New horizon in lattice flavor physics

冯 旭

北京大學
PEKING UNIVERSITY

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How LQCD calculations are calibrated

LQCD calculation starts from the QCD Lagrangian

$$L = \frac{1}{4g_s^2}\,\mathrm{Tr}[F_{\mu\nu}F_{\mu\nu}] + \sum_f \bar{q}_f(\not{D} + m_f)q_f$$

Input parameters

- strong coupling constant $\alpha_s^{lat} = g_s^2/4\pi$
- quark masses m_f, for each flavor
  - isospin limit $m_u = m_d \quad \Rightarrow \quad N_f = 2$
  - strange and charm quark, $N_f = 2+1$, $N_f = 2+1+1$
  - bottom: only in the valence
  - top: decays before it can hadronize, too heavy thus ignored
- For each value of α_s^{lat}, choose physical bare quark masses by requiring e.g.

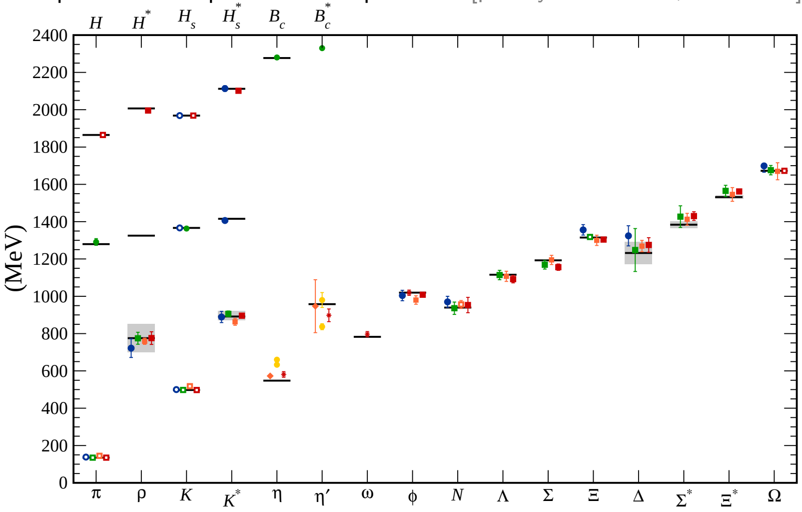
  $$\frac{am_\pi}{am_\Omega}, \frac{am_K}{am_\Omega}, \frac{am_D}{am_\Omega}, \frac{am_B}{am_\Omega} \Rightarrow \text{take physical values}$$

  Define lattice spacing by imposing, e.g.

  $$a = \frac{am_\Omega}{m_\Omega^{\mathrm{phys}}}$$

Hadron spectrum from lattice QCD

- Input: α_s, quark masses; set by π, K, $\cdots$ (empty symbols in the plot)
- Output: hadron spectrum vs experiment [plot by A. Kronfeld, 1209.3468]

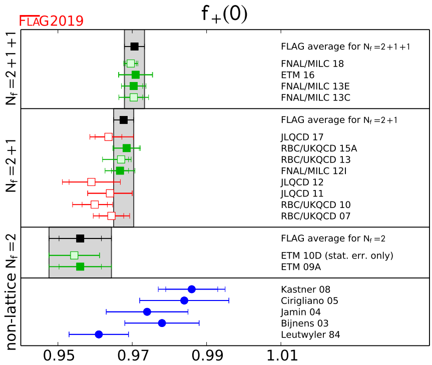


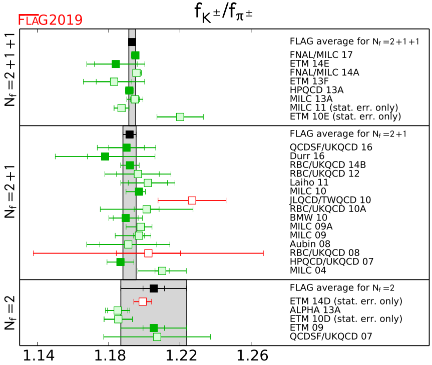
Mile stone: $f_{K^\pm}/f_{\pi^\pm}$ and $f_+(0)$

Flavor Lattice Averaging Group (FLAG) average 2019

$$f_+^{K\pi}(0) = 0.9706(27) \quad \Rightarrow \quad 0.28\% \text{ error}$$

$$f_{K^\pm}/f_{\pi^\pm} = 1.1932(19) \quad \Rightarrow \quad 0.16\% \text{ error}$$





Experimental information [arXiv:1411.5252, 1509.02220]

$$K_{\ell 3} \quad \Rightarrow \quad |V_{us}| f_+(0) = 0.2165(4) \quad \Rightarrow \quad |V_{us}| = 0.2231(7)$$

$$K_{\mu 2}/\pi_{\mu 2} \quad \Rightarrow \quad \left|\frac{V_{us}}{V_{ud}}\right| \frac{f_{K^\pm}}{f_{\pi^\pm}} = 0.2760(4) \quad \Rightarrow \quad \left|\frac{V_{us}}{V_{ud}}\right| = 0.2313(5)$$

Flag average 2019

Error < 1%

	N_f	FLAG average	Frac. Err.
f_K/f_π	$2+1+1$	1.1932(19)	0.16%
$f_+(0)$	$2+1+1$	0.9706(27)	0.28%
f_D	$2+1+1$	212.0(7) MeV	0.33%
f_{D_s}	$2+1+1$	249.9(5) MeV	0.20%
f_{D_s}/f_D	$2+1+1$	1.1783(16)	0.13%
f_B	$2+1+1$	190.0(1.3) MeV	0.68%
f_{B_s}	$2+1+1$	230.3(1.3) MeV	0.56%
f_{B_s}/f_B	$2+1+1$	1.209(5)	0.41%

Error < 5%

	N_f	FLAG average	Frac. Err.
$\hat{B}_K$	$2+1$	0.7625(97)	1.3%
$f_+^{D\pi}(0)$	$2+1$	0.666(29)	4.4%
$f_+^{DK}(0)$	$2+1$	0.747(19)	2.5%
$\hat{B}_{B_s}$	$2+1$	1.35(6)	4.4%
B_{B_s}/B_{B_d}	$2+1$	1.032(28)	3.7%
…			

Precision beyond 1% $\Rightarrow$ time to include QED

Full QCD + QED action is

$$S^{\rm full} = \frac{1}{4g_s^2} S^{\rm YM} + \sum_f \left\{ S_f^{\rm kim} + m_f S_f^{\rm m} \right\} + S^A + \sum_\ell \left\{ S_\ell^{\rm kim} + m_\ell S_\ell^{\rm m} \right\}$$

- Only in full theory, phys. results $O^{\rm phys}$ can be reproduced unambiguously
- $O^{\rm phys} = O^{\rm QCD} + \delta Q^{\rm QED}$ is prescription dependent
- "Hadronic scheme" recently proposed by [Di Carlo et. al., arXiv:1904.08731]
  - impose same condition as in pure QCD

    $$\frac{am_\pi}{am_\Omega}, \frac{am_{K^0}}{am_\Omega}, \frac{am_{K^+}}{am_\Omega}, \frac{am_D}{am_\Omega}, \frac{am_B}{am_\Omega} \Rightarrow \text{take physical values}$$

  - Add mass counterterms $m_f = m_f^0 + \delta m_f$ and $a = a_0 + \delta a$
  - For a general observable O

$$O^{\rm phys} = \frac{\langle aO \rangle^{\rm full}}{a} = \underbrace{\frac{\langle a_0 O \rangle^{\rm QCD}}{a_0}}_{O^{\rm QCD} = O^{\rm ISO} + \delta O^{\rm SIB}} + \underbrace{\frac{\delta O}{a_0} - \frac{\delta a}{a_0^2} \langle a_0 O \rangle^{\rm QCD}}_{\delta O^{\rm QED}} + O(\alpha_{EM}^2)$$

- GRS scheme: $g_s^{\overline{\rm MS}}(\mu)$, $m_f^{\overline{\rm MS}}(\mu)$ in pure QCD = that in QCD+QED

  [Gasser, Rusetsky, Scimemi, hep-ph/0305260]

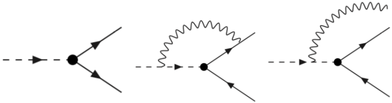
Leptonic decay rates

Consider the leptonic decay in pure QCD

$$\Gamma(P^- \to \ell^- \bar{\nu}_\ell) = \frac{G_F^2 |V_{ij}|^2 f_P^2}{8\pi} m_P m_\ell^2 \left(1 - \frac{m_\ell^2}{m_P^2}\right)^2, \quad \langle 0|A_\mu|P(p)\rangle = i f_P p_\mu$$

- f_P as "pure QCD" quantity, is scheme dependent

Inclusive decay rate



$$\Gamma(P^- \to \ell^- \bar{\nu}_\ell(\gamma)) = \Gamma(P^- \to \ell^- \bar{\nu}_\ell) + \Gamma(\ell^- \bar{\nu}_\ell \gamma)$$

- Define prescription independent quantity, up to $O(\alpha_{EM}^2)$

$$\mathcal{F}_P^2 \equiv \frac{\Gamma(P^- \to \ell^- \bar{\nu}_\ell(\gamma))}{G_F^2 |V_{ij}|^2 m_P^{\mathrm{phys}} m_\ell^2 \left(1 - \frac{m_\ell^2}{m_P^{\mathrm{phys}\,2}}\right)^2} = f_P^2 (1 + \delta R_P)$$

  - f_P only contains part of the QCD effects
  - δR_P depends on the structure of hadron

First lattice study using GRS scheme [Giusti et. al. PRL120 (2018) 072001]

$\delta R_{K\pi} \equiv \delta R_K - \delta R_\pi = -0.0122(16)$ VS $\delta R_{K\pi}^{\mathrm{ChPT}} = -0.0112(21)$ quoted by PDG

$K \to \pi\pi$ decays and direct CP violation

CP violation is first observed in neutral Kaon decays

- CP eigenstates
  - Under CP transform: $CP|K^0\rangle = -|\overline{K^0}\rangle$
  - Define CP eigenstates: $K^0_\pm = (K^0 \mp \overline{K^0})/\sqrt{2}$
- Weak eigenstates
  - $K_S \to 2\pi \; (CP = +)$
  - $K_L \to 3\pi \; (CP = -)$
- Neglecting CP violation, we have $K_S = K^0_+$ and $K_L = K^0_-$

1964, BNL discovered $K_L \to 2\pi \Rightarrow CP$ violation $\Rightarrow$ Nobel prize (1980)

Direct and indirect CP violation

- $K_{L/S}$ are not CP eigenstates

$$|K_{L/S}\rangle = \frac{1}{\sqrt{1+\bar{\epsilon}^2}} \left(|K^0_{\mp}\rangle + \bar{\epsilon}|K^0_{\pm}\rangle\right)$$

- $K_L \to 2\pi$ $(CP = +)$
  - $K^0_+ \to 2\pi$ (indirect CP violation, ϵ or ϵ_K)
  - $K^0_- \to 2\pi$ (direct CP violation, ϵ')

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  - $K^0_+ \to 2\pi$ (indirect CP violation, ϵ or ϵ_K)
  - $K^0_- \to 2\pi$ (direct CP violation, ϵ')
- Experimental measurement

$$\frac{A(K_L \to \pi^+\pi^-)}{A(K_S \to \pi^+\pi^-)} \equiv \eta_{+-} \equiv \epsilon + \epsilon'$$

$$\frac{A(K_L \to \pi^0\pi^0)}{A(K_S \to \pi^0\pi^0)} \equiv \eta_{00} \equiv \epsilon - 2\epsilon'$$

- Using $|\eta_{+-}|$ and $|\eta_{00}|$ as input, PDG quotes

$$|\epsilon| \approx \frac{1}{3}\left(2|\eta_{+-}| + |\eta_{00}|\right) = 2.228(11)\times 10^{-3}, \quad \mathrm{Re}[\epsilon'/\epsilon] \approx \frac{1}{3}\left(1 - \frac{|\eta_{00}|}{|\eta_{+-}|}\right) = 1.66(23)\times 10^{-3}$$

ϵ' is 1000 times smaller than the indirect CP violation ϵ

Thus direct CP violation ϵ' is very sensitive to New Physics

$K \to \pi\pi$ decays and CP violation

- Theoretically, Kaon decays into the isospin $I = 2$ and 0 $\pi\pi$ states

$$\Delta I = 3/2 \text{ transition:} \quad \langle \pi\pi(I=2)|H_W|K^0\rangle = A_2 e^{i\delta_2}$$
$$\Delta I = 1/2 \text{ transition:} \quad \langle \pi\pi(I=0)|H_W|K^0\rangle = A_0 e^{i\delta_0}$$

- If CP symmetry were protected $\Rightarrow$ A_2 and A_0 are real amplitudes

- ϵ and ϵ' depend on the $K \to \pi\pi(I)$ amplitudes A_I

$$\epsilon = \bar{\epsilon} + i\left(\frac{\mathrm{Im}[A_0]}{\mathrm{Re}[A_0]}\right)$$
$$\epsilon' = \frac{ie^{i(\delta_2-\delta_0)}}{\sqrt{2}}\frac{\mathrm{Re}[A_2]}{\mathrm{Re}[A_0]}\left(\frac{\mathrm{Im}[A_2]}{\mathrm{Re}[A_2]} - \frac{\mathrm{Im}[A_0]}{\mathrm{Re}[A_0]}\right)$$

The target for lattice QCD is to calculate both amplitude A_2 and A_0

Results for $K \to \pi\pi(I = 2)$

Results for A_2 [RBC-UKQCD, PRD91 (2015) 074502]

- Use two ensembles (both at $m_\pi = 135$ MeV) for continuum extrapolation

$$48^3 \times 96, \quad a = 0.11 \text{ fm}, \quad L = 5.4 \text{ fm}, \quad N_{\mathrm{conf}} = 76$$
$$64^3 \times 128, \quad a = 0.084 \text{ fm}, \quad L = 5.4 \text{ fm}, \quad N_{\mathrm{conf}} = 40$$

- After continuum extrapolation:

$$\mathrm{Re}[A_2] = 1.50(4)_{\mathrm{stat}}(14)_{\mathrm{syst}} \times 10^{-8} \text{ GeV}$$
$$\mathrm{Im}[A_2] = -6.99(20)_{\mathrm{stat}}(84)_{\mathrm{syst}} \times 10^{-13} \text{ GeV}$$

- Experimental measurement:

$$\mathrm{Re}[A_2] = 1.479(3) \times 10^{-8} \text{ GeV}$$
$$\mathrm{Im}[A_2] \text{ is unknown}$$

- Scattering phase at $E_{\pi\pi} = M_K$

$$\delta_2 = -11.6(2.5)(1.2)^\circ$$

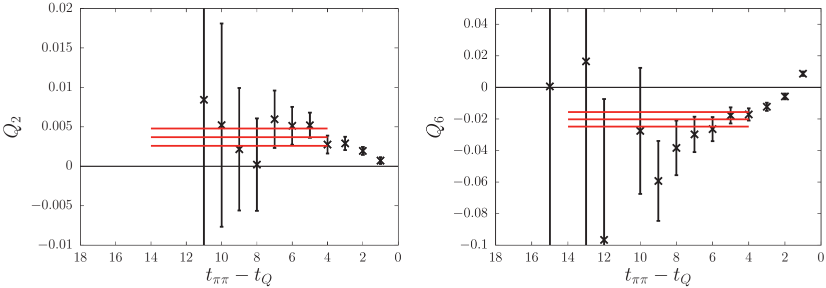
consistent with phenomenological analysis [Schenk, NPB363 (1991) 97]

Results for $K \to \pi\pi(I = 0)$

Results for A_0 [RBC-UKQCD, PRL115 (2015) 212001]

- Use a $32^3 \times 64$ ensemble, $N_{\mathrm{conf}} = 216$, $a = 0.14$ fm, $L = 4.53$ fm

  $M_\pi = 143.1(2.0)$ MeV, $M_K = 490(2.2)$ MeV, $E_{\pi\pi} = 498(11)$ MeV
- G-boundary condition is used: non-trivial to tune the volume $\Rightarrow M_K = E_{\pi\pi}$
- The largest contributions to $\mathrm{Re}[A_0]$ and $\mathrm{Im}[A_0]$ come from Q_2 (current-current) and Q_6 (QCD penguin) operator



- Scattering phase at $E_{\pi\pi} = M_K$: $\delta_0 = 23.8(4.9)(1.2)^\circ$
  - somewhat smaller than phenomenological expectation $\delta_0 = 38.0(1.3)^\circ$

[Courtesy of G. Colangelo]

Results for $\mathrm{Re}[A_0]$, $\mathrm{Im}[A_0]$ and $\mathrm{Re}[\epsilon'/\epsilon]$

[RBC-UKQCD, PRL115 (2015) 212001]

- Determine the $K \to \pi\pi(I = 0)$ amplitude A_0
  - Lattice results

$$\mathrm{Re}[A_0] = 4.66(1.00)_{\mathrm{stat}}(1.26)_{\mathrm{syst}} \times 10^{-7}\ \mathrm{GeV}$$
$$\mathrm{Im}[A_0] = -1.90(1.23)_{\mathrm{stat}}(1.08)_{\mathrm{syst}} \times 10^{-11}\ \mathrm{GeV}$$

  - Experimental measurement

$$\mathrm{Re}[A_0] = 3.3201(18) \times 10^{-7}\ \mathrm{GeV}$$
$$\mathrm{Im}[A_0]\ \text{is unknown}$$

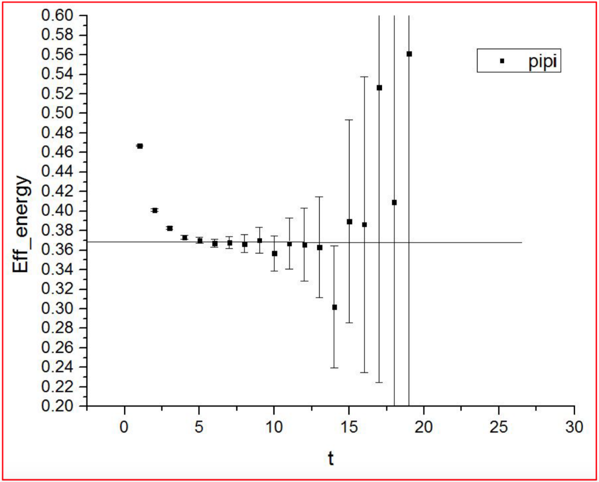
- Determine the direct CP violation $\mathrm{Re}[\epsilon'/\epsilon]$

$$\mathrm{Re}[\epsilon'/\epsilon] = 0.14(52)_{\mathrm{stat}}(46)_{\mathrm{syst}} \times 10^{-3} \quad \text{Lattice}$$
$$\mathrm{Re}[\epsilon'/\epsilon] = 1.66(23) \times 10^{-3} \quad \text{Experiment}$$

2.1 σ deviation $\Rightarrow$ require more accurate lattice results

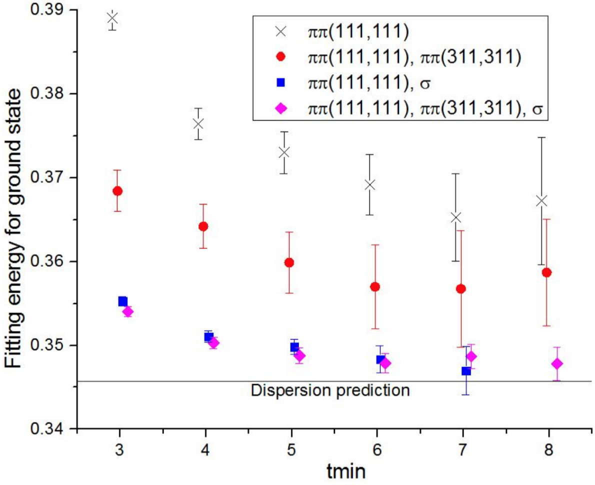
Add more statistics

- Statistics increased: 216 → 1400 configs
  - $\pi\pi$-$\pi\pi$ correlator well-described by a single $\pi\pi$ state
  - $\delta_0 = 23.8(4.9)(1.2)^\circ \rightarrow 19.1(2.5)(1.2)^\circ$??
    $\chi^2/DoF = 1.6$



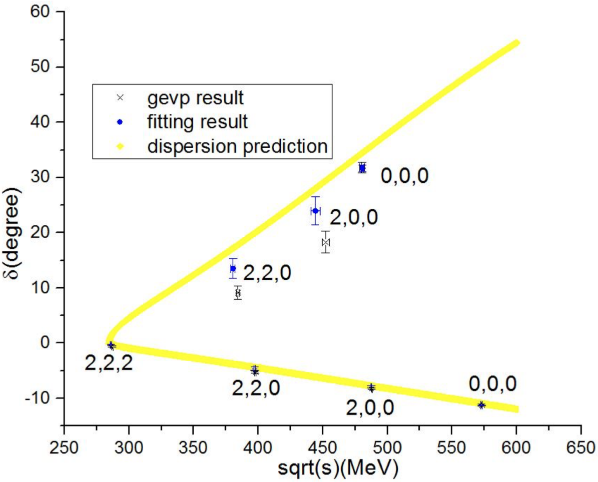
Reported by T. Wang at Lattice 2019

- Systematic effects reduced: using S-wave $\pi\pi(111)$, $\pi\pi(311)$, σ



Helpful to introduce the σ operator

Comparison with Colangelo et. al's dispersion analysis



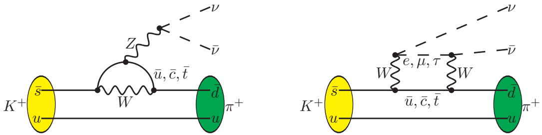
Understanding $I = 0$ $\pi\pi$ system is crucial:

- $E_{\pi\pi}$ needed for extraction of $K \to \pi\pi$ matrix element
- δ enters Lellouch-Luscher finite-volume correction to matrix element

Effect on $K \to \pi\pi$ as yet undetermined, awaiting final $\pi\pi$ numbers

Rare Kaon Decays

$K^+ \to \pi^+ \nu\bar{\nu}$: Experiment vs Standard model



Z-exchange   *W*-*W*

$K^+ \to \pi^+ \nu\bar{\nu}$: largest contribution from top quark loop, thus theoretically clean

$$\mathcal{H}_{eff} \sim \frac{G_F}{\sqrt{2}} \cdot \underbrace{\frac{\alpha_{\rm EM}}{2\pi \sin^2\theta_W} \lambda_t X_t(x_t)}_{\mathcal{N} \sim 2\times 10^{-5}} \cdot (\bar{s}d)_{V-A} (\bar{\nu}\nu)_{V-A}$$

Probe the new physics at scales of $\mathcal{N}^{-\frac{1}{2}} M_W = O(10\ \text{TeV})$

Past experimental measurement is 2 times larger than SM prediction

$$\text{Br}(K^+ \to \pi^+ \nu\bar{\nu})_{\text{exp}} = 1.73^{+1.15}_{-1.05} \times 10^{-10} \qquad \text{[BNL E949, '08]}$$

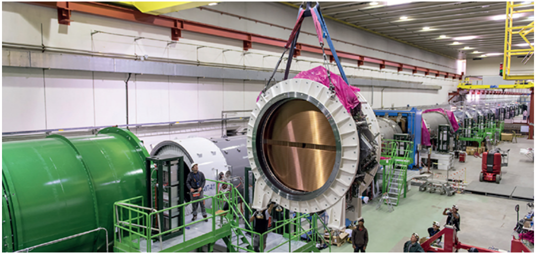
$$\text{Br}(K^+ \to \pi^+ \nu\bar{\nu})_{\text{SM}} = 9.11 \pm 0.72 \times 10^{-11} \qquad \text{[Buras et. al., '15]}$$

but still consistent with $> 60\%$ exp. error

New experiments

New generation of experiment: NA62 at CERN

- aims at observation of $O(100)$ events
- 10%-precision measurement of $\mathrm{Br}(K^+ \to \pi^+ \nu \bar{\nu})$



Recent results [NA62, PLB791 (2019) 156]

- Detector installation completed in 09.2016
- Took data in 2016, 2017, 2018
- Full 2016 data $\to 1.21 \times 10^{11}$ K^+ decays collected $\to \mathrm{Br} < 14 \times 10^{-10}$

$2^{\rm nd}$-order weak interaction and bilocal matrix element

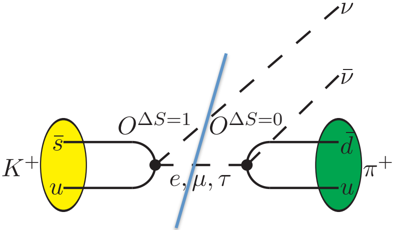
Hadronic matrix element for the $2^{\rm nd}$-order weak interaction

$$\int_{-T}^{T} dt\, \langle \pi^+\nu\bar{\nu}|T\left[Q_A(t)Q_B(0)\right]|K^+\rangle$$
$$= \sum_n \left\{ \frac{\langle \pi^+\nu\bar{\nu}|Q_A|n\rangle\langle n|Q_B|K^+\rangle}{M_K - E_n} + \frac{\langle \pi^+\nu\bar{\nu}|Q_B|n\rangle\langle n|Q_A|K^+\rangle}{M_K - E_n} \right\} \left(1 - e^{(M_K-E_n)T}\right)$$

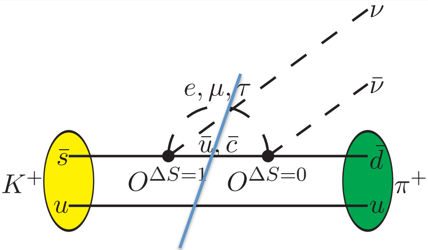
- For $E_n > M_K$, the exponential terms exponentially vanish at large T
- For $E_n < M_K$, the exponentially growing terms must be removed
- $\sum_n$: principal part of the integral replaced by finite-volume summation
    - possible large finite volume correction when $E_n \to M_K$

[Christ, XF, Martinelli, Sachrajda, PRD 91 (2015) 114510]

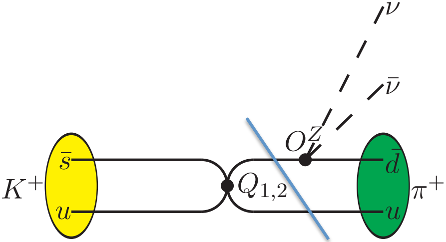
Low lying intermediate states



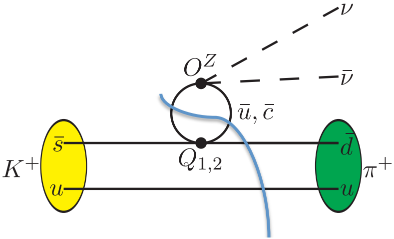
$K^+ \to \ell^+\nu \quad \& \quad \ell^+ \to \pi^+\bar{\nu}$



$K^+ \to \pi^0\ell^+\nu \quad \& \quad \pi^0\ell^+ \to \pi^+\bar{\nu}$



$K^+ \xrightarrow{H_W} \pi^+ \quad \& \quad \pi^+ \xrightarrow{V_\mu} \pi^+$



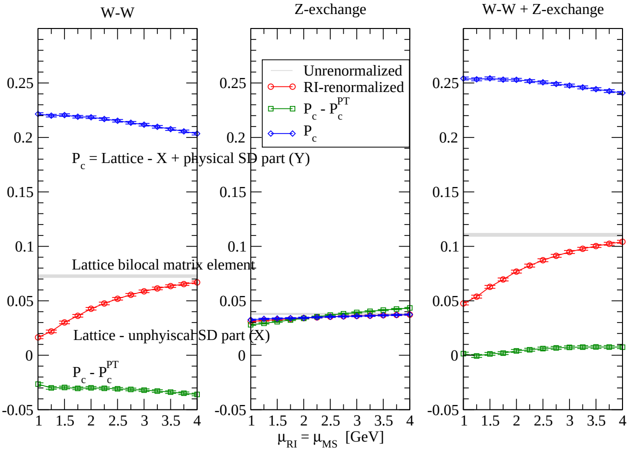
$K^+ \xrightarrow{H_W} \pi^+\pi^0 \quad \& \quad \pi^+\pi^0 \xrightarrow{A_\mu} \pi^+$

Lattice results

First results @ $m_\pi = 420$ MeV, $m_c = 860$ MeV

[Bai, Christ, XF, Lawson, Portelli, Schrajda, PRL 118 (2017) 252001]

$$P_c = 0.2529(\pm 13)_{\rm stat}(\pm 32)_{\rm scale}(-45)_{\rm FV}$$



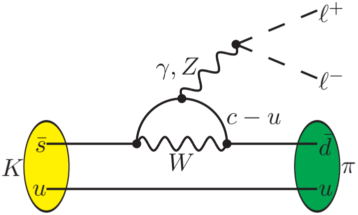
Lattice QCD is now capable of first-principles calculation of rare kaon decay

- The remaining task is to control various systematic effects

$K \to \pi\ell^+\ell^-$: CP conserving chanel

CP conserving decay: $K^+ \to \pi^+\ell^+\ell^-$ and $K_S \to \pi^0\ell^+\ell^-$

- Involve both γ- and Z-exchange diagram, but γ-exchange is much larger



- Unlike Z-exchange, the γ-exchange diagram is LD dominated
  - By power counting, loop integral is quadratically UV divergent
  - EM gauge invariance reduces divergence to logarithmic
  - $c - u$ GIM cancellation further reduces log divergence to be UV finite

Lattice calculation strategy

Focus on γ-exchange

- Hadronic part of decay amplitude is described by a form factor

$$\begin{aligned} T^{\mu}_{+,S}(p_K, p_\pi) &= \int d^4x\, e^{iqx} \langle \pi(p_\pi) | T\{J^{\mu}_{em}(x) \mathcal{H}^{\Delta S=1}(0)\} | K^+/K_S(p_K)\rangle \\ &= \frac{G_F M_K^2}{(4\pi)^2} V_{+,S}(z) \left[z(p_K + p_\pi)^\mu - (1 - r_\pi^2) q^\mu \right] \end{aligned}$$

with $q = p_K - p_\pi$, $z = q^2/M_K^2$, $r_\pi = M_\pi/M_K$

The target for lattice QCD is to calculate the form factor $V_{+,S}(z)$

First exploratory calculation on $K^+ \to \pi^+ \ell^+ \ell^-$

[Christ, XF, Jüttner, et. al. PRD94 (2016) 114516]

Use $24^3 \times 64$ ensemble, $N_{\text{conf}} = 128$

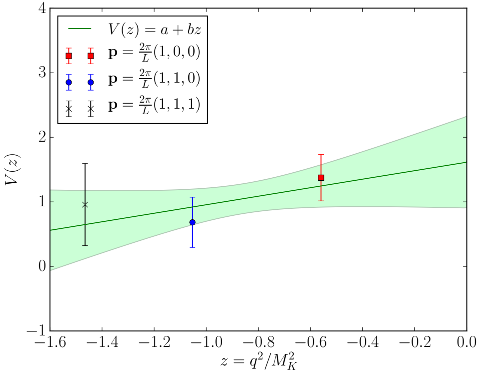
$a^{-1} = 1.78$ GeV, $m_\pi = 430$ MeV

$m_K = 625$ MeV, $m_c = 530$ MeV

Momentum dependence of $V_+(z)$

$$V_+(z) = a_+ + b_+ z$$

$$\Rightarrow \quad a_+ = 1.6(7), \; b_+ = 0.7(8)$$



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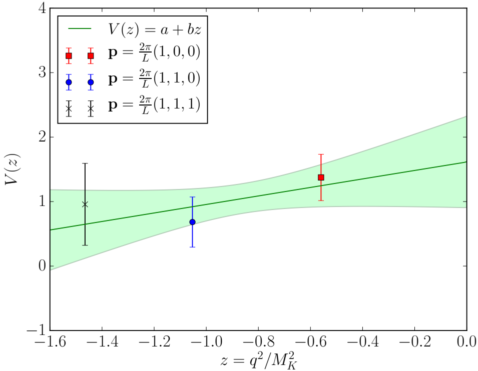


$K^+ \to \pi^+ e^+ e^-$ data + phenomenological analysis: $a_+ = -0.58(2)$, $b_+ = -0.78(7)$

[Cirigliano, et. al., Rev. Mod. Phys. 84 (2012) 399]

$$V_j(z) = a_j + b_j z + \underbrace{\frac{\alpha_j r_\pi^2 + \beta_j (z - z_0)}{G_F M_K^2 r_\pi^4}}_{K\to\pi\pi\pi} \underbrace{\left[1 + \frac{z}{r_V^2}\right]}_{F_V(z)} \underbrace{\left[\phi(z/r_\pi^2) + \frac{1}{6}\right]}_{\text{loop}}, \quad j = +, S$$

- Experimental data only provide $\frac{d\Gamma}{dz} \Rightarrow$ square of form factor $|V_+(z)|^2$
- Need phenomenological knowledge to determine the sign for a_+, b_+

Conclusion

- For quantities such as f_K/f_π, $f_+(0)$, f_D, $\cdots$

	N_f	FLAG average	Frac. Err.
f_K/f_π	$2+1+1$	1.1932(19)	0.16%
$f_+(0)$	$2+1+1$	0.9706(27)	0.28%
f_D	$2+1+1$	212.0(7) MeV	0.33%
f_{D_s}	$2+1+1$	249.9(5) MeV	0.20%
f_{D_s}/f_D	$2+1+1$	1.1783(16)	0.13%
f_B	$2+1+1$	190.0(1.3) MeV	0.68%
f_{B_s}	$2+1+1$	230.3(1.3) MeV	0.56%
f_{B_s}/f_B	$2+1+1$	1.209(5)	0.41%

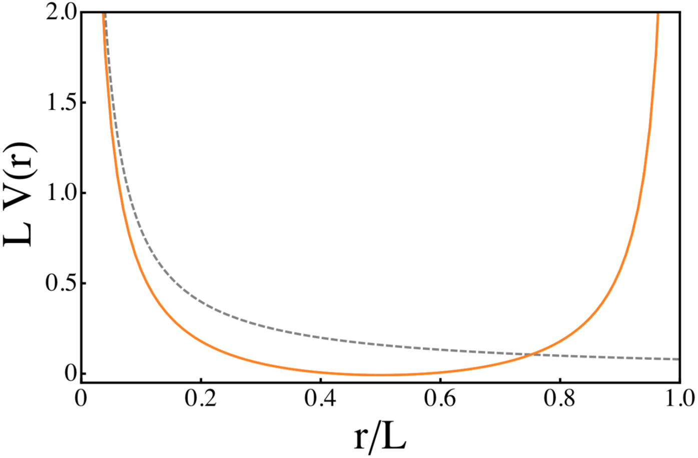
lattice QCD calculations play important role in precision flavor physics

- Horizon expanded to
  - EM corrections to leptonic decays
  - $K \to \pi\pi$ and ϵ'
  - rare kaon decays: $K \to \pi\nu\bar{\nu}$ and $K \to \pi\ell^+\ell^-$
- Develop techniques for the new horizon

Remove zero mode - QED_L

Infinite volume propagator $\Rightarrow$ finite-volume propagator

$$S_\infty(x) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{ipx}}{p^2} = \frac{1}{4\pi^2 x^2} \quad \Rightarrow \quad S_L(x) = \frac{1}{VT}\sum_p{}' \frac{e^{ipx}}{p^2}, \quad p = \frac{2\pi}{L} n \neq 0$$

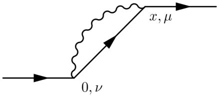


[Davoudi, Savage, PRD90 (2014) 054503]

Power-law ($1/L^n$) finite volume effect as lattice volume L increase

XF, Luchang Jin [arXiv:1812.09817]

QED self energy



- We start with infinite volume

$$\mathcal{I} = \frac{1}{2}\int d^4x\, \mathcal{H}_{\mu,\nu}(x) S^{\gamma}_{\mu,\nu}(x)$$

- Propose to replace $\mathcal{H}_{\mu,\nu}(x)$ by $\mathcal{H}^{\rm lat}_{\mu,\nu}(x)$
  - $\mathcal{H}^{\rm lat}_{\mu,\nu}(x)$ mainly differs from $\mathcal{H}_{\mu,\nu}(x)$ at $x \sim L$
- The hadronic part $\mathcal{H}_{\mu,\nu}(x)$ is given by

$$\mathcal{H}_{\mu,\nu}(x) = \mathcal{H}_{\mu,\nu}(t,\vec{x}) = \langle P|T[J_\mu(t,\vec{x})J_\nu(0)]|P\rangle$$

  - $J_\mu(t,\vec{x})J_\nu(0) \quad \to \quad e^{-M\sqrt{t^2+\vec{x}}}$
  - $\langle P|J_\mu(t,\vec{x}) \quad \to \quad e^{Mt}$

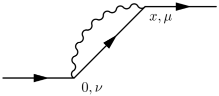
For small $|t|$, we have exponentially suppressed FV effects:

$$\mathcal{H}_{\mu,\nu}(t,\vec{x}) \sim e^{-M\left(\sqrt{t^2+\vec{x}^2}-t\right)} \sim e^{-M|\vec{x}|} \quad \Rightarrow \quad \int d^3\vec{x}\, \mathcal{H}_{\mu,\nu}(x) - \mathcal{H}^{\rm lat}_{\mu,\nu}(x) \sim e^{-ML}$$

Infinite volume reconstruction method

XF, Luchang Jin [arXiv:1812.09817]

QED self energy



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Infinite volume reconstruction method

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For large $|t|$, we shall have:

$$\mathcal{H}_{\mu,\nu}(t,\vec{x}) \sim e^{-M\left(\sqrt{t^2+\vec{x}^2}-t\right)} \sim e^{-M\frac{\vec{x}^2}{2t}} \sim O(1)$$

Realizing at large $t > t_s$ we have ground state dominance:

$$\langle P|J_\mu(t,\vec{x})J_\nu(0)|P\rangle \sim \int \frac{d^3\vec{k}}{(2\pi)^3}\langle P|J_\mu(0)|P(\vec{k})\rangle\langle P(\vec{k})|J_\nu(0)|P\rangle e^{-E_{\vec{k}}t+Mt}e^{-i\vec{k}\cdot\vec{x}}$$

- Reconstruct $\mathcal{H}_{\mu,\nu}(t,\vec{x})$ at large t using $\mathcal{H}_{\mu,\nu}(t_s,\vec{x})$ at modest t_s

$$\mathcal{H}_{\mu,\nu}(t,\vec{x}') \approx \int d^3\vec{x}\,\mathcal{H}_{\mu,\nu}(t_s,\vec{x}) \int \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} e^{-(E_{\vec{k}}-M)(t-t_s)} e^{-i\vec{p}\cdot\vec{x}'}$$

Replace

$$\mathcal{H}_{\mu,\nu}(t,\vec{x}) \quad \Leftarrow \quad \mathcal{H}_{\mu,\nu}(t_s,\vec{x}) \quad \Leftarrow \quad \mathcal{H}^{\mathrm{lat}}_{\mu,\nu}(t_s,\vec{x})$$

The replacement only amounts for exponentially suppressed FV effects