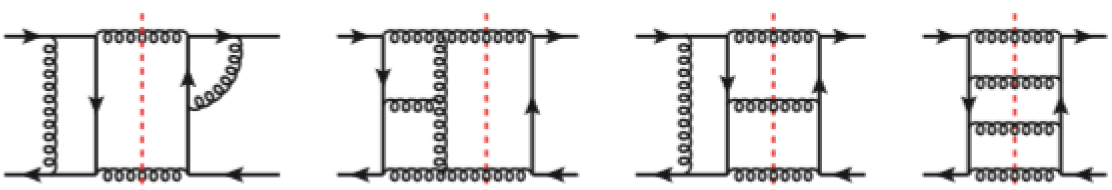
R. Bonciani, et.al 2016



200MB, 10 min

- **Numerical: Quarkonium decay at NNLO**

Feng, Jia, Sang, 1707.05758



NNLO (Virtual Squared) NNLO (Double Virtual) NNLO (Virtual−Real) NNLO (Double Real)

10^5 **CPU core-hour**

Recent developments

➢ Improvements for IBP reduction

- Finite field method Manteuffel, Schabinger, 1406.4513
- Direct solution Kosower, 1804.00131
- Module-intersection IBP method Böhm, Georgoudis, Larsen, Schönemann, Zhang, 1805.01873
- Obtain one coefficient at each step Chawdhry, Lim, Mitov, 1805.09182
- …

➢ Improvements for evaluating scalar integrals

- Quasi-Monte Carlo method Li, Wang, Yan, Zhao, 1508.02512
- Finite basis Manteuffel, Panzer, Schabinger, 1510.06758
- Uniform-transcendental basis Boels, Huber, Yang, 1705.03444
- …

State-of-the-art computation

- 2→2 process with massive particles at two-loop order is already the frontier
  - $g+g\to t+\bar{t},\ g+g\to H+H,\ g+g\to H+g, \ldots$
- Very time-consuming
  - Two-loop $g+g\to H+H\ (g)$: complete IBP reduction cannot be achieved within tolerable time
    Borowka et. al., 1604.06447
    Jones, Kerner, Luisoni, 1802.00349
  - Two-loop decay $Q+\bar{Q}\to g+g$, MIs cost $O(10^5)$ CPU core-hour
    Feng, Jia, Sang, 1707.05758
  - Four-loop nonplanar cusp anomalous dimension, within tolerable computational expense, calculated MIs have 10% uncertainty
    Boels, Huber, Yang, 1705.03444

New ideas are badly needed

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II. A New Representation

III. Reduction

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Introduction of auxiliary variable

➢ **Dimensionally regularized Feynman loop integral with an auxiliary variable η**

$$\mathcal{M}(D,\vec{s},\eta) \equiv \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \prod_{\alpha=1}^{N} \frac{1}{(\mathcal{D}_\alpha + \mathrm{i}\eta)^{\nu_\alpha}} \qquad \mathcal{D}_\alpha \equiv q_\alpha^2 - m_\alpha^2$$

- Think it as an analytical function of η
- Physical result is defined by

$$\mathcal{M}(D,\vec{s},0) \equiv \lim_{\eta \to 0^+} \mathcal{M}(D,\vec{s},\eta)$$

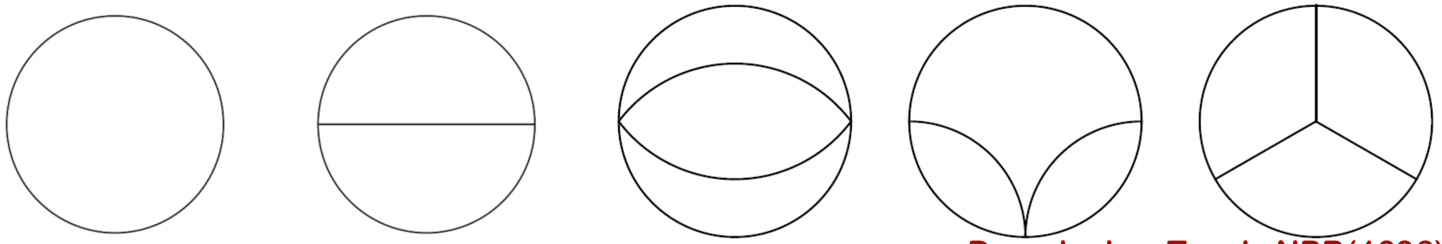
Expansion at infinity

- **Expansion of propagators around $\eta = \infty$**

$$\frac{1}{[(\ell+p)^2 - m^2 + \mathrm{i}\eta]^\nu} = \frac{1}{(\ell^2+\mathrm{i}\eta)^\nu} \sum_{n=0}^{\infty} \frac{(\nu)_n}{n!} \left(\frac{-2\ell\cdot p - p^2 + m^2}{\ell^2 + \mathrm{i}\eta} \right)^n$$

  - Only one region: $l^\mu \sim |\eta|^{1/2}$
  - No external momenta in denominator, vacuum integrals
  - Simple enough to deal with

- **Vacuum MIs with equal internal masses**



  - Analytical results are known up to 3-loop
  - Numerical results are known up to 5-loop

Davydychev, Tausk, NPB(1993)
Broadhurst, 9803091
Kniehl, Pikelner, Veretin, 1705.05136

Schroder, Vuorinen, 0503209
Luthe, PhD thesis (2015)
Luthe, Maier, Marquard, Ychroder, 1701.07068

A new representation

➢ Asymptotic expansion

$$\mathcal{M}(D,\vec{s},\eta) = \eta^{LD/2-\sum_\alpha \nu_\alpha} \sum_{\mu_0=0}^{\infty} \eta^{-\mu_0} \mathcal{M}_{\mu_0}^{\mathrm{bub}}(D,\vec{s})$$

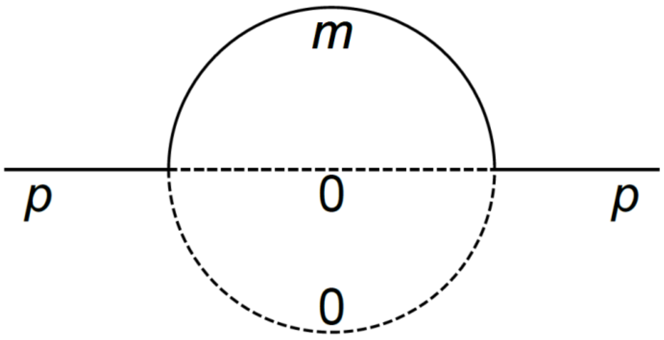
$$\mathcal{M}_{\mu_0}^{\mathrm{bub}}(D,\vec{s}) = \sum_{k=1}^{B_L} I_{L,k}^{\mathrm{bub}}(D) \sum_{\vec{\mu}\in\Omega_{\mu_0}^r} C_k^{\mu_0\cdots\mu_r}(D) s_1^{\mu_1}\cdots s_r^{\mu_r}$$

- $I_{L,k}^{\mathrm{bub}}(D)$: k-th master vacuum integral at L-loop order
- $C_k^{\mu_0\ldots\mu_r}(D)$: rational functions of D
- Physical Feynman integral can be obtained by analytical continuation of this calculable asymptotic series: a new representation

Example

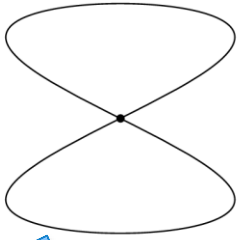
➢ Sunrise integral

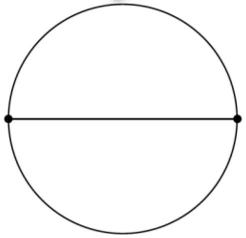
$$\hat{I}_{\nu_1\nu_2\nu_3} \equiv \int \prod_{i=1}^{2} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \frac{1}{\mathcal{D}_1^{\nu_1}\mathcal{D}_2^{\nu_2}\mathcal{D}_3^{\nu_3}}$$



$$\mathcal{D}_1 =(\ell_1+p)^2 - m^2,\ \mathcal{D}_2 = \ell_2^2,\ \mathcal{D}_3 = (\ell_1+\ell_2)^2$$

$$I_{111} = \eta^{D-3}\left\{\left[1 - \frac{D-3}{3}\frac{m^2}{\mathrm{i}\eta} + \frac{(D+4)(D-3)}{9D}\frac{p^2}{\mathrm{i}\eta}\right] I_{2,2}^{\mathrm{bub}} - \mathrm{i}\left[\frac{(D-2)^2}{3D}\frac{p^2}{\mathrm{i}\eta}\right] I_{2,1}^{\mathrm{bub}} + \mathcal{O}(\eta^{-2})\right\}$$

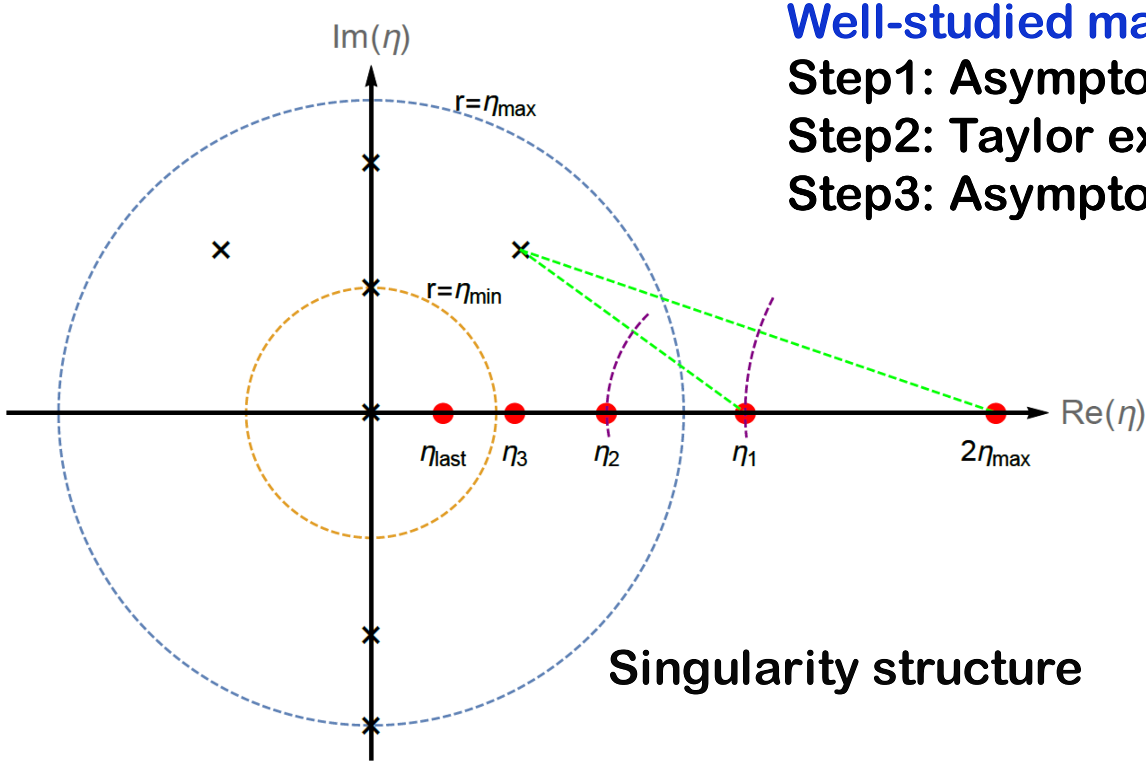




Analytical continuation

- **Reduce all loop integrals to MIs**
  - IBP is hopeless in general, see next section for new reduction
- **Set up and solve DEs of MIs**

$$\frac{\partial}{\partial \eta}\vec{I}(D;\eta) = A(D;\eta)\vec{I}(D;\eta) \quad \text{with known } \vec{I}(D;\infty)$$

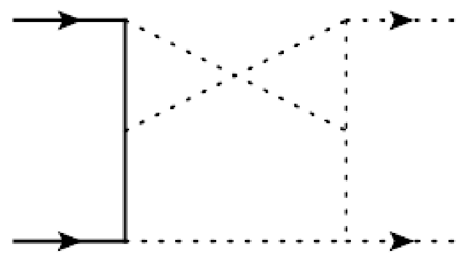


Well-studied mathematic problem:
Step1: Asymptotic expansion at $\eta = \infty$
Step2: Taylor expansion at analytical points
Step3: Asymptotic expansion at $\eta = 0$

Singularity structure

Example

- 2-loop non-planar sector for $\mathrm{Q} + \bar{\mathrm{Q}} \rightarrow g + g$



  - 168 master integrals
  - Traditional method sector decomposition: $O(10^4)$ CPU core-hour
  - Our method: a few minutes

- Faster by 10^5 times!!
- But depends on the existence of efficient reduction method

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What is reduction

➢ Reduction

- Find relations between loop integrals
- Use them to express all loop integrals as linear combinations of MIs

➢ Relations among $G \equiv \{M_1, M_2, \dots, M_n\}$

$$\sum_{i=1}^{n} Q_i(D, \vec{s}, \eta)\, \mathcal{M}_i(D, \vec{s}, \eta) = 0$$

- $Q_i(D, \vec{s}, \eta)$: homogeneous polynomials of $\vec{s}, \eta$ of degree d_i

➢ Constraints from mass dimension

$$2d_1 + \mathrm{Dim}(\mathcal{M}_1) = \cdots = 2d_n + \mathrm{Dim}(\mathcal{M}_n)$$

- Only 1 degree of freedom in $\{d_i\}$, chosen as $d_{\max} \equiv \mathrm{Max}\,\{d_i\}$

Determine relations

- **Decomposition of $Q_i(D,\vec{s},\eta)$**

$$Q_i(D,\vec{s},\eta) = \sum_{(\lambda_0,\vec{\lambda})\in\Omega_{d_i}^{r+1}} Q_i^{\lambda_0\ldots\lambda_r}(D)\,\eta^{\lambda_0} s_1^{\lambda_1}\cdots s_r^{\lambda_r}$$

$$\Rightarrow \sum_{k,\rho_0,\vec{\rho}} f_k^{\rho_0\ldots\rho_r}\, I_{L,k}^{\mathrm{bub}}(D)\;\eta^{\rho_0} s_1^{\rho_1}\cdots s_r^{\rho_r} = 0$$

- **Linear equations:** $f_k^{\rho_0\rho_1\ldots\rho_r}(Q) = 0$
  - With enough constraints $\Rightarrow Q_i^{\lambda_0\ldots\lambda_r}(D)$
  - With finite field technique, only integers in a finite field are involved, equations can be efficiently solved

- **Relations among $G \equiv \{M_1, M_2, \ldots, M_n\}$ with a fixed $d_{\max}$ are fully determined**

Reduction

- **With $G = G_1 \cup G_2$, assume**
  - G_1 is more complicated than G_2
  - G_1 can be reduced to G_2
- **Algorithm** *Search for simplest relations*
  1. Set $d_{\max} = 0$
  2. Find out all reduction relations with fixed $d_{\max}$
  3. If obtained relations are enough to determine G_1, stop; else $d_{\max} + +$ and go to step 2
- **Question: how to choose G_1 and G_2?**
  1. Relations among G_1 and G_2 are not too complicated: relations easy to find
  2. Size of G_1 is not too large: relations can be efficiently used numerically

Scalar reduction

- **Scalar integral:** $\vec{\nu} = (\nu_1, \ldots, \nu_N), \nu_i \geq 0$
  - $\mathbf{0}^{\pm} \equiv \mathbf{Identity},\ \mathbf{m}^{\pm} \equiv (\mathbf{m}-\mathbf{1})^{\pm}\mathbf{1}^{\pm}$
  - $\mathbf{1}^{+}(5,1,0,3) = \{(6,1,0,3), (5,2,0,3), (5,1,0,4)\}$
  - $\mathbf{1}^{-}(5,1,0,3) = \{(4,1,0,3), (5,0,0,3), (5,1,0,2)\}$
- **1-loop:** $G_1 = \mathbf{1}^{+}\vec{\nu}, G_2 = \mathbf{1}^{-}\mathbf{1}^{+}\vec{\nu}$ Duplancic and Nizic, hep-ph/0303184
- **Multi-loop:**

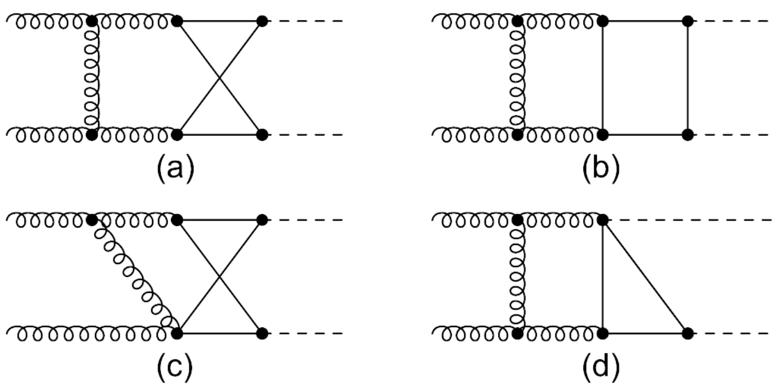
$$G_1 = \mathbf{m}^{+}\vec{\nu}, G_2 = \{\mathbf{1}^{-}\mathbf{m}^{+}, \mathbf{1}^{-}(\mathbf{m}-\mathbf{1})^{+}, \ldots, \mathbf{1}^{-}\mathbf{1}^{+}\}\vec{\nu}$$

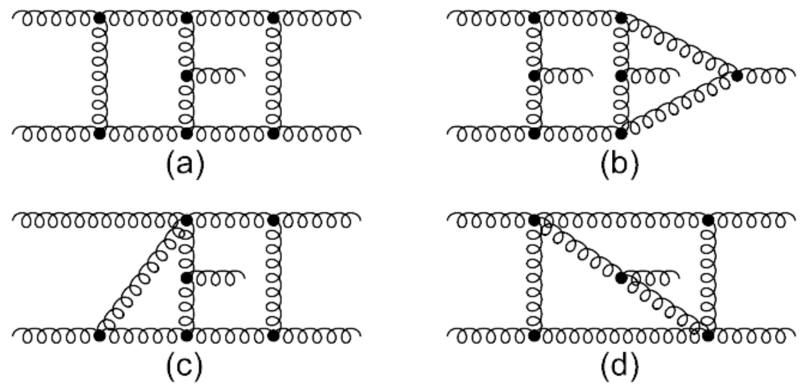
  - The size of G_1 is not too large, about dozens of integrals
  - Relations among G_1 and G_2 are not too complicated, see examples

A step-by-step reduction is realized!

Examples

➤ 2-loop $g + g \to H + H$ and $g + g \to g + g + g$





$g+g\to H+H$				$g+g\to g+g+g$			
Sector	Type	$d_{\max}$	$\mathbf{m}^+$	Sector	Type	$d_{\max}$	$\mathbf{m}^+$
1(a)	7-NP	1	$\mathbf{3}^+$	2(a)	8-NP	1	$\mathbf{3}^+$
1(b)	7-P	1	$\mathbf{3}^+$	2(b)	8-NP	3	$\mathbf{3}^+$
1(c)	6-NP	5	$\mathbf{3}^+$	2(c)	7-NP	4	$\mathbf{3}^+$
1(d)	6-P	4	$\mathbf{2}^+$	2(d)	6-NP	2	$\mathbf{3}^+$

Difficulty:

- More legs > less legs
- Nonplanar > Planar
- $\mathbf{m}^+\vec{e} > \mathbf{m}^+\vec{v}$

- The reduction is obtained by a single-core laptop

Tensor reduction (preliminary)

- **Rank-R tensor integral**

$$\mathcal{M}^{\mu_1\cdots\mu_R} \equiv \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \frac{\ell_{i_1}^{\mu_1}\cdots\ell_{i_R}^{\mu_R}}{(\mathcal{D}_1+\mathrm{i}\eta)^{\nu_1}\cdots(\mathcal{D}_N+\mathrm{i}\eta)^{\nu_N}}$$

- **Tensor decomposition**

$$\mathcal{M}^{\mu_1\cdots\mu_R} = \sum_i A_i(D,\vec{s},\eta)\times T_i^{\mu_1\cdots\mu_R}(g,p)$$

$$\Downarrow$$

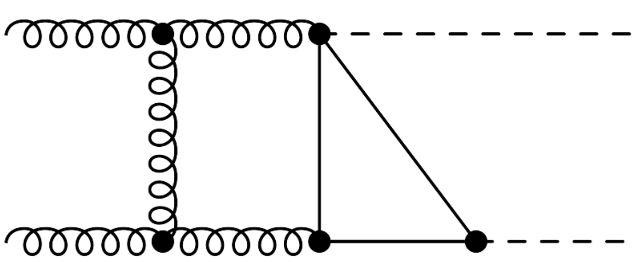
$$A_i = \sum_j (T\cdot T)^{-1}_{ij}\, T_j\cdot\mathcal{M}, \quad \vec{A} = K\,\vec{I} + \vec{J}$$

- $\vec{I}$: rank-R integrals, containing irreducible scalar products (ISP)
- $\vec{J}$: integrals in sub-sectors or with lower rank
- $\vec{A}$ can't be directly reduced to scalar integrals, different from 1-loop

Tensor reduction (preliminary)

- **Goal: to find nontrivial relations among $\vec{A}$ together with trivial relations $\vec{A} = K\,\vec{I} + \vec{J}$, to reduce $\vec{A}$ to simper integrals**
  - $\vec{A}$ in general has lower mass dimension than $\vec{I}$
  - Possibility for simpler relations

- **Example: rank-2 tensors**



$$G_1 = \{\mathbf{1}^+, \mathbf{0}^+\}\,\vec{e} \otimes \{\ell_1^{\mu_1}\ell_1^{\mu_2}, \ell_1^{\mu_1}\ell_2^{\mu_2}, \ell_2^{\mu_1}\ell_2^{\mu_2}\}$$

$$G_2 = \mathbf{1}^+\mathbf{1}^-\vec{e} \otimes \{\ell_1^{\mu_1}\ell_1^{\mu_2}, \ell_1^{\mu_1}\ell_2^{\mu_2}, \ell_2^{\mu_1}\ell_2^{\mu_2}\} \cup \{\mathbf{1}^+, \mathbf{1}^-\mathbf{1}^+\}\,\vec{e} \otimes \{\ell_1^{\mu}, \ell_2^{\mu}\}$$

$$d_{\max} = 2$$

Comparison with IBP reduction

- IBP relations can be obtained very fast, but it is a problem how to use them
  - Analytical: almost impossible for multi-scale problem
  - Numerical: very time consuming because the relations are fully coupled, each phase space point may need hours to days

- Our reduction strategy
  - Needs time to obtain relations, but 1) relations are analytical that can be used for any phase space point; 2) according to cutting-edge examples, the time is tolerable
  - Use our relations numerically: very efficient because relations are decoupled to small blocks, similar to one-loop case

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Summary

- Find a new representation for Feynman integrals, conceptually translates the loop calculation to the problem of performing analytical continuations

- Propose a new reduction strategy, which may overcome difficulties encountered in IBP reduction

- Two-loop example $gg \to \mathrm{HH},\ \mathrm{gg} \to \mathrm{ggg}$: correctness and efficiency of our reduction method