Master Integrals for Higher Order QCD Corrections to Heavy Quarks Production and Decay

Based on
arXiv:1801.01033
arXiv:1810.04328

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Top quark

1. The heaviest particle in standard model

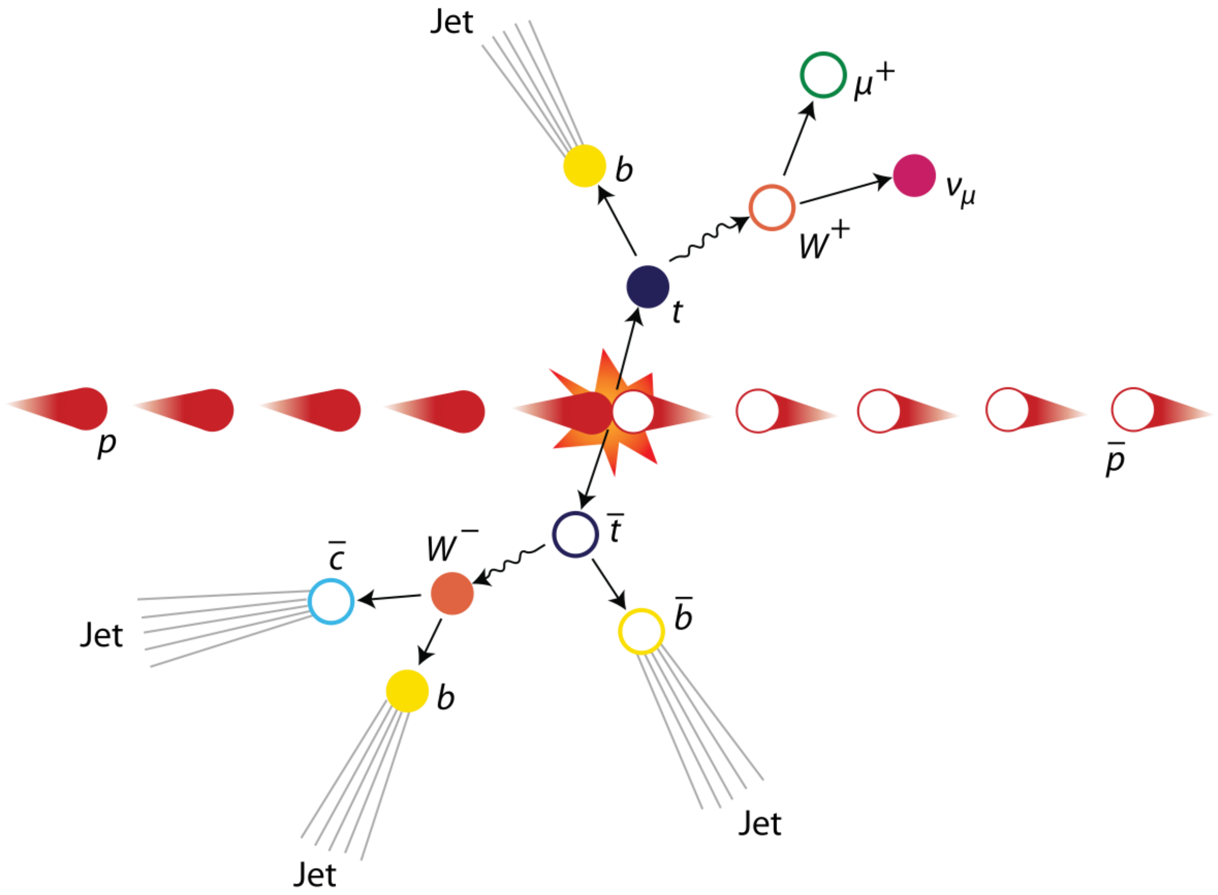
2. Suitable for QCD perturbative calculations

3. Huge samples of top quarks at the LHC

4. Study of CKM matrix element

5. Search for new physics beyond standard model

…………………………….



Top pairs production at hadron colliders

The need for higher order QCD corrections

QED $\alpha \sim O(0.01)$

QCD $\alpha_s \sim O(0.1)$ $(\mu \gg \Lambda_{QCD})$

QED are more convergence
in perturbation expansion than **QCD**

1. Two-loop master integrals for heavy-to-light form factors of two different massive fermions

Massive quark decays to massive quark

$$(t \to b + W^+(l + \bar{\nu}), b \to c + l + \bar{\nu})$$

Massive lepton decays to massive lepton

$$(\mu \to e + \nu_\mu + \bar{\nu}_e, \tau \to \mu + \nu_\tau + \bar{\nu}_\mu)$$

[1] K.G. Chetyrkin, R. Harlander, T. Seidensticker and M. Steinhauser, *Second order QCD corrections to* $\Gamma(t \to Wb)$, *Phys. Rev.* **D 60** (1999) 114015 [hep-ph/9906273] [INSPIRE].

[2] I.R. Blokland, A. Czarnecki, M. Slusarczyk and F. Tkachov, *Heavy to light decays with a two loop accuracy*, *Phys. Rev. Lett.* **93** (2004) 062001 [hep-ph/0403221] [INSPIRE].

[3] A. Czarnecki, M. Ślusarczyk and F.V. Tkachov, *Enhancement of the hadronic b quark decays*, *Phys. Rev. Lett.* **96** (2006) 171803 [hep-ph/0511004] [INSPIRE].

[4] M. Brucherseifer, F. Caola and K. Melnikov, $\mathcal{O}(\alpha_s^2)$ *corrections to fully-differential top quark decays*, *JHEP* **04** (2013) 059 [arXiv:1301.7133] [INSPIRE].

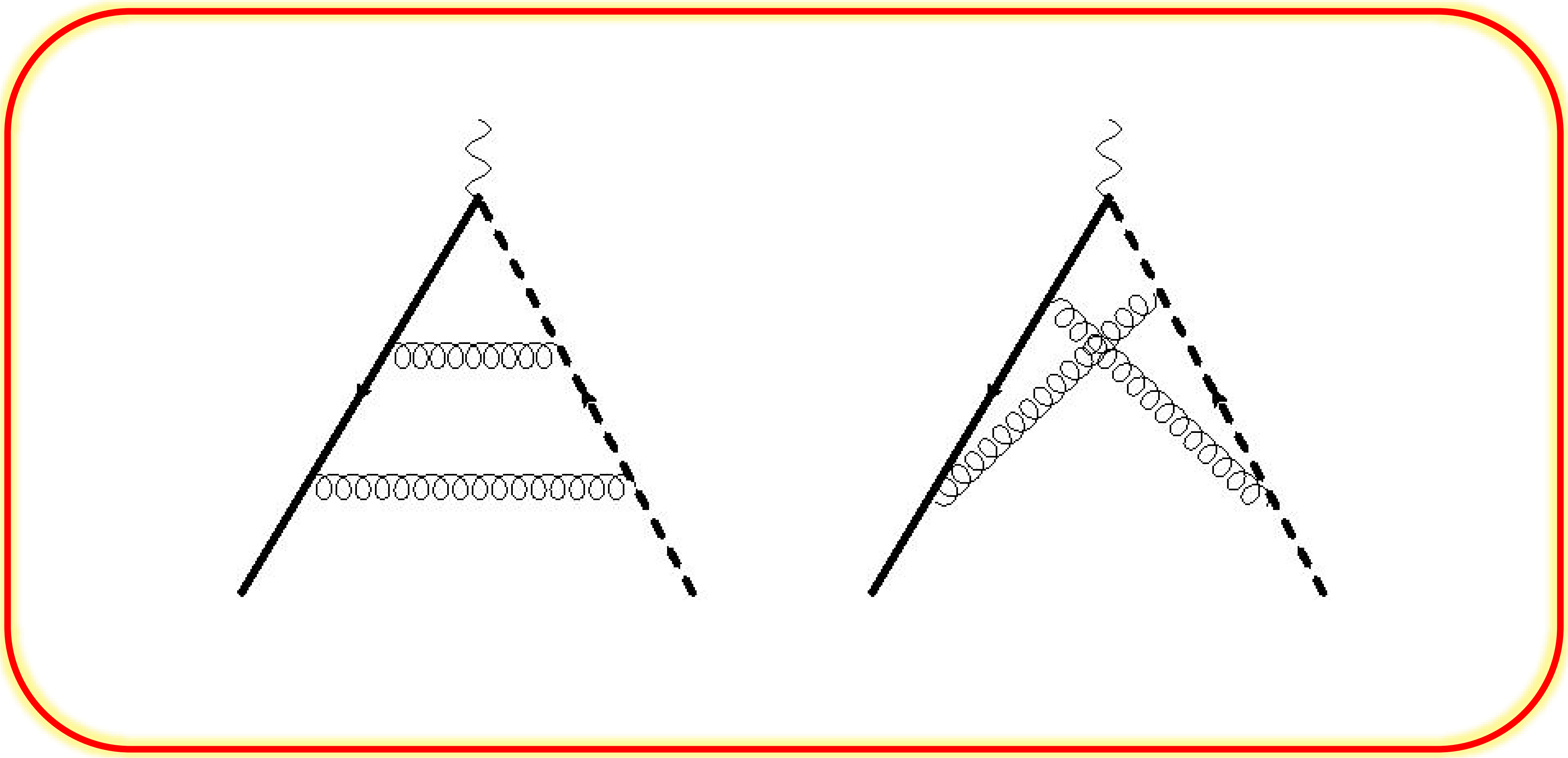
[5] J. Gao, C.S. Li and H.X. Zhu, *Top quark decay at next-to-next-to leading order in QCD*, *Phys. Rev. Lett.* **110** (2013) 042001 [arXiv:1210.2808] [INSPIRE].

[6] R. Bonciani and A. Ferroglia, *Two-loop QCD corrections to the heavy-to-light quark decay*, *JHEP* **11** (2008) 065 [arXiv:0809.4687] [INSPIRE].

………………

$$\ln^2\left(\frac{m_{\text{heavy}}}{m_{\text{light}}}\right) \qquad \ln\left(\frac{m_{\text{heavy}}}{m_{\text{light}}}\right)$$

Large logarithms will appear when calculating the differential decay rates



Sample of Two-loop Feynman diagrams

All two-loop integrals can be reduced to 40 master Integrals

Integration-By-Parts (IBP) reduction

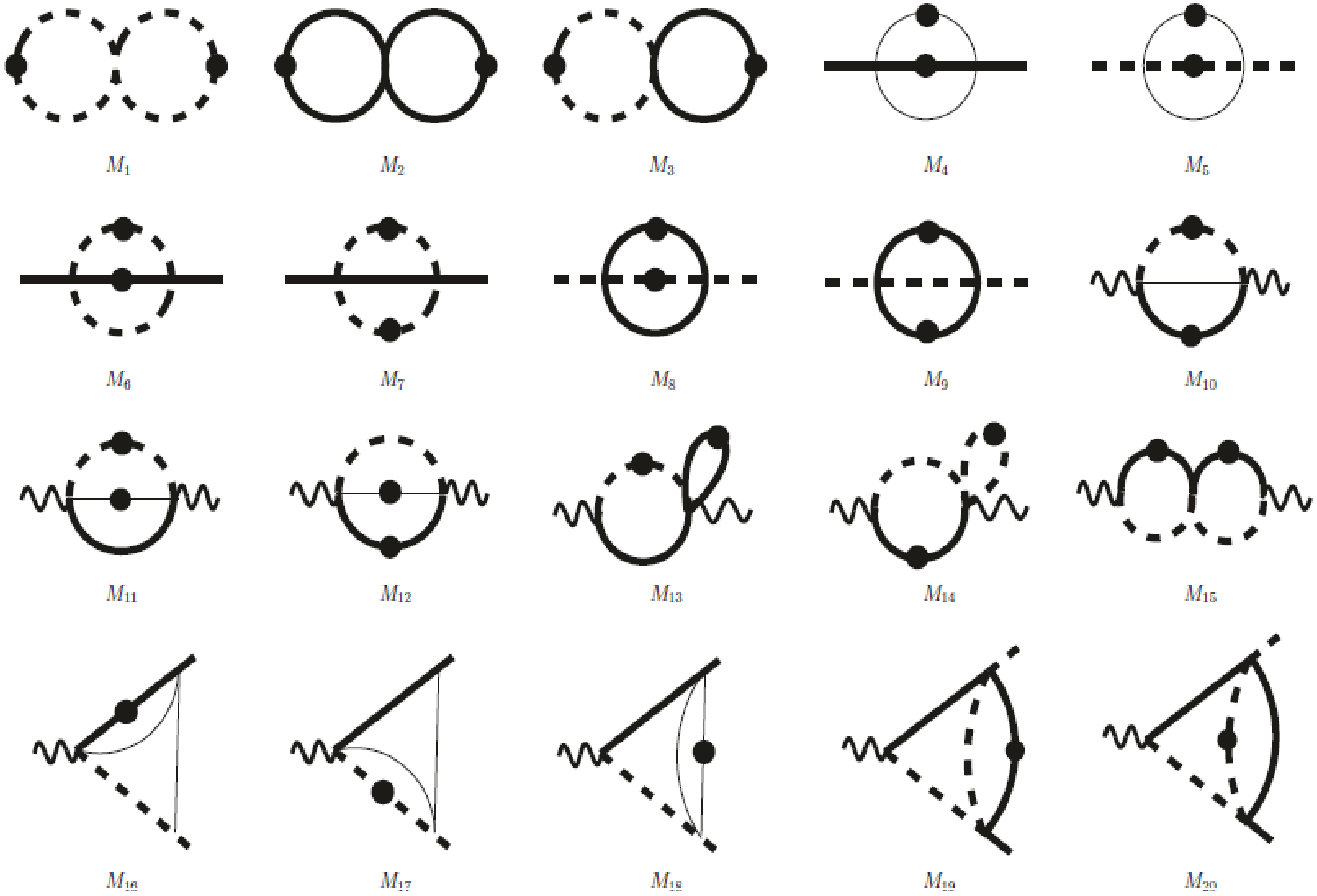
(FIRE, Reduze, Kira⋯)

$$F(a_1, a_2) = \int \frac{\mathrm{d}^d k}{(k^2)^{a_1}[(q-k)^2]^{a_2}}.$$

Example

$$\int \mathrm{d}^d k \frac{\partial}{\partial k} \cdot k \frac{1}{(k^2)^{a_1}[(q-k)^2]^{a_2}} = 0$$

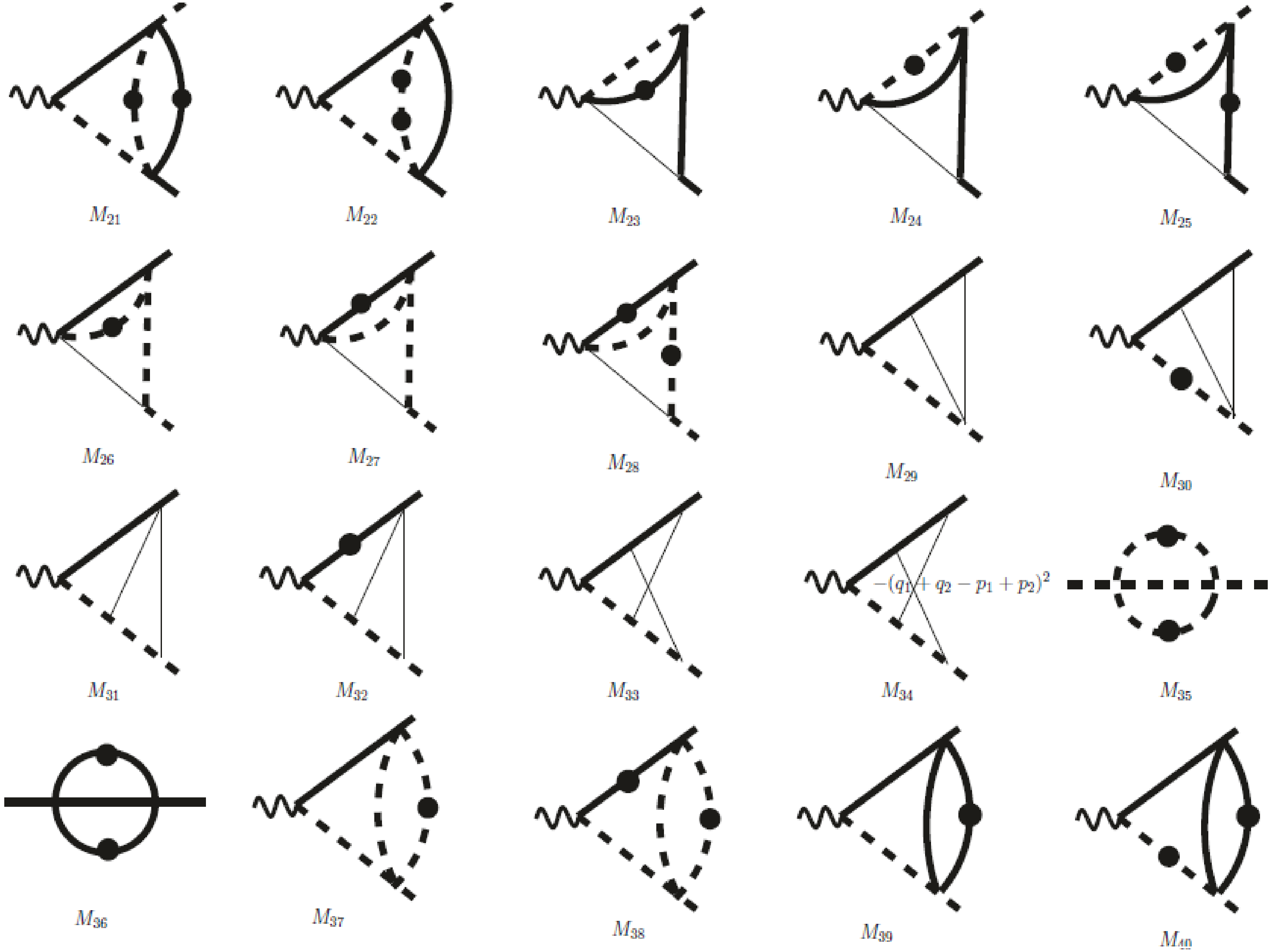
$$F(a_1, a_2) = -\frac{1}{(a_2-1)q^2}[(d-2a_1-a_2+1)F(a_1, a_2-1) - (a_2-1)F(a_1-1, a_2)].$$



40 Master Integrals

Solid: Heavy; Dash solid: Light;

Thick: massless.



Three scales: s,m1,m2

Calculations of Master Integrals

- Evaluating by Feynman Parameters

- Evaluating by Mellin-Barnes Integrals

$$\frac{1}{(X+Y)^{\lambda}} = \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \, \Gamma(\lambda+z) \Gamma(-z) \frac{Y^{z}}{X^{\lambda+z}}.$$

- Sector Decompositions (Numeric Calculations)

- ……………………

Differential Equations (DE)

$$\frac{d}{dx}\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} A_{11} & \dots & A_{1,n} \\ \vdots & & \vdots \\ A_{n1} & \dots & A_{n,n} \end{pmatrix}\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$

x are Lorentz invariant kinematics

A suitable choice of basis (canonical basis) arXiv:1304.1806

Multiloop integrals in dimensional regularization made simple
Johannes M. Henn (Princeton, Inst. Advanced Study). Apr 5, 2013. 4 pp.
Published in **Phys.Rev.Lett. 110 (2013) 251601**
DOI: 10.1103/PhysRevLett.110.251601
e-Print: **arXiv:1304.1806** **[hep-th]** | **PDF**
References | BibTeX | LaTeX(US) | LaTeX(EU) | Harvmac | EndNote
ADS Abstract Service; OSTI.gov Server
Detailed record - Cited by 297 records 250+

$$\partial_x \vec{g}(x;\epsilon) = B(x,\epsilon)\,\vec{g}(x;\epsilon)$$

$$\vec{f} = T\vec{g},$$

$$B = T^{-1}AT - T^{-1}\partial_x T$$

$$\boxed{d\,\vec{f}(x,\epsilon) = \epsilon\,\left(d\,\tilde{A}\right)\,\vec{f}(x;\epsilon)}$$

$$\tilde{A} = \left[\sum_k A_k \log \alpha_k(x)\right].$$

$$
\begin{aligned}
F_{16} &= \epsilon^3 \sqrt{s-(m_1+m_2)^2}\sqrt{s-(m_1-m_2)^2}\, M_{16}\,, \\
F_{17} &= \epsilon^3 \sqrt{s-(m_1+m_2)^2}\sqrt{s-(m_1-m_2)^2}\, M_{17}\,, \\
F_{18} &= \epsilon^3 \sqrt{s-(m_1+m_2)^2}\sqrt{s-(m_1-m_2)^2}\, M_{18}\,, \\
F_{19} &= \epsilon^3 \sqrt{s-(m_1+m_2)^2}\sqrt{s-(m_1-m_2)^2}\, M_{19}\,, \\
F_{20} &= \epsilon^3 \sqrt{s-(m_1+m_2)^2}\sqrt{s-(m_1-m_2)^2}\, M_{20}\,, \\
F_{21} &= \epsilon^2 \, \Big(2s\,(\epsilon(M_{19}+2M_{20}) - m_2^2 M_{21} - 2m_1^2 M_{22}) \\
&\quad +2(m_2^2-m_1^2)(\epsilon(M_{19}+2M_{20}) + m_2^2 M_{21} - 2m_1^2 M_{22})\Big) + 2\frac{m_2}{m_1}F_9\,, \\
F_{22} &= \epsilon^2 \, \Big(2(m_1^2-m_2^2)(2\epsilon(M_{19}+2M_{20}) + (m_1^2+m_2^2-s)M_{21} - 4m_1^2 M_{22})\Big) \\
&\quad +2\frac{m_1}{m_2}F_7 - 2\frac{m_2}{m_1}F_9\,, \\
F_{23} &= \epsilon^3 \sqrt{s-(m_1+m_2)^2}\sqrt{s-(m_1-m_2)^2}\, M_{23}\,, \\
F_{24} &= \epsilon^3 \sqrt{s-(m_1+m_2)^2}\sqrt{s-(m_1-m_2)^2}\, M_{24}\,, \\
F_{25} &= \epsilon^2 \, \frac{s\, m_2^2 (s-(m_1+m_2)^2)(s-(m_1-m_2)^2)}{(s-m_1^2+m_2^2)^2}\, M_{25} \\
&\quad +\epsilon^3 \, \frac{(s-(m_1+m_2)^2)(s-(m_1-m_2)^2)}{2(s-m_1^2+m_2^2)}\,(M_{23}-M_{24}) \\
&\quad -\frac{m_2\, s\,(s-m_1^2-m_2^2)}{m_1(s-m_1^2+m_2^2)^2}F_9 - \frac{\sqrt{s-(m_1+m_2)^2}\sqrt{s-(m_1-m_2)^2}}{4(s-m_1^2+m_2^2)}F_{11} \\
&\quad +\frac{s\, m_2^2}{(s-m_1^2+m_2^2)^2}(F_2 - F_3 - 6F_8 + 2F_{12})\,,
\end{aligned}
$$

Canonical Basis

Differential Equations In Canonical Form:

$$\mathrm{d}\mathbf{F}(x,y;\epsilon) = \epsilon\, \mathrm{d}\,\tilde{A}(x,y)\,\mathbf{F}(x,y;\epsilon)$$

$$\begin{aligned}\tilde{A}(x,y) = {} & A_1 \ln(x) + A_2 \ln(x+1) + A_3 \ln(x-1) + A_4 \ln(x+y) + A_5 \ln(x-y) \\ & + A_6 \ln(x\,y+1) + A_7 \ln(x\,y-1) + A_8 \ln(y) + A_9 \ln(y+1) + A_{10} \ln(y-1) \\ & + A_{11} \ln(x^2\,y - 2x + y) + A_{12} \ln(x^2 - 2y\,x + 1).\end{aligned} \tag{4.41}$$

A_i are rational matrices

$$s = m_1^2 \frac{(x-y)(1-x\,y)}{x}, \quad \text{and} \quad m_2 = m_1\,y.$$

Boundary Conditions

$$\frac{\partial F_{12}}{\partial x} = \epsilon \left(-\frac{F_{11} + F_{12}}{x - y} + y\,\frac{F_{11} - F_{12}}{x\,y - 1} + \frac{F_{12}}{x} \right)$$

$$F_{11}|_{x=y} = -F_{12}|_{x=y}.$$

$$F_{11}|_{x=\frac{1}{y}} = F_{12}|_{x=\frac{1}{y}}.$$

Goncharov Polylogarithms

$$G_{a_1,a_2,\ldots,a_n}(x) \equiv \int_0^x \frac{\mathrm{d}t}{t-a_1} G_{a_2,\ldots,a_n}(x)\,,$$

GINAC

$$G_{\vec{0}_n}(x) \equiv \frac{1}{n!}\log^n x\,.$$

$$G_{a_1,\ldots,a_m}(x) G_{b_1,\ldots,b_n}(x) = \sum_{c\in a \amalg b} G_{c_1,c_2,\ldots,c_{m+n}}(x)\,.$$

A. B. Goncharov, *Multiple polylogarithms, cyclotomy and modular complexes*, Math. Res. Lett. **5**, (1998) 497–516, [arXiv:1105.2076].

Results:

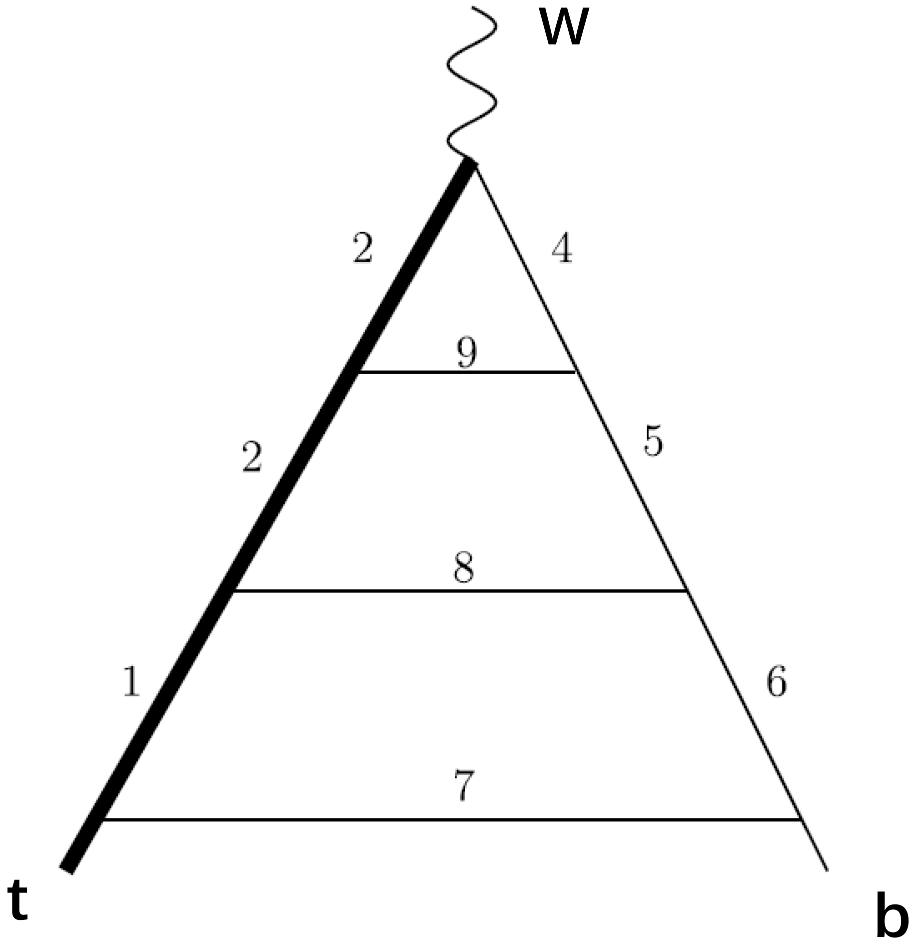
$$
\begin{aligned}
F_{10} =\; & 2\epsilon^2 \left[G_{0,0}(y) - G_{0,0}(x)\right] + \epsilon^3 \Big[\frac{G_{0,0}(y)}{2}(G_{\frac{1}{y}}(x) + G_y(x) + 4G_y(1) - 4G_{\frac{1}{y}}(1) - G_{\frac{1}{y}}(y)) \\
& + 4G_0(y)(G_{0,y}(x) - G_{0,\frac{1}{y}}(x) + G_{0,0}(x) + G_{0,\frac{1}{y}}(y) - G_{\frac{1}{y},0}(1) - G_{y,0}(1) + \frac{\pi^2}{3}) \\
& + G_0(x)(\frac{G_{0,0}(y)}{2} - 4(G_{\frac{1}{y}}(1) - G_y(1))G_0(y) - 4G_{\frac{1}{y},0}(1) - 4G_{y,0}(1) + \pi^2) - 6G_{0,0,0}(x) \\
& + 12(G_{0,-1,0}(x) + G_{0,1,0}(x) - G_{0,-1,0}(y) - G_{0,1,0}(y)) - 2(2G_{0,\frac{1}{y},0}(x) + 2G_{0,y,0}(x) \\
& + G_{\frac{1}{y},0,0}(x) + G_{y,0,0}(x)) + 2G_{\frac{1}{y},0,0}(y) + 4G_{0,\frac{1}{y},0}(y) + 6\zeta(3)\Big] + \mathcal{O}(\epsilon^4), \\
F_{33} =\; & \epsilon^3 \left[2G_{0,0,0}(x) - 2G_{0,1,0}(x) - 2G_{0,-1,0}(x) + \frac{1}{6}\pi^2 G_0(x) + \zeta(3)\right] + \mathcal{O}(\epsilon^4).
\end{aligned}
$$

Goncharov Polylogarithms

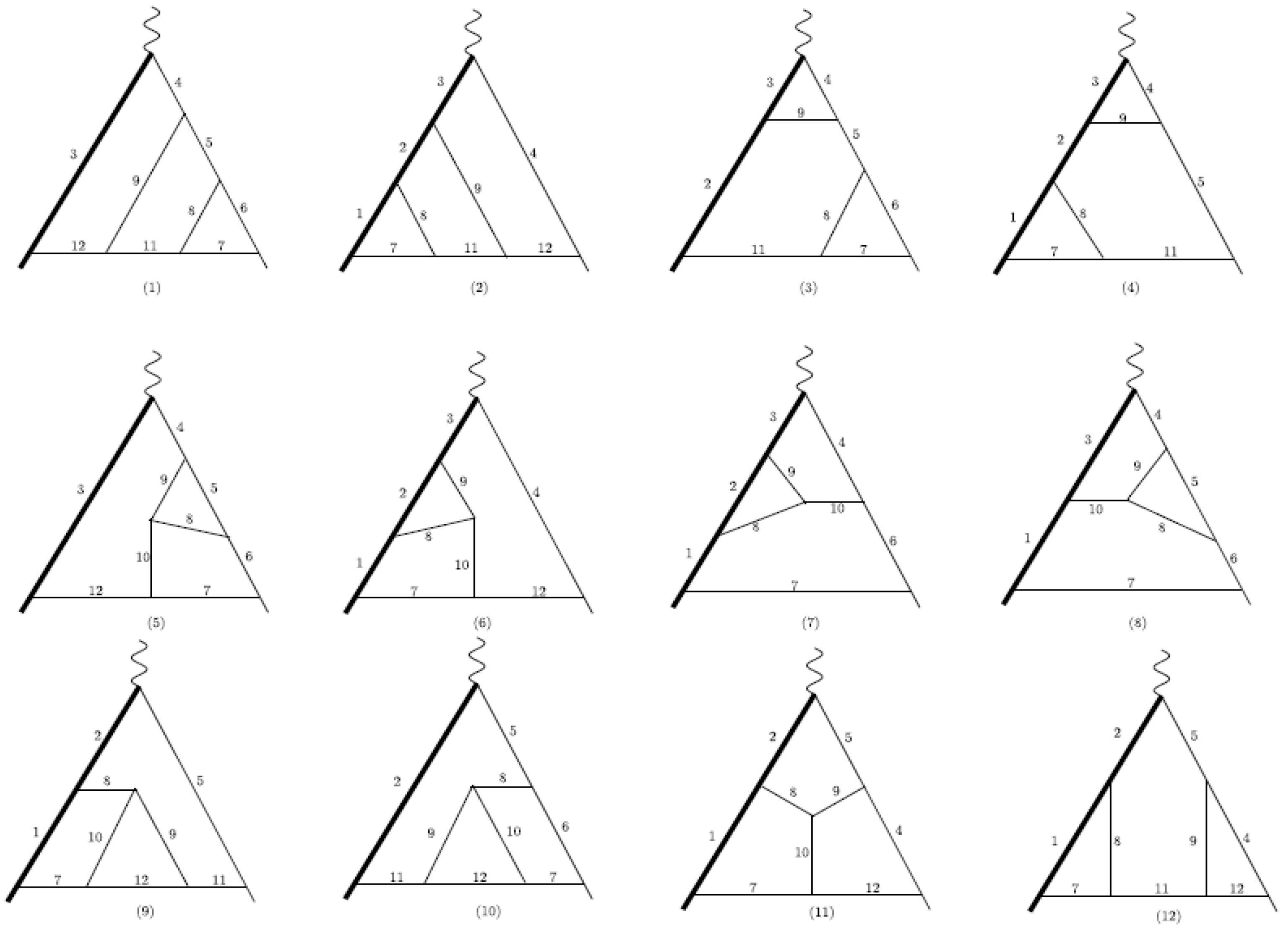
Check:

$$
\begin{aligned}
M_{33}^{\mathrm{SecDec}}(-5.4, 1.0, 0.2) &= \frac{-0.4466129 \pm 0.0000004}{\epsilon} - 0.507366 \pm 0.000006, \\
M_{33}^{\mathrm{FIESTA}}(-5.4, 1.0, 0.2) &= \frac{-0.446613 \pm 0.000005}{\epsilon} - 0.507387 \pm 0.000049, \\
M_{33}^{\mathrm{Ours}}(-5.4, 1.0, 0.2) &= \frac{-0.4466129967\ldots}{\epsilon} - 0.5073683817\ldots.
\end{aligned}
$$

2. Three-Loop Heavy-to-light Form factors



Color-planar diagrams (Leading Color Contribution)



Integrals can be parameterized by

d=4-2ϵ

$$
I_{n_1,n_2,\ldots,n_{12}} = \int \frac{\mathcal{D}^d k_1\, \mathcal{D}^d k_2\, \mathcal{D}^d k_3}{D_1^{n_1}\, D_2^{n_2}\, D_3^{n_3}\, D_4^{n_4}\, D_5^{n_5}\, D_6^{n_6}\, D_7^{n_7}\, D_8^{n_8}\, D_9^{n_9}\, D_{10}^{n_{10}}\, D_{11}^{n_{11}}\, D_{12}^{n_{12}}}
$$

$$
\begin{aligned}
&D_1 = -(k_1+p_1)^2+m^2,\ D_2 = -(k_2+p_1)^2+m^2,\\
&D_3 = -(k_3+p_1)^2+m^2,\ D_4 = -(k_3+p_2)^2,\\
&D_5 = -(k_2+p_2)^2,\ D_6 = -(k_1+p_2)^2,\ D_7 = -k_1^2,\\
&D_8 = -(k_1-k_2)^2,\ D_9 = -(k_2-k_3)^2,\ D_{10} = -(k_1-k_3)^2,\\
&D_{11} = -k_2^2,\ D_{12} = -k_3^2,
\end{aligned}
$$

All integrals can be reduced to 71 Master Integrals

$$\boxed{\frac{\partial}{\partial s} = \frac{1}{s-m^2} p_2 \cdot \frac{\partial}{\partial p_2}}$$

Canonical Basis:

$$\begin{aligned}
F_1 &= m^6\, I_{3,3,3,0,0,0,0,0,0,0,0,0}\,, \\
F_2 &= \epsilon^2 m^4\, I_{0,2,3,0,0,0,1,2,0,0,0,0}\,, \\
F_3 &= \epsilon^3 m^2\, I_{0,0,2,0,0,0,2,2,1,0,0,0}\,, \\
F_4 &= (\epsilon-1)(1+4\epsilon)\epsilon\, m^2\, I_{2,0,2,0,0,0,0,2,1,0,0,0}\,, \\
F_5 &= \epsilon\, s\, m^4\, I_{3,3,2,1,0,0,0,0,0,0,0,0}\,, \\
F_6 &= \epsilon^3 s\, I_{2,0,0,2,0,0,0,2,1,0,0,0}\,, \\
F_7 &= \epsilon^2 m^2\, (2\epsilon\, I_{2,0,0,2,0,0,0,2,1,0,0,0} \\
&\quad + (s-m^2)\, I_{3,0,0,2,0,0,0,2,1,0,0,0})\,, \\
&\quad \cdots\cdots
\end{aligned}$$

$$
\begin{aligned}
F_{67} &= \epsilon^5\,(s-m^2)\,I_{1,1,0,1,1,-1,1,1,0,2,0,0}\,,\\
F_{68} &= \epsilon^5\,(s-m^2)^2\,I_{1,1,0,1,1,0,1,1,0,2,0,0}\,,\\
F_{69} &= \epsilon^6\,(s-m^2)\,I_{1,1,0,1,0,0,1,1,1,0,0,1}\,,\\
F_{70} &= \epsilon^6\,(s-m^2)^2\,I_{1,1,0,1,1,0,1,1,1,1,-1,1}\,,\\
F_{71} &= \epsilon^6\,(s-m^2)\,I_{1,1,0,1,1,-1,1,1,1,1,-1,1}\\
&\quad+\frac{1}{12(1-2\epsilon)}(12F_2+6F_3+3F_4-2F_7+6F_9\\
&\quad-18F_{14}+2F_{24}+12F_{25})\,.
\end{aligned}
$$

$$\frac{\partial \mathbf{F}(x,\epsilon)}{\partial x} = \epsilon \left(\frac{\mathbf{P}}{x} + \frac{\mathbf{Q}}{x-1} \right) \mathbf{F}(x,\epsilon).$$

P and Q are 71*71
rational matrices

DE for non-canonical basis

$$\frac{1}{s-m^2}\Bigg(\frac{(2d-7)(5d-18)\left(7m^2+5s\right)G(1,\{1,0,0,0,1,0,0,1,1,1,0,1\})(d-4)^2}{(d-5)(d-3)\left(m^2-s\right)s\left(19dm^2-72m^2-3ds+12s\right)}+$$

$$\frac{G(1,\{1,0,1,1,1,0,1,1,1,0,0,0\})(d-4)^2}{2(d-3)s}+\frac{6(d-6)m^2\left(9dm^2-33m^2+23ds-87s\right)G(1,\{0,1,0,1,0,0,1,1,1,2,0,0\})(d-4)}{(d-5)(d-3)\left(m^2-s\right)s\left(19dm^2-72m^2-3ds+12s\right)}-$$

$$\Big((2d-7)\left(88d^2m^4-663dm^4+1242m^4+9d^2sm^2-73dsm^2+150sm^2-d^2s^2+16ds^2-48s^2\right)$$
$$G(1,\{1,0,1,0,1,0,1,1,1,0,0,0\})(d-4)\Big)\Big/\left(3(d-6)(d-3)\left(m^2-s\right)^2s\left(dm^2-3m^2-2ds+7s\right)\right)+$$

$$\Big(\big(167d^3m^6-1858d^2m^6+6864dm^6-8424m^6+111d^3sm^4-1256d^2sm^4+4762dsm^4-6036sm^4+$$
$$232d^3s^2m^2-2563d^2s^2m^2+9388ds^2m^2-11412s^2m^2-30d^3s^3+349d^2s^3-1334ds^3+1680s^3\big)$$
$$G(1,\{1,0,1,0,1,0,1,1,2,0,0,0\})(d-4)\Big)\Big/\left(3(d-6)(d-3)(3d-10)\left(m^2-s\right)^2s\left(dm^2-3m^2-2ds+7s\right)\right)+$$

$$\frac{3\left(5dm^4-18m^4+22dsm^2-84sm^2+5ds^2-18s^2\right)G(1,\{1,1,0,1,0,0,1,1,2,0,0,0\})(d-4)}{2(d-6)(d-3)\left(m^2-s\right)^2s}-$$

$$\frac{\left(dm^2-6m^2+5ds-24s\right)G(1,\{1,1,0,1,1,0,1,0,1,2,0,0\})(d-4)}{(d-6)(d-3)s}-$$

$$\frac{3G(1,\{1,1,0,1,1,0,1,1,0,1,0,0\})(d-4)}{2s}+\frac{5\left(m^2+s\right)G(1,\{1,1,0,1,1,0,1,1,0,2,0,0\})(d-4)}{2(d-3)s}+$$

$$\Big(2(d-3)^2\big(85d^3m^6-879d^2m^6+3074dm^6-3648m^6-1149d^3sm^4+13999d^2sm^4-56198dsm^4+74472sm^4-517d^3s^2m^2+$$
$$6775d^2s^2m^2-28786ds^2m^2+39936s^2m^2+45d^3s^3-567d^2s^3+2358ds^3-3240s^3\big)G(1,\{0,1,0,0,0,1,0,1,1,0,0,1\})\Big)\Big/$$
$$\left((d-6)(d-5)(5d-18)m^2\left(m^2-s\right)^3s\left(19dm^2-72m^2-3ds+12s\right)\right)-$$

$$\Big(\big(85d^4m^8-1219d^3m^8+6590d^2m^8-15944dm^8+14592m^8-6024d^4sm^6+96352d^3sm^6-572876d^2sm^6+1502664dsm^6-$$
$$1468704sm^6-13186d^4s^2m^4+214990d^3s^2m^4-1299328d^2s^2m^4+3456840ds^2m^4-3421152s^2m^4+648d^4s^3m^2-$$
$$9856d^3s^3m^2+55932d^2s^3m^2-140360ds^3m^2+131424s^3m^2+45d^4s^4-747d^3s^4+4626d^2s^4-12672ds^4+12960s^4\big)$$
$$G(1,\{0,1,0,1,0,0,1,1,2,0,0,0\})\Big)\Big/\left(2(d-6)(d-5)(5d-18)m^2\left(m^2-s\right)^3s\left(19dm^2-72m^2-3ds+12s\right)\right)-$$

$$G(1, \{0, 1, 0, 1, 0, 0, 1, 1, 2, 0, 0, 0\}) \big/ \left(2 (d-6)(d-5)(5 d-18) m^2 \left(m^2-s\right)^3 s \left(19 d m^2-72 m^2-3 d s+12 s\right)\right) -$$

$$\big(2 \big(15\,136 d^6 m^8-351\,926 d^5 m^8+3\,397\,098 d^4 m^8-17\,429\,796 d^3 m^8+50\,143\,848 d^2 m^8-76\,708\,008 d m^8+48\,755\,520 m^8- 27\,710 d^6 s m^6+617\,510 d^5 s m^6-5\,702\,807 d^4 s m^6+27\,953\,891 d^3 s m^6-76\,744\,006 d^2 s m^6+111\,931\,368 d s m^6- 67\,780\,800 s m^6+11\,804 d^6 s^2 m^4-259\,522 d^5 s^2 m^4+2\,380\,136 d^4 s^2 m^4-11\,644\,177 d^3 s^2 m^4+32\,021\,430 d^2 s^2 m^4- 46\,895\,544 d s^2 m^4+28\,553\,760 s^2 m^4+44\,978 d^6 s^3 m^2-1\,032\,122 d^5 s^3 m^2+9\,785\,211 d^4 s^3 m^2-49\,119\,463 d^3 s^3 m^2+ 137\,821\,510 d^2 s^3 m^2-205\,099\,608 d s^3 m^2+126\,547\,200 s^3 m^2-2688 d^6 s^4+59\,496 d^5 s^4-546\,258 d^4 s^4+ 2\,663\,577 d^3 s^4-7\,276\,014 d^2 s^4+10\,558\,992 d s^4-6\,360\,480 s^4\big) G(1, \{1, 0, 0, 0, 1, 0, 0, 1, 1, 0, 0, 2\})\big) \big/$$

$$\left(3 (d-6)(d-5)(d-3)(3 d-10) \left(m^2-s\right)^3 s \left(19 d m^2-72 m^2-3 d s+12 s\right) \left(d m^2-3 m^2-2 d s+7 s\right)\right) +$$

$$\big(\big(633 d^4 m^6-9198 d^3 m^6+50\,083 d^2 m^6-121\,116 d m^6+109\,764 m^6+816 d^4 s m^4-12\,011 d^3 s m^4+66\,179 d^2 s m^4- 161\,814 d s m^4+148\,176 s m^4+249 d^4 s^2 m^2-3600 d^3 s^2 m^2+19\,597 d^2 s^2 m^2-47\,580 d s^2 m^2+ 43\,452 s^2 m^2+30 d^4 s^3-439 d^3 s^3+2381 d^2 s^3-5682 d s^3+5040 s^3\big) G(1, \{1, 0, 1, 0, 1, 0, 1, 2, 1, 0, 0, 0\})\big) \big/$$

$$\left(3 (d-6)(d-3)(3 d-10) \left(m^2-s\right)^2 s \left(d m^2-3 m^2-2 d s+7 s\right)\right) - 1 \big/ \left(2 (d-6)(d-3) \left(m^2-s\right)^2 s \left(m^2+s\right)\right)$$

$$\left(3 d^2 m^6-30 d m^6+72 m^6-4 d^2 s m^4+50 d s m^4-132 s m^4+87 d^2 s^2 m^2-676 d s^2 m^2+1308 s^2 m^2+10 d^2 s^3-88 d s^3+192 s^3\right)$$

$$G(1, \{1, 0, 1, 1, 0, 0, 1, 1, 2, 0, 0, 0\}) - \frac{3 (3 d-10)^2 \left(m^2+s\right) G(1, \{1, 1, 0, 1, 0, 0, 1, 1, 1, 0, 0, 0\})}{2 (d-6) \left(m^2-s\right)^2 s} -$$

$$\big(4 (d-3)^2 (2 d-7)(3 d-10) \big(34 d^2 m^6-294 d m^6+624 m^6+61 d^2 s m^4-421 d s m^4+720 s m^4-156 d^2 s^2 m^2+ 1184 d s^2 m^2-2244 s^2 m^2-3 d^2 s^3+27 d s^3-60 s^3\big) G(1, \{0, 1, 0, 1, 0, 0, 1, 1, 0, 1, 0, 0\})\big) \big/$$

$$\left((d-5)(5 d-18)(5 d-16) m^2 \left(m^2-s\right)^3 s \left(19 d m^2-72 m^2-3 d s+12 s\right)(d-4)\right) +$$

$$\big((d-3) \left(3 d^3 m^4-38 d^2 m^4+157 d m^4-210 m^4+20 d^3 s m^2-324 d^2 s m^2+1648 d s m^2-2688 s m^2+d^3 s^2-30 d^2 s^2+211 d s^2-430 s^2\right)$$

$$G(1, \{0, 1, 0, 1, 0, 1, 0, 1, 1, 0, 0, 0\})\big) \big/ \left(2 (d-5)(3 d-14) m^2 \left(m^2-s\right)^3 s (d-4)\right) -$$

$$\big(\big(3 d^4 m^6-50 d^3 m^6+309 d^2 m^6-838 d m^6+840 m^6+79 d^4 s m^4-1466 d^3 s m^4+9925 d^2 s m^4-29\,250 d s m^4+31\,808 s m^4+ 61 d^4 s^2 m^2-1330 d^3 s^2 m^2+10\,155 d^2 s^2 m^2-32\,958 d s^2 m^2+38\,848 s^2 m^2+d^4 s^3-34 d^3 s^3+331 d^2 s^3-1274 d s^3+1720 s^3\big)$$

$$G(1, \{0, 1, 0, 1, 0, 1, 0, 1, 2, 0, 0, 0\})\big) \big/ \left(4 (d-5)(d-3)(3 d-14) m^2 \left(m^2-s\right)^3 s (d-4)\right) -$$

Boundary Conditions

Known

$$F_1 = \frac{1}{8},$$

$$F_2 = \frac{1}{8} + \epsilon^2 \frac{\pi^2}{12} + \epsilon^3 \zeta(3) + \epsilon^4 \frac{4\pi^4}{45} + 2\epsilon^5 \frac{27\zeta(5) + \pi^2\zeta(3)}{3} + \epsilon^6 \left(\frac{229\pi^6}{1890} + 4\zeta^2(3) \right) + \mathcal{O}(\epsilon^7),$$

$x \equiv \frac{s}{m^2}.$

$$\frac{\partial F_{38}}{\partial x} = \epsilon \left(\frac{-4F_2 - F_3 + 6F_{11} + 2F_{19} - 6(3F_{38} - 2F_{39})}{6x} + \frac{2F_{38}}{x-1} \right),$$

Regular at x=0

$$-6(3F_{38} - 2F_{39})|_{x=0} = (-4F_2 - F_3 + 6F_{11} + 2F_{19})|_{x=0}$$

$$
\boxed{\begin{aligned}
F_{38} &= \epsilon^5 \,(s-m^2)\, I_{0,1,1,1,0,0,1,2,1,0,0,0}\,,\\
F_{39} &= \epsilon^4\, m^2\, (s-m^2)\, I_{0,1,2,1,0,0,1,2,1,0,0,0}\,,
\end{aligned}}
$$

$$
\begin{aligned}
\frac{\partial F_{38}}{\partial x} &= \epsilon\left(\frac{-4F_2 - F_3 + 6F_{11} + 2F_{19} - 6(3F_{38} - 2F_{39})}{6x} + \frac{2F_{38}}{x-1}\right),\\
\frac{\partial F_{39}}{\partial x} &= \epsilon\left(\frac{-20F_2 + 4F_3 + 3F_{11} - 12F_{16} + 30F_{18} + 16F_{19} - 30(3F_{38} - 2F_{39})}{12x}\right.\\
&\quad \left. -2\frac{3F_{39} - 4F_{38}}{x-1}\right).
\end{aligned}
$$

$$
\begin{aligned}
-6(3F_{38} - 2F_{39})|_{x=0} &= (-4F_2 - F_3 + 6F_{11} + 2F_{19})|_{x=0}\,,\\
-30(3F_{38} - 2F_{39})|_{x=0} &= (-20F_2 + 4F_3 + 3F_{11} - 12F_{16} + 30F_{18} + 16F_{19})|_{x=0}\,.
\end{aligned}
$$

$$
\begin{aligned}
F_{37}|_{x=0} &= -\frac{11}{240} - \epsilon^2 \frac{\pi^2}{180} + \epsilon^3 \frac{3\zeta(3)}{4} + \epsilon^4 \frac{379\pi^4}{5400} \\
&\quad - \epsilon^5 \left(\frac{4\pi^2\zeta(3)}{3} - \frac{1107\zeta(5)}{20} \right) \\
&\quad + \epsilon^6 \left(\frac{901\pi^6}{4725} + \frac{21\zeta^2(3)}{2} \right) + \mathcal{O}(\epsilon^7)\,, \\
F_{39}|_{x=0} &= -\epsilon^2 \frac{\pi^2}{36} + \epsilon^4 \frac{79\pi^4}{1080} + \epsilon^5 \left(\frac{143\pi^2\zeta(3)}{18} + \frac{5\zeta(5)}{2} \right) \\
&\quad + \epsilon^6 \left(\frac{18737\pi^6}{22680} + 48\zeta^2(3) \right) + \mathcal{O}(\epsilon^7)\,, \\
F_{41}|_{x=0} &= \frac{7}{180} - \epsilon^2 \frac{7\pi^2}{270} - \epsilon^3 \frac{89\zeta(3)}{45} - \epsilon^4 \frac{139\pi^4}{900} \\
&\quad - \epsilon^5 \frac{353\pi^2\zeta(3) + 8469\zeta(5)}{135} \\
&\quad - \epsilon^6 \left(\frac{92077\pi^6}{170100} + \frac{2503\zeta^2(3)}{45} \right) + \mathcal{O}(\epsilon^7)\,,
\end{aligned}
$$

$$
\begin{aligned}
F_{71} \;=\;& \epsilon^4\left(H_{0,1,0,1}(x) - H_{0,0,1,1}(x) + \frac{\pi^2}{6}H_{0,1}(x) - \frac{\pi^4}{30}\right) \\
+\;& \epsilon^5\Bigg(-2H_{0,0,0,0,1}(x) - 2H_{0,0,0,1,1}(x) - 2H_{0,0,1,0,1}(x) - 10H_{0,0,1,1,1}(x) \\
+\;& 2H_{0,1,0,1,1}(x) + 6H_{0,1,1,0,1}(x) - \frac{\pi^2}{6}H_{0,0,1}(x) + \pi^2 H_{0,1,1}(x) + 2\zeta(3)H_{0,1}(x) \\
-\;& \frac{7\pi^2\zeta(3)}{6} - \zeta(5)\Bigg) \\
+\;& \epsilon^6\Bigg(-\left(2\zeta(5) + \frac{\pi^2\zeta(3)}{3}\right)H_1(x) + \frac{9\pi^4}{40}H_{0,1}(x) \\
+\;& \zeta(3)(-13H_{0,0,1}(x) + 9H_{0,1,1}(x) - 2H_{1,0,1}(x)) - \pi^2\Bigg(-H_{0,0,0,1}(x) - \frac{5}{6}H_{0,0,1,1}(x) \\
+\;& H_{0,1,0,1}(x) + 6H_{0,1,1,1}(x) + \frac{1}{3}H_{1,0,0,1}(x)\Bigg) - 11H_{0,0,0,0,0,1}(x) - 11H_{0,0,0,0,1,1}(x) \\
-\;& 20H_{0,0,0,1,0,1}(x) - 20H_{0,0,0,1,1,1}(x) - 16H_{0,0,1,0,0,1}(x) - 26H_{0,0,1,0,1,1,}(x) \\
-\;& 29H_{0,0,1,1,0,1}(x) - 76H_{0,0,1,1,1,1}(x) - 14H_{0,1,0,0,0,1}(x) - 12H_{0,1,0,0,1,1}(x) \\
+\;& 2H_{0,1,0,1,0,1}(x) - 4H_{0,1,0,1,1,1}(x) + 3H_{0,1,1,0,0,1}(x) + 12H_{0,1,1,0,1,1}(x) \\
+\;& 36H_{0,1,1,1,0,1}(x) + 4H_{1,0,0,0,0,1}(x) + 2H_{1,0,0,1,0,1}(x) + 2H_{1,0,1,0,0,1}(x) \\
-\;& \frac{1219\pi^6}{15120}\Bigg) + \mathcal{O}(\epsilon^7), \qquad (14)
\end{aligned}
$$

H are Harmonic Polylogarithms

Check: (s=-1.3,m=1.0)

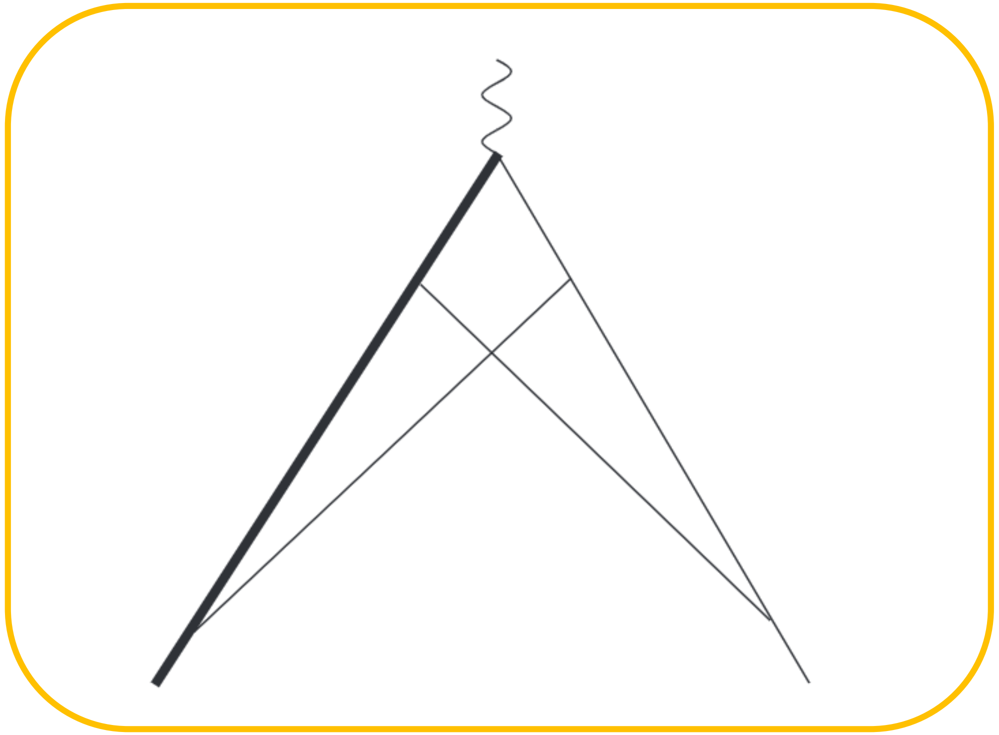
$$I^{\text{analytic}}_{1,1,0,1,1,0,1,1,1,1,-1,1} = \frac{0.00078765}{\epsilon^6} - \frac{0.00393624}{\epsilon^5} + \frac{0.0190587}{\epsilon^4} - \frac{0.0151068}{\epsilon^3} + \frac{0.290244}{\epsilon^2} + \frac{1.37654}{\epsilon} + 4.82542,$$

$$I^{\text{analytic}}_{1,1,0,1,1,-1,1,1,1,1,-1,1} = \frac{-6.69426}{\epsilon} - 63.1207.$$

$$I^{\text{numeric}}_{1,1,0,1,1,0,1,1,1,1,-1,1} = \frac{0.000788}{\epsilon^6} - \frac{0.003936}{\epsilon^5} + \frac{0.019058 \pm 0.000002}{\epsilon^4} - \frac{0.015109 \pm 0.000035}{\epsilon^3} + \frac{0.290192 \pm 0.000756}{\epsilon^2} + \frac{1.37606 \pm 0.01581}{\epsilon} + 4.80886 \pm 0.31758,$$

$$I^{\text{numeric}}_{1,1,0,1,1,-1,1,1,1,1,-1,1} = \frac{-6.69429 \pm 0.00003}{\epsilon} - 63.1213 \pm 0.0004,$$

FIESTA packages



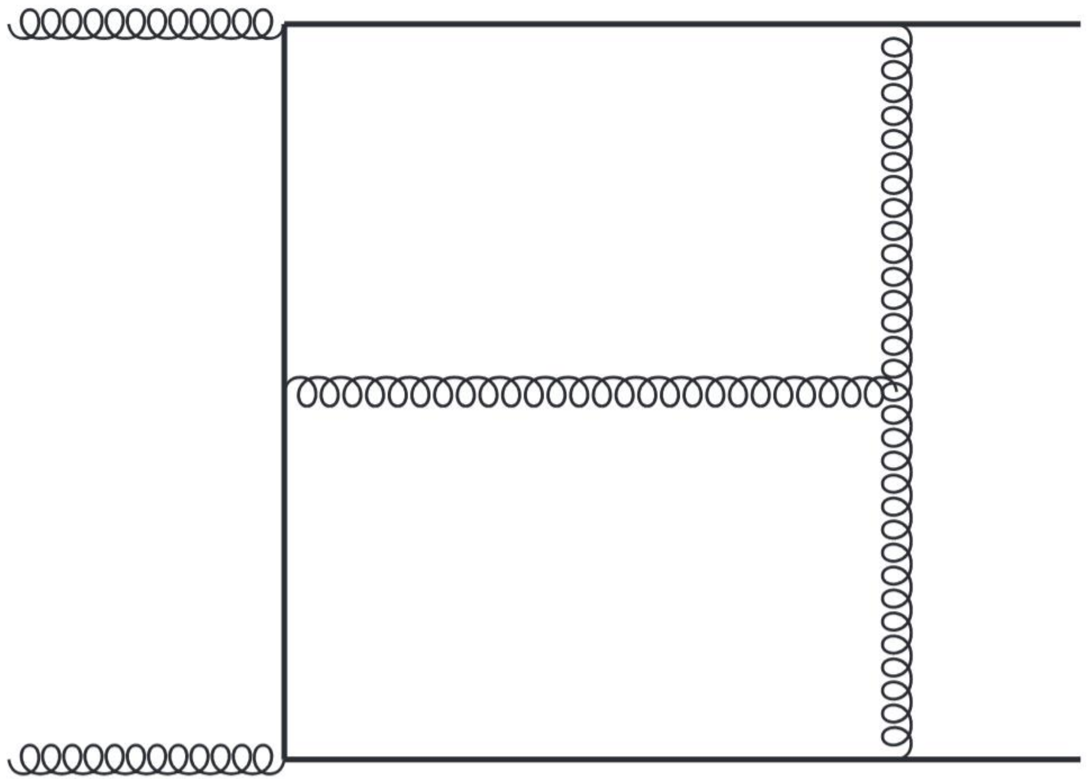
$$J_{n_1,\ldots,n_7} = \int \frac{\mathcal{D}^d k_1 \, \mathcal{D}^d k_2}{[-(k_1+p_1)^2+m^2]^{n_1}[-(k_2+p_1)^2+m^2]^{n_2}[-k_1^2]^{n_3}} \times \frac{1}{[-(k_2+p_2)^2]^{n_4}[-(k_1-k_2)^2]^{n_5}[-(k_2-k_1+p_2)^2]^{n_6}[-k_2^2]^{n_7}}.$$

$$\frac{\partial \mathbf{K}(y,\epsilon)}{\partial y} = \epsilon\left(\frac{\mathbf{L}}{y} + \frac{\mathbf{M}}{y-1} + \frac{\mathbf{N}}{y+1}\right)\mathbf{K}(y,\epsilon)$$

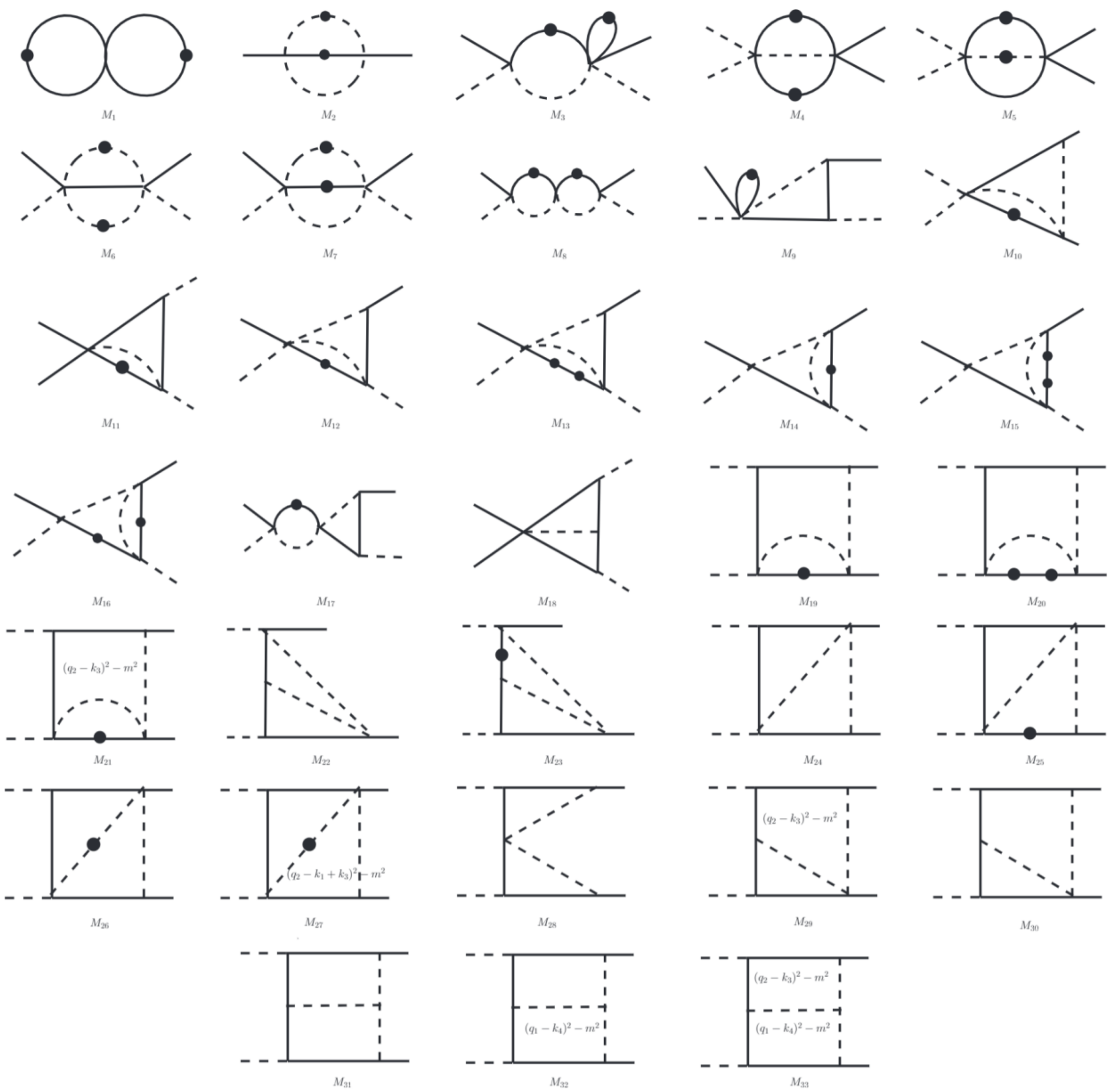
$$
\begin{aligned}
K_1 &= \epsilon^2\, J_{2,2,0,0,0,0,0}\,,\\
K_2 &= \epsilon^2\, m^2\, J_{0,2,2,0,1,0,0}\,,\\
K_3 &= \epsilon^2\, s\, J_{2,2,0,1,0,0,0}\,,\\
K_4 &= \epsilon^2\, s\, J_{0,2,2,0,0,1,0}\,,\\
K_5 &= \epsilon^2\,(s-m^2)\, J_{0,1,2,0,0,2,0} - \frac{2m^2}{s} K_4\,,\\
K_6 &= \epsilon^3\,(s-m^2)\, J_{0,1,1,1,2,0,0}\,,\\
K_7 &= \epsilon^3\,(s-m^2)\, J_{1,2,1,0,0,1,0}\,,\\
K_8 &= \epsilon^2\,\frac{m^2\, s\,(s-m^2)}{s+m^2}\, J_{2,2,1,0,0,1,0} - \frac{m^2}{2(s+m^2)}(K_1 - 4K_4 + K_5)\,,\\
K_9 &= \epsilon^3\,(s-m^2)\, J_{2,1,0,1,0,1,0}\,,\\
K_{10} &= \epsilon^2\, m^2\,(s-m^2)\, J_{3,1,0,1,0,1,0}\,,\\
K_{11} &= \epsilon^2\, m^2(s-2m^2) J_{2,2,0,1,0,1,0} + \frac{(s-2m^2)}{s-m^2}(2K_{10} - 3K_9)\,,\\
K_{12} &= \epsilon^4\,(s-m^2)\, J_{1,1,1,1,1,0,0}\,,\\
K_{13} &= \epsilon^3\,(s-m^2)^2\, J_{1,1,1,2,1,0,0}\,,\\
K_{14} &= \epsilon^4\,(s-m^2)\, J_{1,1,1,1,1,1,-1}\,.
\end{aligned}
$$

K_{14}=0 at s=0.

3. A planar double box for top pair Hadron Production in d-log form



33 Master Integrals



Four squared Roots

$$\sqrt{s}, \quad \sqrt{s-4m^2}, \quad \sqrt{t-m^2}, \quad \sqrt{t(s-m^2)^2-(s^2-6sm^2+m^4)m^2}.$$

Rationalize

$$y=-\frac{\sqrt{s}-\sqrt{s-4m^2}}{\sqrt{s}+\sqrt{s-4m^2}}, \qquad x=(y^2-y+1)\frac{\sqrt{t-m^2\frac{s^2-6s\,m^2+m^4}{(s-m^2)^2}}}{\sqrt{t-m^2}}.$$

$$\boxed{d\,\mathbf{F}(x,y;\epsilon) = \epsilon\,(d\,\tilde{A})\,\mathbf{F}(x,y;\epsilon),}$$

$$d\,\tilde{A} = \sum_{i=1}^{13} \mathrm{R}_i\, d\log(l_i)$$

$$l_1 = x - (y^2 + y - 1)\,, \qquad l_2 = x + (y^2 + y - 1)\,,$$
$$l_3 = x - (y^2 - y - 1)\,, \qquad l_4 = x + (y^2 - y - 1)\,,$$
$$l_5 = x - (y^2 - y + 1)\,, \qquad l_6 = x + (y^2 - y + 1)\,,$$
$$l_7 = x - (y^2 - 3y + 1)\,, \qquad l_8 = x + (y^2 - 3y + 1)\,,$$
$$l_9 = x^2 - [(y-3)y + 1]\left(y^2 + y + 1\right)\,, \qquad l_{10} = x^2 - (y-1)y[y(y+3) - 2] - 1\,,$$
$$l_{11} = y\,, \qquad l_{12} = y + 1\,,$$
$$l_{13} = y - 1\,.$$

Results:

$$
\begin{aligned}
\mathrm{F}_{31} &= \epsilon^3[2G_{0,0,0}(y) + \frac{\pi^2}{3}G_0(y)] + \epsilon^4[-\frac{5\pi^4}{18} + \frac{\pi^2}{3}(2G_{1,0}(y) + 2G_{-1,0}(y) - 7G_{0,0}(y)) \\
&+ 4G_{-1,0,0,0}(y) - 12G_{0,0,-1,0}(y) - 4G_{0,0,0,0}(y) + 4G_{0,0,1,0}(y) + 8G_{0,1,0,0}(y) \\
&+ 4G_{1,0,0,0}(y) - 8G_{0,0,0}(y)G_1(z) + \frac{2}{3}G_0(y)(\pi^2(3G_0(z_1) - 2G_{\frac{1}{z}}(z_1) - 2G_1(z)) \\
&- 6(G_{\frac{1}{z},0,0}(z_1) - G_{\frac{1}{z},1,0}(z_1) + G_{z,1,0}(z_1) + G_{1,0,1}(z) + G_{\frac{1}{z},1,0}(1) - G_{z,1,0}(1) \\
&+ 2G_{0,1,0}(z_1) + 3G_{1,0,0}(z_1) - G_{0,0,1}(z) - 3G_{0,0,0}(z_1)) - 9\zeta(3))] + \mathcal{O}(\epsilon^5), \qquad (18)
\end{aligned}
$$

Thanks!