The NNLO soft function for top-quark pair production

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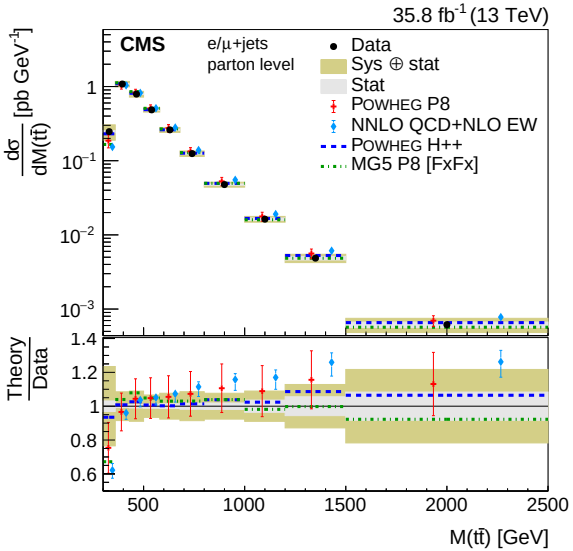
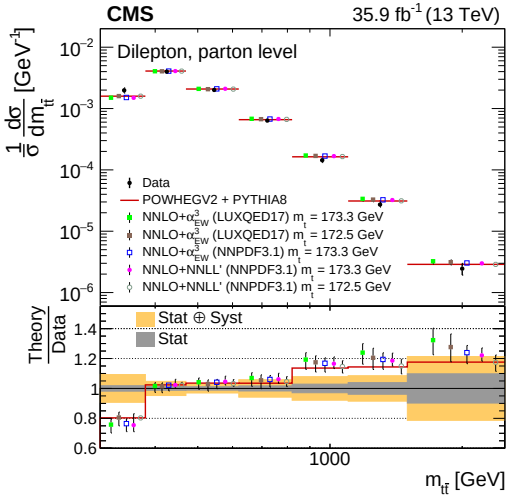
December 30, 2018

Outline

Top quark

- The heaviest particle in SM
  - Perturbative QCD and EWSB;
  - Large background to many processes ($t\bar{t} + X(\mathrm{Higgs}, W, Z, \gamma)$);
  - New physics(heavy particles, $t\bar{t} + \mathrm{DM}, \cdots$);
- Experiment*
  - $\sigma_{t\bar{t}}$: 2014($\sqrt{s}$=8 TeV): 3.5%, RUN 2(2015): 4.4% @LHC;
  - Differential cross section

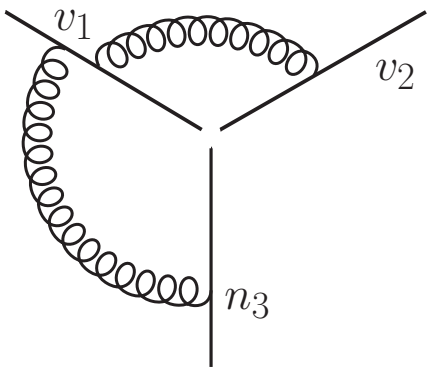




* Ulrich Husemann :1704.01356, CMS. :1803.08856, 1811.06625

Massive soft function

- Phenomenology
  - Large logarithm $\log(\omega/\mu_s), \log(m_t/M_{t\bar{t}})$;
  - Major bottleneck in extending threshold resummation to $\mathrm{N^3LL}$.
- Theoretical interests
  - Non-trivial correlation among three partons;
  - Cancellation of soft singularities with three parton correlation.



Example for three-parton-correlation diagram

Factorization

- Kinematics of $N(P_1) + N(P_2) \to t(p_3) + \bar{t}(p_4) + X_s(p_s)$:

$$s = (P_1 + P_2)^2\,, \quad \hat{s} = (p_1 + p_2)^2\,, \quad M^2 = (p_3 + p_4)^2\,,$$
$$z = \frac{M^2}{\hat{s}}\,, \quad \tau = \frac{M^2}{s}\,, \quad \beta = \sqrt{1 - \frac{4m_t^2}{M^2}}\,, \quad t_1 = -\frac{M^2}{2}(1 - \beta\cos\theta)\,.$$

- QCD factorization theorm*:

$$\frac{d^2\sigma}{dM\,d\cos\theta} = \frac{8\pi\beta}{3sM}\sum_{i,j}\int_\tau^1 \frac{dz}{z}\, ff_{ij}(\tau/z, \mu_f)\, C_{ij}(z, M, m_t, \cos\theta, \mu_f)\,,$$

- Threshold limit $z \to 1^\dagger$:

$$C_{ij} = \mathrm{Tr}\left[\boldsymbol{H}_{ij}(M, m_t, \cos\theta, \mu_f)\, \boldsymbol{S}_{ij}(\sqrt{\hat{s}}(1-z), M, m_t, \cos\theta, \mu_f)\right] + \mathcal{O}(1-z)\,. \tag{1}$$

* John C. Collins, *et al.* arXiv:hep-ph/0409313

† V. Ahrens, *et al.* arXiv:1003.5827

Soft Function

- Soft Wilson line:

$$\boldsymbol{S}_{v_i}(x) = \mathcal{P}\exp\left(ig_s\int_{-\infty}^{0} ds\, v_i\cdot A^a(x+sv_i)\,\boldsymbol{T}_i^a\right), \tag{2}$$

where $v_i^2 = 0$ for massless partons and $v_i^2 > 0$ for massive partons.

- Soft function:

$$\begin{aligned}\boldsymbol{S} =& \frac{1}{d_R}\sum_{X_s}\langle 0|\bar{\mathbf{T}}\left[\boldsymbol{O}_s^\dagger(0)\right]|X_s\rangle\langle X_s|\mathbf{T}\left[\boldsymbol{O}_s(0)\right]|0\rangle\delta\left(\omega - v_0\cdot p_{X_s}\right),\\ \boldsymbol{O}_s(x) =& \left[\boldsymbol{S}_{v_1}\boldsymbol{S}_{v_2}^\dagger\boldsymbol{S}_{v_3}^\dagger\boldsymbol{S}_{v_4}\right](x),\end{aligned} \tag{3}$$

where $v_0 = (2,0,0,0)$.

- Convenient parameterization:

$$\begin{aligned} v_1 &= (1,0,0,1), & v_3 &= -(1,0,\beta\sin\theta,\beta y),\\ v_2 &= (1,0,0,-1), & v_4 &= -(1,0,-\beta\sin\theta,-\beta y). \end{aligned} \tag{4}$$

Main ingredients

- Integration by parts(IBP)

$$\int \left(\prod_i d^d k_i\right) \frac{\partial}{\partial k_j^\mu} q_\mu \prod_l \frac{1}{D_l^{a_l}} = 0; \tag{5}$$

- Differential equation(DE):

$$\partial_x \vec{g}(\epsilon, x) = \epsilon \hat{B}(x) \cdot \vec{g}(\epsilon, x)\,, \tag{6}$$
$$\vec{g}^{(n+1)}(x) = \int^x dx'\, \hat{B}(x')\, \vec{g}^{(n)}(x')\,;$$

- Generalized polylogarithms(GPLs)

$$G(a_1, \ldots, a_n; x) \equiv \int_0^x \frac{dx'}{x' - a_1} G(a_2, \ldots, a_n; x')\,, \tag{7}$$

with $G(;x) \equiv 1$ and $G(0, \ldots, 0; x) \equiv \log^n(x)/n!$.

Workflow

- Generate Feynman diagrams and amplitudes and square the amplitudes to get soft function, at NLO:

$$\boldsymbol{S}^{(1)}_{\text{bare}} \sim \sum_{i,j} \boldsymbol{w}^{(1)}_{ij} \int d^d k\, \delta^+(k^2)\, \frac{v_i \cdot v_j}{v_i \cdot k\, v_j \cdot k}\, \delta(\omega - v_0 \cdot k)\,; \tag{8}$$

- Simplification and finding integral families;
- IBP reduction with Cutkosky rule

$$\delta^+(k^2) \equiv \delta(k^2)\,\theta(k^0) = \frac{1}{2\pi i}\left(\frac{1}{k^2 + i0} - \frac{1}{k^2 - i0}\right),$$
$$\delta(\omega - v_0 \cdot k) = \frac{1}{2\pi i}\left(\frac{1}{\omega - v_0 \cdot k + i0} - \frac{1}{\omega - v_0 \cdot k - i0}\right); \tag{9}$$

- Differential equations and boundary conditions
- `QGRAF`, `FORM`, `Reduze2` and `FIRE5`.

Calculation of NLO soft function

- Only one integral family in NLO case:

$$F_{a_1,a_2} \equiv \int [dk]\, \delta(\omega - v_0 \cdot k)\, \frac{1}{(v_1 \cdot k)^{a_1}\, (-v_3 \cdot k)^{a_2}}\,; \tag{10}$$

- Master integrals: $\vec{f}(\epsilon, \beta, y, \omega) \equiv (F_{0,0}, F_{0,1}, F_{1,1})^{\mathsf{T}}$;
- Differential equation:

$$\partial_\beta \vec{f}(\epsilon, \beta, y, \omega) = \hat{A}(\epsilon, \beta, y, \omega)\, \vec{f}(\epsilon, \beta, y, \omega)\,; \tag{11}$$

- Canonical basis: $\vec{g}(\epsilon, \beta, y) = \hat{T}(\epsilon, \beta, y, \omega)\, \vec{f}(\epsilon, \beta, y, \omega)$, where

$$\hat{T} = \frac{2\,\Gamma(1-2\epsilon)}{\pi^{1-\epsilon}\,\omega^{1-2\epsilon}\,\Gamma(1-\epsilon)} \begin{pmatrix} 1-2\epsilon & 0 & 0 \\ 0 & \epsilon\,\omega\,\beta & 0 \\ 0 & 0 & \epsilon\,\omega^2\,(1-\beta y) \end{pmatrix}. \tag{12}$$

Calculation of NLO soft function

- The new vector $\vec{g}$ satisfies $\partial_\beta \vec{g}(\epsilon,\beta,y) = \epsilon\, \hat{B}(\beta,y)\, \vec{g}(\epsilon,\beta,y)$, with

$$\hat{B}(\beta,y) = -\frac{\hat{a}}{\beta-1} + \frac{\hat{b}}{\beta} + \frac{\hat{c}}{\beta+1} - \frac{\hat{d}}{\beta-1/y}\,, \tag{13}$$

where

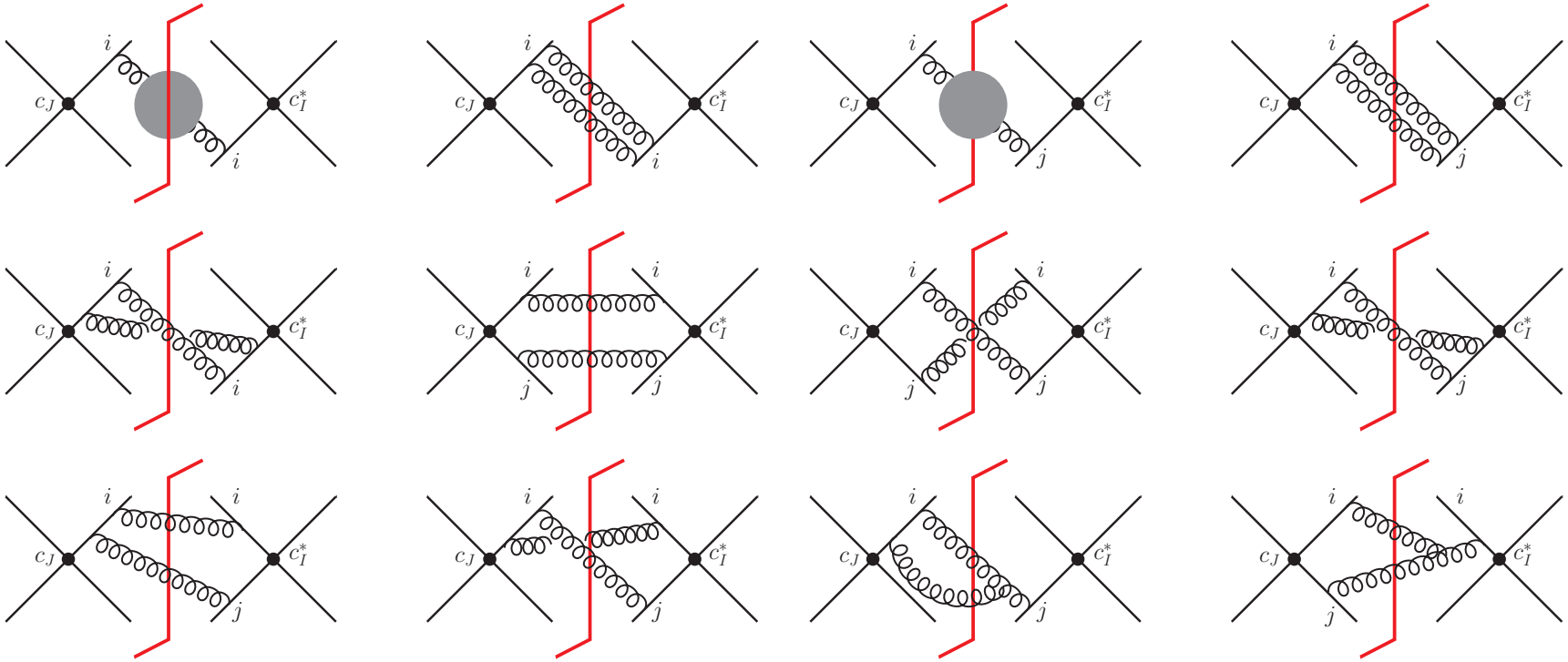
$$\hat{a} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 0 \end{pmatrix}, \hat{b} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 4 & 0 & 2 \end{pmatrix}, \hat{c} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ -2 & 2 & 0 \end{pmatrix}, \hat{d} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix};$$

- $\lim_{\beta\to 0} \beta\, \partial_\beta \vec{g}(\epsilon,\beta,y) = \lim_{\beta\to 0} \beta\, \epsilon\, \hat{B}(\beta,y)\, \vec{g}(\epsilon,\beta,y) = 0$;
- Boundary conditions:

$$g_2(\beta=0) = 0\,, \qquad g_3(\beta=0) = -2g_1(\beta=0)\,,$$
$$g_1(\epsilon,0,y) = \frac{2\,\Gamma(2-2\epsilon)}{\pi^{1-\epsilon}\,\omega^{1-2\epsilon}\,\Gamma(1-\epsilon)} \int [dk]\, \delta(\omega - v_0\cdot k) = 1\,; \tag{14}$$

Double-real corrections

One- or two-Wilson-line diagrams



Double-real corrections

One- or two-Wilson-line diagrams

- Two integral families

$$\{(k_1+k_2)^2,\, v_1\cdot k_2,\, v_1\cdot(k_1+k_2),\, v_2\cdot k_1,\, v_3\cdot k_1,\, v_3\cdot(k_1+k_2)\}\,, \tag{15}$$

$$\{(k_1+k_2)^2,\, v_1\cdot k_2,\, v_1\cdot(k_1+k_2),\, v_4\cdot k_1,\, v_3\cdot k_1,\, v_3\cdot(k_1+k_2)\}\,; \tag{16}$$

- General feynman integral for double real case:

$$F^{(n)}_{a_1,a_2,a_3,a_4,a_5,a_6} \equiv \int [dk_1]\,[dk_2]\,\delta\big(\omega - v_0\cdot(k_1+k_2)\big)\prod_{i=1}^{6}(D_i)^{-a_i}\,, \tag{17}$$

- Master integrals in the first integral family:

$$\begin{aligned}\vec{f}^{(1)} \equiv \big(&F^{(1)}_{0,0,0,0,0,0}, F^{(1)}_{1,0,0,0,-1,2}, F^{(1)}_{0,0,0,0,1,0}, F^{(1)}_{0,0,0,0,1,1}, F^{(1)}_{0,1,0,0,0,2},\\ &F^{(1)}_{0,0,1,0,0,2}, F^{(1)}_{0,1,0,0,1,1}, F^{(1)}_{1,1,0,0,1,-1}, F^{(1)}_{1,1,-1,0,1,0}, F^{(1)}_{1,1,0,0,1,0},\\ &F^{(1)}_{1,0,1,0,1,0}, F^{(1)}_{0,0,1,0,1,0}, F^{(1)}_{0,1,1,0,1,1}, F^{(1)}_{1,1,0,1,0,0}\big)^{\mathsf{T}}\,.\end{aligned} \tag{18}$$

Double-real corrections

- The linear transformation matrix:

$$\hat{T}^{(1)} = \frac{8\,\Gamma(1-4\epsilon)}{\pi^{2-2\epsilon}\,\omega^{3-4\epsilon}\,\Gamma^2(1-\epsilon)}$$
$$\times\,\mathrm{Diag}\Big\{(1-2\epsilon)(1-4\epsilon)(3-4\epsilon),\ \epsilon^2(1-2\epsilon)\omega^3\beta,\ \epsilon(1-2\epsilon)(1-4\epsilon)\omega\beta,$$
$$\epsilon^2(1-4\epsilon)\omega^2\beta^2,\ \epsilon^2\omega^3\beta^2,\ \epsilon(1-2\epsilon)\omega^3\beta^2,\ \epsilon^3\omega^3\beta(1-\beta y),\ \epsilon^3\omega^3,$$
$$\epsilon^3\omega^3\beta,\ \epsilon^3\omega^4(1-\beta y),\ -\epsilon^3\omega^4(1-\beta y),\ \epsilon^2(1-4\epsilon)\omega^2(1+y\beta),$$
$$\epsilon^3\omega^4(1-\beta y)^2,\ -\epsilon^3\omega^4\Big\}\,; \tag{19}$$

- The simplified differential equation:

$$\partial_\beta \vec{g}^{(1)} = \epsilon\left(-\frac{\hat{a}^{(1)}}{\beta-1} + \frac{\hat{b}^{(1)}}{\beta} + \frac{\hat{c}^{(1)}}{\beta+1} + \frac{\hat{d}^{(1)}}{\beta+1/y} - \frac{\hat{e}^{(1)}}{\beta-1/y}\right)\vec{g}^{(1)}\,. \tag{20}$$

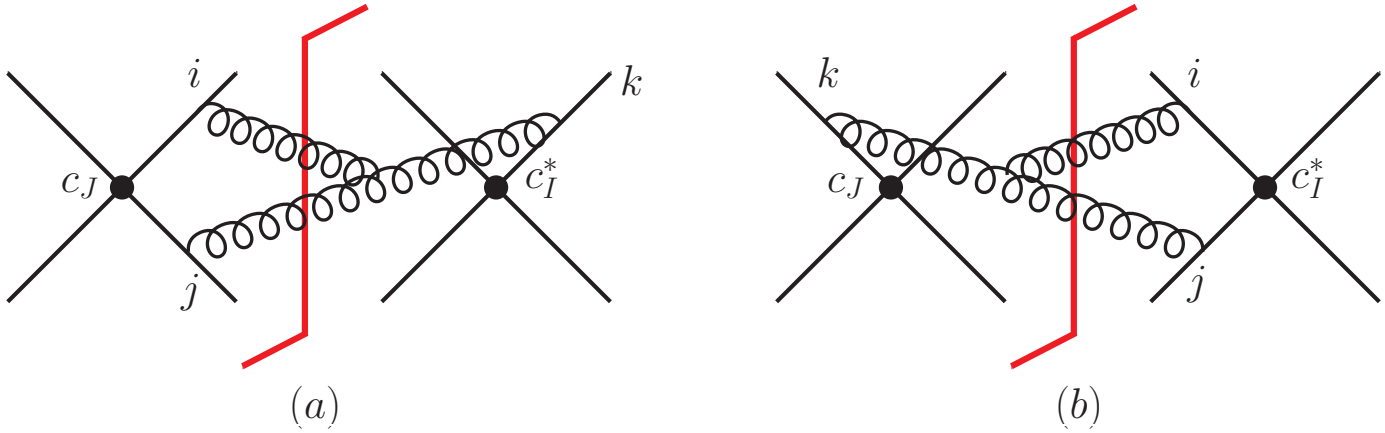
Double-real corrections

- Boundary conditions:

$$
\begin{aligned}
g_1^{(1)}(\epsilon,0,y) &= g_1^{(2)}(\epsilon,0,y) \\
&= \frac{4\,\Gamma(4-4\epsilon)}{\pi^{2-2\epsilon}\,\omega^{3-4\epsilon}\,\Gamma(1-\epsilon)^2}\int[dk_1]\,[dk_2]\,\delta[\omega - v_0\cdot(k_1+k_2)] \\
&= 1\,, \\
g_{12}^{(1)}(\epsilon,0,y) &= \frac{8\,\Gamma(2-4\epsilon)}{\pi^{2-2\epsilon}\,\omega^{1-4\epsilon}\,\Gamma(-\epsilon)^2}\int[dk_1]\,[dk_2]\,\frac{\delta[\omega - v_0\cdot(k_1+k_2)]}{v_1\cdot(k_1+k_2)\;v_3\cdot k_1}\,, \\
&= \frac{2\epsilon^2}{3}\left(\pi^2 + 42\,\epsilon\,\zeta(3) + 2\,\epsilon^2\,\pi^4 + \mathcal{O}(\epsilon^3)\right) \\
g_{14}^{(1)}(\epsilon,0,y) &= \frac{8\,\epsilon^2\,\Gamma(-4\epsilon)\,\omega^{1+4\epsilon}}{\pi^{2-2\epsilon}\,\Gamma(-\epsilon)^2}\int[dk_1]\,[dk_2]\,\frac{\delta[\omega - v_0\cdot(k_1+k_2)]}{(k_1+k_2)^2\;v_1\cdot k_2\;v_2\cdot k_1}\,, \\
&= -\frac{4\Gamma(-2\epsilon)\,\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)\,\Gamma(-3\epsilon)}\;{}_3F_2(-\epsilon,-\epsilon,-\epsilon;1-\epsilon,-3\epsilon;1)\,. \qquad (21)
\end{aligned}
$$

Double-real corrections

Three- and four-Wilson-line diagrams



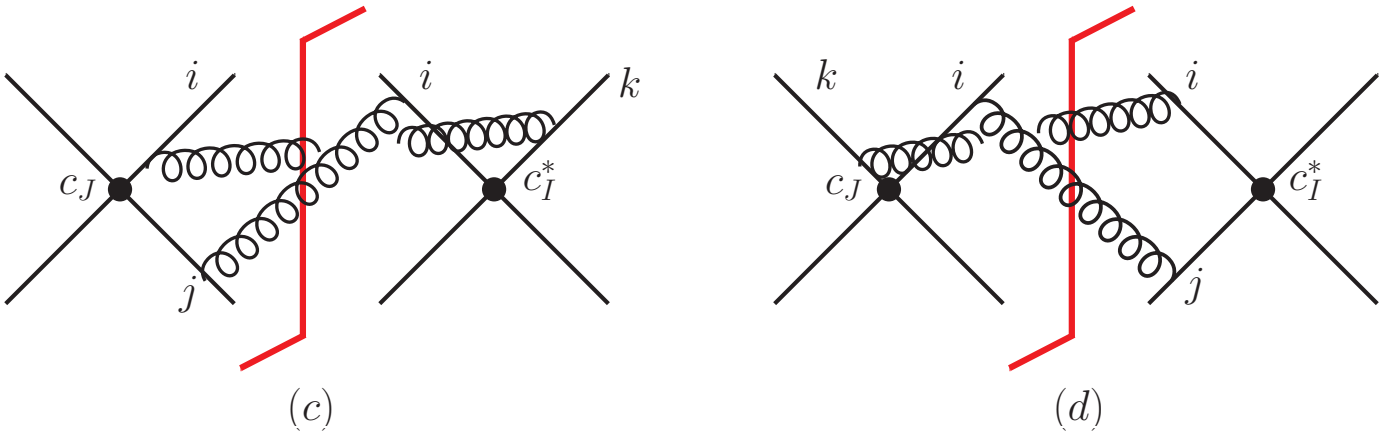
- The contributions of the above two non-Abelian diagrams:

$$D^{(a)} \to +if^{abc}\,\boldsymbol{T}_i^a\,\boldsymbol{T}_j^b\,\boldsymbol{T}_k^c\,I\,, \quad D^{(b)} \to -if^{abc}\,\boldsymbol{T}_i^a\,\boldsymbol{T}_j^b\,\boldsymbol{T}_k^c\,I^*\,, \tag{22}$$

$$I = \int [dk_1]\,[dk_2]\,\frac{v_i^\mu\,v_j^\nu\,v_k^\rho\,[(2k_1+k_2)_\nu\,g_{\mu\rho}+\cdots]\,\delta[\omega - v_0\cdot(k_1+k_2)]}{(k_1+k_2)^2\,v_i\cdot k_1\,v_j\cdot k_2\,v_k\cdot(k_1+k_2)}\,.$$

- Total contribution of non-Abelian three-Wilson-line diagrams is pure imaginary.

Double-real corrections



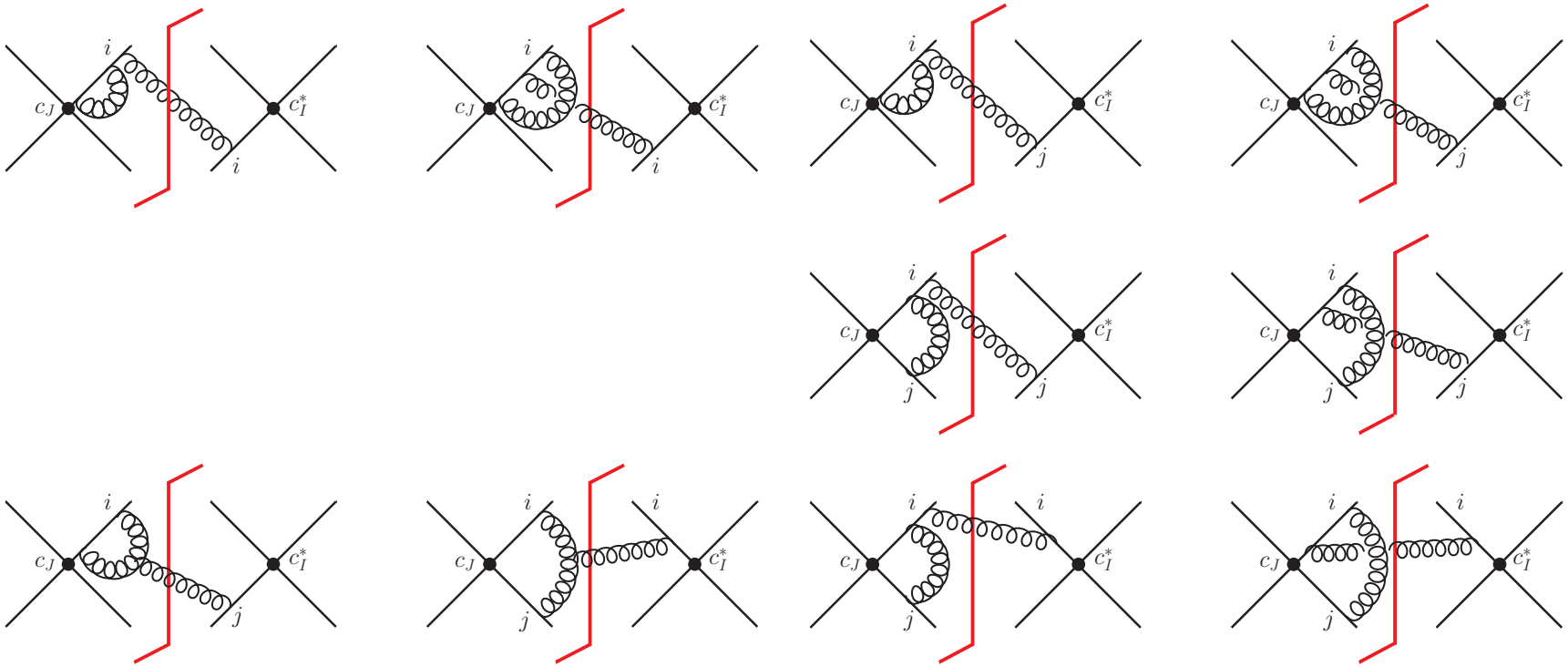
- The summation of the above two Abelian diagrams:

$$D^{(c)+(d)} \to \boldsymbol{w}_{ijk} \int [dk_1]\,[dk_2]\, \frac{v_i \cdot v_j\; v_i \cdot v_k\; \delta(\omega - v_0 \cdot (k_1 + k_2))}{v_i \cdot k_1\; v_i \cdot k_2\; v_j \cdot k_1\; v_k \cdot k_2}\,. \tag{23}$$

- All the reassigned integrals from the Abelian three- and four-Wilson-line diagrams relate to NLO integrals by convolution.

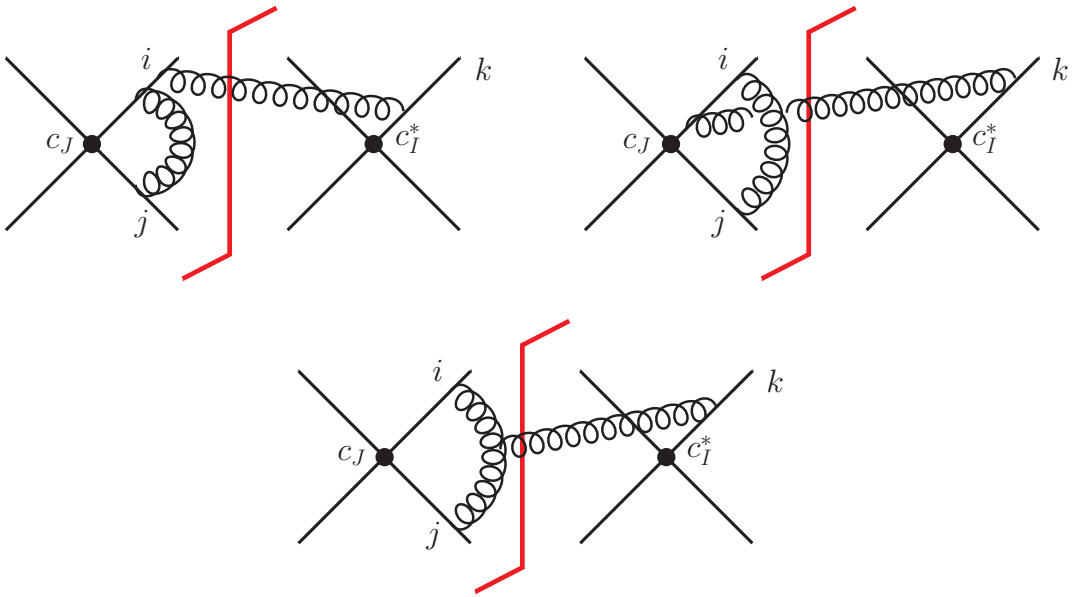
Virtual-real corrections

Feynman diagrams for Virtual-real corrections



One and two-Wilson-line diagrams for virtual-real contributions.

Virtual-real corrections



Three-Wilson-line diagrams for virtual-real contributions.

$$\frac{1}{v_i \cdot k \; v_i \cdot (k+l) \; v_j \cdot l \; v_k \cdot k} + \frac{1}{v_i \cdot l \; v_i \cdot (k+l) \; v_j \cdot l \; v_k \cdot k}$$
$$= \frac{1}{v_i \cdot k \; v_i \cdot l \; v_j \cdot l \; v_k \cdot k},$$
$$\boldsymbol{T}_i^a \, \boldsymbol{T}_i^b \, \boldsymbol{T}_j^b \, \boldsymbol{T}_k^a - \boldsymbol{T}_i^b \, \boldsymbol{T}_i^a \, \boldsymbol{T}_j^b \, \boldsymbol{T}_k^a = -i f^{abc} \, \boldsymbol{T}_i^a \, \boldsymbol{T}_j^b \, \boldsymbol{T}_k^c \tag{24}$$

Virtual-real corrections

- Two integral families

$$\{k^2,\ l^2,\ (k+l)^2,\ v_1\cdot k,\ v_1\cdot l,\ v_2\cdot(k+l),\ v_3\cdot k,\ v_3\cdot l\}\,, \tag{25}$$

$$\{k^2,\ l^2,\ (k+l)^2,\ v_1\cdot k,\ v_1\cdot l,\ v_4\cdot(k+l),\ v_3\cdot k,\ v_3\cdot l\}\,; \tag{26}$$

- General feynman integral:

$$F^{(n')}_{a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8} = \int d^dk\, d^dl\, \delta\big(\omega - v_0\cdot k\big)\, \mathrm{Disc}\big[(k^2)^{-a_1}\big] \prod_{i=2}^{8} (D_i)^{-a_i}\,,$$

$$\mathrm{Disc}\big[(k^2)^{-a_1}\big] \equiv \frac{1}{2\pi i}\left[(k^2+i0)^{-a_1} - (k^2-i0)^{-a_1}\right]. \tag{27}$$

- The forth integral family is the most complicated.

Virtual-real corrections

- The forth integral family:

$$\begin{aligned}\vec{f}^{(4)} \equiv \big(&F^{(4)}_{1,1,1,0,0,0,0,1}, F^{(4)}_{2,0,1,0,0,0,0,2}, F^{(4)}_{1,2,0,0,0,1,1,0}, F^{(4)}_{1,1,0,1,0,2,0,0},\\ &F^{(4)}_{1,0,1,1,0,0,0,2}, F^{(4)}_{2,1,0,0,0,1,0,1}, F^{(4)}_{1,1,0,0,0,2,0,1}, F^{(4)}_{1,1,0,0,0,1,1,1},\\ &F^{(4)}_{1,0,1,1,0,1,0,1}, F^{(4)}_{1,1,1,0,0,1,0,1}, F^{(4)}_{1,1,0,1,0,1,0,1}, F^{(4)}_{1,1,1,0,1,1,0,0},\\ &F^{(4)}_{1,1,1,-1,1,1,0,0}, F^{(4)}_{1,1,1,0,1,1,-1,0}, F^{(4)}_{1,1,1,-1,1,1,1,0}, F^{(4)}_{1,1,1,0,1,1,1,0},\\ &F^{(4)}_{1,1,1,0,1,1,1,-1}, F^{(4)}_{1,1,1,1,-1,1,0,1}, F^{(4)}_{1,1,1,1,0,1,-1,1}, F^{(4)}_{1,1,1,1,0,1,0,1},\\ &F^{(4)}_{1,1,1,1,-1,1,-1,1}\big)^{\mathsf{T}},\end{aligned} \tag{28}$$

- The linear transformation:

$$\vec{g}^{(4)}(\epsilon,\beta,y) = \hat{T}^{(4)}(\epsilon,\beta,y,\omega)\cdot\vec{f}^{(4)}(\epsilon,\beta,y,\omega). \tag{29}$$

Virtual-real corrections

- The simplified differential equation:

$$\partial_\beta \vec{g}^{(4)} = \epsilon \left(-\frac{\hat{a}^{(4)}}{\beta - 1} + \frac{\hat{b}^{(4)}}{\beta} + \frac{\hat{c}^{(4)}}{\beta + 1} - \frac{\hat{d}^{(4)}}{\beta - 1/y} + \frac{\hat{e}^{(4)}}{\beta + 1/y} \right) \vec{g}^{(4)} . \quad (30)$$

- Loop integrals for boundaries*:

$$\begin{aligned} M_i^{(a)} &= \int d^d l \, \frac{1}{(l^2 + i0)\,[(k+l)^2 + i0]\,(-v_i \cdot l + i0)} , \\ M_{ij}^{(b)} &= \int d^d l \, \frac{1}{(l^2 + i0)\,(-v_i \cdot l + i0)\,[-v_j \cdot (k+l) + i0]} \to M_{34}^{(b)} \supset \beta^{2\epsilon} , \\ M_{ij}^{(c)} &= \int d^d l \, \frac{1}{(l^2 + i0)\,[(l+k)^2 + i0]\,(-v_i \cdot l + i0)\,[-v_j \cdot (k+l) + i0]} . \end{aligned} \quad (31)$$

* I. Bierenbaum, *et al.* arXiv:1107.4384

Virtual-real corrections

Boundary conditions:

$$g_6^{(4)}(\epsilon,\beta,y) = \frac{ie^{-2i\pi\epsilon}\,\Gamma(-2\epsilon)\,\omega^{-1+4\epsilon}\,\beta}{2\pi^{3-2\epsilon}\,\Gamma^2(-\epsilon)\,\Gamma(2\epsilon)} \int [dk]\, d^d l\, \frac{\delta(\omega - v_0\cdot k)}{l^2\; v_3\cdot l\,[-v_4\cdot(k+l)]} \times \left[\frac{\omega}{-v_4\cdot(k+l)} + 2(1-4\epsilon)\right]$$

$$\approx \frac{(e^{-2i\pi\epsilon}-1)\,\beta^{2\epsilon}\,\Gamma(1-2\epsilon)\,\Gamma(1+\epsilon)}{4^{1-2\epsilon}\,\Gamma(1-\epsilon)}\,,$$

$$g_9^{(4)}(\epsilon,\beta,y) = -\frac{ie^{-2i\pi\epsilon}\,\Gamma(1-2\epsilon)\,\omega^{4\epsilon}\,\beta}{4\pi^{3-2\epsilon}\,\Gamma^2(-\epsilon)\,\Gamma(2\epsilon)} \int [dk]\, d^d l\, \frac{\delta(\omega - v_0\cdot k)}{l^2\;(-v_1\cdot k)\; v_3\cdot l} \times \frac{1}{-v_4\cdot(k+l)}$$

$$\approx \frac{(e^{-2i\pi\epsilon}-1)\,\beta^{2\epsilon}\,\Gamma(1-2\epsilon)\,\Gamma(1+\epsilon)}{2^{3-4\epsilon}\,\Gamma(1-\epsilon)}$$

$$= \frac{1}{2}\, g_6^{(4)}(\epsilon,\beta\to 0,y)\,. \tag{32}$$

Virtual-real corrections

Boundary conditions:

$$
\begin{aligned}
g_2^{(3)}(\epsilon,0,y) = g_2^{(4)}(\epsilon,0,y) &= -\frac{ie^{-2i\pi\epsilon}\,(1-2\epsilon)\,\Gamma(2-2\epsilon)}{\pi^{3-2\epsilon}\,\omega^{2-4\epsilon}\,\Gamma^2(1-\epsilon)\Gamma(2\epsilon)}\int[dk]\,d^dl\,\frac{\delta(\omega-v_0\cdot k)}{(k+l)^2\;v_3\cdot l} \\
&= 1\,, \\
g_4^{(3)}(\epsilon,0,y) &= \frac{ie^{-2i\pi\epsilon}\,(1-4\epsilon)\,\Gamma(-2\epsilon)}{2\pi^{3-2\epsilon}\,\omega^{1-4\epsilon}\,\Gamma^2(-\epsilon)\,\Gamma(2\epsilon)}\int[dk]\,d^dl\,\frac{\delta(\omega-v_0\cdot k)}{l^2\;[-v_1\cdot(k+l)]\;v_3\cdot l} \\
&= -\frac{e^{-2i\pi\epsilon}\,\Gamma(1-3\epsilon)\,\Gamma^2(-2\epsilon)\,\Gamma(1+\epsilon)}{\Gamma(1-4\epsilon)\,\Gamma^2(-\epsilon)}\,, \\
g_{10}^{(3)}(\epsilon,0,y) &= \frac{ie^{-2i\pi\epsilon}\,\omega^{1+4\epsilon}\,\Gamma(1-2\epsilon)}{4\pi^{3-2\epsilon}\,\Gamma^2(-\epsilon)\,\Gamma(2\epsilon)}\int[dk]\,d^dl\,\frac{\delta(\omega-v_0\cdot k)}{l^2\;(k+l)^2\;[-v_1\cdot(k+l)]\;v_2\cdot l} \\
&= \frac{e^{-3i\pi\epsilon}\,\Gamma^2(1-2\epsilon)\,\Gamma^2(1+\epsilon)}{2\,\Gamma(1-4\epsilon)\,\Gamma(1+2\epsilon)}\,,
\end{aligned}
\tag{33}
$$

Virtual-real corrections

Boundary conditions:

$$g_9^{(3)}(\epsilon,0,y) = \frac{ie^{-2i\pi\epsilon}\,\omega^{1+4\epsilon}\,\Gamma(1-2\epsilon)}{4\pi^{3-2\epsilon}\,\Gamma^2(-\epsilon)\,\Gamma(2\epsilon)} \int [dk]\, d^d l\, \frac{\delta(\omega - v_0\cdot k)}{l^2\,(k+l)^2\,[-v_2\cdot(k+l)]\,v_3\cdot l} \times \left[\frac{\omega}{-v_1\cdot k + i0} + 2\right]$$

$$= \frac{1}{3} - \frac{i\pi}{6}\epsilon - \frac{\pi^2}{6}\epsilon^2 + \left(\frac{38\zeta(3)}{9} + \frac{4i\pi^3}{3}\right)\epsilon^3 - \left(\frac{209\pi^4}{240} + \frac{79i\pi\zeta(3)}{9}\right)\epsilon^4 + \mathcal{O}\left(\epsilon^5\right) . \tag{34}$$

Renormalized soft function

- Expansion of bare soft function in $d = 4 - 2\epsilon$ dimension:

$$\boldsymbol{S}_{\text{bare}} = \boldsymbol{S}^{(0)}_{\text{bare}} + \frac{Z_\alpha \alpha_s}{4\pi} \boldsymbol{S}^{(1)}_{\text{bare}} + \left(\frac{Z_\alpha \alpha_s}{4\pi}\right)^2 \boldsymbol{S}^{(2)}_{\text{bare}} + \cdots ; \tag{35}$$

- Renormalized coupling constant α_s relates to the bare one by $\overline{\text{MS}}$ scheme:

$$Z_\alpha \alpha_s \mu^{2\epsilon} = e^{-\epsilon\gamma_E} (4\pi)^\epsilon \alpha_s^{(0)}$$
$$Z_\alpha = 1 - \frac{\beta_0 \alpha_s}{4\pi\epsilon}, \quad \beta_0 = \frac{11}{3} C_A - \frac{4}{3} T_F n_f \tag{36}$$

- In Laplace space:

$$\tilde{\boldsymbol{s}}(\Lambda) = \int_0^\infty d\omega \, \exp\left(-\frac{\omega}{\Lambda e^{\gamma_E}}\right) \boldsymbol{S}(\omega),$$
$$\tilde{\boldsymbol{s}}(L, \mu) = \boldsymbol{Z}_s^\dagger(L, \mu) \, \tilde{\boldsymbol{s}}_{\text{bare}}(\Lambda) \, \boldsymbol{Z}_s(L, \mu) \tag{37}$$

Soft anomalous dimension matrix

- RGE for renormalization factor:

$$\frac{d}{d\mu}\boldsymbol{Z}_s(L,\beta,\cos\theta,\mu) = -\boldsymbol{Z}_s(L,\beta,\cos\theta,\mu)\,\boldsymbol{\Gamma}_s(L,\beta,\cos\theta,\mu)\,, \tag{38}$$

- Soft anomalous dimension matrix for $q\bar{q}$ channel:

$$\begin{aligned}\boldsymbol{\Gamma}_s^{q\bar{q}} = &\left[C_F\,\gamma_{\text{cusp}}(\alpha_s)\,(L-i\pi) + C_F\,\gamma_{\text{cusp}}(\beta_{34},\alpha_s) + 2\gamma_s^q(\alpha_s) + 2\gamma^Q(\alpha_s)\right]\mathbf{1}\\ &+\frac{N}{2}\left[\gamma_{\text{cusp}}(\alpha_s)\left(\ln\frac{t_1^2}{M^2m_t^2}+i\pi\right)-\gamma_{\text{cusp}}(\beta_{34},\alpha_s)\right]\begin{pmatrix}0&0\\0&1\end{pmatrix}\\ &+\gamma_{\text{cusp}}(\alpha_s)\,\ln\frac{t_1^2}{u_1^2}\left[\begin{pmatrix}0&\frac{C_F}{2N}\\1&-\frac{1}{N}\end{pmatrix}+\frac{\alpha_s}{4\pi}\,g(\beta_{34})\begin{pmatrix}0&\frac{C_F}{2}\\-N&0\end{pmatrix}\right].\end{aligned} \tag{39}$$

- Decompose $\boldsymbol{\Gamma}_s$ into the form:

$$\boldsymbol{\Gamma}_s(L,\{v\},\mu) \equiv \frac{\alpha_s}{4\pi}\left(A_0L_s+\boldsymbol{\Gamma}_0\right)+\left(\frac{\alpha_s}{4\pi}\right)^2\left(A_1L_s+\boldsymbol{\Gamma}_1\right), \tag{40}$$

where $L_s = L - i\pi$, $L = \ln(\Lambda^2/\mu^2)$

A. Ferroglia, *et al.* arXiv:0908.3676

Renormalization

- Renormalization factor up to $\mathcal{O}(\alpha_s^2)$:

$$\ln \boldsymbol{Z}_s = \frac{\alpha_s}{4\pi}\left(-\frac{A_0}{2\epsilon^2} + \frac{A_0 L_s + \boldsymbol{\Gamma}_0}{2\epsilon}\right) + \left(\frac{\alpha_s}{4\pi}\right)^2 \left[\frac{3A_0\beta_0}{8\epsilon^3} + \frac{-A_1 - 2\beta_0(A_0 L_s + \boldsymbol{\Gamma}_0)}{8\epsilon^2} + \frac{A_1 L_s + \boldsymbol{\Gamma}_1}{4\epsilon}\right]. \quad (41)$$

- The renormalized NNLO soft function

$$\begin{aligned}\tilde{\boldsymbol{s}}^{(2)}(L,\mu) &= \tilde{\boldsymbol{s}}^{(2)}_{\text{bare}} + \boldsymbol{Z}_s^{\dagger(2)}\tilde{\boldsymbol{s}}^{(0)} + \tilde{\boldsymbol{s}}^{(0)}\boldsymbol{Z}_s^{(2)} \\ &\quad + \boldsymbol{Z}_s^{\dagger(1)}\tilde{\boldsymbol{s}}^{(1)}_{\text{bare}} + \tilde{\boldsymbol{s}}^{(1)}_{\text{bare}}\boldsymbol{Z}_s^{(1)} + \boldsymbol{Z}_s^{\dagger(1)}\tilde{\boldsymbol{s}}^{(0)}\boldsymbol{Z}_s^{(1)} - \frac{\beta_0}{\epsilon}\tilde{\boldsymbol{s}}^{(1)}_{\text{bare}} \\ &\quad \to \text{finite} \end{aligned} \quad (42)$$

Check with known results

- Our renormalized soft function satisfies the RGE

$$\frac{d}{d\mu}\tilde{\boldsymbol{s}}(L,\mu) = -\boldsymbol{\Gamma}_s^\dagger(L,\mu)\,\tilde{\boldsymbol{s}}(L,\mu) - \tilde{\boldsymbol{s}}(L,\mu)\,\boldsymbol{\Gamma}_s(L,\mu)\,. \tag{43}$$

- In the limit of $\beta \to 0$, our result is consistent with [1]* for color-singlet state of $t\bar{t}$ and [2]† for color-octet state of $t\bar{t}$
- In the limit of $\beta \to 1$, our result agrees with [3]‡

$$\tilde{\boldsymbol{s}}_{\text{massive}}\left(\ln\frac{\Lambda^2}{\mu^2},\beta,\cos\theta,\mu\right) = \tilde{\boldsymbol{s}}_{\text{massless}}\left(\ln\frac{\Lambda^2}{\mu^2},\cos\theta,\mu\right)\,\tilde{s}_D^2\left(\ln\frac{m_t^2\Lambda^2}{M^2\mu^2},\mu\right) + \mathcal{O}(m_t^2/M^2)\,. \tag{44}$$

* A. V. Belitsky, arXiv:hep-ph/9808389

† M. Czakon, *et al.* arXiv:1311.2541

‡ A. Ferroglia, *et al.* arXiv:1205.3662

Definition for numerical calculation

- The numerical impact of the NNLO correction

$$\mathcal{S}_{ij}^{(n)}(\beta,\mu/\mu_{\mathsf{def}}) = \int_{-1}^{1} d\cos\theta \left(\frac{\alpha_s}{4\pi}\right)^n \mathrm{Tr}\left[\boldsymbol{H}_{ij}^{(0)}(\beta,\cos\theta)\,\tilde{\boldsymbol{s}}_{ij}^{(n)}\left(\ln\frac{\Lambda^2}{\mu^2},\beta,\cos\theta\right)\right]$$

$$R_{ij}^{\mathsf{NLO}}(\beta,\mu/\mu_{\mathsf{def}}) = \frac{\mathcal{S}_{ij}^{(1)}(\beta,\mu/\mu_{\mathsf{def}})}{\mathcal{S}_{ij}^{(0)}(\beta,\mu/\mu_{\mathsf{def}})}\,,$$

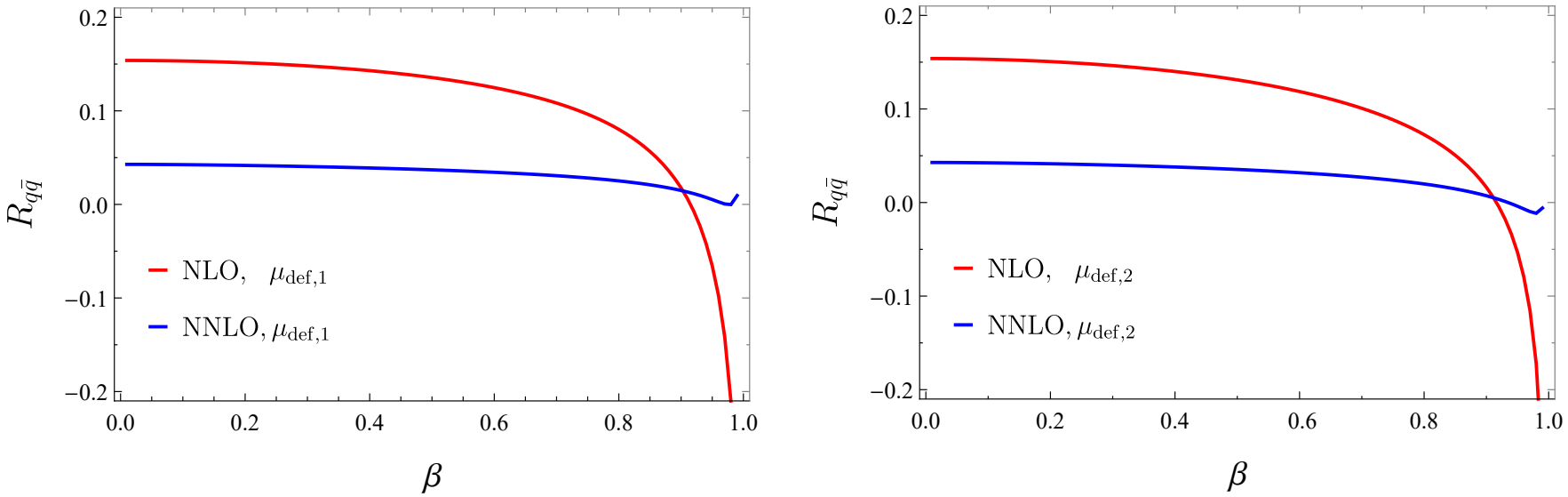
$$R_{ij}^{\mathsf{NNLO}}(\beta,\mu/\mu_{\mathsf{def}}) = \frac{\mathcal{S}_{ij}^{(2)}(\beta,\mu/\mu_{\mathsf{def}})}{\mathcal{S}_{ij}^{(0)}(\beta,\mu/\mu_{\mathsf{def}})}\,, \tag{45}$$

- Two kind of choices for default soft scales*

$$\mu_{\mathsf{def},1} = \Lambda, \qquad \mu_{\mathsf{def},2} = \Lambda\sqrt{1-\beta^2\cos^2\theta}\,, \tag{46}$$

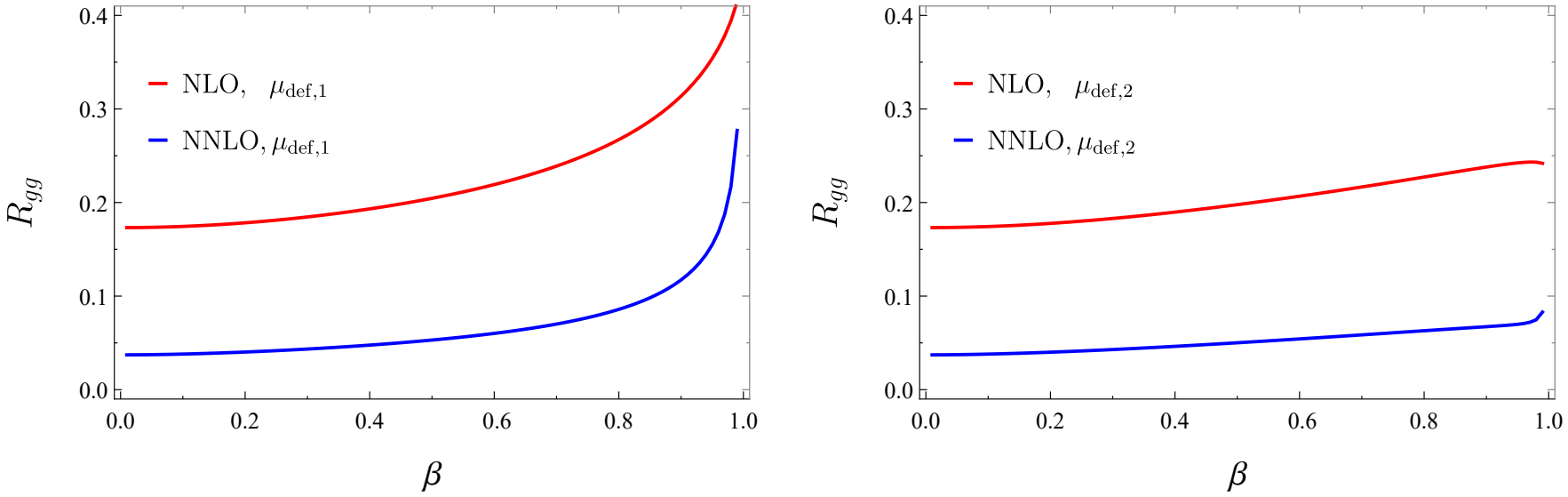
* B. D. Pecjak, *et al.* arXiv:1601.07020, M. Czakon, *et al.* arXiv:1803.07623

Numerical results for $q\bar{q}$ channel



Relative soft corrections as a function of β for two choices of the default scale: $\mu_{\mathrm{def},1} = \Lambda$ (left) and $\mu_{\mathrm{def},2} = \Lambda\sqrt{1-\beta^2\cos^2\theta}$ (right) for $q\bar{q}$ channel

Numerical results for gg channel



Relative soft corrections as a function of β for two choices of the default scale: $\mu_{\text{def},1} = \Lambda$ (left) and $\mu_{\text{def},2} = \Lambda\sqrt{1-\beta^2\cos^2\theta}$ (right) for gg channel

Conclusion

- IBP and DE are very powerful methods for calculation of phase space integrals;
- Our result of NNLO soft function is checked in different aspects and reliable;
- It's a good verification of the cancellation of the IR singularity with three parton correlation;
- Our result gives a chance to extend the theoretical precision to $\mathrm{NNLO} + \mathrm{NNNLL}$ in soft limit in the future.

Thanks!